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Some Congruences for the Coefficients of Rogers–Ramanujan Type Identities

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Abstract: We examine a few mathematical characteristics of Rogers–Ramanujan type identities as a follow-up work. Recently authors interpreted Rogers–Ramanujan type identities combinatorially using signed color partitions. In the present study, we discovered several congruences for the coefficients of powers of q that are in arithmetic progressions modulo powers of 2 and 3.

Keywords: combinatorics; combinatorial congruences; Rogers–Ramanujan type identities

MSC: 05A17; 11P83; 11P84; 03E05

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Citation: Gupta, V.; Rana, M. Some Congruences for the Coefficients of Rogers–Ramanujan Type Identities. *Mathematics* **2022**, *10*, 3582. https://doi.org/10.3390/math10193582

Academic Editor: Abdelmejid Bayad

Received: 7 September 2022
Accepted: 25 September 2022
Published: 1 October 2022

1. Introduction

The Rogers–Ramanujan identities (RRI) are the two most well-known q-series identities that have impacted studies in many branches of mathematics and science, and are given as

$$\sum_{n=0}^{\infty} \frac{q^{n^2}}{(q;q)_n} = \prod_{n=0}^{\infty} (1-q^{5n-1})^{-1}(1-q^{5n-4})^{-1}, \tag{1}$$

$$\sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(q;q)_n} = \prod_{n=0}^{\infty} (1-q^{5n-2})^{-1}(1-q^{5n-3})^{-1}. \tag{2}$$

where

$$(a;q)_n = (1-a)(1-aq)(1-aq^2)\cdots(1-aq^{n-1}),\ \forall\ n \geq 1,$$

$$(a;q)_\infty = \lim_{n\to\infty}(a;q)_n,\ |q| < 1.$$

These identities were first discovered by Rogers [1] but were appreciated only after Ramanujan rediscovered these sometime before 1913. Despite being over a century old, the Rogers–Ramanujan identities are still the focus of ongoing research. RRI played a major role in algebraic characters [2], partition theory [3], and statistical mechanics [4]. Some of the useful texts on the history of these identities are found in Refs. [3,5–7]. These identities are of the form 'Sum=Product'; therefore, they are sometimes called sum-product identities. MacMahon [8] provided the partition–theoretic interpretations of the RRI given by (1) and (2) as:

Theorem 1. *The number of partitions of n into parts with minimal difference* 2 *equals the number of partitions of n into parts which are congruent to* ± 1 (mod 5).

Theorem 2. *The number of partitions of n with minimal part* 2 *and minimal difference* 2 *equals the number of partitions of n into parts which are congruent to* ± 2 (mod 5).

Recently, P. Afsharijoo [9] added a new companion to the Rogers–Ramanujan identities. This new companion counts partitions with different types of constraints on even and odd parts. Bailey [10] systematically explored the Rogers–Ramanujan type identities (RRTIs). Additionally, a list of 130 identities of the Rogers–Ramanujan type identities was provided by Slater [11]. Furthermore, Chu and Zhang [12] found many RRTIs using certain transformations. The hard hexagon model in statistical mechanics, a specific instance of a solvable family of hard-square-type models, naturally incorporated many Rogers–Ramanujan type identities. Baxter [13] explained that a number of Rogers–Ramanujan type identities occur in the determination of sub-lattice densities and order parameters. Kedem et al. [14] believed that the Rogers–Ramanujan identities represent the partition function of a physical system with quasiparticles that adhered to specific exclusion statistics. The relationship between RRI and fractional statistics is developed by these exclusion statistics, which are related to fractional statistics. Furthermore, the combinatorial interpretations of many RRTIs were studied using different combinatorial tools, and are available in Refs. [15–18]. Recently, we have found the combinatorial interpretations of many RRTIs, some of which are listed in Tables 1–3, using signed partitions (for signed partitions, readers are referred to Ref. [19]). Additionally, many mathematicians were interested in finding the arithmetic properties of some restricted partition functions [20,21].

The purpose of this paper is to explore the congruences for RRTI, as given in Section 3, Tables 1–3. We have arranged 17 Rogers–Ramanujan type identities into three groups: Group 1 contains 10 RRTIs, which are listed in Table 1; Group 2 contains 3 identities, which are listed in Table 2; and Group 3 contains 4 identities, which are listed in Table 3.

2. Preliminaries

We require the following definitions and lemmas to prove the main results in the next section. For $|ab| < 1$, Ramanujan's general theta function $f(a,b)$ is defined as

$$f(a,b) = \sum_{m=-\infty}^{\infty} a^{\frac{m(m+1)}{2}} b^{\frac{m(m-1)}{2}}. \tag{3}$$

Using Jacobi's triple-product identity [22] (entry 19, p. 35), (3) becomes

$$f(a,b) = (-a;ab)_\infty(-a;ab)_\infty(ab;ab)_\infty. \tag{4}$$

The special cases of $f(a,b)$ are

$$\varphi(q) = f(q;q) = 1 + 2\sum_{m=1}^{\infty} q^{m^2} = (-q;q^2)^2_\infty(q^2;q^2)_\infty = \frac{f_2^5}{f_1^2 f_4^2}, \tag{5}$$

$$\psi(q) = f(q;q^3) = \sum_{m=1}^{\infty} q^{\frac{m(m+1)}{2}} = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty} = \frac{f_2^2}{f_1}. \tag{6}$$

In some of the proofs, we also employ Jacobi's identity from Ref. [23] as Equation (1.7.1):

$$f_1^3 = \sum_{n=0}^{\infty}(-1)^n(2n+1)q^{n(n+1)/2}. \tag{7}$$

Lemma 1. *We have*

$$\frac{1}{f_1^2} = \frac{f_8^5}{f_2^5 f_{16}^2} + 2q\frac{f_4^2 f_{16}^2}{f_2^5 f_8}, \tag{8}$$

$$f_1^2 = \frac{f_2 f_8^5}{f_4^2 f_{16}^2} - 2q\frac{f_2 f_{16}^2}{f_8}, \tag{9}$$

$$\frac{1}{f_1^4} = \frac{f_4^{14}}{f_2^{14} f_8^4} + 4q\frac{f_4^2 f_8^4}{f_2^{10}}, \tag{10}$$

$$f_1^4 = \frac{f_4^{10}}{f_2^2 f_8^4} - 4q\frac{f_2^2 f_8^4}{f_4^2}, \tag{11}$$

$$\frac{1}{f_1 f_3} = \frac{f_8^2 f_{12}^5}{f_2^2 f_4 f_6^4 f_{24}^2} + q\frac{f_4^5 f_{24}^2}{f_2^4 f_6^2 f_8^2 f_{12}}, \tag{12}$$

$$\frac{f_1^3}{f_3} = \frac{f_4^3}{f_{12}} - 3q\frac{f_2^2 f_{12}^3}{f_4 f_6^2}, \tag{13}$$

$$\frac{f_3}{f_1^3} = \frac{f_4^6 f_6^3}{f_2^9 f_{12}^2} + 3q\frac{f_4^2 f_6 f_{12}^2}{f_2^7}, \tag{14}$$

$$\frac{f_3^3}{f_1} = \frac{f_4^3 f_6^2}{f_2^2 f_{12}} + q\frac{f_{12}^3}{f_4}, \tag{15}$$

$$\frac{f_3^2}{f_1^2} = \frac{f_4^4 f_6 f_{12}^2}{f_2^5 f_8 f_{24}} + 2q\frac{f_4 f_6^2 f_8 f_{24}}{f_2^4 f_{12}}, \tag{16}$$

$$\frac{f_3}{f_1} = \frac{f_4 f_6 f_{16} f_{24}^2}{f_2^2 f_8 f_{12} f_{48}} + q\frac{f_6 f_8^2 f_{48}}{f_2^2 f_{16} f_{24}}. \tag{17}$$

Proof. Using the two-dissection of $\varphi(q)$ and $\varphi(q^2)$ from (Ref. [23], Equation (1.9.4) and (1.10.1)), we obtain (8) and (10). On replacing q by $-q$ in (5), we obtain (9) and (11). Furthermore, (12), (13), (15), (16), and (17) are Equations (30.12.3), (22.1.13), (22.1.14), (30.10.4), and (30.10.3), respectively, in Ref. [23]. Next, (14) follows from (13) by using q instead of $-q$. □

