

The Dynamics of a Continuous Newton-like Method

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Abstract: The objective of the current work is to invent and introduce the continuous version of Newton's method. This scheme is used to establish some interesting properties with examples. We have plotted the fractal pattern graphs for a Newton-like method and a Damped Newton-like method in the discrete case and hence we have introduced a new concept of streamline for the continuous version of the Newton-like method. The graph and streamlines of these patterns are in agreement with numerical results and describe the convergence and stability of the proposed method to different roots when the Newton method fails.

Keywords: continuation Newton-like method; iteration function; order of convergence; basin of attraction; fractal

MSC: 30C15; 37F50; 37N30; 65F10; 65H04; 65H05



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1. Introduction

This work deals with the continuous Newton-like method and related deformations for the approximate numerical solution of the nonlinear and transcendental equation of the type

$$F(x) = 0, \quad (1)$$

where F is a function such that $F : X \rightarrow X$. X is a subset of real space \mathbb{R} or Complex space \mathbb{C} . Normally we use Newton's method, secant method, and Chebyshev method, etc. for an approximate solution of (1) (see [1–6]). Among these methods, Newton's iterative method is one of the most well known and basic method and it is defined by

$$z_{n+1} = z_n - \frac{F(z_n)}{F'(z_n)}, \quad n = 0, 1, 2, \dots \quad (2)$$

It has quadratic convergence under some conditions. Several modifications and refinements have been made to it through various means (see [7–11]). However, the ambiguity and conflicts about the convergence and stability of the methods to the desired roots still persist, as these methods are unable to explain the following questions:

1. What is the connection between initial approximation and roots?
2. What is the reason that iteration process fail or diverge?
3. What is the significance of the denominator part in the method?

To answer the questions 2 and 3, Wu [12], suggested a second-order converging method for solution of (1) in weak condition as follows:

$$z_{n+1} = z_n - \frac{F(z_n)}{\lambda_n F(z_n) + F'(z_n)}, \quad n = 0, 1, 2, \dots, \quad (3)$$

where $\lambda_n \in (-\infty, +\infty)$. In 2010, Wang and Liu [13], suggested a third-order converging variant of Newton’s method for solution of (1) under the weak condition as follows:

$$\begin{cases} y_n = z_n + \frac{F(z_n)}{\lambda_n F(z_n) - F'(z_n)}, \\ z_{n+1} = y_n + \frac{F(y_n)}{\lambda_n F(z_n) - F'(z_n)}, \end{cases} \quad n = 0, 1, 2, \dots, \tag{4}$$

where $\lambda_n \in \mathbb{R}$, such that $\lambda_n = -\text{sign}(F(z_n)F'(z_n))\min(1, |F(z_n)|)$.

Similarly, in order to solve the system of nonlinear equations, Kou [14] gave a third-order converging Newton-like method. But study of these methods are only rectifying Newton’s method by enhancing its order or making it compatible in weak conditions. Although these methods remove some limitations of Newton’s method, they are unable to give the complete answer for questions 2 and 3.

In view of the questions 1 and 2, Hetzler developed in [9] a continuous version of Newton’s method in 1997, which is defined by

$$\begin{cases} \frac{dz}{dt} = -\frac{F(z)}{F'(z)}, \\ z(0) = z_0. \end{cases} \tag{5}$$

Again in 2000, Wu discussed a new continuation Newton-like method and its deformation in [12] as follows:

$$z_{n+1} = z_n - \frac{F(z_n)}{F'(z_n) + \lambda_n F(z_n)} \tag{6}$$

where, $\lambda_n \in \mathbb{R}$ and $\lambda_n \in (-\infty, +\infty)$. Wu also considered the Liapunov function for this method. Both the authors Wu and Hetzler discussed the continuous Newton-like method but none of them described the fractal patterns and dynamics of the method. Hetzler only said that the continuous version of Newton’s method does not exhibit the fractal pattern graphs. Hence, several modifications and rectifications of Newton’s method have been suggested by researchers in different spaces. However, in the study of Newton-like methods, the problem persists in the approximation of solution and convergence until the numerical results of iterative method is discussed along with the dynamics.

In the present paper, we study the convergence and stability of the Newton-like methods by depicting the dynamics of the continuous version. We know that it is an Euler’s like initial value problem whose solution flow to the root of the equation. Firstly, we have considered the fractal patterns of different discrete Newton-like methods and then we have studied the interchange between Newton-like method, Damped Newton’s method, and continuous Newton’s method. Secondly, we have considered the complete analysis of discrete and continuous Newton’s method by describing the fractals and the stream-lines for the discrete and continuous Newton’s method respectively. In other words, we have described the discrete fractal graphs of the Newton’s method (2), Damped Newton method (6), proposed Newton method (10), and Damped proposed Newton method (11). The continuous motion of fluid flowing to the roots does not show the fractal patterns. Hence, we have introduced a new concept of the stream-line of continuous motion analogous to a discrete Newton-like method. Thus, we have plotted the stream-line graphs for the continuous motion of solutions flowing to the different roots and discrete fractal graphs of the discrete Newton-like methods.

2. Preliminary

Now, we will define the following definitions in the extended complex plane.

Definition 1. (see [15,16]) Let I be a subset of the complex numbers \mathbb{C} . Let us consider $g : I \rightarrow \mathbb{C}$ be a rational map on the Riemann sphere, then a point z_0 is said to be a fixed point of g if

$$[g(z_0) = z_0.]$$

Again, for any point $z \in \mathbb{C}$, the Orbit of the point z can be defined as the set

$$[Orb(z) = \{z, g(z), g^2(z), \dots, g^n(z), \dots\}.]$$

Definition 2. (see [15,16]) A periodic point z_0 is said to be of period k if \exists a smallest positive integer k i.e., $g^k(z_0) = z_0$.

