

Article

Steady-State Solutions for Two Mixed Initial-Boundary Value Problems Which Describe Isothermal Motions of Burgers' Fluids: Application

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Abstract: Steady-state solutions of two mixed initial-boundary value problems are presented in equivalent forms. They describe isothermal permanent motions of incompressible Burgers' fluids over an infinite flat plate that applies time-dependent shear stresses to the fluid. More exactly, they are the first exact solutions for motions of Burgers' fluids with differential expressions of the shear stress or velocity on the boundary. The obtained results are designed to make equivalent solutions for motions caused by an infinite plate moving in its plane at velocities that seem to be similar to previous shear stresses. It is simple to limit all results for the purpose of providing efficient results for incompressible Oldroyd-B, Maxwell, second grade and Newtonian fluids undergoing comparable motions. They may also be used to estimate how long it will take to get to a steady or permanent state.

Keywords: steady-state solutions; mixed initial-boundary value problems; Burgers' fluids

MSC: 35F16; 76A05

1. Introduction

The one-dimensional rate type fluid model proposed by Burgers [\[1\]](#page-8-0) has often been used to describe the behavior of different viscoelastic materials such as polymeric liquids, cheese, soil and asphalt [\[2](#page-8-1)[,3\]](#page-8-2). For instance, Lee and Markwick [\[4\]](#page-8-3) reported that the behavior of asphalt and sand-asphalt and the predictions of their model agreed well. Additionally, this model was designed to describe the earth's mantle's ephemeral creep tendencies [\[5,](#page-8-4)[6\]](#page-8-5) and the fine-grained polycrystalline olivine's high-temperature viscoelasticity [\[7](#page-8-6)[,8\]](#page-8-7). Krishnan and Rajagopal have given a detailed analysis of the modeling, use and applications of asphalt concrete from antiquity to the present [\[9\]](#page-8-8). The same authors explored the expansion of Burgers' model to a frame-indifferent three-dimensional form [\[10\]](#page-8-9).

First exact steady solutions for isothermal motions of incompressible Burgers' fluids are those obtained by Ravindran et al. [\[11\]](#page-8-10) in an orthogonal rheometer. Hayat et al. [\[12\]](#page-8-11) have derived steady-state solutions for periodic motions of the same fluid over an infinite plate or between parallel plates. Starting solutions for oscillatory motions (the second problem of Stokes) of incompressible Burgers' fluids over an infinite moving plate, for instance, can be found in the references [\[13](#page-8-12)[–15\]](#page-8-13). However, none of these solutions correspond to a motion in which a differential expression of the shear stress is given on the boundary.

The main purpose of this note is to present the first closed-form formulations for the dimensionless steady-state solutions corresponding to some motions of the incompressible Burgers' fluids over an infinite flat plate. The novelty of this work consists in the consideration of the shear stress or velocity on the boundary of the flow domain as a

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time-dependent function defined as the sum between a decreasing exponential function and an oscillatory function with a given pulsation. The steady-state (permanent) solutions which are determined in this article are presented in elegant forms and are new in the literature. They correspond to isothermal motions of incompressible Burgers' fluids for which differential expressions of shear stress or velocity are given on the boundary.

The solutions that have been obtained for motions with shear on the boundary are given in their most basic form and are easily particularized to provide the comparable responses for incompressible Oldroyd-B, Maxwell, second grade and Newtonian fluids flowing in a similar way. They may also be used to determine the time needed to attain the steady or permanent state, which is essential for experimentalists' researchers in practice. Additionally, new precise solutions are developed for motions of the same fluids caused by the infinite plate that moves in its plane at velocities of the same form as the previous shear stresses. It is exploited by the fact that the governing equations of the fluid velocity and shear stress have identical forms.

2. Constitutive and Governing Equations

The constitutive equations of the incompressible Burgers' fluids (IBF) are given by Equations.

$$
T = -pI + \hat{S}_1, \ \hat{S}_1 + \alpha \frac{D\hat{S}_1}{D\hat{t}_1} + \beta \frac{D^2\hat{S}_1}{D\hat{t}_1^2} = \mu \left(\hat{A}_1 + \gamma \frac{D\hat{A}_1}{D\hat{t}_1}\right), \tag{1}
$$

where $T-$ Cauchy stress tensor, \hat{S}_1 —extra-stress tensor, $I-$ unit tensor, $\hat{A}_1 = \hat{L}_1 + \hat{L}_1{}^T$ is the first Rivlin–Ericksen tensor (*L*ˆ ¹ being grad υ), *p*—hydrostatic pressure, *µ*—fluid viscosity, *α*, *β* and *γ* (\lt *α*) are material constants and *D*/*D* \hat{t}_1 —upper-convected derivative. The model defined by Equation (1) contains as particular cases the Oldroyd-B fluids if $\beta = 0$, Maxwell fluids if $\beta = \gamma = 0$ and Newtonian fluids if $\alpha = \beta = \gamma = 0$. In some motions, the governing equations of second grade fluids can also be obtained as particular cases of the present equations. Since the incompressible fluids undergo isochoric motions only, it results in the following condition

$$
tr \hat{A}_1 = 0 \text{ or equivalently } div \upsilon = 0,
$$
 (2)

having to be identically satisfied.

In the following, we shall consider isothermal unsteady motions of IBF over an infinite flat plate with velocity field:

$$
\mathbf{v} = \mathbf{v}(\hat{y}_1, \hat{t}_1) = \hat{u}_1(\hat{y}_1, \hat{t}_1) \mathbf{e}_{\hat{x}_1},
$$
(3)

where $e_{\hat{x}_1}$ —unit vector along the \hat{x}_1 —direction of the Cartesian coordinate system (CCS) \hat{x}_1 , \hat{y}_1 and \hat{z}_1 whose \hat{y}_1 —axis is perpendicular to the plate. At $\hat{t}_1 = 0$, the fluid is at rest. We also assume that $\hat{\bm{S}}_1$, as well as $\bm{\upsilon}$, are functions of \hat{y}_1 and \hat{t}_1 only. Substituting the fluid velocity in Equation $(1)_2$ and bearing in mind that the fluid has been at rest at the initial moment $\hat{t}_1 = 0$, it is easy to show that the components $\hat{S}_{\hat{y}_1\hat{y}_1}$, $\hat{S}_{\hat{y}_1\hat{z}_1}$, $\hat{S}_{\hat{z}_1\hat{z}_1}$ and $\hat{S}_{\hat{z}_1\hat{x}_1}$ of the \hat{S}_1 are zero while the non-trivial shear stress $\hat{\tau}_1(\hat{y}_1,\hat{t}_1)=\hat{S}_{\hat{x}_1\hat{y}_1}(\hat{y}_1,\hat{t}_1)$ has to satisfy the partial differential equation

$$
\left(1+\alpha\frac{\partial}{\partial\hat{t}_1}+\beta\frac{\partial^2}{\partial\hat{t}_1^2}\right)\hat{\tau}_1(\hat{y}_1,\hat{t}_1)=\mu\left(1+\gamma\frac{\partial}{\partial\hat{t}_1}\right)\frac{\partial\hat{u}_1(\hat{y}_1,\hat{t}_1)}{\partial\hat{y}_1};\ \ \hat{y}_1>0,\ \ \hat{t}_1>0.\tag{4}
$$

The incompressibility condition (2) is identically satisfied. When the body forces are conservative and there is no pressure gradient in the flow direction, the motion equations reduce to the following relevant partial differential equation

$$
\frac{\partial \hat{\tau}_1(\hat{y}_1, \hat{t}_1)}{\partial \hat{y}_1} = \rho \frac{\partial \hat{u}_1(\hat{y}_1, \hat{t}_1)}{\partial \hat{t}_1}; \quad \hat{y}_1 > 0, \quad \hat{t}_1 > 0,
$$
\n(5)

where *ρ*—constant density. The boundary conditions that will be used here are:

$$
\left(1+\alpha\frac{\partial}{\partial\hat{t}_1}+\beta\frac{\partial^2}{\partial\hat{t}_1^2}\right)\hat{\tau}_1(0,\hat{t}_1)=\mu\left(1+\gamma\frac{\partial}{\partial\hat{t}_1}\right)\frac{\partial\hat{u}_1(\hat{y}_1,\hat{t}_1)}{\partial\hat{y}_1}\bigg|_{\hat{y}_1=0}=\hat{S}_1\cos(\omega\,\hat{t}_1),\quad\lim_{\hat{y}_1\to\infty}\hat{u}_1(\hat{y}_1,\hat{t}_1)=0,\tag{6}
$$

$$
\left(1+\alpha\frac{\partial}{\partial\hat{t}_1}+\beta\frac{\partial^2}{\partial\hat{t}_1^2}\right)\hat{\tau}_1(0,\hat{t}_1)=\mu\left(1+\gamma\frac{\partial}{\partial\hat{t}_1}\right)\frac{\partial\hat{u}_1(\hat{y}_1,t_1)}{\partial\hat{y}_1}\bigg|_{\hat{y}_1=0}=\hat{S}_1\sin(\omega\,\hat{t}_1),\quad\lim_{\hat{y}_1\to\infty}\hat{u}_1(\hat{y}_1,\hat{t}_1)=0.\tag{7}
$$