Lemma 2. *We have*

$$\frac{f_4}{f_1} = \frac{f_{12} f_{18}^4}{f_3^3 f_{36}^2} + q\frac{f_6^2 f_9^3 f_{36}}{f_3^4 f_{18}^2} + 2q^2\frac{f_6 f_{18} f_{36}}{f_3^3}, \tag{18}$$

$$\frac{f_1^2}{f_2} = \frac{f_9^2}{f_{18}} - 2q\frac{f_3 f_{18}^2}{f_6 f_9}, \tag{19}$$

$$\frac{f_2}{f_1^2} = \frac{f_6^4 f_9^6}{f_3^8 f_{18}^3} + 2q\frac{f_6^3 f_9^3}{f_3^7} + 4q^2\frac{f_6^2 f_{18}^3}{f_3^6}, \tag{20}$$

$$f_1^3 = f_3 c(q^3) - 3q f_9^3. \tag{21}$$

where

$$c(q) = \sum_{m,n=-\infty}^{\infty} q^{m^2+mn+n^2}.$$

Proof. The first identity follows from Equations (33.2.1) and (33.2.5) in Ref. [23]. The second identity is equivalent to the three-dissection of $\varphi(-q)$ (see Ref. [23] Equation (14.3.2)). We obtained (20) by replacing q with ωq and $\omega^2 q$ and multiplying the two results, where ω is the primitive cube root of unity. □

The three-dissection of $\psi(q)$ follows as:

Lemma 3. *We have*

$$\psi(q) = \frac{f_2^2}{f_1} = \frac{f_6 f_9^2}{f_3 f_{18}} + q\frac{f_{18}^2}{f_9}. \tag{22}$$

Proof. Identity (22) is Equation (14.3.3) of Ref. [23]. □

Lemma 4. *In Ref. [24], for any prime* $p \geq 5$,

$$f_1 = \sum_{\substack{k=\frac{-(p-1)}{2} \\ k \neq (\pm p-1)/6}}^{\frac{(p-1)}{2}} (-1)^k q^{\frac{3k^2+k}{2}} f(-q^{\frac{(3p^2+(6k+1)p)}{2}}, -q^{\frac{(3p^2-(6k+1)p)}{2}}) + (-1)^{\frac{(\pm p-1)}{6}} q^{\frac{(p^2-1)}{24}} f_{p^2}, \quad (23)$$

where

$$\frac{\pm p-1}{6} = \begin{cases} \frac{(p-1)}{6} & \text{if } p \equiv \ \ 1 \pmod 6, \\ \frac{(-p-1)}{6} & \text{if } p \equiv -1 \pmod 6. \end{cases} \quad (24)$$

If $\frac{-p-1}{2} \leq k \leq \frac{p-1}{2}$ *and* $k \neq \frac{\pm p-1}{2}$, *then* $\frac{3k^2+k}{2} \not\equiv \frac{p^2-1}{24} \pmod p$.

3. Main Results

In Tables 1–3, the sum sides of RRTIs are the generators for the partitions written in the second column, and the product sides of the RRTIs are written in the third column.

Group 1

We now present 10 RRTIs in this group from Ref. [12] with identity nos. 8, 9, 10, 33, 45, 70, 98, 104, 111, and 112, as shown below.

Table 1. Rogers–Ramanujan type identities.

Function	Sum Side	=	Product Side
$A_1(q)$	$\sum_{m=0}^{\infty} \frac{(-q;q)_{m+1} q^{\frac{m(m+1)}{2}}}{(q;q)_m}$	=	$\frac{(-q;q)_\infty}{(q;q)_\infty}[q^4, q^2, q^2; q^4]_\infty$
$A_2(q)$	$\sum_{m=0}^{\infty} \frac{(-1;q)_{2m} q^m}{(q;q)_{2m}}$	=	$\frac{(-q;q)_\infty}{(q;q)_\infty}[q^4, -q^2, -q^2; q^4]_\infty$
$A_3(q)$	$\sum_{m=0}^{\infty} \frac{(-q;q)_{2m} q^m}{(q;q)_{2m+1}}$	=	$\frac{(-q;q)_\infty}{(q;q)_\infty}[q^4, -q^4, -q^4; q^4]_\infty$
$A_4(q)$	$\sum_{m=0}^{\infty} \frac{(q;q^2)_m^2 q^{2m^2}}{(q^2;q^2)_{2m}}$	=	$\frac{[q^6, q^3, q^3; q^6]_\infty}{(q^2;q^2)_\infty}$
$A_5(q)$	$\sum_{m=0}^{\infty} \frac{(-1;q^2)_m q^{m(m+1)}}{(q;q)_{2m}}$	=	$\frac{(-q^2;q^2)_\infty}{(q^2;q^2)_\infty}[q^6, -q^3, -q^3; q^6]_\infty$
$A_6(q)$	$\sum_{m=0}^{\infty} \frac{(-1;q^2)_m q^{\frac{m(m+1)}{2}}}{(q;q)_m (q;q^2)_m}$	=	$\frac{(-q;q)_\infty}{(q;q)_\infty}[q^8, q^4, q^4; q^8]_\infty$
$A_7(q)$	$\sum_{m=0}^{\infty} \frac{(-1;q)_m q^{\frac{m(m+1)}{2}}}{(q;q^2)_m (q;q)_m}$	=	$\frac{(-q;q)_\infty}{(q;q)_\infty}[q^{10}, q^5, q^5; q^{10}]_\infty$
$A_8(q)$	$\sum_{m=0}^{\infty} \frac{(-q;q^2)_m q^{m(m+1)}}{(q;q^2)_{2m+1}(q^2;q^2)_m}$	=	$\frac{[q^{12}, q^4, q^8; q^{12}]_\infty}{(q;q)_\infty}$
$A_9(q)$	$\sum_{m=0}^{\infty} \frac{(-q;q^2)_m q^m}{(q;q)_{2m+1}}$	=	$\frac{(-q;q)_\infty}{(q;q)_\infty}[q^{12}, q^3, q^9; q^{12}]_\infty$
$A_{10}(q)$	$\sum_{m=0}^{\infty} \frac{(-1;q^2)_m q^m}{(q;q)_{2m}}$	=	$\frac{(-q;q)_\infty}{(q;q)_\infty}[q^{12}, q^6, q^6; q^{12}]_\infty$

Throughout the remainder of this paper, we use

$$f_k = (q^k; q^k)_\infty,$$

for positive integer k.

From the binomial theorem, we have

$$f_1^{2^k} \equiv f_2^{2^{k-1}} \pmod{2^k}. \tag{25}$$

Before stating the main results, we define

$$A_i(q) = \sum_{m=0}^{\infty} a_i(m) q^m.$$

Theorem 3. *For $m \geq 0$, we have*

$$a_1(4m+2) \equiv 0 \pmod 2, \tag{26}$$
$$a_1(4m+3) \equiv 0 \pmod 4, \tag{27}$$
$$a_1(8m+5) \equiv 0 \pmod 8, \tag{28}$$
$$a_1(8m+6) \equiv 0 \pmod 4, \tag{29}$$
$$a_1(8m+7) \equiv 0 \pmod{16}, \tag{30}$$
$$a_1(16m+t) \equiv 0 \pmod 4, \textit{for } t \in \{10, 12\} \tag{31}$$
$$a_1(32m+20) \equiv 0 \pmod 4, \tag{32}$$
$$a_1(48m+34) \equiv 0 \pmod 4. \tag{33}$$

Theorem 4. *For $m \geq 0$, we have*

$$a_2(4m+3) \equiv 0 \pmod 4, \tag{34}$$
$$a_2(8m+t) \equiv 0 \pmod 4, \textit{for } t \in \{3, 6\} \tag{35}$$
$$a_2(8m+7) \equiv 0 \pmod 8, \tag{36}$$
$$a_2(16m+10) \equiv 0 \pmod 4. \tag{37}$$

Theorem 5. *For $m \geq 0$, we have*

$$a_3(3m+2) \equiv 0 \pmod 2, \tag{38}$$
$$a_3(4m+2) \equiv 0 \pmod 4, \tag{39}$$
$$a_3(4m+3) \equiv 0 \pmod 8, \tag{40}$$
$$a_3(12m+t) \equiv 0 \pmod 4, \textit{for } t \in \{2, 3, 6, 11\}. \tag{41}$$

Theorem 6. *For $m \geq 0$, we have*

$$a_4(2m+1) \equiv 0 \pmod 2, \tag{42}$$
$$a_4(32m+t) \equiv 0 \pmod 4, \textit{for } t \in \{6, 30\} \tag{43}$$
$$a_4(64m+50) \equiv 0 \pmod 4. \tag{44}$$