Remark 1. If z_0 is periodic point of period k , then clearly it is a fixed point for g^k .

Definition 3. (see [15,16]) Let z^* be a zero of the function F , then the basin of attraction of the zero z^* is defined as the set of collection of all initial approximations z_0 such that any numerical iterative method with z_0 converges to z^* . This can be written as

$$B(z^*) = [z_0 : z_n = g^n(z_0) \text{ converges } \rightarrow z^*]. \tag{7}$$

Here, g^n is any fixed point iterative method.

Remark 2. For example in case of Newton’s method

$$[z_{n+1} = g(z_n),]$$

$$[g(z_n) = z_n - \frac{F(z_n)}{F'(z_n)}, \quad n = 0, 1, 2, \dots.]$$

We can write the basin of attraction of the zero z^* for the Newton’s method as follows:

$$[B(z^*) = [z_0 : z_n = g^n(z_0) \text{ converges } \rightarrow z^*].]$$

Definition 4. (see [15]) Let I be a subset of \mathbb{C} . Let us consider the map $g : I \rightarrow \mathbb{C}$ be a rational map on the Riemann sphere. Then, fixed point z_0 of map g is said to be a

1. attracting if and only if $|g'(z_0)| < 1$
2. super-attracting if and only if $|g'(z_0)| = 0$
3. repelling if and only if $|g'(z_0)| > 1$

Remark 3. From the above definition, it is clear that all the fixed points of the Newton’s method are super attracting fixed points, because if $z_0 = z^*$ then for Newton’s method we have

$$g'(z^*) = 1 - \frac{F'(z^*)F'(z^*) - F(z^*)F''(z^*)}{F'(z^*)^2} = 0$$

Hence, z^* is super attracting fixed points for the Newton’s method.

Definition 5. (see [15,16]) The Julia set of a nonlinear map $g(z)$ is denoted as $J(g)$ and is defined as a set consisting of the closure of its repelling periodic points. The complement of the Julia set $J(g)$ is called as the Fatou set $F(g)$.

Remark 4.

- (i) Sometimes the Julia set of a nonlinear map may also be defined as the common boundary shared by basins of the roots and the Fatou set may also be defined as the set which contains the basin of attraction.

- (ii) The Fatou set of a nonlinear map may also be defined as the solution space and the Julia set of a nonlinear map may also be defined as error space.
- (iii) Fractals are very complicated phenomenon that may be defined as a self-similar surprising geometric object which repeated at every small scale ([17]). Weierstrass function (1872) which is continuous everywhere, but nowhere differentiable and the Cantor set (1883) are probably the two simplest mathematical examples of a fractal.

3. Description of Method

We know that in the case of starlike functions for example $F(z) = |z|^\alpha$ where $0 < \alpha < 1/2$ and $z \in \mathbb{R}$, Newton’s method fails or diverges for any suitable initial approximations. For such functions, Newton’s method increases the distance from the solution at every iteration. To overcome this situation, many researchers considered the Damped version of Newton’s method (2), for a small $h > 0$, as follows:

$$z_{n+1} = z_n - h \frac{F(z_n)}{F'(z_n)}, \quad n = 0, 1, 2, \dots \tag{8}$$

Now, plotting the basins of attraction for the Newton’s method (2) and its Damped version (8), we get a remarkable result that the basins of attraction of Damped Newton’s method (8) get larger and their fractal boundary structure shrinks away as $h \rightarrow 0$ (see Figure 1) and it leads the continuous version of Newton’s method (9), which does not show any fractal boundary structure. This result also supports the fact that smaller step sizes in numerical approximations always improve the accuracy of the solutions.

$$z(t+h) - z(t) = -h \frac{F(z(t))}{F'(z(t))},$$

$$\lim_{h \rightarrow 0} \frac{z(t+h) - z(t)}{h} = - \frac{F(z(t))}{F'(z(t))}.$$

Here on replacing t by a natural number $n \in \mathbb{N}$ and letting $h = 1$, the above equation reduces to (2). Again, by the above equation we have

$$\begin{cases} \frac{dz}{dt} = - \frac{F(z)}{F'(z)}, \\ z(0) = z_0. \end{cases} \tag{9}$$

This is called the continuous version of Newton’s method. Applying the Euler’s method to solve (9), we get

$$z_{n+1} - z_n = -h \frac{F(z_n)}{F'(z_n)}. \tag{10}$$

Clearly, Euler’s method becomes Newton’s method with an initial approximation z_0 when stepsize $h = 1$. Thus, we have seen an inter-conversion between Newton, Damped Newton, and continuous Newton’s methods, but when discrete Newton’s method ceases to work properly then its continuous version is also affected by this drawback. At this moment, the following question naturally arises.

Question 1 Can we obtain a Newton-like method (continuous and discrete) which may work effectively in these conditions? We have tried to answer this question in the rest of the paper by considering the continuous version. For this purpose we have followed the same approach of previous authors.