The second condition from the relations (6) and (7) tell us that the fluid is quiet far away from the plate. We also assume that there is no shear in the free stream, i.e.,

$$
\lim_{\hat{y}_1 \to \infty} \hat{\tau}_1(\hat{y}_1, \hat{t}_1) = 0.
$$
\n(8)

The initial conditions $\hat{\tau}_1(0,0) = \frac{\partial \hat{\tau}_1(0,\hat{t}_1)}{\partial \hat{t}_1}$ *∂*ˆ*t*1 $\left| \right|_{\hat{t}_1=0} = 0$ and the boundary conditions (6) and (7) imply for $\hat{\tau}_1(0, \hat{t}_1)$ the following expressions

$$
\hat{\tau}_1(0,\hat{t}_1) = \frac{(1-\beta\omega^2)\cos(\omega\hat{t}_1) + \alpha\omega\sin(\omega\hat{t}_1)}{(\alpha\omega)^2 + (1-\beta\omega^2)^2}\hat{S}_1 \n+ \frac{\hat{S}_1}{r_2-r_1} \left[\frac{\alpha\omega^2 - r_2(1-\beta\omega^2)}{(\alpha\omega)^2 + (1-\beta\omega^2)^2} e^{r_1\hat{t}_1} - \frac{\alpha\omega^2 - r_1(1-\beta\omega^2)}{(\alpha\omega)^2 + (1-\beta\omega^2)^2} e^{r_2\hat{t}_1} \right],
$$
\n(9)

respectively,

$$
\hat{\tau}_1(0,\hat{t}_1) = \frac{(1-\beta\omega^2)\sin(\omega \hat{t}_1) - \alpha\omega\cos(\omega \hat{t}_1)}{(\alpha\omega)^2 + (1-\beta\omega^2)^2} \hat{S}_1 \n+ \frac{\omega \hat{S}_1}{r_2 - r_1} \left[\frac{\alpha r_2 + 1 - \beta\omega^2}{(\alpha\omega)^2 + (1-\beta\omega^2)^2} e^{r_1\hat{t}_1} - \frac{\alpha r_1 + 1 - \beta\omega^2}{(\alpha\omega)^2 + (1-\beta\omega^2)^2} e^{r_2\hat{t}_1} \right],
$$
\n(10)

where $r_{1,2} = \left(-\alpha \pm \sqrt{\alpha^2 - 4\beta}\right)/(2\beta).$

Consequently, the result is that the fluid motion is generated by the flat plate that applies a shear stress $\hat{\tau}_1(0, \hat{t}_1)$ of the form (9) or (10) to the fluid. If $\beta \to 0$, then $r_2 \to -\infty$, $r_1 \to -1/\alpha$ and the previous expressions take the simpler forms (see [16], Equations (5) and (6)) ious expressions take the simpler forms (see [16], Equations (5) and (6))
 $\left[\cos(\omega \hat{t}_1) + \alpha \omega \sin(\omega \hat{t}_1) - 1 - (\hat{t}_1) \right]$.

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$$
\hat{\tau}_1(0,\hat{t}_1) = \left[\frac{\cos(\omega \,\hat{t}_1) + \alpha \omega \sin(\omega \hat{t}_1)}{(\alpha \omega)^2 + 1} - \frac{1}{(\alpha \omega)^2 + 1} \exp\left(-\frac{\hat{t}_1}{\alpha}\right) \right] \hat{S}_1,\tag{11}
$$

$$
\hat{\tau}_1(0,\hat{t}_1) = \left[\frac{\sin(\omega \,\hat{t}_1) - \alpha \omega \cos(\omega \hat{t}_1)}{(\alpha \omega)^2 + 1} + \frac{\alpha \omega}{(\alpha \omega)^2 + 1} \exp\left(-\frac{\hat{t}_1}{\alpha}\right) \right] \hat{S}_1,\tag{12}
$$

corresponding to similar motions of incompressible Maxwell and Oldroyd-B fluids. If both *α* and *β* tend to zero, the plate applies an oscillatory shear stress corresponding to similar motions of incompressible Maxwell and Oldroyc

$$
\hat{\tau}_1(0, t_1) = \hat{S}_1 \cos(\omega \hat{t}_1) \text{ or } \hat{\tau}_1(0, \hat{t}_1) = \hat{S}_1 \sin(\omega \hat{t}_1),
$$
\n(13)

to the fluid. Such shear stresses are applied by the flat plate to incompressible second grade to the fluid. Such shear stresses are applied by the flat plate to incompressible second fluids if γ is different to zero or to Newtonian fluids if $\gamma = 0$. for the fluid. Such shear stresses are applied by the flat plate to incompre

3. Exact Steady-State Solutions 3. Exact Steady-State Solutions

Let us introduce the following non-dimensional variables, functions and parameters Let us introduce the following non-dimensional variables, functions and parameters in order to get exact results that are independent of the flow geometry. in order to get exact results that are independent of the flow geometry.

$$
\begin{split} \Upsilon &= \hat{y}_1 \sqrt{\frac{\hat{S}_1}{\mu \nu}}, \quad \tilde{\mathfrak{J}} = \frac{\hat{S}_1}{\mu} \hat{t}_1, \quad \omega = \hat{u}_1 \sqrt{\frac{\rho}{\hat{S}_1}}, \quad \tau^* = \frac{\hat{\tau}_1}{\hat{S}_1}, \quad \omega^* = \frac{\mu}{\hat{S}_1} \omega, \\ \alpha^* &= \frac{\hat{S}_1}{\mu} \alpha, \quad \beta^* = \frac{\hat{S}_1^2}{\mu^2} \beta, \quad \gamma^* = \frac{\hat{S}_1}{\mu} \gamma \,. \end{split} \tag{14}
$$

The dimensionless forms of the relations (4) and (5), namely in order to get exact results that are independent of the flow geometry. \mathcal{L} us introduce the following non-dimensional variables, functions and parameters and parameter in order to get exact results that are independent of the flow geometry. \mathcal{L} us introduce the following non-dimensional variables, functions and parameters and parameter in order to get exact results that are independent of the flow geometry. t_{c} sheared at the fluid sheared $\left(1\right)$ and $\left(5\right)$ period to incompressible second ioniess forms or the relations (\pm) and (\cup) , namely $\cos(\theta)$ and $(\overline{\mathsf{E}})$ namely mensionless forms of the relations (4) and (5), namely \mathbf{t} to the fluid stresses are applied by the flat plate to incompressible second s and (5) , namely s and (5) , namely $\overline{1}$ 1 $\overline{2}$ 1 r ms of the relations (4) and (5), namely cos() sin() 1 ˆ ˆ ^ˆ ^ˆ ^ˆ ^ˆ (0,) exp , ()1 ()1

to the fluid. Such shear stresses are applied by the flat plate to incompressible second

3. Exact Steady-State Solutions

The dimensionless forms of the relations (4) and (5), namely
\n
$$
\left(1 + \alpha \frac{\partial}{\partial \Im} + \beta \frac{\partial^2}{\partial \Im^2}\right) \tau(\Upsilon, \Im) = \left(1 + \gamma \frac{\partial}{\partial \Im} \right) \frac{\partial \omega(\Upsilon, \Im)}{\partial \Upsilon}; \quad \frac{\partial \tau(\Upsilon, \Im)}{\partial \Upsilon} = \frac{\partial \omega(\Upsilon, \Im)}{\partial \Im}, \quad (15)
$$
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between the two relations (15), one obtains the next governing equation Equation (14) and dropping out the star notation. Eliminat-
tions (15), one obtains the next governing equation one obtains the next governing equation are immediately obtained using Equation (14) and dropping out the star notation. Eliminating $\tau(\Upsilon,\tilde{\mathfrak{J}})$ between the two relations (15), one obtains the next governing equation