Theorem 7. *For $m \geq 0$, we have*

$$a_5(6m+2) \equiv 0 \pmod 2, \tag{45}$$
$$a_5(6m+4) \equiv 0 \pmod 4, \tag{46}$$
$$a_5(18m+12) \equiv 0 \pmod 4, \tag{47}$$
$$a_5(54m+42) \equiv 0 \pmod 4, \tag{48}$$
$$a_5(162m+114) \equiv 0 \pmod 4. \tag{49}$$

Theorem 8. *For $m \geq 0$, we have*

$$a_6(3m+1) \equiv 0 \pmod{2}, \tag{50}$$
$$a_6(3m+2) \equiv 0 \pmod{4}, \tag{51}$$
$$a_6(4m+3) \equiv 0 \pmod{8}, \tag{52}$$
$$a_6(8m+t) \equiv 0 \pmod{4}, \textit{for } t \in \{4,5\} \tag{53}$$
$$a_6(8m+t) \equiv 0 \pmod{16}, \textit{for } t \in \{6,7\} \tag{54}$$
$$a_6(12m+3) \equiv 0 \pmod{8}, \tag{55}$$
$$a_6(16m+8) \equiv 0 \pmod{4}, \tag{56}$$
$$a_6(24m+6) \equiv 0 \pmod{8}, \tag{57}$$
$$a_6(80m+t) \equiv 0 \pmod{4}, \textit{for } t \in \{18,64\}. \tag{58}$$

Theorem 9. *For $m \geq 0$, we have*

$$a_7(6m+4) \equiv a_7(24m+16) \pmod{4}, \tag{59}$$
$$a_7(9m+t) \equiv 0 \pmod{4}, \textit{for } t \in \{3,6\} \tag{60}$$
$$a_7(15m+t) \equiv 0 \pmod{4}, \textit{for } t \in \{2,8,11,14\} \tag{61}$$
$$a_7(12m+t) \equiv 0 \pmod{4}, \textit{for } t \in \{7,10\} \tag{62}$$
$$a_7(24m+13) \equiv 0 \pmod{4}, \tag{63}$$
$$a_7(48m+28) \equiv 0 \pmod{4}. \tag{64}$$

Theorem 10. *For prime $p \geq 5$*

$$a_8\left(3p^2m + 3pi + \frac{p^2-1}{8}\right) \equiv 0 \pmod{2}, \tag{65}$$

where $i = 1, 2, \cdots, (p-1)$.

Theorem 11. *For $m \geq 0$, we have*

$$a_8(9m+1) \equiv a_8(m) \pmod{4}, \tag{66}$$
$$a_8(9m+4) \equiv 0 \pmod{4}, \tag{67}$$
$$a_8(9m+7) \equiv 0 \pmod{4}. \tag{68}$$

Theorem 12. *For $m \geq 0$, we have*

$$a_9(3m+1) \equiv 0 \pmod{2}, \tag{69}$$
$$a_9(3m+2) \equiv 0 \pmod{4}, \tag{70}$$
$$a_9(18m+9) \equiv 0 \pmod{3}, \tag{71}$$
$$a_9(18m+15) \equiv 0 \pmod{3}. \tag{72}$$

Theorem 13. *For $m \geq 0$, we have*

$$a_{10}(4m+3) \equiv 0 \pmod{4}, \tag{73}$$
$$a_{10}(8m+t) \equiv 0 \pmod{4}, \textit{for } t \in \{2,5\}, \tag{74}$$
$$a_{10}(16m+t) \equiv 0 \pmod{4}, \textit{for } t \in \{9,12,14\}, \tag{75}$$
$$a_{10}(32m+20) \equiv 0 \pmod{4}, \tag{76}$$
$$a_{10}(48m+t) \equiv 0 \pmod{4}, \textit{for } t \in \{22,38\}. \tag{77}$$

Group 2

In this group, we have the following RRTIs with identity nos. 1, 36, and 37 in Ref. [12]. These RRTIs have the same congruences.

Table 2. Rogers–Ramanujan type identities.

Function	Sum Side	=	Product Side
$A_{11}(q)$	$\sum_{m=0}^{\infty} \frac{(-1;q)_m q^{m^2}}{(q;q^2)_{2m}}$	=	$\frac{[q^3,-q,-q^2;q^3]_\infty}{(q;q)_\infty}$
$A_{12}(q)$	$\sum_{m=0}^{\infty} \frac{(-1;q)_{2m} q^{m}}{(q^2;q^2)_{m}}$	=	$\frac{(-q;q)_\infty}{(q;q)_\infty}[q^6,q^3,q^3;q^6]_\infty$
$A_{13}(q)$	$\sum_{m=0}^{\infty} \frac{(-1;q)_m q^{m^2}}{(q;q^2)_m (q;q)_m}$	=	$\frac{(-q;q)_\infty}{(q;q)_\infty}[q^6,q^3,q^3;q^6]_\infty$

Theorem 14. *For $m \geq 0$ and $i = 11, 12,$ and 13, we have*

$$a_i(3m+1) \equiv 0 \pmod{2}, \tag{78}$$
$$a_i(3m+2) \equiv 0 \pmod{4}, \tag{79}$$
$$a_i(4m+2) \equiv 0 \pmod{4}, \tag{80}$$
$$a_i(4m+3) \equiv 0 \pmod{6}, \tag{81}$$
$$a_i(6m+5) \equiv 0 \pmod{16}, \tag{82}$$
$$a_i(8m+4) \equiv 0 \pmod{2}, \tag{83}$$
$$a_i(8m+5) \equiv 0 \pmod{4}, \tag{84}$$
$$a_i(8m+t) \equiv 0 \pmod{12}, \textit{for } t \in \{6,7\} \tag{85}$$
$$a_i(24m+14) \equiv 0 \pmod{8}, \tag{86}$$
$$a_i(24m+20) \equiv 0 \pmod{16}, \tag{87}$$
$$a_i(32m+24) \equiv 0 \pmod{8}, \tag{88}$$
$$a_i(40m+t) \equiv 0 \pmod{4}, \textit{for } t \in \{17,33\} \tag{89}$$
$$a_i(40m+t) \equiv 0 \pmod{12}, \textit{for } t \in \{11,19\} \tag{90}$$
$$a_i(64m+40) \equiv 0 \pmod{8}. \tag{91}$$

Group 3

In this group, we use the following RRTIs from Ref. [12] with identity nos. 3, 39, 46, and 103. The identities $A_{14}(q)$, $A_{15}(q)$ and $A_{16}(q)$, $A_{17}(q)$ have the same congruences.

Table 3. Rogers–Ramanujan type identities.

Function	Sum Side	=	Product Side
$A_{14}(q)$	$\sum_{m=0}^{\infty} \frac{(-q;q)_m q^{m(m+1)}}{(q;q)_m (q;q^2)_{m+1}}$	=	$\frac{[q^3,-q^3,-q^3;q^3]_\infty}{(q;q)_\infty}$
$A_{15}(q)$	$\sum_{m=0}^{\infty} \frac{(-q;q)_{2m} q^{m}}{(q^2;q^2)_{m}}$	=	$\frac{(-q;q)_\infty}{(q;q)_\infty}[q^6,q,q^5;q^6]_\infty$
$A_{16}(q)$	$\sum_{m=0}^{\infty} \frac{(-q^2;q^2)_m q^{m(m+1)}}{(q;q)_{2m+1}}$	=	$\frac{(-q^2;q^2)_\infty}{(q^2;q^2)_\infty}[q^6,-q,-q^5;q^6]_\infty$
$A_{17}(q)$	$\sum_{m=0}^{\infty} \frac{(-q^2;q^2)_m q^{m(m+1)}}{(q;q)_{2m+1}}$	=	$\frac{[q^6,-q,-q^5;q^6]_\infty}{(q;q)_\infty}$

Theorem 15. *For $m \geq 0$ and $i = 14, 15$, we have*

$$a_i(4m+2) \equiv 0 \pmod{2}, \tag{92}$$
$$a_i(4m+3) \equiv 0 \pmod{4}, \tag{93}$$
$$a_i(16m+13) \equiv 0 \pmod{4}, \tag{94}$$
$$a_i(16m+t) \equiv 0 \pmod{8}, \text{for } t \in \{11, 15\}. \tag{95}$$

Theorem 16. *For $m \geq 0$ and $i = 16, 17$, we have*

$$a_i(4m+1) \equiv a_i(m) \pmod{4}, \tag{96}$$
$$a_i(8m+4) \equiv 0 \pmod{4}, \tag{97}$$
$$a_i(8m+6) \equiv 0 \pmod{8}, \tag{98}$$
$$a_i(16m+t) \equiv 0 \pmod{8}, \text{for } t \in \{11, 15\}. \tag{99}$$