Consider a modified version of Newton’s method, which still can give the solution when Newton’s method fails.

$$z_{n+1} = z_n - \frac{F(z_n)}{\lambda_n F(z_n) + F'(z_n)}, \quad n = 0, 1, 2, \dots \tag{11}$$

An error equation for above quadratically convergent method (11) is given by $e_{n+1} = (A_2 + \lambda_n)e_n^2 + O(e_n^3)$, where $A_2 = \frac{1}{2} \frac{F''(z_n)}{F'(z_n)}$ (see [11]). Hence, on choosing $\lambda_n = -A_2$, we get the better approximation to the solution or a minimum error. Now, neglecting the derivative terms, we have $\lambda_n = -\frac{1}{2}$. Keeping this in mind, we have taken the value of arbitrary parameter λ_n in the closed interval $[-1,1]$ with the purpose that the denominator term in (11) is non-zero. Hence, the damped version of the above method (11) will have bigger basins and less chaotic behavior for $h > 0$. Thus, the continuous version of the method (11) will converge quickly to the solution for any starting point (see Figure 1). Therefore, we consider the Damped and continuous version of the proposed method (11) as follows:

$$z_{n+1} - z_n = -h \frac{F(z_n)}{\lambda_n F(z_n) + F'(z_n)}, \quad n = 0, 1, 2, \dots, \tag{12}$$

$$z(t+h) - z(t) = -h \frac{F(z(t))}{\lambda_n F(z(t)) + F'(z(t))},$$

$$\lim_{h \rightarrow 0} \frac{z(t+h) - z(t)}{h} = - \frac{F(z(t))}{\lambda_n F(z(t)) + F'(z(t))}.$$

From above, we get

$$\begin{cases} \frac{dz}{dt} = - \frac{F(z)}{\lambda_n F(z) + F'(z)}, \\ z(0) = z_0. \end{cases} \tag{13}$$

Equation (13) is the continuous version of the proposed method (11). Applying the Euler’s like method to solve (13), we get

$$z_{n+1} = z_n - \frac{hF(z_n)}{\lambda_n F(z_n) + F'(z_n)}, \quad n = 0, 1, 2, \dots,$$

where λ_n is a parameter such that $\lambda_n \in [-1, 1]$ and z_n is an approximate solution of $z(t)$ at $t = nh$. Thus, for the step size $h = 1$, an improved version of Euler’s like method produces an improved Newton-like method (11).

Now, we first study the Newton method (2) and the proposed Newton-like method (11) locally by an example $F(z) = 10z \exp(-z^2) - 1$, $z^* = 1.67963061042845$, for two arbitrary initial points 0.7 and 2.0 (see Table 1). Clearly, Newton’s method is divergent for the point 0.7 while the proposed method converges to the root much faster among the methods. For detailed analysis, please see [10].

Next, we discuss the Newton method (2), proposed Newton-like method (11) and their Damped versions (e.g., Damped Newton method (8), Damped proposed Newton-like method (12)) globally by plotting the fractal pattern graph of these methods where the basin of attraction of proposed method and its damped version is much larger than Newton’s method. Lastly, on the basis of performances of the proposed method and its Damped version, we have concluded that the continuous versions of the proposed method will be much better.

We consider 700×700 points in a square $\mathbb{R} \times \mathbb{R} = [-5.5, 5.5] \times [-5.5, 5.5]$ for the study of the dynamics with tolerance- $|F(z_n)| < 5 \times 10^{-2}$ and a maximum of 30 iterations (see [15,16]). We have obtained the basin of attraction for the considered example by Newton method (2), proposed Newton-like method ($h = 1$) (11) and its Damped versions for $h = 3/4$ in Equations (8) and (12) (see Figure 1). We can deduce from the Figure 1, that the proposed method (11) and the Damped proposed method ($h = 3/4$) (12) contains Fatou set with blue color having a larger basins (solution space) and Julia set with red color having less chaotic behavior (error space) in the right-hand side of the fractal patterns (see [17,18]). In summary, using the discrete proposed method when we move from $h = 1$ to $h = 3/4$

solution space is maximized and error space is minimized (see Figure 1). Therefore, the continuous version ($h \rightarrow 0$) of the proposed method will produce much better results.

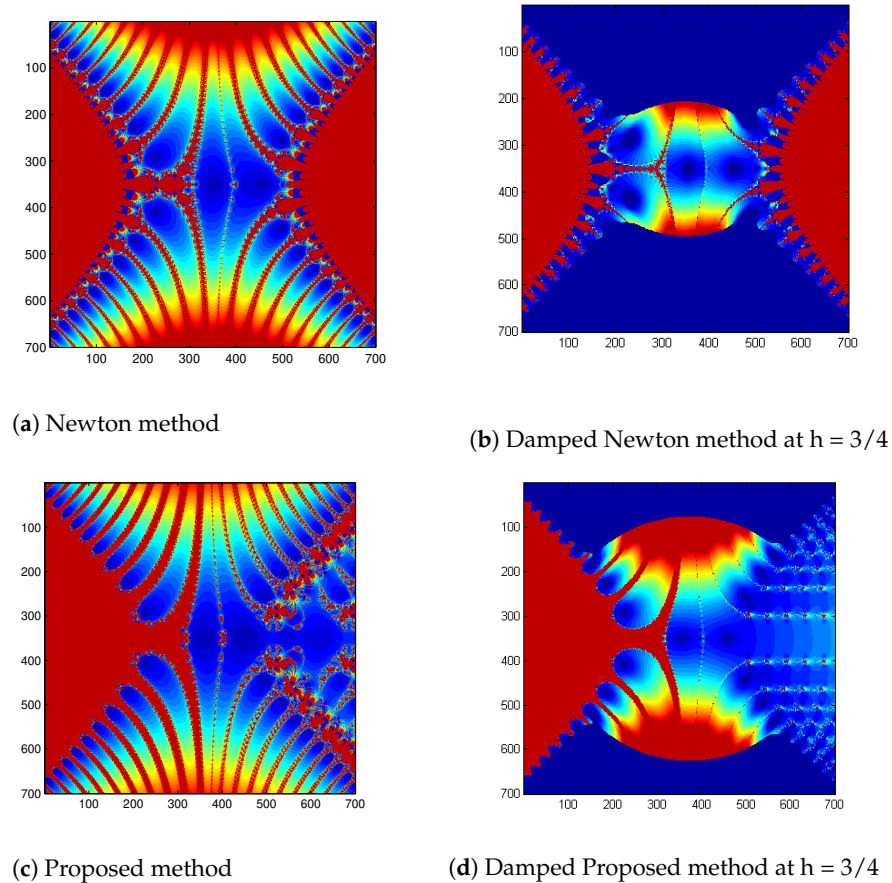


Figure 1. Basin of attraction for function $F(z) = 10z \exp(-z^2) - 1$ by different methods.