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$$
\left(1 + \alpha \frac{\partial}{\partial \mathfrak{J}} + \beta \frac{\partial^2}{\partial \mathfrak{J}^2}\right) \frac{\partial \omega(\Upsilon, \mathfrak{J})}{\partial \mathfrak{J}} = \left(1 + \gamma \frac{\partial}{\partial \mathfrak{J}}\right) \frac{\partial^2 \omega(\Upsilon, \mathfrak{J})}{\partial \Upsilon^2}; \quad \Upsilon > 0, \quad \mathfrak{J} > 0,
$$
 (16)
the dimensionless velocity field $\omega(\Upsilon, \mathfrak{J}).$
the corresponding dimensionless boundary conditions are

2 for the dimensionless velocity field $\varpi(\Upsilon, \mathfrak{J})$. $(\gamma, \mathfrak{J}).$ corresponding to similar motions of incompressible Maxwell and Oldroyd-B fluids. If \mathcal{A} fluids. If \mathcal{A} \mathfrak{a} $(11 - (22)^2)$ α α)

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The corresponding dimensions boundary conditions are
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$$
\left(1 + \alpha \frac{\partial}{\partial \mathfrak{J}} + \beta \frac{\partial^2}{\partial \mathfrak{J}^2}\right) \tau(0, \mathfrak{J}) = \left(1 + \gamma \frac{\partial}{\partial \mathfrak{J}}\right) \frac{\partial \omega(\Upsilon, \mathfrak{J})}{\partial \Upsilon}\Big|_{\Upsilon=0} = \cos(\omega \mathfrak{J}), \quad \lim_{\Upsilon \to \infty} \omega(\Upsilon, t) = 0; \quad \mathfrak{J} > 0,
$$
\nor

or
\n
$$
\left(1 + \alpha \frac{\partial}{\partial \mathfrak{J}} + \beta \frac{\partial^2}{\partial \mathfrak{J}^2}\right) \tau(0, \mathfrak{J}) = \left(1 + \gamma \frac{\partial}{\partial \mathfrak{J}}\right) \frac{\partial \omega(\mathbf{\Upsilon}, \mathfrak{J})}{\partial \mathbf{\Upsilon}}\Big|_{\mathbf{\Upsilon} = 0} = \sin(\omega \mathfrak{J}), \quad \lim_{\mathbf{\Upsilon} \to \infty} \omega(\mathbf{\Upsilon}, \mathfrak{J}) = 0; \quad \mathfrak{J} > 0.
$$
\n(18)
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the transient or steady-state components of sta permanent or steady state. To determine this time, exact expressions have to be known for
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motions in consideration. These solutions are independent of the initial conditions, but
they satisfy the governing equations and bou λ they satisfy the governing equations and boundary conditions. In order to avoid a possible
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or steady state. To determine this time, exact expressions hav they satisfy the governing equations and boundary conditions. In order to avoid a possible confusion, we denote by $\omega_{cp}(\Upsilon, \mathfrak{J})$, $\tau_{cp}(\Upsilon, \mathfrak{J})$ and $\omega_{sp}(\Upsilon, \mathfrak{J})$, $\tau_{sp}(\Upsilon, \mathfrak{J})$ the permanent solu- (17) , respectively (18) . when the need three to reach the (two unsteady motions become permanent or steady

such motions is to know the need time to reach the

inne this time, exact expressions have to be known for $\ddot{}$ *n*. These solutions are independent of the finital conditions, but g equations and boundary conditions. In order to avoid a possible $\omega_{cp}(\Upsilon, \tilde{\jmath})$, $\tau_{cp}(\Upsilon, \tilde{\jmath})$ and $\omega_{sp}(\Upsilon, \tilde{\jmath})$, $\tau_{sp}(\Upsilon, \tilde{\jmath})$ the p to know the need time to reac. $\ddot{}$ corresponding to the partial differential Equation (16) with the boundary conditions
respectively (18).
Direct computations show that the dimensionless velocity fields $\varpi_{cp}(\Upsilon,\mathfrak{J})$ and $\varpi_{sp}(\Upsilon,\mathfrak{J})$ nt problem for such motions is to know the need time to reach the sions have to be known for (otions is to know the need time to reach the
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I differential Equation (16) with the boundary conditions orresponding to the partial differential Equation (16) with the boundary conditions
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y the correctness of the transient solutions. This is the reason that we shall
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verify the correctness of the transient solutions. This is the reason that we shall ant problem for such motions is to know the need time to reach the \mathbf{r} time. An important problem for such motions is to know the need time to reach the
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sion, we denote by $\omega_{cp}(\Upsilon, \mathfrak{J})$, $\tau_{cp}(\Upsilon, \mathfrak{J})$ and $\omega_{sp}(\Upsilon, \mathfrak{J})$, $\tau_{sp}(\Upsilon, \mathfrak{J})$ the permanent solu-
corresponding t permanent or steady state. To determine this time, exact expressions have to be known for) and (18) and the fact that the fluid was at rest at the 100 and the fact that the fluid was at rest at the e transient or steady-state components of starting solutions. Unfortunately, there is no
odality to verify the correctness of the transient solutions. This is the reason that we shall
ovide closed-form expressions for the (\mathcal{X}, \mathcal{X}), $\tau_{cp}(\mathcal{Y}, \mathcal{X})$ and $\omega_{sp}(\mathcal{Y}, \mathcal{X})$, $\tau_{sp}(\mathcal{Y}, \mathcal{X})$ the permanent soluthe two unsteady motions become permanent or steady
for such motions is to know the need time to reach the
ermine this time, exact expressions have to be known for the governing equations and boundary conditions. In order to avoid a possible
we denote by $\omega_{cp}(\Upsilon, \tilde{\jmath})$, $\tau_{cp}(\Upsilon, \tilde{\jmath})$ and $\omega_{sp}(\Upsilon, \tilde{\jmath})$, $\tau_{sp}(\Upsilon, \tilde{\jmath})$ the permanent solutions corresponding to the partial differential Equation (16) with the boundary conditions (17), respectively (18). $\left| \begin{array}{ccc} \n\frac{1}{1} & -\sin(\alpha, y), & \frac{1}{1} & \cos(\alpha, y) & \cos(\alpha, y) \\
\frac{1}{1} & -\cos(\alpha, y), & \frac{1}{1} & \cos(\alpha, y) & \cos(\alpha, y)\n\end{array} \right|$ initial moment $\mathfrak{J} = 0$, tell us that the two unsteady motions become permanent or steady
in time. An important problem for such motions is to know the need time to reach the bovide closed-form expressions for the steady-state solutions corresponding to the two
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we satisfy the governing equations and boundar The boundary conditions (17) and (18) and the fact that the fluid was at rest at the $\frac{1}{2}$ These solutions are independent of the initial conditions, but the transient or steady-state components or starting solutions. Unfortunately, there is no
modality to verify the correctness of the transient solutions. This is the reason that we shall
provide closed-form expressions for they satisfy the governing equations and boundary conditions. In order to avoid a possible
confusion, we denote by $\omega_{cp}(\Upsilon, \mathfrak{J})$, $\tau_{cp}(\Upsilon, \mathfrak{J})$ and $\omega_{sp}(\Upsilon, \mathfrak{J})$, $\tau_{sp}(\Upsilon, \mathfrak{J})$ the permanent solutions modality to verify the correctness of the transient solutions. This is the reason that we shall
provide closed-form expressions for the steady-state solutions corresponding to the two
motions in consideration. These soluti σ_{ν}
The boundary conditions (17) and (18) and the fact that the fluid was at rest at the
initial moment $\tilde{\tau} = 0$ tell us that the two unsteady motions become permanent or steady provide closed-form expressions for the steady-state solutions corresponding to the two
motions in consideration. These solutions are independent of the initial conditions, but μ $\frac{10150}{20}$ confusion, we denote by $\mathcal{O}_{cp}(\Upsilon, \mathfrak{J})$, $\tau_{cp}(\Upsilon, \mathfrak{J})$ and $\mathcal{O}_{sp}(\Upsilon, \mathfrak{J})$, $\tau_{sp}(\Upsilon, \mathfrak{J})$ the permanent solutions corresponding to the partial differential Equation (16) with the boundary conditions 18) and the fact that the fluid was at rest at the
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state to profile the project solution. This is the masses that we shall state components of starting solutions. Unfortunately, there is no
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the series of the transient solutions. This is the reason that we shall
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ons (17) and (18) and the fact that the fluid was at rest at the
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 T (19).
And the set of the boundary conditions (17) and (18) and the fact that the fact that the fluid was at the following T (17), respectively (18).
Direct computations show that the dimensionless velocity fields $\omega_{cp}(\Upsilon, \mathfrak{J})$ and $\omega_{sp}(\Upsilon, \mathfrak{J})$
corresponding to the two motions can be presented in the simple forms corresponding to the two motions can be presented in the simple forms
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corresponding to the two motions can be presented in the simple forms
 $\omega_{cp}(\mathbf{1}, \mathbf{J})$ and $\omega_{sp}(\mathbf{1}, \mathbf{J})$ utations show that the dimensionless velocity fields $\omega_{cp}(\Upsilon,\mathfrak{J})$ and $\omega_{sp}(\Upsilon,\mathfrak{J})$
b the two motions can be presented in the simple forms ding to the two motions can be presented in the simple forms espectively (18).
Direct computations show that the dimensionless velocity fields $\omega_{cp}(\Upsilon, \mathfrak{J})$ and $\omega_{sp}(\Upsilon, \mathfrak{J})$
sponding to the two motions can be presented in the simple forms Direct computations show that the dimensionless velocity fields $\varpi_{cp}(\Upsilon, \mathfrak{J})$ and $\varpi_{sp}(\Upsilon, \mathfrak{J})$ ponding to the two motions can be presented in the simple forms s *show* that the diffi corresponding to the two motions can be presented in the simple forms respectively (18).
Direct computations show that the dimensionless velocity fields $\omega_{cp}(\Upsilon, \mathfrak{J})$ and $\omega_{sp}(\Upsilon, \mathfrak{J})$
sponding to the two motions can be presented in the simple forms 2 \mathbf{u} and $\mathcal{O}_{sp}(\Upsilon,\mathfrak{J})$ (17), respectively (16).
Direct computations show that the dimensionless velocity fields $\varpi_{cp}(\Upsilon, \mathfrak{J})$ and $\varpi_{sp}(\Upsilon, \mathfrak{J})$
corresponding to the two motions can be presented in the simple forms *y tu SS S* (Υ, \mathfrak{J}) mputations show that the dimensionless velocity fields $\omega_{cp}(\Upsilon,\mathfrak{J})$ and $\omega_{sp}(\Upsilon,\mathfrak{J})$ $\frac{c}{\sqrt{1-\left(\frac{1}{\sqrt{1-\left(\frac{1}{\sqrt{1-\left(\frac{1}{\sqrt{1-\left(\frac{1}{\sqrt{1-\left(\frac{1}{\sqrt{1-\left(\frac{1}{\sqrt{1-\left(\frac{1}{\sqrt{1-\left(\frac{1}{\sqrt{1-\left(\frac{1}{\sqrt{1-\left(\frac{1}{\sqrt{1-\left(\frac{1}{\sqrt{1-\left(\frac{1}{\sqrt{1-\left(\frac{1}{\sqrt{1-\left(\frac{1}{\sqrt{1-\left(\frac{1}{\sqrt{1-\left(\frac{1}{\sqrt{1-\left(\frac{1}{\sqrt{1-\left(\frac{1}{\sqrt{1-\left(\frac{1}{\sqrt{1-\left$ in the simple forms $\frac{d}{dr}$ and $\frac{d}{dr}$ and $\frac{d}{dr}$ \frac $\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{d\mu}{d\mu} \, d\mu$ $\int_{-\infty}^{\infty} \frac$

