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Improved Hille-Type Oscillation Criteria for Second-Order Quasilinear Dynamic Equations

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Abstract: In this work, we develop enhanced Hille-type oscillation conditions for arbitrary-time, second-order quasilinear functional dynamic equations. These findings extend and improve previous research that has been published in the literature. Some examples are given to demonstrate the importance of the obtained results.

Keywords: oscillation behavior; second-order; quasilinear; differential equation; dynamic equation; time scale

MSC: 34K11; 34N05; 39A10; 39A99

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1. Introduction

Oscillation phenomena take part in different models from real world applications; we refer to the papers [1,2] for models from mathematical biology where oscillation and/or delay actions may be formulated by means of cross-diffusion terms. The study of nonlinear dynamic equations is dealt within this paper because these equations arise in various real-world problems such as non-Newtonian fluid theory, the turbulent flow of a polytrophic gas in a porous medium, and in the study of $p-$Laplace equations; see, e.g., the papers [3–10] for more details. Therefore, we are interested in the oscillatory behaviour of the second-order quasilinear functional dynamic equation

$$\left[a(\xi)\left|z^{\Delta}(\xi)\right|^{\alpha-1}z^{\Delta}(\xi)\right]^{\Delta}+p(\xi)|z(g(\xi))|^{\beta-1}z(g(\xi))=0 \tag{1}$$

on an arbitrary unbounded above time scale $\mathbb{T}$, where $\xi\in[\xi_0,\infty)_{\mathbb{T}}$, $\xi_0\geq 0$, $\xi_0\in\mathbb{T}$, $\alpha,\beta>0$, $a(\xi)$ and $p(\xi)$ are positive rd-continuous functions on $\mathbb{T}$ such that $a^{\Delta}\geq 0$ and $\int_{\xi_0}^{\infty}a^{-\frac{1}{\alpha}}(\tau)\Delta\tau=\infty$, $g:\mathbb{T}\to\mathbb{T}$ is a rd-continuous function satisfying $\lim_{\xi\to\infty}g(\xi)=\infty$, and $l:=\liminf_{\xi\to\infty}\dfrac{\xi}{\sigma(\xi)}>0$.

By a solution of Equation (1), we mean a nontrivial real–valued function $z\in C^1_{\mathrm{ad}}[T_z,\infty)_{\mathbb{T}}$ for some T_z in $[\xi_0,\infty)_{\mathbb{T}}$ for a positive constant $\xi_0\in\mathbb{T}$ such that z satisfies Equation (1) on $[T_z,\infty)_{\mathbb{T}}$ and $a(\xi)\left|z^{\Delta}(\xi)\right|^{\alpha-1}z^{\Delta}(\xi)\in C^1_{\mathrm{ad}}[T_z,\infty)_{\mathbb{T}}$ where C_{ad} is the space of right-dense continuous functions.

We shall not investigate solutions which vanish in the neighborhood of infinity. A solution z of (1) is said to be oscillatory if it is neither eventually positive nor negative; otherwise, it is said to be nonoscillatory. We assume that the reader is already familiar with the fundamentals of time scales; for a very useful introduction to time scale calculus, see [11–14].

In the following, we present some oscillation results for dynamic equations that are connected to our oscillation results for (1) on time scales and explain the significant contributions of this paper. Karpuz [15] presented a Hille–Nehari test for nonoscillation/oscillation of the second order dynamic equations

$$\left[a(\xi)z^{\Delta}(\xi)\right]^{\Delta}+p(\xi)z(\xi)=0$$

and

$$\left[a(\xi)z^{\Delta}(\xi)\right]^{\Delta}+p(\xi)z(\sigma(\xi))=0.$$

and showed that the critical constant for these dynamic equations is $\frac{1}{4}$ as in the well-known cases $\mathbb{T}=\mathbb{R}$ and $\mathbb{T}=\mathbb{Z}$. Erbe et al. [16] derived a Hille-type oscillation criterion for the half-linear second order dynamic equation

$$\left(\left(z^{\Delta}(\xi)\right)^{\alpha}\right)^{\Delta}+p(\xi)z^{\alpha}(g(\xi))=0, \tag{2}$$

where $\alpha\geq 1$ is a quotient of odd positive integers and $g(\xi)\leq\xi$ for $\xi\in\mathbb{T}$, and showed that, if

$$\int_{\xi_0}^{\infty}g^{\alpha}(\tau)p(\tau)\Delta\tau=\infty, \tag{3}$$

and

$$\liminf_{\xi\to\infty}\xi^{\alpha}\int_{\sigma(\xi)}^{\infty}\left(\frac{g(\tau)}{\sigma(\tau)}\right)^{\alpha}p(\tau)\Delta\tau>\frac{\alpha^{\alpha}}{l^{\alpha^2}(\alpha+1)^{\alpha+1}}, \tag{4}$$

then all solutions to (2) oscillate.

Erbe et al. [17] established the Hille-type oscillation criterion for half-linear second order dynamic equation

$$\left(a(\xi)\left(z^{\Delta}(\xi)\right)^{\alpha}\right)^{\Delta}+p(\xi)z^{\alpha}(g(\xi))=0, \tag{5}$$

where $0<\alpha\leq 1$ is a ratio of odd positive integers and $g(\xi)\leq\xi$ for $\xi\in\mathbb{T}$, and proved that, if (3) holds and

$$\liminf_{\xi\to\infty}\frac{\xi^{\alpha}}{a(\xi)}\int_{\sigma(\xi)}^{\infty}\left(\frac{g(\tau)}{\sigma(\tau)}\right)^{\alpha}p(\tau)\Delta\tau>\frac{\alpha^{\alpha}}{l^{\alpha^2}(\alpha+1)^{\alpha+1}}, \tag{6}$$

then all solutions to (5) oscillate. Bohner et al. [3] improved conditions (4) and (6) without restricted condition (3) for half-linear second order dynamic equation

$$\left[a(\xi)\left|z^{\Delta}(\xi)\right|^{\alpha-1}z^{\Delta}(\xi)\right]^{\Delta}+p(\xi)|z(g(\xi))|^{\alpha-1}z(g(\xi))=0 \tag{7}$$

and obtained that if

$$\liminf_{\xi\to\infty}\frac{\xi^{\alpha}}{a(\xi)}\int_{\xi}^{\infty}\bar{\varphi}(\tau)p(\tau)\Delta\tau>\frac{\alpha^{\alpha}}{l^{\alpha(1-\alpha)}(\alpha+1)^{\alpha+1}},\quad 0<\alpha\leq 1 \tag{8}$$

and

$$\liminf_{\xi\to\infty}\frac{\xi^{\alpha}}{a(\xi)}\int_{\xi}^{\infty}\varphi(\tau)p(\tau)\Delta\tau>\frac{\alpha^{\alpha}}{l^{\alpha(\alpha-1)}(\alpha+1)^{\alpha+1}},\quad \alpha\geq 1, \tag{9}$$

where

$$\bar{\varphi}(\xi) := \begin{cases} \left(\frac{g(\xi)}{\sigma(\xi)}\right)^{\alpha}, & g(\xi) \le \sigma(\xi), \\ 1, & g(\xi) \ge \sigma(\xi) \end{cases}$$

and

$$\varphi(\xi) := \begin{cases} \left(\frac{g(\xi)}{\xi}\right)^{\alpha}, & g(\xi) \le \xi, \\ 1, & g(\xi) \ge \xi. \end{cases}$$

We seal by noting that Agarwal et al. [18–20], Erbe et al. [21,22], Hassan [23,24], Li and Saker [25], Saker [26], and Zhang and Li [27] established a number of Kamenev-type and Philos-type oscillation results for various classes of second-order dynamic equations. The reader is directed to papers [28–40] as well as the sources listed therein.

The goal of this paper is to find some improved Hille-type oscillation criteria for the generalized quasilinear second-order dynamic equation (1) in the cases where $\alpha \ge \beta$, $\alpha \le \beta$, $g(\xi) \le \xi$, $g(\xi) \le \sigma(\xi)$, $g(\xi) \ge \xi$, and $g(\xi) \ge \sigma(\xi)$, which improve and extend relevant significant contributions reported in [3,16,17] without the condition (3) or extra time scale constraints. In the next results, we use the notation $\gamma := \max\{\alpha, \beta\}$ and we assume that the improper integrals are convergent in the following theorems. Otherwise, we find that Equation (1) oscillates, see [41].

The content of the paper is as follows: In Section 2, we present the main results for Equation (1) for the delayed case. In Section 3, we provide the main results for Equation (1) for the advanced case, and to illustrate the significance of the results, we provide several examples on an arbitrary time scale.

2. Hille-Type Oscillation Criteria for the Delay Case

The next two theorems deal with the Hille-type oscillation criteria of the second-order quasilinear dynamic Equation (1) when $g(\xi) \le \xi$ and $g(\xi) \le \sigma(\xi)$ on $[\xi_0, \infty)_{\mathbb{T}}$, respectively.

Theorem 1. *Let $g(\xi) \le \xi$ on $[\xi_0, \infty)_{\mathbb{T}}$. If*

$$\liminf_{\xi \to \infty} \frac{\xi^{\alpha}}{a(\xi)} \int_{\xi}^{\infty} \frac{g^{\beta}(\tau)}{\tau^{\gamma}} p(\tau) \Delta\tau > \frac{\alpha^{\alpha}}{l^{\alpha|1-\alpha|}(\alpha+1)^{\alpha+1}}, \tag{10}$$

then all solutions to Equation (1) oscillate.