Table 1. Comparison of methods for $F(z) = 10z \exp(-z^2) - 1$, $z^* = 1.67963061042845$ “IT” is the number of iterations.

$F(z)$	z_0	Proposed method		Wang et al. [13]		Newton Method	
		IT	$ F(z_n) $	IT	$ F(z_n) $	IT	$ F(z_n) $
$F(z)$	0.7	5	2.220×10^{-16}	5	2.220×10^{-16}	-	Divergent
	2	3	2.220×10^{-16}	4	2.220×10^{-16}	6	2.220×10^{-16}

Thus, we have introduced a continuous version of the method (13) which not only relax some conditions (Theorem 1) to give the affirmative answer to the **question 1** but also increases the velocity of the convergence towards the roots of $F(z) = 0$. Later on, we have shown that the dynamics of the continuous version of Newton’s or Newton-like method does not show the fractal boundary and chaotic behavior. Hence, the new concept streamline of the solution flowing the roots has been introduced.

Theorem 1. [9]: Let I be an open subset of \mathbb{R} and $F : I \rightarrow \mathbb{R}$ be a function such that

(D1) it has simple root $z^* \in I$,

(D2) F and F' be continuous in I , and z^* is an end point of I .

Then for any point z_0 the iterative scheme (13) has a unique solution which converges to the root z^* as $t \rightarrow \infty$.

Proof. Let us consider the proposed iterative method (13) in an inverted approach

$$\frac{dt}{dz} = -\frac{\lambda_n F(z) + F'(z)}{F(z)}, \quad t(z_0) = 0.$$

By condition D_2 , of Theorem 1, F and F' are continuous on I (F' may be zero). Therefore, by the existence and uniqueness theorem, there exists a unique solution to the above initial value problem, namely

$$\begin{aligned} -t(z) &= \lambda_n z + \ln|F(z)| - \lambda_n z_0 - \ln|F(z_0)|, \\ -t(z) &= \lambda_n(z - z_0) + \ln|F(z)| - \ln|F(z_0)|, \\ -t(z) &= \lambda_n(z - z_0) + \ln\left(\frac{|F(z)|}{|F(z_0)|}\right), \\ F(z)\exp(\lambda_n z) &= F(z_0)\exp(\lambda_n z_0 - t). \end{aligned} \tag{14}$$

From above we have $\lim_{z \rightarrow z^*} t(z) = \infty$. Since F and F' have constant sign on open interval I , so does

$$-\frac{F(z)}{\lambda_n F(z) + F'(z)},$$

which means that $t(z)$ is invertible. Hence, the inverse function $z(t)$ is the unique solution of (13). Since z and t are inverse functions; therefore, the relation $\lim_{z \rightarrow z^*} t(z) = \infty$, implies that

$$\lim_{t \rightarrow \infty} z(t) = z^*.$$

Hence, proved. \square

Remark 5. Using Equation (14), the solution of (13) can be given as

$$\exp(\lambda_n z)F(z) = \exp(\lambda_n z_0 - t)F(z_0),$$

or

$$F(z) = F(z_0)e^{-\lambda_n(z-z_0)-t}.$$

Since $F(z)$ and $F(z_0)$ have same sign on the open interval I ; therefore, F decays exponentially to 0 along the solution curve $z(t)$ as $t \rightarrow \infty$. Clearly solving (13) for $z(t)$ and taking the limit as $t \rightarrow \infty$ is more lengthy and difficult than solving the equation $F(z) = 0$ directly.

Numerical Results

Example 1.

$$F_1(z) = |z|^\alpha, \quad \text{for } z \in \mathbb{R}, \quad 0 < \alpha < 1/2. \tag{15}$$

It has the single root $z^* = 0$. We can see that the sequence generated by Newton's method (2) will diverge for any starting value $z_0 \neq 0$. Solving it by the proposed continuous version of Newton-like method (13) we have

$$\begin{aligned} -t &= \lambda z + \alpha \log z + c, \\ z(t) &= z_0 e^{-\frac{\lambda}{\alpha}(z-z_0)} e^{(-\frac{t}{\alpha})}. \end{aligned}$$

Taking the limit $t \rightarrow \infty$ we have $z(t) = 0$. Clearly $z(t) \rightarrow z^* = 0$ as $t \rightarrow \infty$ for any initial estimates z_0 for $F_1(z) = z^\alpha$.

Example 2.

$$F_2(z) = \frac{(2z - 1)}{(z - 3)}. \tag{16}$$

It has a single root $z^* = 1/2$. Newton’s method with $z_0 = 3.0$ produces the same point. Solving the test example by the proposed continuous version of Newton-like method (13), we have

$$\begin{aligned} \exp(\lambda z)F_2(z) &= \exp(\lambda z_0 - t)F_2(z_0), \\ \exp(\lambda z)\frac{(2z - 1)}{(z - 3)} &= \exp(\lambda z_0 - t)\frac{(2z_0 - 1)}{(z_0 - 3)}. \end{aligned}$$

Taking the limit $t \rightarrow \infty$ we have $\exp(\lambda z)\frac{(2z-1)}{(z-3)} = 0$. Since $\exp(\lambda z) \neq 0$. Hence we have $\frac{(2z-1)}{(z-3)} = 0$, or $z = \frac{1}{2}$. Clearly $z(t) \rightarrow z^* = \frac{1}{2}$ as $t \rightarrow \infty$ for any initial estimates z_0 for the function $F_2(z) = \frac{(2z-1)}{(z-3)}$.