corresponding to the two motions can be presented in the simple forms
\n
$$
\omega_{cp}(\Upsilon, \mathfrak{J}) = -\Re e \left\{ \frac{1}{(1 + i\omega \gamma)\delta} e^{-\delta \Upsilon + i\omega \mathfrak{J}} \right\}, \quad \omega_{sp}(\Upsilon, \mathfrak{J}) = -\text{Im} \left\{ \frac{1}{(1 + i\omega \gamma)\delta} e^{-\delta \Upsilon + i\omega \mathfrak{J}} \right\}, \tag{19}
$$
\nor equivalently

 \mathbf{r} states. To determine this time, exact expressions have to be known t (a) or equivalently *SS S* $\sum_{i=1}^{\infty}$ $\frac{1}{2}$ $\frac{1}{2}$

or equivalently
\n
$$
\omega_{cp}(\Upsilon, \mathfrak{J}) = -\sqrt{p^2 + q^2} e^{-\wp \Upsilon} \cos(\omega \mathfrak{J} - \eta \Upsilon + \phi),
$$
\n
$$
\omega_{sp}(\Upsilon, \mathfrak{J}) = -\sqrt{p^2 + q^2} e^{-\wp \Upsilon} \sin(\omega \mathfrak{J} - \eta \Upsilon + \phi),
$$
\n(20)

te the real and the imaginary part, respectively, of that which follows. for the transient or steady-state components of starting solutions. Unfortunately, there is \mathcal{P} states. To determine this time, exact expressions have to be known t The boundary conditions (17) and the fact that the fact that the fluid was at the fluid was at α and Im denote the real and the imaginary part, respectively, of that which follows. \overline{a} the real and the imaginary part, respectively, of that which for tively, of that which follows. 11 11 11 11 11 where $\Re e$ and Im denote the real and the imaginary part, respectively, of that which follows.
 \Box σ shear stresses are applied by the flat plate to incompressible second by the flat plate to incompressible second σ shear stresses are applied by the flat plate to incompressible second by the flat plate to incompressible second note the real and the imaginary part, respectively, of that which follows. − та се представа на селото на с
Стата на селото на

$$
\wp = \sqrt{\frac{\omega}{2}} \sqrt{\frac{a\omega + \sqrt{(a\omega)^2 + b^2}}{1 + (\gamma\omega)^2}}, \quad \eta = \sqrt{\frac{\omega}{2}} \sqrt{\frac{-a\omega + \sqrt{(a\omega)^2 + b^2}}{1 + (\gamma\omega)^2}},
$$
(21)

$$
p = \frac{\omega \eta \gamma - \wp}{(\omega \wp \gamma + \eta)^2 + (\omega \eta \gamma - \wp)^2}, \quad q = \frac{\omega \wp \gamma + \eta}{(\omega \wp \gamma + \eta)^2 + (\omega \eta \gamma - \wp)^2},
$$

$$
\delta = \sqrt{i\omega(1 - \beta\omega^2 + i\omega\alpha)/(1 + i\omega\gamma)}, \quad a = \gamma(1 - \beta\omega^2) - \alpha, \quad b = 1 - \beta\omega^2 + \alpha\gamma\omega^2, \quad \phi = \arctg(q/p)
$$
 (22)

$$
\delta = \sqrt{i\omega(1 - \beta\omega^2 + i\omega\alpha)/(1 + i\omega\gamma)}, \ a = \gamma(1 - \beta\omega^2) - \alpha, \ b = 1 - \beta\omega^2 + \alpha\gamma\omega^2, \ \phi = \arctg(q/p)
$$

The corresponding shear stresses $\tau_{cp}(\Upsilon, \mathfrak{J})$ and $\tau_{sp}(\Upsilon, \mathfrak{J})$ are given by the relations

rresponding shear stresses $\tau_{cp}(\Upsilon \mathcal{,} \mathfrak{J})$ and $\tau_{sp}(\Upsilon \mathcal{,} \mathfrak{J})$ are given by the relations orresponding shear stresses $\tau_{cp}(\Upsilon \mathfrak{J})$ and $\tau_{sp}(\Upsilon \mathfrak{J})$ are given by the relations

$$
\tau_{cp}(\Upsilon,\mathfrak{J}) = \omega \mathfrak{Re}\left\{\frac{i}{(1+i\omega\gamma)\delta^2}e^{-\delta \Upsilon + i\omega\mathfrak{J}}\right\}, \quad \tau_{sp}(\Upsilon,\mathfrak{J}) = \omega \mathrm{Im}\left\{\frac{i}{(1+i\omega\gamma)\delta^2}e^{-\delta \Upsilon + i\omega\mathfrak{J}}\right\},\tag{23}
$$