4. Proofs of Main Results

Proof of Theorem 3. Consider

$$\sum_{m=0}^{\infty} a_1(m)q^m = \frac{(-q;q)_\infty}{(q;q)_\infty}[q^4, q^2, q^2; q^4]_\infty = \frac{f_2^3}{f_4}\frac{1}{f_1^2}$$

$$\sum_{m=0}^{\infty} a_1(m)q^m = \frac{f_2^3}{f_4}\left(\frac{f_8^5}{f_2^5 f_{16}^2} + 2q\frac{f_4^2 f_{16}^2}{f_2^5 f_8}\right).$$

Extracting even and odd terms, we obtain

$$\sum_{m=0}^{\infty} a_1(2m)q^m = \frac{f_4^5}{f_2 f_8^2}\frac{1}{f_1^2}, \tag{100}$$
$$\sum_{m=0}^{\infty} a_1(2m+1)q^m = \frac{2f_2 f_8^2}{f_4}\frac{1}{f_1^2}. \tag{101}$$

Substituting (8) in (100), on extracting even and odd terms, we obtain

$$\sum_{m=0}^{\infty} a_1(4m)q^m = \frac{f_2^5 f_4^3}{f_1^6 f_8^2}, \tag{102}$$
$$\sum_{m=0}^{\infty} a_1(4m+2)q^m = \frac{2f_2^7 f_8^2}{f_1^6 f_4^3}. \tag{103}$$

From (103), we also reach (26)

$$\sum_{m=0}^{\infty} a_1(4m+2)q^m \equiv 2\frac{f_2^4 f_8^2}{f_4^3} \pmod{4},$$

and we extract even terms to reach (29). On bringing out the odd terms, from the above equation and using (25), we have

$$\sum_{m=0}^{\infty} a_1(8m+2)q^m \equiv 2\frac{f_1^4 f_4^2}{f_2^3} \equiv 2\frac{f_4^2}{f_2} \pmod{4}.$$

Extracting odd terms from the above equation to obtain (31) for $t = 10$ and on extracting even terms, we obtain

$$\sum_{m=0}^{\infty} a_1(16m+2)q^m \equiv 2\frac{f_2^2}{f_1} \pmod 4.$$

Using (22) in the above equation and extracting the terms involving q^{3m+2}, we divide by q^2 and replace q^3 by q to obtain (33).

On substituting (8) in (101), extracting even and odd terms, we obtain

$$\sum_{m=0}^{\infty} a_1(4m+1)q^m = 2\frac{f_4^9}{f_2 f_8^2}\frac{1}{f_1^4}, \tag{104}$$

$$\sum_{m=0}^{\infty} a_1(4m+3)q^m = 4f_4^3 f_2 f_8^2 \frac{1}{f_1^4}. \tag{105}$$

From (105), we readily reach (27). Putting (10) in (104) and (105), then extracting odd terms from both equations, we obtain (28) and (30), respectively. Consider (102),

$$\sum_{m=0}^{\infty} a_1(4m)q^m = \frac{f_2^5 f_4^3}{f_1^6 f_8^2} \equiv \frac{f_2^3 f_4^3}{f_8^2}\frac{1}{f_1^2} \pmod 4.$$

Applying (8) in the above relation, extracting odd terms, we have

$$\sum_{m=0}^{\infty} a_1(8m+4)q^m \equiv 2\frac{f_2^5 f_8^2}{f_4^3}\frac{1}{f_1^2} \pmod 4,$$

and again putting (8) then extracting odd terms gives (31) for $t = 12$, and extracting even terms gives

$$\sum_{m=0}^{\infty} a_1(16m+4)q^m \equiv 2\frac{f_4^7}{f_2^3 f_8^2} \pmod 4.$$

Extracting the odd terms from the above equation, we reach (32). □

Proof of Theorem 4.

$$\sum_{m=0}^{\infty} a_2(m)q^m = \frac{(-q;q)_\infty}{(q;q)_\infty}[q^4,-q^2,-q^2;q^4]_\infty = \frac{f_4^5}{f_8^2 f_2}\frac{1}{f_1^2}.$$

Using (8) in above equation, by extracting even and odd terms, we have

$$\sum_{m=0}^{\infty} a_2(2m)q^m = \frac{f_2^5 f_4^3}{f_1^6 f_8^2}, \tag{106}$$

$$\sum_{m=0}^{\infty} a_2(2m+1)q^m = 2\frac{f_2^7 f_8^2}{f_1^6 f_4^3}. \tag{107}$$

Putting (8) in (106) and extracting odd terms, we obtain

$$\sum_{m=0}^{\infty} a_2(4m+2)q^m \equiv 6\frac{f_4^7}{f_8^2} \pmod 4.$$

Again, extracting odd terms from the above equation to obtain (35), and then extracting even terms, we obtain

$$\sum_{m=0}^{\infty} a_2(8m+2)q^m \equiv \frac{6f_2^7}{f_4^2} \pmod 4.$$

On extracting odd terms, we obtain (37).

If we consider (107) and then substitute (8), we have

$$\sum_{m=0}^{\infty} a_2(2m+1)q^m = 2\frac{f_2^7 f_8^2}{f_4^3}\left(\frac{f_8^5}{f_2^5 f_{16}^2} + 2q\frac{f_4^2 f_{16}^2}{f_2^5 f_8}\right)^3$$

Then, by extracting odd terms to obtain (34), and on taking modulo 8, we have

$$\sum_{m=0}^{\infty} a_2(4m+3)q^m \equiv 4\frac{f_4^{11}}{f_1^8 f_2 f_8^2} \equiv 4\frac{f_4^{11}}{f_2^5 f_4^2} \pmod 8.$$

By extracting odd terms from above we obtain (36). □

Proof of Theorem 5. Consider

$$\sum_{m=0}^{\infty} a_3(m)q^m = \frac{(-q;q)_\infty}{(q;q)_\infty}[q^4,-q^4,-q^4;q^4]_\infty = \frac{f_2 f_8^2}{f_4}\frac{1}{f_1^2}. \tag{108}$$

Substituting the value from (8), we obtain

$$\sum_{m=0}^{\infty} a_3(m)q^m = \frac{f_2 f_8^2}{f_4}\left(\frac{f_8^5}{f_2^5 f_{16}^2} + 2q\frac{f_4^2 f_{16}^2}{f_2^5 f_8}\right).$$

Extracting even and odd terms, we obtain

$$\sum_{m=0}^{\infty} a_3(2m)q^m = \frac{f_4^7}{f_2 f_8^2}\frac{1}{f_1^4}, \tag{109}$$

$$\sum_{m=0}^{\infty} a_3(2m+1)q^m = 2f_4 f_2 f_8^2 \frac{1}{f_1^4}. \tag{110}$$

Using (10) in both (109) and (110), we then extract the odd terms from both of them to obtain (39) and (40), respectively. Again, from (108),

$$\sum_{m=0}^{\infty} a_3(m)q^m = \frac{f_2}{f_1^2}\frac{f_8^2}{f_4}.$$

Using (20) and (22), we have

$$\sum_{m=0}^{\infty} a_3(m)q^m = \left(\frac{f_6^4 f_9^6}{f_3^8 f_{18}^3} + 2q\frac{f_6^3 f_9^3}{f_3^7} + 4q^2\frac{f_6^2 f_{18}^3}{f_3^6}\right)\left(\frac{f_{24} f_{36}^2}{f_{12} f_{72}} + q^4\frac{f_{72}^2}{f_{36}}\right) \tag{111}$$

Then, we extract the terms involving q^{3m} and replace with q^3 by q to obtain

$$\sum_{m=0}^{\infty} a_3(3m)q^m \equiv \frac{f_3^2 f_8 f_{12}^2}{f_4 f_6 f_{24}} \pmod 4. \tag{112}$$

Next, extract the terms involving q^{3m+2} from (111), dividing both sides by q^2 and replacing q^3 by q to obtain

$$\sum_{m=0}^{\infty} a_3(3m+2)q^m = 4\frac{f_2^2 f_6^3 f_8 f_{12}^2}{f_1^6 f_4 f_{24}} + 2q\frac{f_2^3 f_3^3 f_{24}^2}{f_1^7 f_{12}}. \tag{113}$$

From the above equation, we readily reach (38). Now, using (9) in (112), we have

$$\sum_{m=0}^{\infty} a_3(3m)q^m \equiv \frac{f_8 f_{12}^2}{f_4 f_6 f_{24}}\left(\frac{f_6 f_{24}^5}{f_{12}^2 f_{48}^2} - 2q^3\frac{f_6 f_{48}^2}{f_{24}}\right) \pmod{4}.$$