Proof. Suppose that (1) has a nonoscillatory solution z on $[\xi_0, \infty)_{\mathbb{T}}$. Without loss of generality, let $z(\xi) > 0$ and $z(g(\xi)) > 0$ on $[\xi_0, \infty)_{\mathbb{T}}$. According to [3] [Lemmas 2.1 and 2.2], there exists a $\xi_1 \in (\xi_0, \infty)_{\mathbb{T}}$ such that $z(\xi)$ is strictly increasing and $\frac{z(\xi)}{\xi - \xi_0}$ is strictly decreasing on $[\xi_1, \infty)_{\mathbb{T}}$. Define

$$w(\xi) := \frac{a(\xi)\left(z^{\Delta}(\xi)\right)^{\alpha}}{z^{\alpha}(\xi)}. \tag{11}$$

Hence,

$$\begin{aligned} w^{\Delta}(\xi) &= \left[a(\xi)\left(z^{\Delta}(\xi)\right)^{\alpha}\right]^{\Delta} \frac{1}{z^{\alpha}(\xi)} + \left[a(\xi)\left(z^{\Delta}(\xi)\right)^{\alpha}\right]^{\sigma} \left(\frac{1}{z^{\alpha}(\xi)}\right)^{\Delta} \\ &= \frac{\left[a(\xi)\left(z^{\Delta}(\xi)\right)^{\alpha}\right]^{\Delta}}{z^{\alpha}(\xi)} - \left[a(\xi)\left(z^{\Delta}(\xi)\right)^{\alpha}\right]^{\sigma} \frac{(z^{\alpha}(\xi))^{\Delta}}{z^{\alpha}(\xi) z^{\alpha}(\sigma(\xi))}. \end{aligned}$$

In view of (1) and (11), we have

$$w^{\Delta}(\xi) = -\frac{z^{\beta}(g(\xi))}{z^{\alpha}(\xi)} p(\xi) - \frac{(z^{\alpha}(\xi))^{\Delta}}{z^{\alpha}(\xi)} w(\sigma(\xi)). \tag{12}$$

If $\beta \leq \alpha$, by the fact that $\frac{z(\xi)}{\xi - \xi_0}$ is strictly decreasing on $[\xi_1, \infty)_{\mathbb{T}}$, we obtain for $\xi \in [\xi_1, \infty)_{\mathbb{T}}$,

$$\begin{aligned}\frac{z^{\beta}(g(\xi))}{z^{\alpha}(\xi)} &\geq \left(\frac{g(\xi)-\xi_0}{\xi-\xi_0}\right)^{\beta} z^{\beta-\alpha}(\xi) \\ &\geq \frac{(g(\xi)-\xi_0)^{\beta}}{(\xi-\xi_0)^{\alpha}}(\xi_1 - \xi_0)^{\alpha-\beta} z^{\beta-\alpha}(\xi_1),\end{aligned}$$

whereas, if $\beta \geq \alpha$, by the fact that $z(\xi)$ is nondecreasing on $[\xi_1, \infty)_{\mathbb{T}}$ as well, we obtain for $\xi \in [\xi_1, \infty)_{\mathbb{T}}$,

$$\begin{aligned}\frac{z^{\beta}(g(\xi))}{z^{\alpha}(\xi)} &\geq \left(\frac{g(\xi)-\xi_0}{\xi-\xi_0}\right)^{\beta} z^{\beta-\alpha}(\xi) \\ &\geq \left(\frac{g(\xi)-\xi_0}{\xi-\xi_0}\right)^{\beta} z^{\beta-\alpha}(\xi_1).\end{aligned}$$

Let $0 < k_1 < 1$ be arbitrary. There exists a $\xi_{k_1} \in [\xi_1, \infty)_{\mathbb{T}}$ such that

$$\frac{z^{\beta}(g(\xi))}{z^{\alpha}(\xi)} \geq k_1 \frac{g^{\beta}(\xi)}{\xi^{\gamma}}. \tag{13}$$

Substituting (13) into (12), we obtain for $\xi \in [\xi_{k_1}, \infty)_{\mathbb{T}}$,

$$w^{\Delta}(\xi) \leq -k_1 \frac{g^{\beta}(\xi)}{\xi^{\gamma}} p(\xi) - \frac{(z^{\alpha}(\xi))^{\Delta}}{z^{\alpha}(\xi)} w(\sigma(\xi)). \tag{14}$$

(I) $0 < \alpha \leq 1$. The result of applying the Pötzsche chain rule (see [13] [Theorem 1.90]) is

$$\frac{(z^{\alpha}(\xi))^{\Delta}}{z^{\alpha}(\xi)} \geq \alpha \left(\frac{z(\xi)}{z(\sigma(\xi))}\right)^{1-\alpha} \frac{z^{\Delta}(\xi)}{z(\xi)}. \tag{15}$$

In addition,

$$\left(\frac{z(\xi)}{z(\sigma(\xi))}\right)^{1-\alpha} \geq \left(\frac{\xi - \xi_0}{\sigma(\xi) - \xi_0}\right)^{1-\alpha},$$

by dint that $\frac{z(\xi)}{\xi - \xi_0}$ is strictly decreasing. Let $0 < k_2 < 1$ be arbitrary. There exists a $\xi_{k_2} \in [\xi_{k_1}, \infty)_{\mathbb{T}}$ such that

$$\left(\frac{z(\xi)}{z(\sigma(\xi))}\right)^{1-\alpha} \geq k_2 \left(\frac{\xi}{\sigma(\xi)}\right)^{1-\alpha} \tag{16}$$

Hence, (15) becomes

$$\frac{(z^{\alpha}(\xi))^{\Delta}}{z^{\alpha}(\xi)} \geq \alpha k_2 \left(\frac{\xi}{\sigma(\xi)}\right)^{1-\alpha} \frac{z^{\Delta}(\xi)}{z(\xi)} \tag{17}$$

Substituting (17) into (14), we obtain for $\xi \in [\xi_{k_2}, \infty)_{\mathbb{T}}$,

$$w^{\Delta}(\xi) \leq -k_1 \frac{g^{\beta}(\xi)}{\xi^{\gamma}} p(\xi) - \alpha k_2 \left(\frac{\xi}{\sigma(\xi)}\right)^{1-\alpha} a^{-\frac{1}{\alpha}}(\xi) w^{\frac{1}{\alpha}}(\xi) w(\sigma(\xi)), \tag{18}$$

which yields that $w^{\Delta} < 0$. Now, for any $\epsilon > 0$, there exists a $\xi \in [\xi_{k_2}, \infty)_{\mathbb{T}}$ such that, for $\xi \in [\xi, \infty)_{\mathbb{T}}$,

$$\frac{\xi}{\sigma(\xi)} \geq l - \epsilon \quad \text{and} \quad \frac{\xi^{\alpha} w(\xi)}{a(\xi)} \geq a_* - \epsilon, \tag{19}$$

where

$$a_* := \liminf_{\xi \to \infty} \frac{\xi^{\alpha} w(\xi)}{a(\xi)}, \quad 0 \leq a_* \leq 1.$$

In view of (18) and (19), we have

$$w^{\Delta}(\xi) \leq -k_1 \frac{g^{\beta}(\xi)}{\xi^{\gamma}} p(\xi) - \alpha k_2 (l - \epsilon)^{1-\alpha} (a_* - \epsilon)^{1+\frac{1}{\alpha}} \frac{a(\sigma(\xi))}{\xi\, \sigma^{\alpha}(\xi)}. \tag{20}$$

Integrating (20) from ξ to v, we conclude that

$$w(v) - w(\xi) \leq -k_1 \int_{\xi}^{v} \frac{g^{\beta}(\tau)}{\tau^{\gamma}} p(\tau) \Delta\tau - \alpha k_2 (l - \epsilon)^{1-\alpha} (a_* - \epsilon)^{1+\frac{1}{\alpha}} \int_{\xi}^{v} \frac{a(\sigma(\tau))}{\tau\, \sigma^{\alpha}(\tau)} \Delta\tau.$$

Taking into consideration that $w > 0$ and passing to the limit as $v \to \infty$, we obtain

$$k_1 \int_{\xi}^{\infty} \frac{g^{\beta}(\tau)}{\tau^{\gamma}} p(\tau) \Delta\tau \leq w(\xi) - \alpha k_2 (l - \epsilon)^{1-\alpha} (a_* - \epsilon)^{1+\frac{1}{\alpha}} \int_{\xi}^{\infty} \frac{a(\sigma(\tau))}{\tau\, \sigma^{\alpha}(\tau)} \Delta\tau. \tag{21}$$

Multiplying both sides of (21) by $\dfrac{\xi^{\alpha}}{a(\xi)}$ and the fact that $a(\xi)$ is nondecreasing, we find that

$$\begin{aligned} k_1 \frac{\xi^{\alpha}}{a(\xi)} \int_{\xi}^{\infty} \frac{g^{\beta}(\tau)}{\tau^{\gamma}} p(\tau) \Delta\tau \leq\ & \frac{\xi^{\alpha}}{a(\xi)} w(\xi) \\ & - \alpha k_2 (l - \epsilon)^{1-\alpha} (a_* - \epsilon)^{1+\frac{1}{\alpha}} \frac{\xi^{\alpha}}{a(\xi)} \int_{\xi}^{\infty} \frac{a(\sigma(\tau))}{\tau\, \sigma^{\alpha}(\tau)} \Delta\tau \\ \leq\ & \frac{\xi^{\alpha}}{a(\xi)} w(\xi) \\ & - k_2 (l - \epsilon)^{1-\alpha} (a_* - \epsilon)^{1+\frac{1}{\alpha}} \xi^{\alpha} \int_{\xi}^{\infty} \frac{\alpha}{\tau\, \sigma^{\alpha}(\tau)} \Delta\tau. \end{aligned} \tag{22}$$

Use the Pötzsche chain, it follows that

$$\left(\frac{-1}{\tau^{\alpha}} \right)^{\Delta} = \frac{(\tau^{\alpha})^{\Delta}}{\tau^{\alpha} \sigma^{\alpha}(\tau)} \leq \frac{\alpha}{\tau \sigma^{\alpha}(\tau)} \tag{23}$$

Substituting (23) into (22), we achieve

$$\begin{aligned} k_1 \frac{\xi^{\alpha}}{a(\xi)} \int_{\xi}^{\infty} \frac{g^{\beta}(\tau)}{\tau^{\gamma}} p(\tau) \Delta\tau \leq\ & \frac{\xi^{\alpha}}{a(\xi)} w(\xi) - k_2 (l - \epsilon)^{1-\alpha} (a_* - \epsilon)^{1+\frac{1}{\alpha}} \xi^{\alpha} \int_{\xi}^{\infty} \left(\frac{-1}{\tau^{\alpha}} \right)^{\Delta} \Delta\tau \\ =\ & \frac{\xi^{\alpha}}{a(\xi)} w(\xi) - k_2 (l - \epsilon)^{1-\alpha} (a_* - \epsilon)^{1+\frac{1}{\alpha}}. \end{aligned}$$