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corresponding to similar motions of incompressible Maxwell and Oldroyd-B fluids. If

= +− + + (12)

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or equivalently exact results that are independent of the flow get exact results that are independent of the flow geometry. The flow geometry α 1 1 1 1 to the single state of \mathbb{R}^n or \mathbb{R}^n \mathbb{R}^n

3. Exact Steady-State Solutions

 \mathbf{b} tend to zero, the plate applies and \mathbf{b} tend to zero, the plate applies and \mathbf{b}

$$
\tau_{cp}(\Upsilon,\mathfrak{J}) = \omega \sqrt{\frac{p^2 + q^2}{\wp^2 + \eta^2}} e^{-\wp \Upsilon} \cos(\omega \mathfrak{J} - \eta \Upsilon + \phi + \psi), \tag{24}
$$

to the fluid shear stresses are applied by the flat plate to incompressible second by the flat plate to incompressible second

3. Exact Steady-State Solutions

$$
\tau_{sp}(\Upsilon,\mathfrak{J}) = \omega \sqrt{\frac{p^2 + q^2}{\wp^2 + \eta^2}} \,\mathrm{e}^{-\wp \Upsilon} \sin(\omega \,\mathfrak{J} - \eta \Upsilon + \phi + \psi),\tag{25}
$$

 \sqrt{v} \sqrt{v} \sqrt{q}
= arctg(\wp/η). Equivalence of the expressions of $\omega_{cp}(\Upsilon,\tilde{J})$, $\omega_{sp}(\Upsilon,\tilde{J})$ and $\mathcal{M}(\eta)$. Equivalence of the expressions of $\omega_{cp}(\Upsilon, \mathfrak{J})$, $\omega_{sp}(\Upsilon, \mathfrak{J})$ and ven by Equation (19), respectively (23) to those from the equalities (20), (24) and (25), is graphically proved by Figures 1 and 2. where $\psi = \arctg(\wp/\eta)$. Equivalence of the expressions of $\omega_{cp}(\Upsilon, \mathfrak{J})$, $\omega_{sp}(\Upsilon, \mathfrak{J})$ and
 $\tau_{cr}(\Upsilon, \mathfrak{J})$, $\tau_{cr}(\Upsilon, \mathfrak{J})$ given by Equation (19), respectively (23) to those from the equalities where $\psi = \arctg(\wp/\eta)$. Equivalence of the expressions of $\omega_{cp}(\Upsilon, \mathfrak{J})$, $\omega_{sp}(\Upsilon, \mathfrak{J})$ and $\tau_{cp}(\Upsilon, \mathfrak{J})$, $\tau_{sp}(\Upsilon, \mathfrak{J})$ given by Equation (19), respectively (23) to those from the equalities (20), (24) a $\frac{1}{1}$ and
lities *S S y* and
s SSS SSSSSSSSSSSSSSSSSS $\sqrt{\wp^2 + \eta^2}$
ivalence of the expressions of $\omega_{cp}(\Upsilon, \tilde{\jmath})$, $\omega_{sp}(\Upsilon, \tilde{\jmath})$ and $-\frac{1}{2}$ − $-\frac{1}{2}$ = $-\frac{1}{2}$ = $-\frac{1}{2}$ $\alpha^2 + \eta^2$
nce of the expressions of $\omega_{cp}(\Upsilon, \tilde{\jmath}), \omega_{sp}(\Upsilon, \tilde{\jmath})$ and are expressions of $\varpi_{cp}(\Upsilon, \mathfrak{J})$, $\varpi_{sp}(\Upsilon, \mathfrak{J})$ and − *Mathematics* **2022**, *10*, 3681 6 of 11

Figure 1. Profiles of $\omega_{cp}(\Upsilon, \mathfrak{J})$ and $\omega_{sp}(\Upsilon, \mathfrak{J})$ given by Equations (19)₁ and (20)₁, respectively, (19)₂ and (20)₁, respectively, \tilde{J}) and $\varpi_{sp}(\Upsilon, \tilde{J})$ given by Equations $(19)_1$ and $(20)_1$, respectively,
 $\tilde{J} = 0.7$ $\tilde{\omega} = 0.6$ $\tilde{\omega} = \pi/12$ and $\tilde{\omega} = 5$ ϒ= \int , \int (12 and $\tilde{J} = 5$. **Figure 1.** Profiles of $\omega_{cp}(\Upsilon, \mathfrak{J})$ and $\omega_{sp}(\Upsilon, \mathfrak{J})$ given by Equations (19)₁ and (20)₁, respectively, (19)₂ and (20)₂ for $\alpha = 0.8$, $\beta = 0.7$, $\gamma = 0.6$, $\omega = \pi/12$ and $\mathfrak{J} = 5$. 14 - Johann Figure 1. Fromes of $\omega_{cp}(\mathbf{1},\mathbf{J})$ and $\omega_{sp}(\mathbf{1},\mathbf{J})$ given by Equations $(19)_1$

Figure 2. Profiles of $\tau_{cp}(\Upsilon, \mathfrak{J})$ and $\tau_{sp}(\Upsilon, \mathfrak{J})$ given by Equations (23)₁ and (24), respectively, (23)₂ and (25) for $\alpha = 0.8$, $\beta = 0.7$, $\gamma = 0.6$, $\omega = \pi/12$ and $\mathfrak{J} = 5$. $(23)_2$ and (25) for $\alpha = 0.8$, $\beta = 0.7$, $\gamma = 0.6$, $\omega = \pi/12$ and $\mathfrak{J} = 5$.

Additionally, the exact solutions corresponding to incompressible Oldroyd-B, Maxwell,
Newtonian or even second-grade fluids performing the same motions may be found Newtonian or even second-grade fluids performing the same motions may be found straight immediately as limiting instances of the earlier discoveries. As an illustration, the Additionally, the exact solutions corresponding to incompressible Oldroyd-B, Maxwell,
Newtonian or even second-grade fluids performing the same motions may be found act solutions corresponding to incomp *Additionally, the exact solutions corresponding to incompressible Oldroyd-B, Maxwell,*
tonian ar avery seened and a fluide performing the same mations may be found

dimensionless steady-state solutions for the isothermal motions of incompressible Newtonian
Guide large https://www.html and the base of the continuous to the Guide Claus (12) fluids brought on by the flat plate that applies shear stresses to the fluid of type (13) dimensionless steady-state solutions for the isothermal motions of incompressible Newtonian fluids brought on by the flat plate that applies shear stresses to the fluid of typ the flat plate that applies shear stresses to the fluid of type (13) 31 11 11 11 11 1×7 type (13) *n* and the solutions for the isothermal motions of incompressible Newtonian set the flat plate that complements the flat of time (12). αω αω α ⁺ = −− + + (11) 1 12 2 1 $\frac{1}{2}$ (10) ssible Newtonian
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$$
\varpi_{Ncp}(\Upsilon, \mathfrak{J}) = -\frac{1}{\sqrt{\omega}} \exp\left(-\Upsilon \sqrt{\frac{\omega}{2}}\right) \cos\left(\omega \mathfrak{J} - \Upsilon \sqrt{\frac{\omega}{2}} - \frac{\pi}{4}\right),
$$

$$
\varpi_{Nsp}(\Upsilon, \mathfrak{J}) = -\frac{1}{\sqrt{\omega}} \exp\left(-\Upsilon \sqrt{\frac{\omega}{2}}\right) \sin\left(\omega \mathfrak{J} - \Upsilon \sqrt{\frac{\omega}{2}} - \frac{\pi}{4}\right),
$$
 (26)

$$
\tau_{Ncp}(\Upsilon, \mathfrak{J}) = \exp\left(-\Upsilon \sqrt{\frac{\omega}{2}}\right) \cos\left(\omega \mathfrak{J} - \Upsilon \sqrt{\frac{\omega}{2}}\right) ,
$$