On extracting even and odd terms, we obtain

$$\sum_{m=0}^{\infty} a_3(6m)q^m \equiv \frac{f_4 f_{12}^4}{f_2 f_{24}^2} \pmod{4}, \tag{114}$$

$$\sum_{m=0}^{\infty} a_3(6m+3)q^m \equiv 2q\frac{f_4 f_6^2 f_{24}^2}{f_2 f_{12}^2} \pmod{4}. \tag{115}$$

We extract the odd and even terms from (114) and (115), respectively, to obtain (41) for $t = 3, 6$. From (113), we have

$$\sum_{m=0}^{\infty} a_3(3m+2)q^m \equiv 2q\frac{f_{24}^2}{f_{12}}\frac{f_3^3}{f_1} \pmod{4}.$$

Using (15), then extracting the even and odd terms, we obtain

$$\sum_{m=0}^{\infty} a_3(6m+2)q^m \equiv 2q\frac{f_{12}^2 f_6^3}{f_2 f_6} \pmod{4}, \tag{116}$$

$$\sum_{m=0}^{\infty} a_3(6m+5)q^m \equiv 2\frac{f_2^2 f_{12}^2}{f_6} \pmod{4}. \tag{117}$$

We extract even and odd terms from (116) and (117), respectively, to obtain (41) for $t = 2, 11$. □

Proof of Theorem 6.

$$\sum_{m=0}^{\infty} a_4(m)q^m = \frac{[q^6, q^3, q^3; q^6]_\infty}{(q^2; q^2)_\infty} = f_3^2\frac{1}{f_2 f_6}.$$

Substituting (9) in the above equation to obtain

$$\sum_{m=0}^{\infty} a_4(m)q^m = \frac{1}{f_2 f_6}\left(\frac{f_6 f_{24}^5}{f_{12}^2 f_{48}^2} - 2q^3\frac{f_6 f_{48}^2}{f_{24}}\right),$$

then extracting even terms, we obtain (118) and on extracting odd terms we obtain (42).

$$\sum_{m=0}^{\infty} a_4(2m)q^m = \frac{f_{12}^5}{f_{24}^2}\frac{1}{f_1 f_3}. \tag{118}$$

Using (12) in (118), and on extracting odd terms and taking modulo 4, we have

$$\sum_{m=0}^{\infty} a_4(4m+2)q^m \equiv \frac{f_6^4 f_2^3}{f_4^2}\frac{1}{f_3^2} \pmod{4}.$$

Using (8), and again on extracting even and odd terms, we obtain

$$\sum_{m=0}^{\infty} a_4(8m+2)q^m \equiv \frac{f_{12}^5}{f_2^2 f_{24}^2}\frac{f_1^3}{f_3} \pmod{4}, \tag{119}$$

$$\sum_{m=0}^{\infty} a_4(8m+6)q^m \equiv 2q\frac{f_6^2 f_{24}^2}{f_2^2 f_{12}}\frac{f_1^3}{f_3} \pmod{4}. \tag{120}$$

Substituting (13) in (119), extracting even terms, we obtain

$$\sum_{m=0}^{\infty} a_4(16m+2)q^m \equiv \frac{f_6^4 f_2^3}{f_{12}^2}\frac{1}{f_1^2} \pmod 4.$$

Using (8) and extracting odd terms we have

$$\sum_{m=0}^{\infty} a_4(32m+18)q^m \equiv 2\frac{f_2 f_8^2}{f_4} \pmod 4.$$

We extract odd terms to obtain (44). Consider (120), and by using (13) upon bringing out the even and odd terms, we obtain

$$\sum_{m=0}^{\infty} a_4(16m+6)q^m \equiv 2q\frac{f_{12}^2 f_6^2}{f_2} \pmod 4, \tag{121}$$

$$\sum_{m=0}^{\infty} a_4(16m+14)q^m \equiv 2\frac{f_{12}^2 f_2^2}{f_6} \pmod 4. \tag{122}$$

We extract the even and odd terms from (121) and (122), respectively, to obtain (43) for $t = 6, 30$. □

Proof of Theorem 7.

$$\sum_{m=0}^{\infty} a_5(m)q^m = \frac{(-q^2;q^2)_\infty}{(q^2;q^2)_\infty}[q^6, -q^3, -q^3; q^6]_\infty = \frac{f_4 f_6^5}{f_2^2 f_{12}^2}\frac{1}{f_3^2}.$$

Using (8) in the above equation,

$$\sum_{m=0}^{\infty} a_5(m)q^m = \frac{f_4 f_6^5}{f_2^2 f_{12}^2}\left(\frac{f_{24}^5}{f_6^5 f_{48}^2} + 2q^3\frac{f_{12}^2 f_{48}^2}{f_6^5 f_{24}}\right)$$

Extracting even terms then using (20), we extract the terms involving the q^{3m}, q^{3m+1}, and q^{3m+2} terms, and we obtain (123), (45), and (46), respectively.

$$\sum_{m=0}^{\infty} a_5(6m)q^m \equiv \frac{f_2^4 f_3^6}{f_1^5 f_6^3}. \tag{123}$$

Taking modulo 4,

$$\sum_{m=0}^{\infty} a_5(6m)q^m \equiv \frac{f_1^3 f_3^6}{f_6^3} \pmod 4.$$

Using (21), we have

$$\sum_{m=0}^{\infty} a_5(6m)q^m \equiv \frac{f_3^6}{f_6^3}(f_3 a(q^3) - 3q f_9^3) \pmod 4.$$

Extracting the term involving q^{3m+2} and q^{3m+1}, we have (47) and

$$\sum_{m=0}^{\infty} a_5(18m+6)q^m \equiv \frac{f_3^3 f_1^2}{f_2} \pmod 4.$$

Using (22) in the above equation, extracting the terms involving q^{3m+1} and q^{3m}, we obtain (48) and

$$\sum_{m=0}^{\infty} a_5(54m+6)q^m \equiv \frac{f_3^2 f_1^3}{f_6} \pmod 4.$$

Using (21) and extracting the terms involving q^{3m+2}, we obtain (49). □

Proof of Theorem 8.

$$\sum_{m=0}^{\infty} a_6(m)q^m = \frac{f_2 f_4^2}{f_8}\frac{1}{f_1^2}. \tag{124}$$

Using (8) and extracting even and odd terms, we have

$$\sum_{m=0}^{\infty} a_6(2m)q^m = \frac{f_2^2 f_4^4}{f_8^2}\frac{1}{f_1^4}, \tag{125}$$

$$\sum_{m=0}^{\infty} a_6(2m+1)q^m = \frac{f_2^4 f_8^2}{f_4^2}\frac{1}{f_1^4}. \tag{126}$$

Substituting the value from (8) in (125), on extracting the even and odd terms, we have

$$\sum_{m=0}^{\infty} a_6(4m)q^m = \frac{f_2^{18}}{f_4^6 f_1^{12}}, \tag{127}$$

$$\sum_{m=0}^{\infty} a_6(4m+2)q^m = 4\frac{f_2^6 f_4^2}{f_1^8}. \tag{128}$$

Taking modulo 4 in (127),

$$\sum_{m=0}^{\infty} a_6(4m)q^m \equiv \frac{f_2^{12}}{f_4^6} \pmod 4.$$

Extracting the odd terms gives (53) (for $t = 4$), and on extracting even terms gives

$$\sum_{m=0}^{\infty} a_6(8m)q^m \equiv f_2^3 \pmod 4.$$

Extracting the odd terms gives (56), and on extracting even terms gives

$$\sum_{m=0}^{\infty} a_6(16m)q^m \equiv f_1^3 \pmod 4.$$

By using Jacobi's triple-product identity, we have

$$\sum_{m=0}^{\infty} a_6(16m)q^m \equiv \sum_{n=0}^{\infty} (-1)^m(2m+1)q^{m(m+1)/2} \pmod 4.$$