Next, to describe the dynamics of Example 1 and Example 2 by discrete Newton’s and proposed Newton-like method, we have investigated the dynamics by plotting $F_1(z) = z^{1/3}$ and $F_2(z) = \frac{(2z-1)}{(z-3)}$ in the square $\mathbb{R} \times \mathbb{R} = [-3.5, 3.5] \times [-3.5, 3.5]$ of 700×700 points with a tolerance- $|F(z_n)| < 5 \times 10^{-2}$ and a maximum of 30 iterations. Figures 2 and 3 demonstrate the basin of attraction of root in the described region for Newton’s method (8), Damped Newton’s method (8) and the Damped proposed Newton-like method (12). It is clear from Figure 2 that the basins for Newton’s method do not exist except for a point indicating the root due to the critical points, whereas the Damped Newton method (8) and Damped proposed method (12) show the smooth basins of attraction with no chaotic behavior and fractal boundaries.

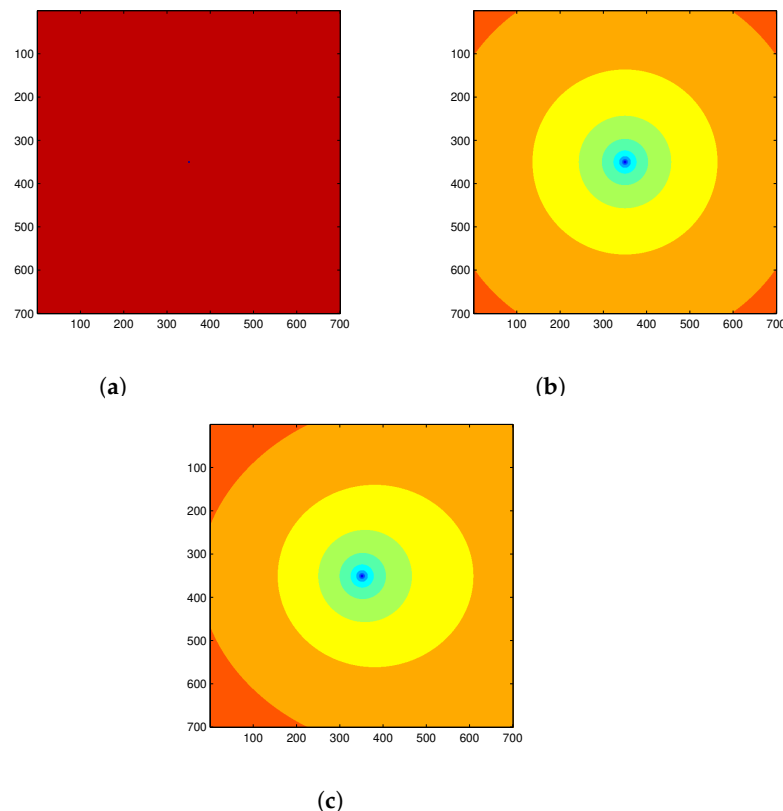


Figure 2. Basin of attraction for function Example 1, $F_1(z) = z^\alpha$ ($\alpha = 1/3$) by different methods. (a) Newton method (2), (b) Damped Newton method (8) at $h = 0.50$, (c) Damped Proposed method (12) at $h = 0.50$ and $\lambda = 0.01$.

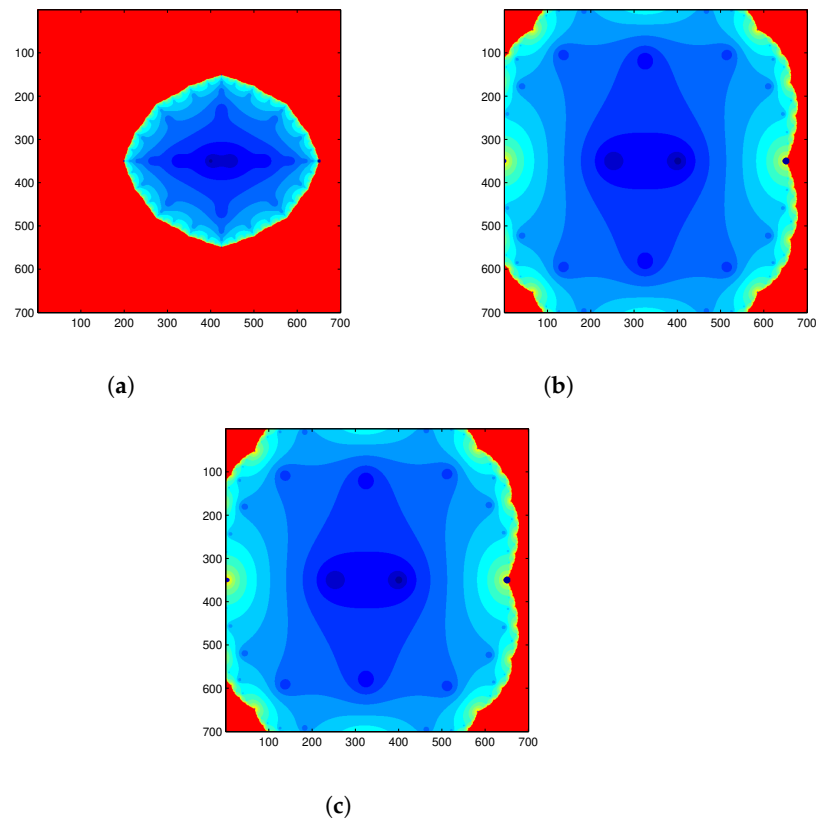


Figure 3. Basin of attraction for function Example 2, $F_2(z) = \frac{(2z-1)}{(z-3)}$ by different methods. (a) Newton method (2), (b) Damped Newton method (8) at $h = 0.50$, (c) Damped Proposed method (12) at $h = 0.50$ and $\lambda = 0.01$.