\n
$$
\tau_{Nsp}(\Upsilon, \mathfrak{J}) = \exp\left(-\Upsilon \sqrt{\frac{\omega}{2}}\right) \sin\left(\omega \mathfrak{J} - \Upsilon \sqrt{\frac{\omega}{2}}\right) ,
$$
\nor equivalently\n(27)

or equivalently T and T $\mathbf i$ y *Mathematics* **2022**, *10*, 3681 4 of 11

or equivalently
\n
$$
\varpi_{Ncp}(\Upsilon, \mathfrak{J}) = -\mathfrak{Re}\left\{\frac{1}{\sqrt{i\omega}}e^{\sqrt{i\omega}(\mathfrak{J}\sqrt{i\omega}-\Upsilon)}\right\}, \quad \varpi_{Nsp}(\Upsilon, \mathfrak{J}) = -\text{Im}\left\{\frac{1}{\sqrt{i\omega}}e^{\sqrt{i\omega}(\mathfrak{J}\sqrt{i\omega}-\Upsilon)}\right\},
$$
\n(28)

$$
\tau_{Ncp}(\Upsilon, \mathfrak{J}) = \mathfrak{Re}\left\{e^{\sqrt{i\omega}(\mathfrak{J}\sqrt{i\omega}-\Upsilon)}\right\}, \quad \tau_{Nsp}(\Upsilon, \mathfrak{J}) = \text{Im}\left\{e^{\sqrt{i\omega}(\mathfrak{J}\sqrt{i\omega}-\Upsilon)}\right\}, \quad (29)
$$
\n
$$
\text{are immediately obtained taking } \alpha = \beta = \gamma = 0 \text{ in Equations (19), (20) and (23)–(25).}
$$

 $\frac{1}{2}$ are immediately obtained taking $\alpha = \beta = \gamma = 0$ in Equations (19), (20) and (23)–(25). are immediately obtained taking $\alpha = \beta = \gamma = 0$ in Equations (19), (20) and (23)–(25). bbtained taking $\alpha = \beta = \gamma = 0$ in Equations (19), (20) and (23)–(25). $= 0$ in Equations (19), (20) and (23)–(25).
o develop dimensionless steady-state solutions.

ci to use previous results to develop are immediately obtained taking $\alpha = \beta = \gamma = 0$ in Equations (19), (20) and (23)–(25).
Finally, in order to use previous results to develop dimensionless steady-state solutions
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optial equation identical in form with that of the fluid velocity namely T ans wacverop annensioniess sicacy ained taking $\alpha = \beta = \gamma = 0$ in Equations (19), (20) and (23)–(25).
to use previous results to develop dimensionless steady-state solutions
totions of the IBF, let us bring to light the fact that eliminating $\omega(\Upsilon, \mathfrak{J})$ The corresponding dimensionless boundary conditions are chsionicss sicacy-siale solutions re immediately obtained taking $\alpha = \beta = \gamma = 0$ in Equations (19), (20) and (23)–(25).
Finally, in order to use previous results to develop dimensionless steady-state solutions
or other unsteady motions of the IBF, let us br The corresponding dimensionless boundary conditions are (for other unsteady motions of the IBF, let us bring to light the fact that eliminating $\varpi(\Upsilon, \mathfrak{J})$) between Equation (15), one obtains for the corresponding shear stress $\tau(\Upsilon, \mathfrak{J})$, a partial the corresponding shear stress τ . (or the previous textual to the track matrix of evently contributed evently contributed by the set of $(\Upsilon, \tilde{\mathfrak{J}})$, one obtains for the corresponding shear stress $\tau(\Upsilon, \tilde{\mathfrak{J}})$, a partial σ choar ctross $\tau(Y, \tilde{\tau})$ a partial (a) the corresponding shear stress $\tau(\Upsilon, \mathfrak{J})$, a partial $\mathfrak{J}(\mathfrak{J})$ is the corresponding shear stress $\tau(\Upsilon, \mathfrak{J})$, a partial for other unsteady motions of the IBF, let us bring to light the fact that eliminating $\mathcal{O}(\Upsilon, \mathfrak{J})$ between Equation (15), one obtains for the corresponding shear stress $\tau(Y,\mathfrak{J})$, a partial
differential equation identical in form with that of the fluid velocity, namely
 $(\mathfrak{J} - \mathfrak{J} - \mathfrak{J} - \mathfrak{J} - \mathfrak{J} - \mathfrak{J$ between Equation (15), one obtains for the corresponding shear stress $\tau(\Upsilon, \mathfrak{J})$, a partial differential equation identical in form with that of the fluid velocity, namely nally, in order to use previous results to develop dimensionless steady-s
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otions of the IBE, let us bring to light the fact that eliminating $\varpi(Y, \mathfrak{J})$ Finally, in order to use previous results to develop dim
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other unsteady motions of the IBF, let us bring to light the fact that eli Finally, in order to use previous results to develop dimensionless steady-state solutions
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for other unsteady motions of the IBF, let us bring to light the fact that eliminating $\omega(Y, \mathfrak{J})$ *y tu SS S* (Υ, \mathfrak{J}) $\arg \varpi(\Upsilon, \mathfrak{J})$, a partial *y* a partial for other unsteady motions of the IBF, let us bring to light the fact that eliminating $\omega(Y, \mathfrak{J})$ finany, in order to use previous results to develop unnerstoncess steady-state solutions
for other unsteady motions of the IBF, let us bring to light the fact that eliminating $\varphi(Y, \mathfrak{J})$
between Equation (15), one obt the fluid velocity, namely
 λ ² $(0, 1)$ 11 1 α velocity, namely
 α α

$$
\left(1 + \alpha \frac{\partial}{\partial \mathfrak{J}} + \beta \frac{\partial^2}{\partial \mathfrak{J}^2}\right) \frac{\partial \tau(\Upsilon, \mathfrak{J})}{\partial \mathfrak{J}} = \left(1 + \gamma \frac{\partial}{\partial \mathfrak{J}}\right) \frac{\partial^2 \tau(\Upsilon, \mathfrak{J})}{\partial \Upsilon^2}; \quad \Upsilon > 0, \quad \mathfrak{J} > 0. \tag{30}
$$

0 ϒ= 0 ϒ= 0 ϒ= es $\tau_{cp}(\Upsilon, \mathfrak{J})$ and $\tau_{sp}(\Upsilon, \mathfrak{J})$ given by the relations $(23)_1$, (24) , respectively $(23)_2$, (25) can Consequently, the result is that the expressions of the dimensionless steady-state shear stresses $\tau_{cp}(\gamma, \tilde{\jmath})$ and $\tau_{sp}(\gamma, \tilde{\jmath})$ given by the relations (23)₁, (24), respectively (23)₂, (25) can be obtained solving the dimensionless differential Equation (30) with the boundary conditions equently, the result is that the expressions of the dimensionless steady-state shear $p(\Upsilon, \mathfrak{J})$ and $\tau_{sp}(\Upsilon, \mathfrak{J})$ given by the relations (23)₁, (24), respectively (23)₂, (25) can t is that the expressions of the dimensionless steady-state shear