Since $m(m+1)/2 \not\equiv 2, 4 \pmod 5$, we obtain (58). Consider (128) and taking modulo 16,

$$\sum_{m=0}^{\infty} a_6(4m+2)q^m \equiv 4f_2^2 f_4^2 \pmod 4.$$

Extracting odd terms from the above equation gives (54). Consider (126) and using (10), by extracting even and odd terms, we obtain

$$\sum_{m=0}^{\infty} a_6(4m+1)q^m = 2\frac{f_2^{12}}{f_2^4 f_4^2}\frac{1}{f_1^2}, \tag{129}$$

$$\sum_{m=0}^{\infty} a_6(4m+3)q^m = 8\frac{f_4^4}{f_1^6}. \tag{130}$$

Consider (129) and using (8), extracting odd terms gives (53) (for $t = 5$). Similarly, taking modulo 16 in (130), we obtain

$$\sum_{m=0}^{\infty} a_6(4m+3)q^m \equiv 8\frac{f_4^4}{f_2^3} \pmod{16}.$$

Extracting odd terms gives (54) (for $t = 16$). Consider (124) and using (19) and (20)

$$\sum_{m=0}^{\infty} a_6(m)q^m = \left(\frac{f_{36}^2}{f_{72}} - 2q^4\frac{f_{12}f_{72}^2}{f_{24}f_{36}}\right)\left(\frac{f_6^4f_9^6}{f_3^8f_{18}^3} + 2q\frac{f_6^3f_9^3}{f_3^7} + 4q^2\frac{f_6^2f_{18}^3}{f_3^6}\right).$$

Extracting the terms involving q^{3m}, q^{3m+1}, and q^{3m+2} gives (131), (50), and (51), respectively.

$$\sum_{m=0}^{\infty} a_6(3m)q^m \equiv \frac{f_{12}^2f_6^4}{f_{24}f_6^3}\frac{1}{f_3^2} \pmod{8}. \tag{131}$$

Using (8) and extracting even and odd terms, we have

$$\sum_{m=0}^{\infty} a_6(6m)q^m \equiv \frac{f_6^2f_{12}^4}{f_{24}^2}\frac{1}{f_3^4} \pmod{4}, \tag{132}$$

$$\sum_{m=0}^{\infty} a_6(6m+3)q^m \equiv 2q\frac{f_6^4f_{24}^2}{f_{12}^2}.\frac{1}{f_3^4} \pmod{8}. \tag{133}$$

Using (10) in (132) and extracting odd terms, we have

$$\sum_{m=0}^{\infty} a_6(12m+6)q^m \equiv 4qf_6^2f_{12}^2 \pmod{8}.$$

On extracting even terms we arrive at (57). Consider (133) and using (10), which, on extracting even terms, gives (55). □

Proof of Theorem 9.

$$\sum_{m=0}^{\infty} a_7(m)q^m = \frac{f_2}{f_1^2}\frac{f_5^2}{f_{10}}.$$

Using (19) and (20), we have

$$\sum_{m=0}^{\infty} a_7(m)q^m = \left(\frac{f_6^4f_9^6}{f_3^8f_{18}^3} + 2q\frac{f_6^3f_9^3}{f_3^7} + 4q^2\frac{f_6^2f_{18}^3}{f_3^6}\right)\left(\frac{f_{45}^2}{f_{90}} - 2q^5\frac{f_{15}f_{90}^2}{f_{30}f_{45}}\right).$$

By extracting the terms involving q^{3m}, q^{3m+1}, and q^{3m+2} and taking modulo 4, we have

$$\sum_{m=0}^{\infty} a_7(3m)q^m \equiv \frac{f_3^6f_{15}^2}{f_6^3f_{30}} \pmod{4}, \tag{134}$$

$$\sum_{m=0}^{\infty} a_7(3m+1)q^m \equiv 2f_6\frac{f_3}{f_1} \pmod{4}, \tag{135}$$

$$\sum_{m=0}^{\infty} a_7(3m+2)q^m \equiv 2q\frac{f_5f_{30}^2}{f_7f_{45}} \pmod{4}. \tag{136}$$

Extracting the terms involving q^{3m+1}, q^{3m+2} and dividing by q, q^2, respectively, by replacing q^3 by q, we obtain (60). Consider (135) and using (17), on extracting even and odd terms, we obtain

$$\sum_{m=0}^{\infty} a_7(6m+1)q^m \equiv 2\frac{f_8 f_{12}^2}{f_4 f_{24}} \pmod 4, \tag{137}$$

$$\sum_{m=0}^{\infty} a_7(6m+4)q^m \equiv 2\frac{f_4^2 f_6 f_{24}}{f_2 f_8 f_{12}} \pmod 4. \tag{138}$$

Consider (137), on extracting even and odd terms, we obtain (139) and (62) (for $t = 7$), respectively.

$$\sum_{m=0}^{\infty} a_7(12m+1)q^m \equiv 2\frac{f_4 f_6^2}{f_2 f_{12}} \pmod 4. \tag{139}$$

Extracting odd terms from the above equation gives (63). From (138), extracting even and odd terms gives (140) and (62) (for $t = 10$), respectively.

$$\sum_{m=0}^{\infty} a_7(12m+4)q^m \equiv 2\frac{f_2^2 f_{12}}{f_4 f_6}\frac{f_3}{f_1} \pmod 4. \tag{140}$$

Using (17), extracting even and odd terms, we have

$$\sum_{m=0}^{\infty} a_7(24m+4)q^m \equiv 2\frac{f_8 f_{12}}{f_4 f_{24}} \pmod 4, \tag{141}$$

$$\sum_{m=0}^{\infty} a_7(24m+16)q^m \equiv 2\frac{f_4^2 f_6 f_{24}}{f_2 f_8 f_{12}} \pmod 4. \tag{142}$$

From (136) and (142), we have $a_7(6m+4) \equiv a_7(24m+16) \pmod 4$. Extracting odd terms from (141), we arrive at (64). Consider (136), and by extracting the terms involving q^{5m}, q^{5m+2}, q^{5m+3}, and q^{5m+4}, we obtain (61). □

Proof of Theorems 10 and 11.

$$\sum_{m=0}^{\infty} a_8(m)q^m = \frac{[q^{12}, q^4, q^8; q^{12}]_\infty}{(q;q)_\infty} = \frac{f_4}{f_1}.$$

Using (18), by extracting the terms involving q^{3m} and q^{3m+1}, we have

$$\sum_{m=0}^{\infty} a_8(3m)q^m = \frac{f_4 f_6^4}{f_1^3 f_{12}^2}, \tag{143}$$

$$\sum_{m=0}^{\infty} a_8(3m+1)q^m = \frac{f_2^2 f_3^3 f_{12}}{f_1^4 f_6^2}. \tag{144}$$

Taking modulo 2 in (143),

$$\sum_{m=0}^{\infty} a_8(3m)q^m \equiv \frac{f_2}{f_1} \equiv f_1 \pmod 2.$$

From Lemma 4, we have

$$\sum_{m=0}^{\infty} a_8(3m)q^m \equiv \sum_{\substack{k=\frac{-(p-1)}{2} \\ k\neq(\pm p-1)/6}}^{\frac{(p-1)}{2}} (-1)^k q^{\frac{3k^2+k}{2}} f(-q^{\frac{(3p^2+(6k+1)p)}{2}}, -q^{\frac{(3p^2-(6k+1)p)}{2}}) + (-1)^{\frac{(\pm p-1)}{6}} q^{\frac{(p^2-1)}{24}} f_{p^2} \pmod{2}.$$

By extracting the terms involving $q^{pm+(p^2-1)/24}$, dividing by $q^{(p^2-1)/24}$, and replacing q^{pm} by q^m, we obtain

$$\sum_{m=0}^{\infty} a_8\left(3pm+\frac{p^2-1}{8}\right)q^m \equiv (-1)^{(\pm p-1)/6} f_p \pmod{2}.$$

Extracting the terms involving q^{pm+i} for $i = 1, 2, \cdots, (p-1)$,

$$\sum_{m=0}^{\infty} a_8\left(3p^2m+3pi+\frac{p^2-1}{8}\right)q^m \equiv 0 \pmod{2}.$$

which proves (65). Taking modulo 4 in (144), we have

$$\sum_{m=0}^{\infty} a_8(3m+1)q^m \equiv \frac{f_2^2 f_3^3 f_{12}}{f_2^2 f_6^2} \pmod{4}.$$

Extracting the terms involving q^{3m}, q^{3m+1}, and q^{3m+2} from above, we obtain (145), (67), and (68), respectively.

$$\sum_{m=0}^{\infty} a_8(9m+1)q^m \equiv \frac{f_1^3 f_4}{f_2^2} \equiv \frac{f_4}{f_1} \pmod{4} \tag{145}$$

From above, it is easy to conclude (66). □

Proof of Theorem 12.

$$\sum_{m=0}^{\infty} a_9(m)q^m = \frac{(-q;q)_\infty}{(q;q)_\infty}[q^{12}, q^3, q^9; q^{12}]_\infty = \frac{f_2 f_3 f_{12}}{f_1^2 f_6}.$$