4. Connection between Discrete and Continuous Version Using Liapunov Function

Now, we will discuss the continuous version of the method (13), in terms of Liapunov function $g(z) = \exp(\lambda_n z)F(z)$, where λ_n is a parameter such that $\lambda_n \in [-1, 1]$. Consider a dynamic system

$$\frac{dz}{dt} = G(z) = -\frac{g(z)}{g'(z)} = -\frac{F(z)}{\lambda_n F(z) + F'(z)}, \quad z(0) = z_0, \tag{17}$$

Applying Euler’s method to solve (17), we get the sequence

$$\begin{aligned} z_{n+1} &= z_n + hG(z), \quad n = 0, 1, 2, \dots \\ z_{n+1} &= z_n + h\frac{dz}{dt}, \quad n = 0, 1, 2, \dots \\ z_{n+1} &= z_n - h\frac{F(z)}{\lambda_n F(z) + F'(z)}, \quad n = 0, 1, 2, \dots \end{aligned}$$

For $h = 1$, and $0 < \lambda_n < \infty$, above Newton-like Method reduces to the method obtained by Wu [12].

Theorem 2. Let $(a, b) \in \mathbb{R}$ be an open interval and $F : (a, b) \rightarrow \mathbb{R}$ be a function such that

- (i) $F(a)F(b) < 0$,
- (ii) F is twice continuously differentiable on (a, b) ,
- (iii) $\lambda_n F(z) + F'(z) \neq 0, \forall z \in [a, b]$,
- (iv) $G(z)$ defined by (17) is Lipschitz continuous.

Then, there exists a unique saturated solution or motion in $[0, +\infty)$ of dynamic system (17).

Now, we may admit that the solution in the discrete case of Newton’s or the Newton-like method (for real or complex functions) are a numerical approximation of continuous flow of solution to the roots. Usually, this approximation converges to the roots but sometimes it fails or diverges. We consider the example $F_3(z) = z^3 - 1$, solving it by the continuous version of the proposed method (14), we get

$$z(t) = \sqrt[3]{(z_0^3 - 1)e^{-\lambda_n(z-z_0)}e^{-t} + 1}.$$

In this case, the corresponding basins by proposed method are defined by the ternary division of the plane by the rays $\theta = \pi/3, \theta = \pi,$ and $\theta = -\pi/3$.

From this fact, we can say that the fractal basin boundary and chaotic behavior arise in the discrete Newton’s or Newton-like method due to the discretization of the continuous Newton flow, and the complex fractal boundary arises because we have taken a discrete numerical approximation of continuous flow. We have shown the discrete numerical approximation of $F_3(z) = z^3 - 1$ by proposed damped Newton-like method in a square $\mathbb{R} \times \mathbb{R} = [-3.5, 3.5] \times [-3.5, 3.5]$ of 700×700 points in Figure 4. We have applied the iterative method to find the zero z^* with a tolerance- $|F(z_n)| < 5 \times 10^{-2}$ and a maximum of 30 iterations. We can see that, the associated basin of attraction get larger and their fractal boundary structure shrinks away for smaller stepsize h , and this is in agreement with the numerical fact that the smaller step sizes may lead to improve accuracy in an approximation of the functions.

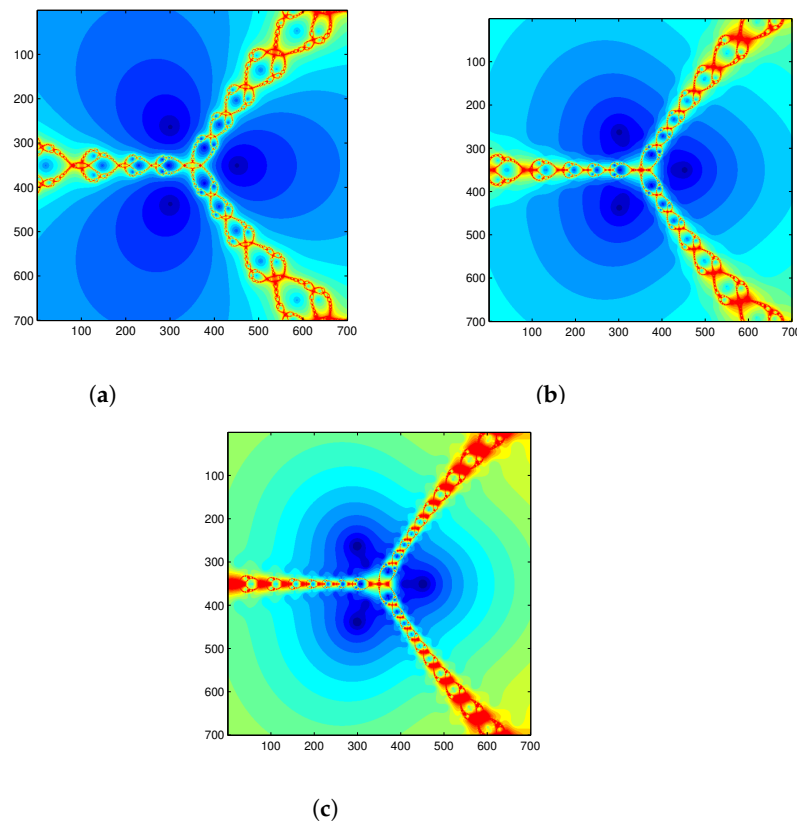


Figure 4. Basin of attraction for function $F_3(z) = z^3 - 1$ by different methods. (a) Modified Newton method $h = 1.0$, (b) Damped Modified Newton method $h = 0.75$, (c) Damped Modified Newton method $h = 0.50$.