We obtain by taking the lim inf on both sides of the latter inequality as $\xi \to \infty$ that

$$\liminf_{\xi \to \infty} k_1 \frac{\xi^{\alpha}}{a(\xi)} \int_{\xi}^{\infty} \frac{g^{\beta}(\tau)}{\tau^{\gamma}} p(\tau) \Delta\tau \leq a_* - k_2 (l - \epsilon)^{1-\alpha} (a_* - \epsilon)^{1+\frac{1}{\alpha}}.$$

Since $0 < k_1, k_2 < 1$ and $\epsilon > 0$ are arbitrary, we deduce that

$$\liminf_{\xi \to \infty} \frac{\xi^{\alpha}}{a(\xi)} \int_{\xi}^{\infty} \frac{g^{\beta}(\tau)}{\tau^{\gamma}} p(\tau) \Delta\tau \leq a_* - l^{1-\alpha} a_*^{1+\frac{1}{\alpha}}.$$

Let

$$A = l^{1-\alpha}, \quad B = 1, \quad \text{and} \quad u = a_*.$$

Using the inequality (see[42])

$$Bu - Au^{1+\frac{1}{\alpha}} \leq \frac{\alpha^{\alpha}}{(\alpha+1)^{\alpha+1}} \frac{B^{\alpha+1}}{A^{\alpha}}, \quad A > 0, \tag{24}$$

we see that

$$\liminf_{\xi \to \infty} \frac{\xi^{\alpha}}{a(\xi)} \int_{\xi}^{\infty} \frac{g^{\beta}(\tau)}{\tau^{\gamma}} p(\tau) \Delta\tau \leq \frac{\alpha^{\alpha}}{l^{\alpha(1-\alpha)}(\alpha+1)^{\alpha+1}},$$

which contradicts (10) with $0 < \alpha \leq 1$.

(II) $\alpha \geq 1$. The result of applying Pötzsche chain rule (see [13] [Theorem 1.90]) is

$$\frac{(z^{\alpha}(\xi))^{\Delta}}{z^{\alpha}(\xi)} \geq \alpha \frac{z^{\Delta}(\xi)}{z(\xi)}.$$

Hence, by (11), (14) becomes

$$w^{\Delta}(\xi) \leq -k_1 \frac{g^{\beta}(\xi)}{\xi^{\gamma}} p(\xi) - \alpha a^{-\frac{1}{\alpha}}(\xi) w^{\frac{1}{\alpha}}(\xi) w(\sigma(\xi)). \tag{25}$$

According to (25) and (19), it follows that, for $\xi \in [\xi_{k_1}, \infty)_{\mathbb{T}}$,

$$w^{\Delta}(\xi) \leq -k_1 \frac{g^{\beta}(\xi)}{\xi^{\gamma}} p(\xi) - \alpha(l-\epsilon)^{\alpha-1}(a_* - \epsilon)^{1+\frac{1}{\alpha}} \frac{a(\sigma(\xi))}{\xi^{\alpha}\, \sigma(\xi)}. \tag{26}$$

Integrating (26) from ξ to v, we have

$$w(v) - w(\xi) \leq -k_1 \int_{\xi}^{v} \frac{g^{\beta}(\tau)}{\tau^{\gamma}} p(\tau) \Delta\tau - \alpha(l-\epsilon)^{\alpha-1}(a_* - \epsilon)^{1+\frac{1}{\alpha}} \int_{\xi}^{v} \frac{a(\sigma(\tau))}{\tau^{\alpha}\, \sigma(\tau)} \Delta\tau.$$

Taking into consideration that $w > 0$ and passing to the limit as $v \to \infty$, we obtain

$$-w(\xi) \leq -k_1 \int_{\xi}^{v} \frac{g^{\beta}(\tau)}{\tau^{\gamma}} p(\tau) \Delta\tau - \alpha(l-\epsilon)^{\alpha-1}(a_* - \epsilon)^{1+\frac{1}{\alpha}} \int_{\xi}^{v} \frac{a(\sigma(\tau))}{\tau^{\alpha}\, \sigma(\tau)} \Delta\tau. \tag{27}$$

Multiplying both sides of (27) by $\dfrac{\xi^{\alpha}}{a(\xi)}$ and the fact that $a(\xi)$ is nondecreasing, we obtain

$$\begin{aligned} k_1 \frac{\xi^{\alpha}}{a(\xi)} \int_{\xi}^{\infty} \frac{g^{\beta}(\tau)}{\tau^{\gamma}} p(\tau) \Delta\tau &\leq \frac{\xi^{\alpha}}{a(\xi)} w(\xi) - \alpha(l-\epsilon)^{\alpha-1}(a_* - \epsilon)^{1+\frac{1}{\alpha}} \frac{\xi^{\alpha}}{a(\xi)} \int_{\xi}^{\infty} \frac{a(\sigma(\tau))}{\tau^{\alpha}\, \sigma(\tau)} \Delta\tau \\ &\leq \frac{\xi^{\alpha}}{a(\xi)} w(\xi) - (l-\epsilon)^{\alpha-1}(a_* - \epsilon)^{1+\frac{1}{\alpha}} \xi^{\alpha} \int_{\xi}^{\infty} \frac{\alpha}{\tau^{\alpha}\, \sigma(\tau)} \Delta\tau. \end{aligned} \tag{28}$$

Applying the Pötzsche chain rule, we obtain

$$\left(\frac{-1}{\tau^{\alpha}}\right)^{\Delta} = \frac{(\tau^{\alpha})^{\Delta}}{\tau^{\alpha}\sigma^{\alpha}(\tau)} \leq \frac{\alpha}{\tau^{\alpha}\sigma(\tau)}. \tag{29}$$

Substituting (29) into (28), we arrive at

$$\begin{aligned} k_1 \frac{\xi^\alpha}{a(\xi)} \int_\xi^\infty \frac{g^\beta(\tau)}{\tau^\gamma} p(\tau)\Delta\tau &\leq \frac{\xi^\alpha}{a(\xi)} w(\xi) - (l-\epsilon)^{\alpha-1}(a_* - \epsilon)^{1+\frac{1}{\alpha}} \xi^\alpha \int_\xi^\infty \left(\frac{-1}{\tau^\alpha}\right)^\Delta \Delta\tau \\ &= \frac{\xi^\alpha}{a(\xi)} w(\xi) - (l-\epsilon)^{\alpha-1}(a_* - \epsilon)^{1+\frac{1}{\alpha}}. \end{aligned}$$

We obtain by taking the lim inf on both sides of the latter inequality as $\xi \to \infty$ that

$$\liminf_{\xi\to\infty} k_1 \frac{\xi^\alpha}{a(\xi)} \int_\xi^\infty \frac{g^\beta(\tau)}{\tau^\gamma} p(\tau)\Delta\tau \leq a_* - (l-\epsilon)^{\alpha-1}(a_* - \epsilon)^{1+\frac{1}{\alpha}}.$$

Since $0 < k_1 < 1$ and $\epsilon > 0$ are arbitrary, we deduce that

$$\liminf_{\xi\to\infty} \frac{\xi^\alpha}{a(\xi)} \int_\xi^\infty \frac{g^\beta(\tau)}{\tau^\gamma} p(\tau)\Delta\tau \leq a_* - l^{\alpha-1} a_*^{1+\frac{1}{\alpha}}.$$

Applying the inequality (24) with

$$A = l^{\alpha-1}, \quad B = 1, \quad \text{and} \quad u = a_*.$$

Hence,

$$\liminf_{\xi\to\infty} \frac{\xi^\alpha}{a(\xi)} \int_\xi^\infty \frac{g^\beta(\tau)}{\tau^\gamma} p(\tau)\Delta\tau \leq \frac{\alpha^\alpha}{l^{\alpha(\alpha-1)}(\alpha+1)^{\alpha+1}},$$

which contradicts (10) with $\alpha \geq 1$. This completes the proof. □

Theorem 2. *Let $g(\xi) \leq \sigma(\xi)$ on $[\xi_0, \infty)_{\mathbb{T}}$. If*

$$\liminf_{\xi\to\infty} \frac{\xi^\alpha}{a(\xi)} \int_\xi^\infty \frac{g^\beta(\tau)}{\sigma^\gamma(\tau)} p(\tau)\Delta\tau > \frac{\alpha^\alpha}{l^{\alpha|\alpha-1|}(\alpha+1)^{\alpha+1}}, \tag{30}$$

then all solutions to Equation (1) oscillate.

Proof. Suppose, on the contrary, that z is a nonoscillatory solution of (1) on $[\xi_0, \infty)_{\mathbb{T}}$. Without loss of generality, we may assume $z(\xi) > 0$ and $z(g(\xi)) > 0$ for $\xi \in [\xi_0, \infty)_{\mathbb{T}}$.