Using (20), extracting the terms involving q^{3m}, q^{3m+1}, and q^{3m+2}, we obtain (146), (69), and (70), respectively.

$$\sum_{m=0}^{\infty} a_9(3m)q^m = \frac{f_2^3 f_3^6 f_4}{f_1^7 f_6^3}. \tag{146}$$

Taking modulo 3, we have

$$\sum_{m=0}^{\infty} a_9(3m)q^m \equiv \frac{f_3^3}{f_6^2}\frac{f_4}{f_1} \pmod{3}.$$

Using (18), we obtain

$$\sum_{m=0}^{\infty} a_9(3m)q^m \equiv \frac{f_3^3}{f_6^2}\left(\frac{f_{12}f_{18}^4}{f_3^3 f_{36}^2} + q\frac{f_6^2 f_9^3 f_{36}}{f_3^4 f_{18}^2} + 2q^2\frac{f_6 f_{18} f_{36}}{f_3^3}\right) \pmod{3}. \tag{147}$$

Extracting the terms involving q^{3m} and replacing q^3 by q, we have

$$\sum_{m=0}^{\infty} a_9(9m)q^m \equiv \frac{f_4 f_6^4}{f_2^2 f_{12}^2} \pmod 3.$$

On extracting odd terms, we obtain (71). Extracting the terms involving q^{3m+1}, dividing by q, and replacing q^3 by q from (147), we have

$$\sum_{m=0}^{\infty} a_9(9m+6)q^m \equiv 2\frac{f_6 f_{12}}{f_2} \pmod 3.$$

On extracting odd terms, we reach (72). □

Proof of Theorem 13.

$$\sum_{m=0}^{\infty} a_{10}(m)q^m = \frac{(-q;q)_\infty}{(q;q)_\infty}[q^{12},q^6,q^6;q^{12}]_\infty = \frac{f_2 f_6^2}{f_1^2 f_{12}}.$$

Using (8), extracting the even and odd terms and taking modulo 4, we have

$$\sum_{m=0}^{\infty} a_{10}(2m)q^m \equiv \frac{f_4^5}{f_2^2 f_6 f_8^2} \cdot f_3^2 \pmod 4, \tag{148}$$

$$\sum_{m=0}^{\infty} a_{10}(2m+1)q^m \equiv 2f_8 \pmod 4. \tag{149}$$

Using (9) in (148), we obtain

$$\sum_{m=0}^{\infty} a_{10}(2m)q^m \equiv \frac{f_4^5}{f_2^2 f_6 f_8^2}\left(\frac{f_6 f_{24}^5}{f_{12}^2 f_{48}} - 2q^3\frac{f_6 f_{48}^2}{f_{24}}\right) \pmod 4.$$

Extracting even and odd terms, we have

$$\sum_{m=0}^{\infty} a_{10}(4m)q^m \equiv \frac{f_2^5 f_{12}^5}{f_4^2 f_6^2 f_{24}^2}\frac{1}{f_1^2} \pmod 4, \tag{150}$$

$$\sum_{m=0}^{\infty} a_{10}(4m+2)q^m \equiv 2q\frac{f_2^4 f_{24}^2}{f_4^2 f_{12}} \pmod 4. \tag{151}$$

Substituting (8) in (150) and extracting odd terms, we have

$$\sum_{m=0}^{\infty} a_{10}(8m+4)q^m \equiv 2\frac{f_6^4 f_8^2}{f_4 f_{12}^2} \pmod 4.$$

On extracting odd and even terms, we reach (75) (for $t = 12$) and

$$\sum_{m=0}^{\infty} a_{10}(16m+4)q^m \equiv 2\frac{f_4^2}{f_2} \pmod 4,$$

respectively. Extracting odd terms from the above equation, we obtain (76). Consider (149) by extracting odd and even terms to obtain (73) and

$$\sum_{m=0}^{\infty} a_{10}(4m+1)q^m \equiv 2f_4 \pmod 4,$$

respectively. Extracting odd and even terms from the above equation, we have (74) (for $t = 5$) and

$$\sum_{m=0}^{\infty} a_{10}(8m+1)q^m \equiv 2f_2 \pmod{4},$$

respectively. On extracting odd terms from the above equation, we arrive at (75) (for $t = 9$). Consider (151), and by extracting odd and even terms, we have (74) and

$$\sum_{m=0}^{\infty} a_{10}(8m+6)q^m \equiv 2\frac{f_{12}^2}{f_6} \pmod{4},$$

respectively. Extracting odd and even terms from the above equation, we obtain (75) (for $t = 14$) and

$$\sum_{m=0}^{\infty} a_{10}(16m+6)q^m \equiv 2\frac{f_6^2}{f_3} \pmod{4},$$

respectively. We extract the terms involving q^{3m+1}, q^{3m+2} from the above equation to obtain (77). □

Proof of Theorem 14. For $i = 11, 12, 13$, we consider

$$\sum_{m=0}^{\infty} a_i(m)q^m = \frac{f_2 f_3^2}{f_6 f_1^2}. \tag{152}$$

Using (16) in the above equation, and extracting even and odd terms, we have

$$\sum_{m=0}^{\infty} a_i(2m)q^m = \frac{f_2^4 f_6^2}{f_4 f_{12}} \frac{1}{f_1^4}, \tag{153}$$

$$\sum_{m=0}^{\infty} a_i(2m+1)q^m = 2\frac{f_2 f_4 f_{12}}{f_6} \frac{f_3}{f_1^3}. \tag{154}$$

Using (10) in (153), again extracting the even and odd terms, we obtain

$$\sum_{m=0}^{\infty} a_i(4m)q^m \equiv \frac{f_2^8}{f_4^4} \pmod{2}, \tag{155}$$

$$\sum_{m=0}^{\infty} a_i(4m+2)q^m \equiv 4\frac{f_2 f_4^4}{f_6} \pmod{12}. \tag{156}$$

From (156), we obtain (80). Now, on extracting odd terms from (155) and (156), we reach (83) and (85) (for $t = 6$), respectively. From (153), we have

$$\sum_{m=0}^{\infty} a_i(4m)q^m \equiv \frac{f_2^9}{f_6 f_4^4} \frac{f_3^2}{f_1^2} \pmod{8}.$$

Using (16), extracting even and odd terms, we have

$$\sum_{m=0}^{\infty} a_i(8m)q^m \equiv \frac{f_6^2}{f_4 f_{12}} f_1^4 \pmod{8}, \tag{157}$$

$$\sum_{m=0}^{\infty} a_i(8m+4)q^m \equiv 2\frac{f_4 f_{12}}{f_2 f_6} . f_1 f_3 \pmod{8}. \tag{158}$$

Substituting the values from (10) in (157) and extracting even terms and odd terms, we obtain

$$\sum_{m=0}^{\infty} a_i(16m)q^m \equiv \frac{f_3^2 f_2^{10}}{f_2 f_6 f_4^4}\frac{1}{f_1^2} \pmod 8, \tag{159}$$

$$\sum_{m=0}^{\infty} a_i(16m+8)q^m \equiv 4\frac{f_4^4}{f_2^2} \pmod 8. \tag{160}$$

On extracting odd terms from (160), we obtain (88), and extracting even terms gives us

$$\sum_{m=0}^{\infty} a_i(32m+8)q^m \equiv 4f_2^3 \pmod 8.$$

Extracting odd terms gives us (91). Now consider (154), and by using (14) and extracting even and odd terms, we have

$$\sum_{m=0}^{\infty} a_i(4m+1)q^m \equiv 2f_2^3 \pmod 4, \tag{161}$$

$$\sum_{m=0}^{\infty} a_i(4m+3)q^m \equiv 6f_6^3 \pmod{12}. \tag{162}$$

Additionally, we obtain (81). From (161), extracting the odd terms gives us (84), while extracting even terms gives

$$\sum_{m=0}^{\infty} a_i(8m+1)q^m \equiv 2f_1^3 \pmod 4.$$

According to Jacobi's triple product

$$\sum_{m=0}^{\infty} a_i(8m+1)q^m \equiv 2\sum_{m=0}^{\infty}(-1)^m(2m+1)q^{m(m+1)/2} \pmod 4.$$

As $m(m+1)/2 \not\equiv 2,4 \pmod 5$, we obtain (90). Consider (162) and extracting the odd terms to obtain (85) (for $t=7$), and by extracting even terms and using Jacobi's triple product, we ultimately reach (89).