At the same time, to describe the continuous Newton flow, we have introduced a new concept streamline in the next section. Streamlines represent the velocity of solution flowing to the different roots. We have shown the streamlines of the continuous flow for the above-mentioned problem $F_3(z) = z^3 - 1$ along with some other problems.

5. Streamline of the Solution Flowing to Roots

Theorem 3. Let discrete Newton’s method (2) and Damped Newton’s method (8) show the discrete fractal patterns graph. Then, its analogous continuous version (9) does not show the fractal pattern graph, but rather represents the velocity of streamlines flow flowing to the roots given by

$$\frac{dz}{dt} = -\frac{F(z)}{F'(z)}, \quad z(0) = z_0.$$

Proof. The damped Newton method (8) for Newton’s method (2) is given by

$$z(t + h) - z(t) = -h \frac{F(z(t))}{F'(z(t))},$$

$$\lim_{h \rightarrow 0} \frac{z(t + h) - z(t)}{h} = -\frac{F(z(t))}{F'(z(t))},$$

where $t > 0 \in \mathbb{R}$. On replacing t by a natural number $n \in \mathbb{N}$ and letting $h = 1$, above equation reduces to (2). We can write above equation as

$$\begin{cases} \frac{dz}{dt} = -\frac{F(z(t))}{F'(z(t))}, \\ z(0) = z_0. \end{cases} \tag{18}$$

Which implies the continuous Newton method (9) given by

$$\frac{dz}{dt} = -\frac{F(z)}{F'(z)}, \quad z(0) = z_0.$$

Furthermore, the continuous version of the Newton method can be taken as an initial value problem in which the solution flows to the different roots. Here, the left-hand side dz/dt in the above IVP (18) may be considered as the streamlines of the velocity for the solution flowing to the roots. Similarly, we may consider the continuous version of proposed Newton-like method (13) as follows

$$\frac{dz}{dt} = -\frac{F(z)}{\lambda_n F(z) + F'(z)}, \quad z(0) = z_0,$$

where $\lambda_n \in [-1, 1]$. Here again, the left-hand side dz/dt may be considered as the streamlines of the continuous motion for the velocity of the fluid flow. Next, we will describe the streamline for the velocity of the fluid flow for Example 2 (F_2) and some other examples by using the continuous Newton method (9) and the proposed continuous Newton-like method (13). □

Remark 6. Streamline can be viewed as the abstract representation of the (continuous) motion of points in fluid flow. Generally, it represents the velocity of fluid flow at that point.

Definition 6. ([4]) A point z is said to be the critical point of F , if F is analytic at z , $F(z) \neq 0$, and $F'(z) = 0$. The value of $F(z)$ is the critical value of F at the point z .

Definition 7. ([4]) A trajectory is said to be a critical trajectory if it terminates at a critical point in-finite time in at least one-time direction.

Streamlines for Example 2, $F_2(z) = \frac{(2z-1)}{(z-3)}$ has been plotted by continuous Newton method (9) and proposed modified continuous method (13) ($\lambda = 0.01$) in Figure 5. It is clear from the figure that the point $1/2$ is an attracting point (root) while point 3 is a repelling point with a pole.

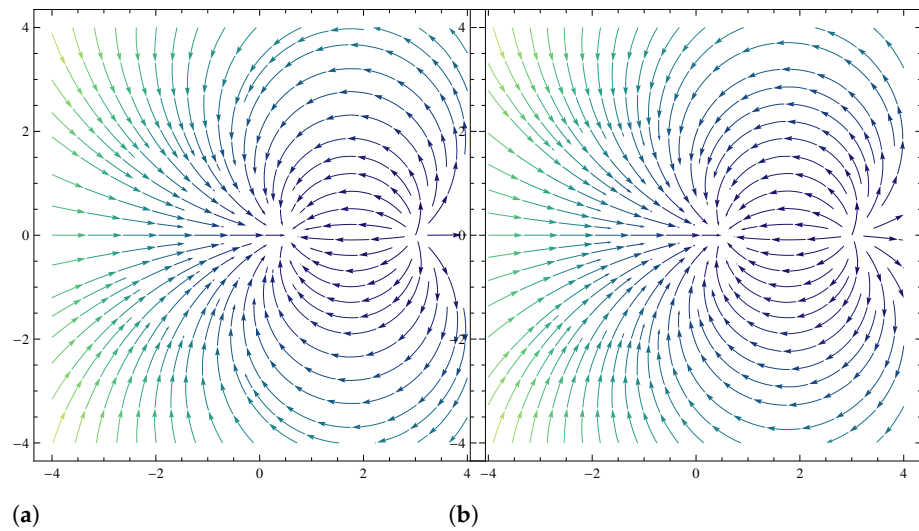


Figure 5. Streamline for Example 2, $F_2(z) = \frac{2z-1}{z-3}$ by different methods. (a) Continuous Newton method (9), (b) Proposed Continuous method (13).

For $F_3(z) = z^2 - 1$, a streamline graph has been shown in Figure 6. Streamline by the continuous Newton method (9), shows that the flow breaks at point 0 and is not transported to any of the roots 1 and -1 (Figure 6a), while for the proposed continuous method (13) at $\lambda = 0.01$ situation is completely different from Newton’s method as all initial points are transported to attracting roots 1 and -1 (Figure 6b).