According to [3] [Lemmas 2.1 and 2.2], there exists a $\xi_1 \in (\xi_0, \infty)_{\mathbb{T}}$ such that $z(\xi)$ is strictly increasing and $\dfrac{z(\xi)}{\xi - \xi_0}$ is strictly decreasing on $[\xi_1, \infty)_{\mathbb{T}}$. Define a function w as in (11). Using the product and quotient rules, we have

$$\begin{aligned} w^\Delta(\xi) &= \left[a(\xi)\left(z^\Delta(\xi)\right)^\alpha\right]^\Delta \frac{1}{z^\alpha(\sigma(\xi))} + a(\xi)\left(z^\Delta(\xi)\right)^\alpha \left(\frac{1}{z^\alpha(\xi)}\right)^\Delta \\ &= \frac{\left[a(\xi)\left(z^\Delta(\xi)\right)^\alpha\right]^\Delta}{z^\alpha(\sigma(\xi))} - a(\xi)\left(z^\Delta(\xi)\right)^\alpha \frac{(z^\alpha(\xi))^\Delta}{z^\alpha(\xi) z^\alpha(\sigma(\xi))}. \end{aligned}$$

By dint of (1) and (11),

$$w^\Delta(\xi) = -\frac{z^\beta(g(\xi))}{z^\alpha(\sigma(\xi))} p(\xi) - \frac{(z^\alpha(\xi))^\Delta}{z^\alpha(\sigma(\xi))} w(\xi). \tag{31}$$

If $\beta \leq \alpha$, by the fact that $\dfrac{z(\xi)}{\xi-\xi_0}$ is strictly decreasing on $[\xi_1,\infty)_{\mathbb{T}}$, we obtain for $\xi \in [\xi_1,\infty)_{\mathbb{T}}$,

$$\begin{aligned}\frac{z^\beta(g(\xi))}{z^\alpha(\sigma(\xi))} &\geq \left(\frac{g(\xi)-\xi_0}{\sigma(\xi)-\xi_0}\right)^\beta z^{\beta-\alpha}(\sigma(\xi))\\ &\geq \frac{(g(\xi)-\xi_0)^\beta}{(\sigma(\xi)-\xi_0)^\alpha}(\xi_1-\xi_0)^{\alpha-\beta} z^{\beta-\alpha}(\xi_1),\end{aligned}$$

whereas, if $\beta \geq \alpha$, by the fact that $z(\xi)$ is nondecreasing on $[\xi_1,\infty)_{\mathbb{T}}$ as well, we obtain for $\xi \in [\xi_1,\infty)_{\mathbb{T}}$,

$$\begin{aligned}\frac{z^\beta(g(\xi))}{z^\alpha(\sigma(\xi))} &\geq \left(\frac{g(\xi)-\xi_0}{\sigma(\xi)-\xi_0}\right)^\beta z^{\beta-\alpha}(\sigma(\xi))\\ &\geq \left(\frac{g(\xi)-\xi_0}{\sigma(\xi)-\xi_0}\right)^\beta z^{\beta-\alpha}(\xi_1).\end{aligned}$$

Let $0<k_1<1$ be arbitrary. There exists a $\xi_{k_1} \in [\xi_1,\infty)_{\mathbb{T}}$ such that

$$\frac{z^\beta(g(\xi))}{z^\alpha(\sigma(\xi))} \geq k_1 \frac{g^\beta(\xi)}{\sigma^\gamma(\xi)}. \tag{32}$$

Substituting (13) into (31), we obtain for $\xi \in [\xi_{k_1},\infty)_{\mathbb{T}}$,

$$w^\Delta(\xi) \leq -k_1 \frac{g^\beta(\xi)}{\sigma^\gamma(\xi)} p(\xi) - \frac{(z^\alpha(\xi))^\Delta}{z^\alpha(\sigma(\xi))} w(\xi). \tag{33}$$

(I) $0<\alpha\leq 1$. Using the Pötzsche chain rule and the fact that $\dfrac{z(\xi)}{\xi-\xi_0}$ is strictly decreasing, we obtain for $\xi \in [\xi_{k_2},\infty)_{\mathbb{T}} \subseteq [\xi_{k_1},\infty)_{\mathbb{T}}$,

$$\begin{aligned}\frac{(z^\alpha(\xi))^\Delta}{z^\alpha(\sigma(\xi))} &\geq \alpha\frac{z^\Delta(\xi)}{z(\sigma(\xi))} \geq \alpha k_2\left(\frac{\xi}{\sigma(\xi)}\right)\frac{z^\Delta(\xi)}{z(\xi)}\\ &= \alpha k_2 a^{-\frac{1}{\alpha}}(\xi)\left(\frac{\xi}{\sigma(\xi)}\right) w^{\frac{1}{\alpha}}(\xi).\end{aligned}$$

Hence,

$$\begin{aligned}w^\Delta(\xi) &\leq -k_1 \frac{g^\beta(\xi)}{\sigma^\gamma(\xi)} p(\xi) - \alpha\frac{z^\Delta(\xi)}{z(\sigma(\xi))} w(\xi)\\ &\leq -k_1 \frac{g^\beta(\xi)}{\sigma^\gamma(\xi)} p(\xi) - \alpha k_2 a^{-\frac{1}{\alpha}}(\xi)\left(\frac{\xi}{\sigma(\xi)}\right) w^{1+\frac{1}{\alpha}}(\xi).\end{aligned} \tag{34}$$

Integrating (34) from ξ to v, we conclude that

$$w(v)-w(\xi) \leq -k_1\int_\xi^v \frac{g^\beta(\tau)}{\sigma^\gamma(\tau)} p(\tau)\Delta\tau - \alpha k_2\int_\xi^v a^{-\frac{1}{\alpha}}(\tau)\left(\frac{\tau}{\sigma(\tau)}\right) w^{1+\frac{1}{\alpha}}(\tau)\Delta\tau,$$

and thus

$$-w(\xi) \leq -k_1\int_\xi^\infty \frac{g^\beta(\tau)}{\sigma^\gamma(\tau)} p(\tau)\Delta\tau - \alpha k_2\int_\xi^\infty a^{-\frac{1}{\alpha}}(\tau)\left(\frac{\tau}{\sigma(\tau)}\right) w^{1+\frac{1}{\alpha}}(\tau)\Delta\tau. \tag{35}$$

Multiplying both sides of (35) by $\dfrac{\xi^\alpha}{a(\xi)}$, we obtain

$$-\frac{\xi^{\alpha}}{a(\xi)}w(\xi) \leq -k_1\frac{\xi^{\alpha}}{a(\xi)}\int_{\xi}^{\infty}\frac{g^{\beta}(\tau)}{\sigma^{\gamma}(\tau)}p(\tau)\Delta\tau - \alpha k_2\frac{\xi^{\alpha}}{a(\xi)}\int_{\xi}^{\infty}a^{-\frac{1}{\alpha}}(\tau)\left(\frac{\tau}{\sigma(\tau)}\right)w^{1+\frac{1}{\alpha}}(\tau)\Delta\tau$$
$$= -k_1\frac{\xi^{\alpha}}{a(\xi)}\int_{\xi}^{\infty}\frac{g^{\beta}(\tau)}{\sigma^{\gamma}(\tau)}p(\tau)\Delta\tau - \alpha k_2\frac{\xi^{\alpha}}{a(\xi)}\int_{\xi}^{\infty}\left(\frac{a(\tau)}{\tau^{\alpha}\sigma(\tau)}\right)\left(\frac{\tau^{\alpha}w(\tau)}{a(\tau)}\right)^{1+\frac{1}{\alpha}}\Delta\tau. \quad (36)$$

Now, for any $\epsilon > 0$, there exists a $\xi \in [\xi_k, \infty)_{\mathbb{T}}$ such that, for $\xi \in [\xi, \infty)_{\mathbb{T}}$,

$$\frac{\xi}{\sigma(\xi)} \geq l - \epsilon \quad \text{and} \quad \frac{\xi^{\alpha}w(\xi)}{a(\xi)} \geq a_* - \epsilon,$$

where

$$a_* := \liminf_{\xi\to\infty}\frac{\xi^{\alpha}w(\xi)}{a(\xi)}, \quad 0 \leq a_* \leq 1.$$

Then, (36) becomes

$$-\frac{\xi^{\alpha}}{a(\xi)}w(\xi) \leq -k_1\frac{\xi^{\alpha}}{a(\xi)}\int_{\xi}^{\infty}\frac{g^{\beta}(\tau)}{\sigma^{\gamma}(\tau)}p(\tau)\Delta\tau$$
$$-k_2\frac{\xi^{\alpha}}{a(\xi)}(l-\epsilon)^{1-\alpha}(a_*-\epsilon)^{1+\frac{1}{\alpha}}\int_{\xi}^{\infty}\frac{\alpha\, a(\tau)}{\tau\sigma^{\alpha}(\tau)}\Delta\tau. \quad (37)$$

Since, by Pötzsche chain rule, we have

$$\left(\frac{-1}{\tau^{\alpha}}\right)^{\Delta} = \frac{(\tau^{\alpha})^{\Delta}}{\tau^{\alpha}\sigma^{\alpha}(\tau)} \leq \frac{\alpha}{\tau\sigma^{\alpha}(\tau)}.$$

It follows now from $a^{\Delta} \geq 0$ and (37) that

$$-\frac{\xi^{\alpha}}{a(\xi)}w(\xi) \leq -k\frac{\xi^{\alpha}}{a(\xi)}\int_{\xi}^{\infty}\frac{g^{\beta}(\tau)}{\sigma^{\gamma}(\tau)}p(\tau)\Delta\tau$$
$$-k_2\xi^{\alpha}(l-\epsilon)^{1-\alpha}(a_*-\epsilon)^{1+\frac{1}{\alpha}}\int_{\xi}^{\infty}\frac{\alpha}{\tau\sigma^{\alpha}(\tau)}\Delta\tau$$
$$\leq -k\frac{\xi^{\alpha}}{a(\xi)}\int_{\xi}^{\infty}\frac{g^{\beta}(\tau)}{\sigma^{\gamma}(\tau)}p(\tau)\Delta\tau$$
$$-k_2(l-\epsilon)^{1-\alpha}(a_*-\epsilon)^{1+\frac{1}{\alpha}}\xi^{\alpha}\int_{\xi}^{\infty}\left(\frac{-1}{\tau^{\alpha}}\right)^{\Delta}\Delta\tau$$
$$= -k\frac{\xi^{\alpha}}{a(\xi)}\int_{\xi}^{\infty}\frac{g^{\beta}(\tau)}{\sigma^{\gamma}(\tau)}p(\tau)\Delta\tau - k_2(l-\epsilon)^{1-\alpha}(a_*-\epsilon)^{1+\frac{1}{\alpha}},$$

which yields that

$$k_1\frac{\xi^{\alpha}}{a(\xi)}\int_{\xi}^{\infty}\frac{g^{\beta}(\tau)}{\sigma^{\gamma}(\tau)}p(\tau)\Delta\tau \leq \frac{\xi^{\alpha}}{a(\xi)}w(\xi) - k_2(l-\epsilon)^{1-\alpha}(a_*-\epsilon)^{1+\frac{1}{\alpha}}.$$

We obtain by taking the lim inf on both sides of the latter inequality as $\xi \to \infty$ that

$$\liminf_{\xi\to\infty}k_1\frac{\xi^{\alpha}}{a(\xi)}\int_{\xi}^{\infty}\frac{g^{\beta}(\tau)}{\sigma^{\gamma}(\tau)}p(\tau)\Delta\tau \leq a_* - k_2(l-\epsilon)^{1-\alpha}(a_*-\epsilon)^{1+\frac{1}{\alpha}}.$$

By virtue of the facts that $0 < k_1, k_2 < 1$ and $\epsilon > 0$ are arbitrary, we conclude that

$$\liminf_{\xi\to\infty}\frac{\xi^{\alpha}}{a(\xi)}\int_{\xi}^{\infty}\frac{g^{\beta}(\tau)}{\sigma^{\gamma}(\tau)}p(\tau)\Delta\tau \leq a_* - l^{1-\alpha}a_*^{1+\frac{1}{\alpha}}.$$

Letting $A = l^{1-\alpha}$, $B = 1$, and $u = a_*$, and using the inequality (24), we arrive at

$$\liminf_{\xi\to\infty} \frac{\xi^\alpha}{a(\xi)} \int_\xi^\infty \frac{g^\beta(\tau)}{\sigma^\gamma(\tau)} p(\tau)\Delta\tau \le \frac{\alpha^\alpha}{l^{\alpha(1-\alpha)}(\alpha+1)^{\alpha+1}},$$

which contradicts (30) with $0 < \alpha \le 1$.