Consider (152), using (20), and extracting the terms involving q^{3m}, q^{3m+1}, and q^{3m+2}, we obtain

$$\sum_{m=0}^{\infty} a_i(3m)q^m \equiv \frac{f_3^4}{f_6^2} \pmod 3, \tag{163}$$

$$\sum_{m=0}^{\infty} a_i(3m+1)q^m \equiv 0 \pmod 2, \tag{164}$$

$$\sum_{m=0}^{\infty} a_i(3m+2)q^m \equiv 0 \pmod 6. \tag{165}$$

Additionally,

$$\sum_{m=0}^{\infty} a_i(3m+2)q^m \equiv 4\frac{f_6^3}{f_2} \pmod{16}.$$

On extracting odd parts, we obtain (82), and on extracting even parts and using (15), we have

$$\sum_{m=0}^{\infty} a_i(6m+2)q^m \equiv 4\left(\frac{f_4^3 f_6^2}{f_2^2 f_{12}} + q\frac{f_{12}^3}{f_4}\right) \pmod{16}. \tag{166}$$

On extracting odd terms, we obtain

$$\sum_{m=0}^{\infty} a_i(12m+8)q^m \equiv 4\frac{f_6^3}{f_2} \pmod{16}.$$

On extracting odd terms, we obtain (87). Taking modulo 8 for (166), using (15), and extracting the even terms gives

$$\sum_{m=0}^{\infty} a_i(12m+2)q^m \equiv 4f_2^2 \pmod{8}.$$

On extracting odd terms, we reach (86). □

Proof of Theorem 15 . For $i = 14, 15$, we consider

$$\sum_{m=0}^{\infty} a_i(m)q^m = f_6^2 \frac{1}{f_1 f_3}$$

Substituting (12) in the above equation and then extracting even and odd terms, we have

$$\sum_{m=0}^{\infty} a_i(2m)q^m = \frac{f_4^2 f_6^5}{f_2 f_3^2 f_{12}^2} \cdot \frac{1}{f_1^2}, \tag{167}$$

$$\sum_{m=0}^{\infty} a_i(2m+1)q^m = \frac{f_2^5 f_{12}^2}{f_4^2 f_6} \cdot \frac{1}{f_1^4}. \tag{168}$$

Taking modulo 2 in (167), we have

$$\sum_{m=0}^{\infty} a_i(2m+1)q^m \equiv \frac{f_4^2 f_6^4}{f_2^2 f_{12}^2} \pmod{2}.$$

On extracting odd terms from the above equation, we readily reach (92). Consider (168) and substituting (10),

$$\sum_{m=0}^{\infty} a_i(2m+1)q^m = \frac{f_2^5 f_{12}^2}{f_4^2 f_6} \left(\frac{f_4^{14}}{f_2^{14} f_8^4} + 4q\frac{f_4^2 f_8^4}{f_2^{10}} \right).$$

On extracting odd terms, we obtain (93) and (170). Similarly, extracting even terms from the same, we have

$$\sum_{m=0}^{\infty} a_i(4m+1)q^m \equiv \frac{f_6^2 f_2^{12}}{f_2^4 f_4^4} \frac{1}{f_1 f_3} \pmod{4}, \tag{169}$$

$$\sum_{m=0}^{\infty} a_i(4m+3)q^m \equiv 4\frac{f_6^2 f_4^4}{f_2^2} \frac{1}{f_1 f_3} \pmod{8}. \tag{170}$$

Using (12) in (169) and extracting odd terms, we obtain

$$\sum_{m=0}^{\infty} a_i(8m+5)q^m \equiv \frac{f_2^3 f_{12}^2}{f_4^2 f_{16}} \pmod{4}.$$

Extracting odd terms to reach (94). Consider (170) and using (12), on extracting even and odd terms, we obtain

$$\sum_{m=0}^{\infty} a_i(8m+3)q^m \equiv 4\frac{f_2 f_4^2 f_6^4}{f_{12}^2} \pmod 8,$$
$$\sum_{m=0}^{\infty} a_i(8m+7)q^m \equiv \frac{f_2^6 f_{12}^2}{f_4^2 f_6} \pmod 8.$$

Extracting odd terms from both the above equations to obtain (95). □

Proof of Theorem 16. For $i = 16, 17$, we consider

$$\sum_{m=0}^{\infty} a_i(m)q^m = \frac{f_{12}}{f_6} \cdot \frac{f_3}{f_1}.$$

Using (17) and extracting even and odd terms, we have

$$\sum_{m=0}^{\infty} a_i(2m)q^m = \frac{f_2 f_8 f_{12}^2}{f_4 f_{24}} \frac{1}{f_1^2}, \tag{171}$$
$$\sum_{m=0}^{\infty} a_i(2m+1)q^m = \frac{f_4^2 f_6 f_{24}}{f_8 f_{12}} \cdot \frac{1}{f_1^2}. \tag{172}$$

We use (8) in (171) and extract even and odd terms to obtain

$$\sum_{m=0}^{\infty} a_i(4m)q^m = \frac{f_4^6 f_6^2}{f_2 f_8^2 f_{12}}, \tag{173}$$
$$\sum_{m=0}^{\infty} a_i(4m+2)q^m = 2\frac{f_2 f_6^2 f_8^2}{f_{12}} \frac{1}{f_1^4}. \tag{174}$$

Extracting odd terms from (173), we readily reach (97). Consider (172) and using (8), and on extracting even and odd terms, we obtain

$$\sum_{m=0}^{\infty} a_i(4m+1)q^m = \frac{f_2^2 f_4^4 f_{12}}{f_1^4 f_6 f_8^2} \frac{f_3}{f_1}, \tag{175}$$
$$\sum_{m=0}^{\infty} a_i(4m+3)q^m = 2\frac{f_2^4 f_3 f_8^2 f_{12}}{f_1^5 f_4^2 f_6}. \tag{176}$$

Taking modulo 4 in (175),

$$\sum_{m=0}^{\infty} a_i(4m+1)q^m \equiv \frac{f_{12}}{f_6} \frac{f_3}{f_1} \pmod 4,$$

which implies $a_i(4m+1) \equiv a_i(m) \pmod 4$. Now consider (174) and using (10), and on extracting odd terms, we obtain (98). Taking modulo 8 in (176), we have

$$\sum_{m=0}^{\infty} a_i(4m+3)q^m = 2\frac{f_2^2 f_8^2 f_{12}}{f_4^2 f_6} \frac{f_3}{f_1} \pmod 8.$$

Using (17) in the above and extracting even and odd terms, we have

$$\sum_{m=0}^{\infty} a_i(8m+3)q^m \equiv 2\frac{f_4 f_8 f_{12}^2}{f_2 f_{24}} \pmod 8,$$
$$\sum_{m=0}^{\infty} a_i(8m+7)q^m \equiv 2\frac{f_4^4 f_6 f_{24}}{f_2^2 f_8 f_{12}} \pmod 8.$$

Extracting odd terms from the above equations, we obtain (99). □

5. Conclusions

This paper provides some congruences for Rogers–Ramanujan type identities to modulo powers of 2, 3, and 6. As mentioned in Section 1, these 'sum-product' identities have been studied by many mathematicians in various contexts (see Refs. [5,18,25,26]). However, in the literature, to the best of our knowledge, we have not found any congruences for Rogers–Ramanujan type identities; instead, there is a huge selection of literature that studies congruences for partition functions. For instance, Ramanujan beautiful congruences for partition functions are shown as:

$$p(5n+4) \equiv 0 \pmod 5,$$
$$p(7n+5) \equiv 0 \pmod 7,$$
$$p(11n+6) \equiv 0 \pmod{11}.$$

These congruences are generalized and written in the form

$$p(\ell n - \delta_\ell) \equiv 0 \pmod \ell,$$

where $\delta_\ell = (\ell^2 - 1)/24$. The above three congruences were further extended to arbitrary powers of 5, 7, and 11 (for instance, see Ref. [27]). Thus, our paper adds one more direction to the study of Rogers–Ramanujan type identities. For future research, one could look for further interesting Rogers–Ramanujan type identities or others available in the literature to find their congruence modulo higher primes. Furthermore, one can think of generalizing the congruences that are proved in this paper. Moreover, it will be fascinating to prove these congruences using some other techniques, such as modular forms.

Author Contributions: Writing—original draft, V.G.; Writing—review & editing, M.R. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Data Availability Statement: Not applicable.

Acknowledgments: The second author is supported by the SERB MATRICS Project Ref. No. MTR/2019/000123.

Conflicts of Interest: The authors have no conflict of interest.

Sample Availability: The manuscript has no associated data.

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