(3) given by the relations (23)₁, (24), respectively (23)₂, (25) can equently, the result is that the expressions of the dimensionless steady-state shear stresses $\tau_{cp}(\Upsilon, \mathfrak{J})$ and $\tau_{sp}(\Upsilon, \mathfrak{J})$ given by the relations (23)₁, (24), resp be obtained solving the dimensionless differential Equation (30) with the tresses $\tau_{cp}(\Upsilon, \mathfrak{J})$ and $\tau_{sp}(\Upsilon, \mathfrak{J})$ given by the relations $(23)_1$, (24) , respectively $(23)_2$, (25) can e obtained solving the dimensionless differential Equation (30) with the boundary conditions stresses $\tau_{cp}(\Upsilon, \mathfrak{J})$ and $\tau_{sp}(\Upsilon, \mathfrak{J})$ given by the relations (23)₁, (24), respectively (23)₂, (25) can
be obtained solving the dimensionless differential Equation (30) with the boundary conditions Consequently, the result is that the expressions of the dimensionless steady-state shear (24), respectively (23)₂, (25) can
(*o* α) with the boundary conditions hat the expressions of the dimensionlessions o by the relations $(23)_1$, (24) , respectively $(23)_2$, (25) can Consequently, the result is that the expressions of the dimensionless steady-state shear
 $\sum_{n=1}^{\infty}$ $\sum_{n=1}^{\infty}$ $\sum_{n=1}^{\infty}$ $\sum_{n=1}^{\infty}$ $\sum_{n=1}^{\infty}$ $\sum_{n=1}^{\infty}$ $\sum_{n=1}^{\infty}$ $\sum_{n=1}^{\infty}$ $\sum_{n=1}^{\in$ (24) , respectively $(23)_2$, (25) can
³⁰) with the boundary conditions μν μ has no boundary conditions $\frac{1}{2}$ $\frac{1}{2}$, $\$ Consequently, the result is that the expressions of the dimensionless steady-state shear
stresses $\tau_{cp}(\Upsilon, \tilde{\mathfrak{J}})$ and $\tau_{sp}(\Upsilon, \tilde{\mathfrak{J}})$ given by the relations (23)₁, (24), respectively (23)₂, (25) can
be o be obtained solving the dimensionless differential Equation (30) with the boundary conditions pressions of the dimensionless steady-state shear itial Equation (30) with the boundary conditions he relations $(23)_1$, (24) , respectively $(23)_2$, (25) can 11 1 1 11 1 1 ^τ ωτ ^ω ˆ ˆ (0,) cos() or (0,) sin(), *tS t tS t* = = ^ˆ ˆ ˆ ^ˆ ^ˆ (13) ntial Equation (30) with the boundary conditions

$$
\left(1 + \alpha \frac{\partial}{\partial \mathfrak{J}} + \beta \frac{\partial^2}{\partial \mathfrak{J}^2}\right) \tau(0, \mathfrak{J}) = \cos(\omega \mathfrak{J}), \lim_{\Upsilon \to \infty} \tau(\Upsilon, \mathfrak{J}) = 0; \ \mathfrak{J} > 0,\tag{31}
$$
 respectively

 $\frac{1}{\sqrt{2}}$ and $\frac{1}{\sqrt{2}}$ and the fact that the fluid was at the fluid was at rest at the fl respectively

respectively
\n
$$
\left(1 + \alpha \frac{\partial}{\partial \mathfrak{J}} + \beta \frac{\partial^2}{\partial \mathfrak{J}^2}\right) \tau(0, \mathfrak{J}) = \sin(\omega \mathfrak{J}), \lim_{\Upsilon \to \infty} \tau(\Upsilon, \mathfrak{J}) = 0; \ \mathfrak{J} > 0.
$$
\n(32)

$f_{\rm{rel}}$ permanent or steady state α states. To determine this time, exact expressions have to be known to b in time. An important problem for such motions is to know the need time to the need time to r permanent or steady state. To determine this time, exact expressions have to be known **4.** Application (14) and dropping out the star notation (14) and dropping out the star notation. Elimiare immediately obtained using E and dropping out the star notation (14) and dropping out the star notation. Elimi-**4. Application**

pplication
Let us again consider an IBF at rest over an infinite flat plate which at the moment
 0^+ begins to move in its plane with a time-dependent velocity $\sum_{i=1}^n \frac{1}{i!}$ gins to move in its plane with a time-dependent velocity $\sum_{i=1}^{n}$ important problem for such motions is to know the need time to reach the need time to reach the need time to reach the need time to $\sum_{i=1}^{n}$ $\frac{1}{\sqrt{1-\frac{1$ \mathbf{r} Let us again consider an IBF at rest over an infinite flat plate $\hat{t}_1 = 0^+$ begins to move in its plane with a time-dependent velocity ϵ σ move in its plane with a time-dependent velocity (a) the matrix of the moment

ndent velocity nt velocity α and α or α and α are the moment
endent velocity (a) at rest over an infinite flat plate which at the moment
with a time-dependent velocity alate which at the moment **4. Application**
Let us again consider an IBF at rest over an infinite flat plate which at the moment $\hat{t}_1 = 0^+$ begins to move in its plane with a time-dependent velocity (, at rest over an infinite flat plate which at the moment
γ with a time-dependent velocity flat plate which at د of an infinite flat plate which at the moment
me-dependent velocity 11 1 at plate which at the moment μμ με το προσπάθει το προσπάθει το προσπάθει το μεγαλύτερο το προσπάθει το μεγαλύτερο το μεγαλύτερο το μεγαλύτ
Προσπάθει το μεγαλύτερο το . .
Let us again consider an IBF at rest over an infinite flat plate which at the moment

$$
i_{1} = 0
$$
 begins to move in its plane with a time-dependent velocity
\n
$$
i_{1}(0, \hat{t}_{1}) = \frac{(1-\beta\omega^{2})\cos(\omega\hat{t}_{1}) + \alpha\omega\sin(\omega\hat{t}_{1})}{(\alpha\omega)^{2} + (1-\beta\omega^{2})^{2}} V + \frac{V}{r_{2}-r_{1}} \left[\frac{\alpha\omega^{2}-r_{2}(1-\beta\omega^{2})}{(\alpha\omega)^{2} + (1-\beta\omega^{2})^{2}} e^{r_{1}\hat{t}_{1}} - \frac{\alpha\omega^{2}-r_{1}(1-\beta\omega^{2})}{(\alpha\omega)^{2} + (1-\beta\omega^{2})^{2}} e^{r_{2}\hat{t}_{1}}\right],
$$
\nor
\n
$$
i_{1} = (1 - \beta\omega^{2})\sin(\omega\hat{t}_{1})
$$

 \mathbf{r} T or

or
\n
$$
\hat{u}_1(0,\hat{t}_1) = \frac{(1-\beta\omega^2)\sin(\omega\hat{t}_1) - \alpha\omega\cos(\omega\hat{t}_1)}{(\alpha\omega)^2 + (1-\beta\omega^2)^2}V + \frac{\omega V}{r_2-r_1} \left[\frac{\alpha r_2+1-\beta\omega^2}{(\alpha\omega)^2 + (1-\beta\omega^2)^2} e^{r_1\hat{t}_1} - \frac{\alpha r_1+1-\beta\omega^2}{(\alpha\omega)^2 + (1-\beta\omega^2)^2} e^{r_2\hat{t}_1} \right],
$$
\nwhere *V* is a dimensional constant velocity.
\n
$$
\text{Cylinder is the shear, the first digit is equal to the area, and the average on the x-axis.}
$$
\n(34)

where *V* is a dimensional constant velocity.

EXECUTE:

The V is a dimensional constant velocity.

Owing to the shear, the fluid is gradually moved and the corresponding velocity
 $\mathbf{F}^{(1)}$ and the shear of CS as before is again given by Equation (2). We vector $v(\hat{y}_1, \hat{t}_1)$, reported to the same CCS as before, is again given by Equation (3). We
also assume that the extra-stress tensor \hat{S}_1 is a function of \hat{y}_1 and \hat{t}_1 only. Its components also assume that the extra-stress tensor \hat{S}_1 is a function of \hat{y}_1 and \hat{t}_1 only. Its components also assume that the extra-stress tensor \hat{S}_1 is a function of \hat{y}_1 and \hat{t}_1 only. Its components $\hat{S}_{\hat{y}_1\hat{y}_1}$, $\hat{S}_{\hat{y}_1\hat{z}_1}$, $\hat{S}_{\hat{z}_1\hat{z}_1}$ and $\hat{S}_{\hat{z}_1\hat{x}_1}$ are again zero and the σ as before, is again given by Equation (b). The V owing to the site of, the fitter is gradually moved and the corresponding velocity vector $v(y_1, \hat{t}_1)$, reported to the same CCS as before, is again given by Equation (3). We $\frac{1}{2}$ and $\frac{1}{2}$ a

αν αναφέρεται αναφέρεται αναφέρεται με το προσωπικό και το προσωπικό και το αναφέρεται το αναφέρεται το προσωπ
Στην αναφέρεται το αναφέρεται το προσωπικό και το προσωπικό και το προσωπικό και το προσωπικό και το προσωπικό