(II) $\alpha \ge 1$. Using the Pötzsche chain rule and the fact that $\frac{z(\xi)}{\xi-\xi_0}$ is strictly decreasing, we obtain for $\xi \in [\xi_{k_2}, \infty)_{\mathbb{T}} \subseteq [\xi_{k_1}, \infty)_{\mathbb{T}}$,

$$\begin{aligned} \frac{(z^\alpha(\xi))^\Delta}{z^\alpha(\sigma(\xi))} &\ge \alpha\left(\frac{z(\xi)}{z(\sigma(\xi))}\right)^\alpha \frac{z^\Delta(\xi)}{z(\xi)} \\ &\ge \alpha k_2 \left(\frac{\xi}{\sigma(\xi)}\right)^\alpha \frac{z^\Delta(\xi)}{z(\xi)} = \alpha k_2 a^{-\frac{1}{\alpha}}(\xi)\left(\frac{\xi}{\sigma(\xi)}\right)^\alpha w^{\frac{1}{\alpha}}(\xi). \end{aligned}$$

Hence,

$$\begin{aligned} w^\Delta(\xi) &\le -k_1 \frac{g^\beta(\xi)}{\sigma^\gamma(\xi)} p(\xi) - \alpha \frac{z^\Delta(\xi)}{z(\sigma(\xi))} w(\xi) \\ &\le -k_1 \frac{g^\beta(\xi)}{\sigma^\gamma(\xi)} p(\xi) - \alpha k_2 a^{-\frac{1}{\alpha}}(\xi)\left(\frac{\xi}{\sigma(\xi)}\right)^\alpha w^{1+\frac{1}{\alpha}}(\xi). \end{aligned} \tag{38}$$

Integrating (38) from ξ to v, we arrive at

$$w(v) - w(\xi) \le -k_1 \int_\xi^v \frac{g^\beta(\tau)}{\sigma^\gamma(\tau)} p(\tau)\Delta\tau - \alpha k_2 \int_\xi^v a^{-\frac{1}{\alpha}}(\tau)\left(\frac{\tau}{\sigma(\tau)}\right)^\alpha w^{1+\frac{1}{\alpha}}(\tau)\Delta\tau,$$

and thus

$$-w(\xi) \le -k_1 \int_\xi^\infty \frac{g^\beta(\tau)}{\sigma^\gamma(\tau)} p(\tau)\Delta\tau - \alpha k_2 \int_\xi^\infty a^{-\frac{1}{\alpha}}(\tau)\left(\frac{\tau}{\sigma(\tau)}\right)^\alpha w^{1+\frac{1}{\alpha}}(\tau)\Delta\tau. \tag{39}$$

Multiplying both sides of (39) by $\frac{\xi^\alpha}{a(\xi)}$, we have

$$\begin{aligned} -\frac{\xi^\alpha}{a(\xi)} w(\xi) &\le -k_1 \frac{\xi^\alpha}{a(\xi)} \int_\xi^\infty \frac{g^\beta(\tau)}{\sigma^\gamma(\tau)} p(\tau)\Delta\tau \\ &\quad -\alpha k_2 \frac{\xi^\alpha}{a(\xi)} \int_\xi^\infty a^{-\frac{1}{\alpha}}(\tau)\left(\frac{\tau}{\sigma(\tau)}\right)^\alpha w^{1+\frac{1}{\alpha}}(\tau)\Delta\tau \\ &= -k_1 \frac{\xi^\alpha}{a(\xi)} \int_\xi^\infty \frac{g^\beta(\tau)}{\sigma^\gamma(\tau)} p(\tau)\Delta\tau \\ &\quad -\alpha k_2 \frac{\xi^\alpha}{a(\xi)} \int_\xi^\infty \left(\frac{a(\tau)}{\tau\,\sigma^\alpha(\tau)}\right)\left(\frac{\tau^\alpha w(\tau)}{a(\tau)}\right)^{1+\frac{1}{\alpha}} \Delta\tau. \end{aligned} \tag{40}$$

Now, for any $\epsilon > 0$, there exists a $\xi \in [\xi_k, \infty)_{\mathbb{T}}$ such that, for $\xi \in [\xi, \infty)_{\mathbb{T}}$,

$$\frac{\xi}{\sigma(\xi)} \ge l - \epsilon \quad \text{and} \quad \frac{\xi^\alpha w(\xi)}{a(\xi)} \ge a_* - \epsilon,$$

where

$$a_* := \liminf_{\xi\to\infty} \frac{\xi^\alpha w(\xi)}{a(\xi)}, \quad 0 \le a_* \le 1.$$

Then, (40) becomes

$$-\frac{\xi^\alpha}{a(\xi)}w(\xi) \le -k_1\frac{\xi^\alpha}{a(\xi)}\int_\xi^\infty \frac{g^\beta(\tau)}{\sigma^\gamma(\tau)}p(\tau)\Delta\tau - k_2\frac{\xi^\alpha}{a(\xi)}(l-\epsilon)^{\alpha-1}(a_*-\epsilon)^{1+\frac{1}{\alpha}}\int_\xi^\infty \frac{\alpha\, a(\tau)}{\tau^\alpha\sigma(\tau)}\Delta\tau. \quad (41)$$

By Pötzsche chain rule, we have

$$\left(\frac{-1}{\tau^\alpha}\right)^\Delta = \frac{(\tau^\alpha)^\Delta}{\tau^\alpha\sigma^\alpha(\tau)} \le \frac{\alpha}{\tau^\alpha\sigma(\tau)}.$$

It follows now from $a^\Delta \ge 0$ and (41) that

$$\begin{aligned}-\frac{\xi^\alpha}{a(\xi)}w(\xi) &\le -k_1\frac{\xi^\alpha}{a(\xi)}\int_\xi^\infty \frac{g^\beta(\tau)}{\sigma^\gamma(\tau)}p(\tau)\Delta\tau \\ &\quad -k_2\xi^\alpha(l-\epsilon)^{\alpha-1}(a_*-\epsilon)^{1+\frac{1}{\alpha}}\int_\xi^\infty \frac{\alpha}{\tau^\alpha\sigma(\tau)}\Delta\tau \\ &\le -k_1\frac{\xi^\alpha}{a(\xi)}\int_\xi^\infty \frac{g^\beta(\tau)}{\sigma^\gamma(\tau)}p(\tau)\Delta\tau \\ &\quad -k_2(l-\epsilon)^{\alpha-1}(a_*-\epsilon)^{1+\frac{1}{\alpha}}\xi^\alpha\int_\xi^\infty \left(\frac{-1}{\tau^\alpha}\right)^\Delta\Delta\tau \\ &= -k_1\frac{\xi^\alpha}{a(\xi)}\int_\xi^\infty \frac{g^\beta(\tau)}{\sigma^\gamma(\tau)}p(\tau)\Delta\tau - k_2(l-\epsilon)^{\alpha-1}(a_*-\epsilon)^{1+\frac{1}{\alpha}},\end{aligned}$$

which implies that

$$k_1\frac{\xi^\alpha}{a(\xi)}\int_\xi^\infty \frac{g^\beta(\tau)}{\sigma^\gamma(\tau)}p(\tau)\Delta\tau \le \frac{\xi^\alpha}{a(\xi)}w(\xi) - k_2(l-\epsilon)^{\alpha-1}(a_*-\epsilon)^{1+\frac{1}{\alpha}}.$$

We obtain by taking the lim inf on both sides of the latter inequality as $\xi \to \infty$ that

$$\liminf_{\xi\to\infty} k_1\frac{\xi^\alpha}{a(\xi)}\int_\xi^\infty \frac{g^\beta(\tau)}{\sigma^\gamma(\tau)}p(\tau)\Delta\tau \le a_* - k(l-\epsilon)^{\alpha-1}(a_*-\epsilon)^{1+\frac{1}{\alpha}}.$$

By means of the facts that $0 < k_1, k_2 < 1$ and $\epsilon > 0$ are arbitrary, we conclude that

$$\liminf_{\xi\to\infty} \frac{\xi^\alpha}{a(\xi)}\int_\xi^\infty \frac{g^\beta(\tau)}{\sigma^\gamma(\tau)}p(\tau)\Delta\tau \le a_* - l^{\alpha-1}a_*^{1+\frac{1}{\alpha}}.$$

Letting $A = l^{\alpha-1}$, $B = 1$, and $u = a_*$, and using the inequality (24), we obtain

$$\liminf_{\xi\to\infty} \frac{\xi^\alpha}{a(\xi)}\int_\xi^\infty \frac{g^\beta(\tau)}{\sigma^\gamma(\tau)}p(\tau)\Delta\tau \le \frac{\alpha^\alpha}{l^{\alpha(\alpha-1)}(\alpha+1)^{\alpha+1}},$$

which contradicts (30) with $\alpha \ge 1$. The proof is complete. □

3. Hille-Type Oscillation Criteria for the Advanced Case

The next two theorems deal with the Hille-type oscillation criteria of the second-order quasilinear dynamic Equation (1) when $g(\xi) \ge \xi$ and $g(\xi) \ge \sigma(\xi)$ on $[\xi_0,\infty)_\mathbb{T}$, respectively.