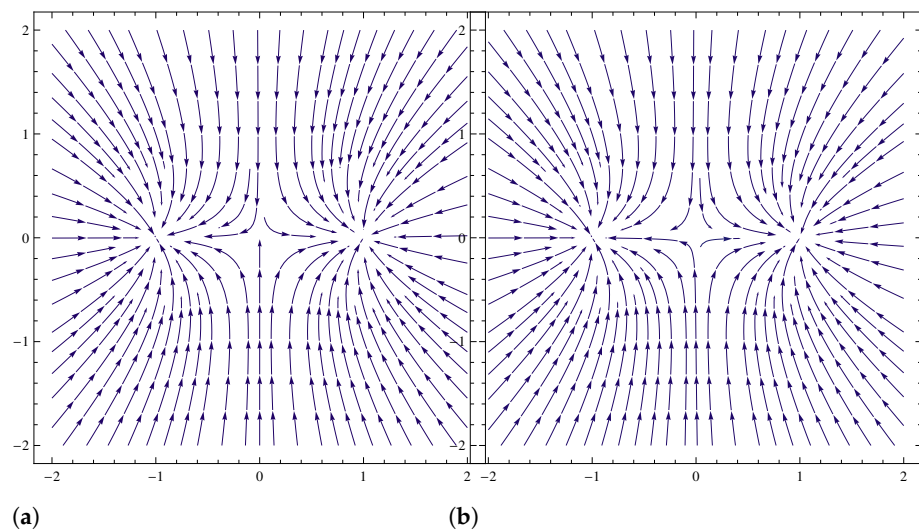


Figure 6. Streamline for $F_3(z) = z^2 - 1$ by different methods. (a) Continuous Newton method (9), (b) Proposed Continuous method (13).

For $F_4(z) = z^3 - 1$ streamlines graph has been shown in Figure 7 for the methods (9) and (13). Streamlines by the continuous Newton method (9) in Figure 7a shows that the point 0 works as a critical point for the solution flowing to the roots 1, w and w^2 as for any initial point the streamline terminates at point 0. For the proposed continuous method (13) at $\lambda = 0.001$, streamlines can be seen at the critical point 0 also and it is transported to attracting roots 1, w and w^2 (Figure 7b).

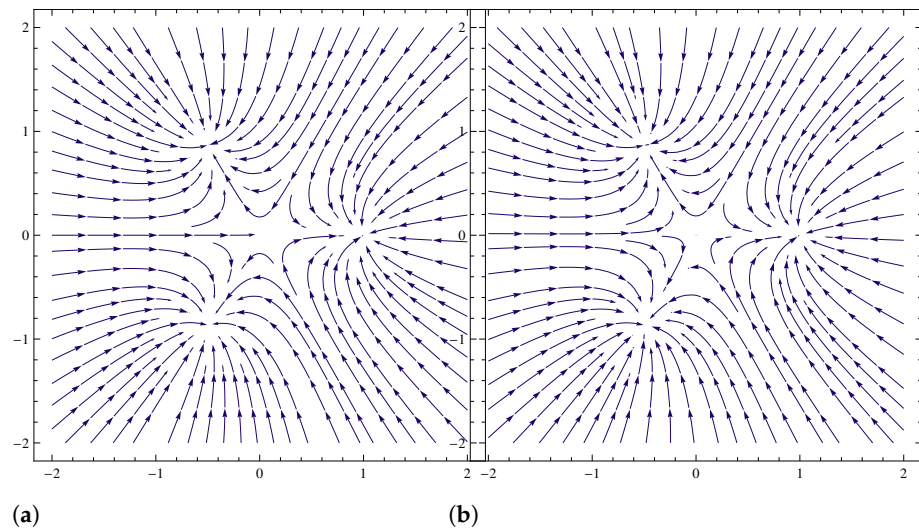


Figure 7. Streamline for $F_4(z) = z^3 - 1$ by different methods. (a) Continuous Newton method (9), (b) Proposed Continuous method (13).

6. Further Approximation of the Continuous Newton’s Method

We have already discussed that, in case of failure of the Newton method (2), sometimes the use of the Damped Newton method (8) may be advantageous.

$$z(t + h) - z(t) = -h \frac{F(z(t))}{F'(z(t))},$$

Above, the Damped newton method leads to the continuous version

$$\begin{cases} \frac{dz}{dt} = -\frac{F(z)}{F'(z)}, \\ z(0) = z_0. \end{cases} \tag{19}$$

Now, solving the initial value problem (9) i.e., (19) using the Euler’s method we get the sequence:

$$z_{n+1} - z_n = -h \frac{F(z_n)}{F'(z_n)}.$$

Further, we may solve the initial value problem (19) with the help of following different numerical methods.

- (1) Refined Euler’s method:
- (2) Heun’s method:
- (3) Second-order Runge-Kutta method:
- (4) Fourth-order Runge-Kutta method:
- (5) Adams-Bashforth method:
- (6) Milne-Simpsons method and many other methods.

These numerical methods allow us to study the continuous version of Newton’s method in a broader sense.

7. Conclusions

We have discussed the continuous version of a new Newton-like method and its different deformations using the various means. We have plotted the fractal pattern graphs for Newton’s method, the proposed Newton-like method and proposed Damped method in a discrete case to show the efficacy of the proposed method. Hence, we have shown that the continuous version does not exhibit the fractal patterns graph and it leads to the introduction of a new concept of the streamline of the velocity flowing to the roots for the continuous version of the Newton-like methods. Thus, we have explained

the transformation between Newton-like method, Damped Newton-like method, and continuous Newton-like method in a very simple and efficient manner.

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