Theorem 3. *Let $g(\xi) \ge \xi$ on $[\xi_0,\infty)_\mathbb{T}$. If*

$$\liminf_{\xi\to\infty} \frac{\xi^\alpha}{a(\xi)}\int_\xi^\infty \tau^{\beta-\gamma}p(\tau)\Delta\tau > \frac{\alpha^\alpha}{l^{\alpha|1-\alpha|}(\alpha+1)^{\alpha+1}}, \quad (42)$$

then all solutions to Equation (1) *oscillate.*

Proof. Suppose, on the contrary, that z is a nonoscillatory solution of (1) on $[\xi_0,\infty)_{\mathbb{T}}$. Without loss of generality, we may assume $z(\xi) > 0$ and $z(g(\xi)) > 0$ for $\xi \in [\xi_0,\infty)_{\mathbb{T}}$. By virtue of Theorem 1, there exists a $\xi_1 \in [\xi_0,\infty)_{\mathbb{T}}$ such that (12) holds for $\xi \in [\xi_1,\infty)_{\mathbb{T}}$. If $\beta \le \alpha$, by the fact that $z(\xi)$ is nondecreasing and $\dfrac{z(\xi)}{\xi-\xi_0}$ is strictly decreasing, we obtain for $\xi \in [\xi_1,\infty)_{\mathbb{T}} \subseteq (\xi_0,\infty)_{\mathbb{T}}$,

$$\frac{z^{\beta}(g(\xi))}{z^{\alpha}(\xi)} \ge z^{\beta-\alpha}(\xi) \ge (\xi-\xi_0)^{\beta-\alpha}(\xi_1-\xi_0)^{\alpha-\beta}z^{\beta-\alpha}(\xi_1),$$

whereas, if $\beta \ge \alpha$, we obtain for $\xi \in [\xi_1,\infty)_{\mathbb{T}}$,

$$\frac{z^{\beta}(g(\xi))}{z^{\alpha}(\xi)} \ge z^{\beta-\alpha}(\xi) \ge z^{\beta-\alpha}(\xi_1).$$

Let $0 < k_1 < 1$ be arbitrary. There exists a $\xi_{k_1} \in [\xi_1,\infty)_{\mathbb{T}}$ such that

$$\frac{z^{\beta}(g(\xi))}{z^{\alpha}(\xi)} \ge k_1\, \xi^{\beta-\gamma}. \tag{43}$$

Substituting (43) into (12),we obtain for $\xi \in [\xi_{k_1},\infty)_{\mathbb{T}}$,

$$w^{\Delta}(\xi) \le -k_1\, \xi^{\beta-\gamma}p(\xi) - \frac{(z^{\alpha}(\xi))^{\Delta}}{z^{\alpha}(\xi)}w(\sigma(\xi)).$$

The remainder of the proof is similar to that of Theorem 1 and is thus omitted. □

Theorem 4. *Let $g(\xi) \ge \sigma(\xi)$ on $[\xi_0,\infty)_{\mathbb{T}}$. If*

$$\liminf_{\xi\to\infty} \frac{\xi^{\alpha}}{a(\xi)} \int_{\xi}^{\infty} \sigma^{\beta-\gamma}(\tau)p(\tau)\Delta\tau > \frac{\alpha^{\alpha}}{l^{\alpha|1-\alpha|}(\alpha+1)^{\alpha+1}}, \tag{44}$$

then all solutions to Equation (1) *oscillate.*

Proof. Suppose on the contrary that z is a nonoscillatory solution of (1) on $[\xi_0,\infty)_{\mathbb{T}}$. Without loss of generality, we may assume $z(\xi) > 0$ and $z(g(\xi)) > 0$ for $\xi \in [\xi_0,\infty)_{\mathbb{T}}$. By virtue of Theorem 2, there exists a $\xi_1 \in [\xi_0,\infty)_{\mathbb{T}}$ such that (31) holds for $\xi \in [\xi_1,\infty)_{\mathbb{T}}$. If $\beta \le \alpha$, by the fact that $z(\xi)$ is nondecreasing and $\dfrac{z(\xi)}{\xi-\xi_0}$ is strictly decreasing, we obtain for $\xi \in [\xi_1,\infty)_{\mathbb{T}} \subseteq (\xi_0,\infty)_{\mathbb{T}}$,

$$\frac{z^{\beta}(g(\xi))}{z^{\alpha}(\sigma(\xi))} \ge z^{\beta-\alpha}(\sigma(\xi)) \ge (\sigma(\xi)-\xi_0)^{\beta-\alpha}(\xi_1-\xi_0)^{\alpha-\beta}z^{\beta-\alpha}(\xi_1),$$

whereas, if $\beta \ge \alpha$, we obtain for $\xi \in [\xi_1,\infty)_{\mathbb{T}}$,

$$\frac{z^{\beta}(g(\xi))}{z^{\alpha}(\sigma(\xi))} \ge z^{\beta-\alpha}(\sigma(\xi)) \ge z^{\beta-\alpha}(\xi_1).$$

Let $0 < k_1 < 1$ be arbitrary. There exists a $\xi_{k_1} \in [\xi_1,\infty)_{\mathbb{T}}$ such that

$$\frac{z^{\beta}(g(\xi))}{z^{\alpha}(\sigma(\xi))} \ge k_1\, \sigma^{\beta-\gamma}(\xi). \tag{45}$$

Substituting (45) into (31), we obtain for $\xi \in [\xi_{k_1}, \infty)_{\mathbb{T}}$,

$$w^{\Delta}(\xi) \leq -k_1\, \sigma^{\beta-\gamma}(\xi) p(\xi) - \frac{(z^{\alpha}(\xi))^{\Delta}}{z^{\alpha}(\sigma(\xi))} w(\xi).$$

The remainder of the proof is similar to that of Theorem 2 and is so omitted. □

Remark 1. *By Theorems 1, 2, 3 and 2, it is clear that the second-order Euler dynamic equations*

$$\xi\sigma(\xi) z^{\Delta\Delta}(\xi) + \lambda z(\xi) = 0 \tag{46}$$

and

$$\xi\sigma(\xi) z^{\Delta\Delta}(\xi) + \lambda z(\sigma(\xi)) = 0, \tag{47}$$

are oscillatory if $\lambda > 1/4$, since, for Equation (46), we have

$$\liminf_{\xi\to\infty} \frac{\xi^{\alpha}}{a(\xi)} \int_{\xi}^{\infty} \frac{g^{\beta}(\tau)}{\tau^{\gamma}} p(\tau)\Delta\tau = \liminf_{\xi\to\infty} \frac{\xi^{\alpha}}{a(\xi)} \int_{\xi}^{\infty} \tau^{\beta-\gamma} p(\tau)\Delta\tau = \lambda \liminf_{\xi\to\infty} \xi \int_{\xi}^{\infty} \frac{\Delta\tau}{\tau\sigma(\tau)} = \lambda,$$

and for Equation (47), we have

$$\liminf_{\xi\to\infty} \frac{\xi^{\alpha}}{a(\xi)} \int_{\xi}^{\infty} \frac{g^{\beta}(\tau)}{\sigma^{\gamma}(\tau)} p(\tau)\Delta\tau = \liminf_{\xi\to\infty} \frac{\xi^{\alpha}}{a(\xi)} \int_{\xi}^{\infty} \sigma^{\beta-\gamma}(\tau) p(\tau)\Delta\tau = \lambda \liminf_{\xi\to\infty} \xi \int_{\xi}^{\infty} \frac{\Delta\tau}{\tau\sigma(\tau)} = \lambda.$$

It is well known that this is the best possible case for the second-order Euler differential equation $\xi^2 z''(\xi) + \lambda z(\xi) = 0$.

4. Examples

The applications of the theoretical findings in this paper are shown in the examples below.

Example 1. *For $\xi \in [\xi_0, \infty)_{\mathbb{T}}$, consider a second-order quasilinear delay dynamic equation*

$$\left[\sqrt[4]{\xi} \frac{z^{\Delta}(\xi)}{\sqrt[4]{|z^{\Delta}(\xi)|}}\right]^{\Delta} + \frac{\lambda}{2\sqrt{\sigma(\xi)}\sqrt[4]{\xi g^3(\xi)}} \frac{z(g(\xi))}{\sqrt[4]{|z(g(\xi))|}} = 0, \quad g(\xi) \leq \xi, \tag{48}$$

where $\lambda > 0$. Here, $\alpha = \beta = \dfrac{3}{4}$, $a(\xi) = \sqrt[4]{\xi}$, and $p(\xi) = \dfrac{\lambda}{2\sqrt{\sigma(\xi)}\sqrt[4]{\xi g^3(\xi)}}$. It is clear that $a^{\Delta}(\xi) \geq 0$ on $[\xi_0, \infty)_{\mathbb{T}}$ and

$$\int_{\xi_0}^{\infty} a^{-\frac{1}{\alpha}}(\tau)\Delta\tau = \int_{\xi_0}^{\infty} \frac{\Delta\tau}{\sqrt[3]{\tau}} = \infty.$$

We will show that the results of this paper improve those reported in [3,17] for Equation (48) for $l < 1$. Now,

$$\begin{aligned} \liminf_{\xi\to\infty} \frac{\xi^{\alpha}}{a(\xi)} \int_{\sigma(\xi)}^{\infty} \left(\frac{g(\tau)}{\sigma(\tau)}\right)^{\alpha} p(\tau)\Delta\tau &= \lambda \liminf_{\xi\to\infty} \sqrt{\xi} \int_{\sigma(\xi)}^{\infty} \frac{\Delta\tau}{2\sqrt[4]{\tau\sigma^5(\tau)}} \\ &\geq \lambda \sqrt[4]{l^3} \liminf_{\xi\to\infty} \sqrt{\xi} \int_{\sigma(\xi)}^{\infty} \left(\frac{-1}{\sqrt{\tau}}\right)^{\Delta} \Delta\tau \\ &= \lambda \sqrt[4]{l^5}, \end{aligned}$$

$$\begin{aligned}\liminf_{\xi\to\infty}\frac{\xi^{\alpha}}{a(\xi)}\int_{\xi}^{\infty}\left(\frac{g(\tau)}{\sigma(\tau)}\right)^{\alpha}p(\tau)\Delta\tau &= \lambda\liminf_{\xi\to\infty}\sqrt{\xi}\int_{\xi}^{\infty}\frac{\Delta\tau}{2\sqrt[4]{\tau\sigma^{5}(\tau)}}\\ &\geq \lambda\sqrt[4]{l^{3}}\liminf_{\xi\to\infty}\sqrt{\xi}\int_{\xi}^{\infty}\left(\frac{-1}{\sqrt{\tau}}\right)^{\Delta}\Delta\tau\\ &= \lambda\sqrt[4]{l^{3}},\end{aligned}$$

and

$$\begin{aligned}\liminf_{\xi\to\infty}\frac{\xi^{\alpha}}{a(\xi)}\int_{\xi}^{\infty}\frac{g^{\beta}(\tau)}{\tau^{\gamma}}p(\tau)\Delta\tau &= \lambda\liminf_{\xi\to\infty}\sqrt{\xi}\int_{\xi}^{\infty}\frac{\Delta\tau}{2\tau\sqrt{\sigma(\tau)}}\\ &\geq \lambda\liminf_{\xi\to\infty}\sqrt{\xi}\int_{\xi}^{\infty}\left(\frac{-1}{\sqrt{\tau}}\right)^{\Delta}\Delta\tau=\lambda.\end{aligned}$$

An application of the results of [17] yields all solutions to Equation (48) oscillating if

$$\lambda>\frac{4}{\sqrt[16]{l^{29}}}\sqrt[4]{\frac{3^{3}}{7^{7}}}, \tag{49}$$

and, by using the results of [3], all solutions to Equation (48) oscillate if

$$\lambda>\frac{4}{\sqrt[16]{l^{15}}}\sqrt[4]{\frac{3^{3}}{7^{7}}}, \tag{50}$$

and also, using Theorem 1, shows that then all solutions to Equation (48) oscillate if

$$\lambda>\frac{4}{\sqrt[16]{l^{3}}}\sqrt[4]{\frac{3^{3}}{7^{7}}}. \tag{51}$$

By comparing (49), (50) and (51), we find that (51) is superior to both (49) and (50). It means that condition (10) improves conditions (6) and (8) to Equation (48).

Example 2. *For $\xi\in[\xi_0,\infty)_{\mathbb{T}}$, consider a second-order quasilinear delay dynamic equation*

$$\left[\left(z^{\Delta}(\xi)\right)^{3}\right]^{\Delta}+\frac{3\lambda}{\xi^{4}}\left(\frac{\sigma(\xi)}{g(\xi)}\right)^{3}z^{3}(g(\xi))=0,\quad g(\xi)\leq\xi, \tag{52}$$

where $\lambda>0$. Here, $\alpha=\beta=3$, $a(\xi)=1$, and $p(\xi)=\frac{3\lambda}{\xi^{4}}\left(\frac{\sigma(\xi)}{g(\xi)}\right)^{3}$. It is clear that the condition (3) holds since

$$\int_{\xi_0}^{\infty}g^{\alpha}(\tau)p(\tau)\Delta\tau=3\lambda\int_{\xi_0}^{\infty}\frac{\sigma^{3}(\tau)}{\tau^{4}}\Delta\tau\geq3\lambda\int_{\xi_0}^{\infty}\frac{\Delta\tau}{\tau}=\infty.$$

We will see that the results of this paper improve those reported in [17] for Equation (52) for $l<1$. Now,

$$\begin{aligned}\liminf_{\xi\to\infty}\xi^{\alpha}\int_{\sigma(\xi)}^{\infty}\left(\frac{g(\tau)}{\sigma(\tau)}\right)^{\alpha}p(\tau)\Delta\tau &= 3\lambda\liminf_{\xi\to\infty}\xi^{3}\int_{\sigma(\xi)}^{\infty}\frac{\Delta\tau}{\tau^{4}}\\ &\geq \lambda\liminf_{\xi\to\infty}\xi^{3}\int_{\sigma(\xi)}^{\infty}\left(\frac{-1}{\tau^{3}}\right)^{\Delta}\Delta\tau\\ &= \lambda l^{3}\end{aligned}$$

and

$$\begin{aligned}\liminf_{\xi\to\infty}\frac{\xi^{\alpha}}{a(\xi)}\int_{\xi}^{\infty}\frac{g^{\beta}(\tau)}{\tau^{\gamma}}p(\tau)\Delta\tau &= 3\lambda\liminf_{\xi\to\infty}\xi^{3}\int_{\xi}^{\infty}\frac{\sigma^{3}(\tau)}{\tau^{7}}\Delta\tau \\ &\geq 3\lambda\liminf_{\xi\to\infty}\xi^{3}\int_{\xi}^{\infty}\frac{\Delta\tau}{\tau^{4}} \\ &\geq \lambda\liminf_{\xi\to\infty}\xi^{3}\int_{\xi}^{\infty}\left(\frac{-1}{\tau^{3}}\right)^{\Delta}\Delta\tau=\lambda.\end{aligned}$$

An application of the results of [16] yields all solutions to Equation (52) oscillating if

$$\lambda > \frac{3^3}{4^4 l^{12}}, \tag{53}$$

and also, using Theorem 1, shows that then all solutions to Equation (52) oscillate if

$$\lambda > \frac{3^3}{4^4 l^{6}}. \tag{54}$$

By comparing (53) and (54), we find that (54) is superior to (53). It means that condition (10) improves condition (4) to Equation (52).

Example 3. *For $\xi\in[\xi_0,\infty)_{\mathbb{T}}$, consider a second-order quasilinear dynamic equation*

$$\left[\sqrt[3]{\xi}\sqrt[3]{(z^{\Delta}(\xi))^{7}}\right]^{\Delta}+\frac{2\lambda\sigma^{2}(\xi)}{\xi^{2}g^{3}(\xi)}z^{3}(g(\xi))=0,\quad g(\xi)\leq\sigma(\xi), \tag{55}$$

where $\lambda>0$. Here, $\alpha=\frac{7}{3}$, $\beta=3$, $a(\xi)=\sqrt[3]{\xi}$, and $p(\xi)=\frac{2\lambda\sigma^{2}(\xi)}{\xi^{2}g^{3}(\xi)}$. Now,

$$\int_{\xi_0}^{\infty}a^{-\frac{1}{\alpha}}(\tau)\Delta\tau=\int_{\xi_0}^{\infty}\frac{\Delta\tau}{\sqrt[7]{\tau}}=\infty$$

and

$$\begin{aligned}\liminf_{\xi\to\infty}\frac{\xi^{\alpha}}{a(\xi)}\int_{\xi}^{\infty}\frac{g^{\beta}(\tau)}{\sigma^{\gamma}(\tau)}p(\tau)\Delta\tau &= \lambda\liminf_{\xi\to\infty}\xi^{2}\int_{\xi}^{\infty}\frac{2}{\tau^{2}\sigma(\tau)}\Delta\tau \\ &\geq \lambda\liminf_{\xi\to\infty}\xi^{2}\int_{\xi}^{\infty}\left(\frac{-1}{\tau^{2}}\right)^{\Delta}\Delta\tau=\lambda.\end{aligned}$$

An application of Theorem 2 shows that then all solutions to Equation (55) oscillate if

$$\lambda > \frac{3}{\sqrt[9]{l^{28}}}\sqrt[3]{\frac{7^7}{10^{10}}}.$$

Example 4. *For $\xi\in[\xi_0,\infty)_{\mathbb{T}}$, consider a second-order quasilinear advanced dynamic equation*

$$\left[\sqrt[3]{\xi^{5}}\left|z^{\Delta}(\xi)\right|z^{\Delta}(\xi)\right]^{\Delta}+\frac{\lambda}{3}\sqrt[3]{\xi}\sqrt[3]{z(g(\xi))}=0,\quad g(\xi)\geq\xi, \tag{56}$$

where $\lambda>0$. Here, $\alpha=2$, $\beta=\frac{1}{3}$, $a(\xi)=\sqrt[3]{\xi^{5}}$, and $p(\xi)=\frac{\lambda}{3}\sqrt[3]{\xi}$. Now,

$$\int_{\xi_0}^{\infty}a^{-\frac{1}{\alpha}}(\tau)\Delta\tau=\int_{\xi_0}^{\infty}\frac{\Delta\tau}{\sqrt[6]{\tau^{5}}}=\infty$$

and

$$\liminf_{\xi\to\infty} \frac{\xi^{\alpha}}{a(\xi)} \int_{\xi}^{\infty} \tau^{\beta-\gamma} p(\tau)\Delta\tau = \lambda \liminf_{\xi\to\infty} \sqrt[3]{\xi} \int_{\xi}^{\infty} \frac{1/3}{\tau\sqrt[3]{\tau}}\Delta\tau$$
$$\geq \lambda \liminf_{\xi\to\infty} \sqrt[3]{\xi} \int_{\xi}^{\infty} \left(\frac{-1}{\sqrt[3]{\tau}}\right)^{\Delta} \Delta\tau = \lambda.$$

An application of Theorem 3 shows then that all solutions to Equation (56) oscillate if

$$\lambda > \frac{4}{27l^2}.$$

The significant point to note here is that the results due to Erbe et al. [16,17] and Bohner et al. [3] do not apply to Equations (55) and (56).

5. Conclusions

(1) In this paper, several Hille-type criteria are presented that can be applied to (1) and are valid for various types of time scales, e.g., $\mathbb{T} = \mathbb{R}$, $\mathbb{T} = \mathbb{Z}$, $\mathbb{T} = h\mathbb{Z}$ with $h > 0$, $\mathbb{T} = q^{\mathbb{T}_0}$ with $q > 1$, etc. (see [13]).

(2) The results in this paper are including the cases where $\alpha \geq \beta$ and $\alpha \leq \beta$ and, for both cases, advanced and delayed dynamic equations without the need to impose condition (3).

(3) In particular, the results of this research are a significant improvement compared to the results of the papers [3,16,17] when $\alpha = \beta$ and $g(\xi) \leq \xi$; see the following details: By dint of

$$\frac{\xi^{\alpha}}{a(\xi)} \int_{\xi}^{\infty} \left(\frac{g(\tau)}{\tau}\right)^{\alpha} p(\tau)\Delta\tau \geq \frac{\xi^{\alpha}}{a(\xi)} \int_{\xi}^{\infty} \left(\frac{g(\tau)}{\sigma(\tau)}\right)^{\alpha} p(\tau)\Delta\tau$$
$$\geq \frac{\xi^{\alpha}}{a(\xi)} \int_{\sigma(\xi)}^{\infty} \left(\frac{g(\tau)}{\sigma(\tau)}\right)^{\alpha} p(\tau)\Delta\tau$$

and

$$\frac{\alpha^{\alpha}}{l^{\alpha|\alpha-1|}(\alpha+1)^{\alpha+1}} < \frac{\alpha^{\alpha}}{l^{\alpha^2}(\alpha+1)^{\alpha+1}} \quad \text{for } \alpha \geq \frac{1}{2} \text{ and } 0 < l < 1,$$

we achieve:

(i) If $\alpha = \beta$ and $g(t) \leq t$, condition (10) improves (8).
(ii) If $\alpha = \beta$ and $g(t) \leq t$, conditions (10) and (30) improve (6).
(iii) If $\alpha = \beta$, $r(t) = 1$, and $g(t) \leq t$, conditions (10) and (30) improve (4).
(iv) If $\alpha = \beta$ and $g(t) \geq t$, condition (42) reduces to (8) for $0 < \alpha \leq 1$ or (9) for $\alpha \geq 1$.
(v) If $\alpha = \beta$ and $g(t) \geq \sigma(t)$, condition (44) reduces to (8) for $0 < \alpha \leq 1$ or (9) for $\alpha \geq 1$.

(4) It would be interesting to establish a Hille-type criterion to Equation (1) assuming that $\int_{t_0}^{\infty} a^{-\frac{1}{\alpha}}(\tau)\Delta\tau < \infty$.

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References

1. Frassu, S.; Viglialoro, G. Boundedness in a chemotaxis system with consumed chemoattractant and produced chemorepellent. *Nonlinear Anal.* **2021**, *213*, 112505. [CrossRef]
2. Li, T.; Viglialoro, G. Boundedness for a nonlocal reaction chemotaxis model even in the attraction-dominated regime. *Differ. Equ.* **2021**, *34*, 315–336.
3. Bohner, M.; Hassan, T.S.; Li, T. Fite-Hille-Wintner-type oscillation criteria for second-order half-linear dynamic equations with deviating arguments. *Indag. Math.* **2018**, *29*, 548–560. [CrossRef]
4. Bohner, M.; Li, T. Oscillation of second-order $p-$Laplace dynamic equations with a nonpositive neutral coefficient. *Appl. Math. Lett.* **2014**, *37*, 72–76. [CrossRef]
5. Bohner, M.; Li, T. Kamenev-type criteria for nonlinear damped dynamic equations. *Sci. China Math.* **2015**, *58* , 1445–1452. [CrossRef]
6. Li, T.; Pintus, N.; Viglialoro, G. Properties of solutions to porous medium problems with different sources and boundary conditions. *Z. Angew. Math. Phys.* **2019**, *70*, 86. [CrossRef]
7. Zhang, C.; Agarwal, R.P.; Bohner, M.; Li, T. Oscillation of second-order nonlinear neutral dynamic equations with noncanonical operators. *Bull. Malays. Math. Sci. Soc.* **2015**, *38*, 761–778. [CrossRef]
8. Agarwal, R.P.; Bazighifan, O.; Ragusa, M.A. Nonlinear neutral delay differential equations of fourth-order: Oscillation of solutions. *Entropy* **2021**, *23*, 129. [CrossRef] [PubMed]
9. Benslimane, O.; Aberqi, A.; Bennouna, J. Existence and uniqueness of weak solution of $p(x)-$Laplacian in Sobolev spaces with variable exponents in complete manifolds. *Filomat* **2021**, *35* , 1453–1463. [CrossRef]
10. Jia, X.Y.; Lou, Z.L. The existence of nontrivial solutions to a class of quasilinear equations. *J. Funct. Spaces* **2021**, *2021*, 9986047. [CrossRef]
11. Hilger, S. Analysis on measure chains—A unified approach to continuous and discrete calculus. *Results Math.* **1990**, *18*, 18–56. [CrossRef]
12. Agarwal, R.P.; Bohner, M.; O'Regan, D.; Peterson, A. Dynamic equations on time scales: A survey. *J. Comput. Appl. Math.* **2002**, *141*, 1–26. [CrossRef]
13. Bohner, M.; Peterson, A. *Dynamic Equations on Time Scales: An Introduction with Applications*; Birkhäuser: Boston, MA, USA, 2001.
14. Bohner, M.; Peterson, A. *Advances in Dynamic Equations on Time Scales*; Birkhäuser: Boston, MA, USA, 2003.
15. Karpuz, B. Hille–Nehari theorems for dynamic equations with a time scale independent critical constant. *Appl. Math. Comput.* **2019**, *346*, 336–351. [CrossRef]
16. Erbe, L.; Hassan, T.S.; Peterson, A.; Saker, S.H. Oscillation criteria for half-linear delay dynamic equations on time scales. *Nonlinear Dyn. Syst. Theory* **2009**, *9*, 51–68.
17. Erbe, L.; Hassan, T.S.; Peterson, A.; Saker, S.H. Oscillation criteria for sublinear half-linear delay dynamic equations on time scales. *Int. J. Differ. Equ.* **2008**, *3*, 227–245.
18. Agarwal, R.P.; Bohner, M.; Li, T. Oscillatory behavior of second-order half-linear damped dynamic equations. *Appl. Math. Comput.* **2015**, *254*, 408–418. [CrossRef]
19. Agarwal, R.P.; Bohner, M.; Li, T.; Zhang, C. Oscillation criteria for second-order dynamic equations on time scales. *Appl. Math. Lett.* **2014**, *31*, 34–40. [CrossRef]
20. Agarwal, R.P.; Zhang, C.; Li, T. New Kamenev-type oscillation criteria for second-order nonlinear advanced dynamic equations. *Appl. Math. Comput.* **2013**, *225*, 822–828. [CrossRef]
21. Erbe, L.; Hassan, T.S.; Peterson, A. Oscillation criteria for nonlinear damped dynamic equations on time scales. *Appl. Math. Comput.* **2008**, *203*, 343–357. [CrossRef]
22. Erbe, L.; Hassan, T.S.; Peterson, A. Oscillation criteria for nonlinear functional neutral dynamic equations on time scales. *J. Differ. Equ. Appl.* **2009**, *15*, 1097–1116. [CrossRef]
23. Hassan, T.S. Oscillation criteria for half-linear dynamic equations on time scales.*J. Math. Anal. Appl.* **2008**, *345*, 176–185. [CrossRef]
24. Hassan, T.S. Kamenev-type oscillation criteria for second order nonlinear dynamic equations on time scales. *Appl. Math. Comput.* **2011**, *217*, 5285–5297. [CrossRef]
25. Li, T.; Saker, S.H. A note on oscillation criteria for second-order neutral dynamic equations on isolated time scales. *Commun. Nonlinear Sci. Numer. Simul.* **2014**, *19*, 4185–4188. [CrossRef]
26. Saker, S.H. Oscillation criteria of second-order half-linear dynamic equations on time scales. *J. Comput. Appl. Math.* **2005** , *177*, 375–387. [CrossRef]
27. Zhang, C.; Li, T. Some oscillation results for second-order nonlinear delay dynamic equations. *Appl. Math. Lett.* **2013**, *26*, 1114–1119. [CrossRef]
28. Agarwal, R.P.; Bohner, M.; Li, T.; Zhang, C. Hille and Nehari type criteria for third order delay dynamic equations. *J. Differ. Equ. Appl.* **2013**, *19*, 1563–1579. [CrossRef]
29. Baculikova, B. Oscillation of second-order nonlinear noncanonical differential equations with deviating argument. *Appl. Math. Lett.* **2019**, *91*, 68–75. [CrossRef]

30. Bazighifan, O.; El-Nabulsi, E.M. Different techniques for studying oscillatory behavior of solution of differential equations. *Rocky Mt. J. Math.* **2021**, *51*, 77–86. [CrossRef]
31. Bohner, M.; Hassan, T.S. Oscillation and boundedness of solutions to first and second order forced functional dynamic equations with mixed nonlinearities. *Appl. Anal. Discret. Math.* **2009**, *3*, 242–252. [CrossRef]
32. Chatzarakis, G.E.; Moaaz, O.; Li, T.; Qaraad, B. Some Oscillation theorems for nonlinear second-order differential equations with an advanced argument. *Adv. Differ. Equ.* **2020**, *2020*, 160. [CrossRef]
33. Džurina, J.; Jadlovská, I. A sharp oscillation result for second-order half-linear noncanonical delay differential equations. *Electron. J. Qual. Theory* **2020**, *46*, 1–14. [CrossRef]
34. Erbe, L. Oscillation criteria for second order nonlinear delay equations. *Canad. Math. Bull.* **1973**, *16*, 49–56. [CrossRef]
35. Grace, S.R.; Bohner, M.; Agarwal, R.P. On the oscillation of second-order half-linear dynamic equations. *J. Differ. Equ. Appl.* **2009**, *15*, 451–460. [CrossRef]
36. Hille, E. Non-oscillation theorems. *Trans. Am. Math. Soc.* **1948**, *64*, 234–252. [CrossRef]
37. Řehak, P. New results on critical oscillation constants depending on a graininess. *Dyn. Syst. Appl.* **2010**, *19* , 271–288.
38. Sun, S.; Han, Z.; Zhao, P.; Zhang, C. Oscillation for a class of second-order Emden-Fowler delay dynamic equations on time scales. *Adv. Differ. Equ.* **2010**, *2010*, 642356. [CrossRef]
39. Sun, Y.; Hassan, T.S. Oscillation criteria for functional dynamic equations with nonlinearities given by Riemann-Stieltjes integral. *Abstr. Appl. Anal.* **2014**, *2014*, 697526. [CrossRef]
40. Zhang, Q.; Gao, L.; Wang, L. Oscillation of second-order nonlinear delay dynamic equations on time scales. *Comput. Math. Appl.* **2011**, *61*, 2342–2348. [CrossRef]
41. Fite, W.B. Concerning the zeros of the solutions of certain differential equations. *Trans. Am. Math. Soc.* **1918**, *19*, 341–352. [CrossRef]
42. Hardy, G.H.; Littlewood, J.E.; Polya, G. *Inequalities*, 2nd ed.; Cambridge University Press: Cambridge, UK, 1988.