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together with the corresponding non-trivial shear stress $\hat{\tau}_1(\hat{y}_1,\hat{t}_1)$ satisfy the same partial differential Equations (4) and (5). The corresponding boundary conditions can be written
in suitable forms in suitable forms of incompressible \sim incompressible Maxwell and Oldroyd-B fluids. If \sim 100 μ responding non-trivial shear stress $\hat{\tau}_1(\hat{y}_1, \hat{t}_1)$ satisfy the same paruai
vritten *tt t t t n* ⁺ = −− + + (11) $\frac{1}{2}$ + $\frac{1$ \overline{c} be written *t t t t t t t s* $\frac{1}{2}$ − cos() sin \mathcal{S} $\mathcal{$ $\frac{1}{2}$ $\frac{1}{2}$ $\frac{1}{2}$ $\frac{1}{2}$ $\frac{1}{2}$ $\frac{1}{2}$

cos() sin() 1 ˆ ˆ ^ˆ ^ˆ ^ˆ ^ˆ (0,) exp , ()1 ()1 *tt t t S* ^ω αω ^ω

$$
\left(1+\alpha\frac{\partial}{\partial \hat{t}_1}+\beta\frac{\partial^2}{\partial \hat{t}_1^2}\right)\hat{u}_1(0,\hat{t}_1)=\hat{V}_1\cos(\omega\hat{t}_1), \quad \lim_{\hat{y}_1\to\infty}\hat{u}_1(\hat{y}_1,\hat{t}_1)=0; \ \hat{t}_1>0,\tag{35}
$$
 respectively

respectively $t_{\rm s}$ shear stresses are applied by the flat plate to incompressible second by the flat plate to incompressible second $\frac{1}{2}$ is different to $\frac{1}{2}$ is different to $\frac{1}{2}$.

$$
\left(1 + \alpha \frac{\partial}{\partial \hat{t}_1} + \beta \frac{\partial^2}{\partial \hat{t}_1^2}\right) \hat{u}_1(0, \hat{t}_1) = \hat{V}_1 \sin(\omega \hat{t}_1), \quad \lim_{\hat{y}_1 \to \infty} \hat{u}_1(\hat{y}_1, \hat{t}_1) = 0; \ \hat{t}_1 > 0. \tag{36}
$$

The following non-dimensional variables, functions and parameters are introduced

The following non-dimensional variables, functions and parameters are introduced

The following non-dimensional variables, functions and parameters are introduced
\n
$$
\Upsilon = \frac{\hat{V}_1}{\nu} \hat{y}_1, \quad \tilde{J} = \frac{\hat{V}_1^2}{\nu} \hat{t}_1, \quad \omega = \frac{\hat{u}_1}{\hat{V}_1}, \quad \tau^* = \frac{\hat{\tau}_1}{\rho \hat{V}_1^2}, \quad \omega^* = \frac{\nu}{\hat{V}_1^2} \omega,
$$
\n
$$
\alpha^* = \frac{\hat{V}_1^2}{\nu} \alpha, \quad \beta^* = \frac{\hat{V}_1^4}{\nu^2} \beta, \quad \gamma^* = \frac{\hat{V}_1^2}{\nu} \gamma
$$
\n(37)

 $\frac{v}{v}$ $\frac{v}{v}$ $\frac{v}{v}$ $\frac{v}{v}$ $\frac{v}{v}$ doning star notation, one obtains f ϵ star notation, one obtain and again, abandoning star notation, one obtains for the two dimensionless entities $\omega(\Upsilon, \mathfrak{J})$
and $\tau(\Upsilon, \mathfrak{J})$ partial differential equations of the forms (15). The velocity field $\varphi(\Upsilon, \mathfrak{J})$ also (Υ, ງ̃) partial differential equations of the forms (15). The v
es the partial differential Equation (16) which is identical in The corresponding non-dimensional boundary conditions are *y tu SS S* (Υ, \mathfrak{J}) and again, abandoning star notation, one obtains for the two dimensionless entities $\omega(\Upsilon, \mathfrak{J})$
and $\tau(\Upsilon, \mathfrak{J})$ partial differential equations of the forms (15). The velocity field $\omega(\Upsilon, \mathfrak{J})$ also
satisfies t $\overline{1}$ $\frac{1100}{200}$ \hat{T} , \hat{J})
also
(30) *S S y* also
y (30). satisfies the partial differential Equation (16) which is identical in form with Equation (30).
The corresponding non-dimensional boundary conditions are satisfies the partial differential Equations of the forms (15) . The velocity field α (170) also satisfies the partial differential Equation (16) which is identical in form with Equation (30). [The number] on ordinary of the forms (15). The velocity field $\omega(\Upsilon, \tilde{\mathfrak{J}})$ also
ntial Equation (16) which is identical in form with Equation (30) ing star notation, one obtains for the two dimensionless entities $\varpi(\Upsilon,\mathfrak{J})$
differential equations of the forms (15). The velocity field $\varpi(\Upsilon,\mathfrak{J})$ also otation, one obtains for the two dimensionless entities $\varpi(\Upsilon, \mathfrak{J})$
ial equations of the forms (15). The velocity field $\varpi(\Upsilon, \mathfrak{J})$ also
al Equation (16) which is identical in form with Equation (30) *Mathematics 2022* **2022 2022** ionless entities $\varpi(\Upsilon,\mathfrak{J})$
ocity field $\varpi(\Upsilon,\mathfrak{J})$ also *contract a contract a singular and the singular process with a particular and the sing non-dimensional boundary conditions are* $\frac{1}{2}$ $\frac{1}{2}$ *t* $\frac{1}{2}$ are

The corresponding non-dimensional boundary conditions are
 $(1 + \theta + \theta^2)$ $(0, \infty)$ $(0, \infty)$ $(0, \infty)$ $(0, \infty)$ $\left(\begin{array}{cc} 0_{\mathbf{U}} & 0_{\mathbf{U}} \end{array}\right)$ $> 0,$ (50) $\begin{pmatrix} a & b^2 \end{pmatrix}$ $(1+\alpha \frac{\theta}{\theta} + \beta \frac{\theta}{\theta}) u(0, \gamma) =$ $\left(1+\alpha\frac{\partial}{\partial x}+\beta\frac{\partial^2}{\partial y^2}\right)u(0,\mathfrak{J})$ (38) $\frac{\partial}{\partial \mathfrak{J}} + \beta \frac{\partial^2}{\partial \mathfrak{J}}$ *∂*J 2 $u(0, \mathfrak{J}) = \cos(\omega \mathfrak{J})$, lim $\lim_{\Upsilon \to \infty} \mathcal{Q}(\Upsilon, \mathfrak{J}) = 0; \mathfrak{J} > 0,$ (38) (38) $(\Upsilon, \mathfrak{J}) = 0; \, \mathfrak{J} > 0,$ (38) **3. Example 3. Example** $\lim_{\alpha \to 0} \varpi(\Upsilon, \mathfrak{J}) = 0; \mathfrak{J} > 0,$ (38) $i \rightarrow \infty$ \mathbf{f} shear stresses are applied by the flat plate to incompressible second The corresponding non-dimensional boundary conditions are
 $\left(1+\alpha\frac{\partial}{\partial x}+\beta\frac{\partial^2}{\partial y^2}\right)u(0,\mathfrak{J})=\cos(\omega \mathfrak{J})$, $\lim \omega(\Upsilon,\mathfrak{J})=0; \mathfrak{J}>0$, (38) \sim such shear stresses are applied by the flat plate to incompressible second $\left(\begin{array}{cc} 1 + \alpha & \lambda \\ 1 + \alpha & \lambda \end{array} \right)$ is $\left(\begin{array}{cc} 0 & \lambda \\ 0 & \lambda \end{array} \right)$ for α fluids. If α (1, 3) α fluids. $\sqrt{2}$ and $\sqrt{2}$ the plate applies and plate applies and plate applies and $\sqrt{2}$ $\cos(\omega \, \mathfrak{J})$, $\lim_{\mathbf{Y} \to \infty} \mathcal{O}(\Upsilon, \mathfrak{J}) = 0; \mathfrak{J} > 0$, (38) \mathbf{r} $\mathbf{$ \mathfrak{g} tend to zero, the plate applies and \mathfrak{g} tend to zero, the plate applies and stress shear $\frac{\partial}{\partial t} + \frac{\partial^2}{\partial t^2}$ $\Big|_{t}$ (0, 2) = $\cos(t, 2)$ $\Big|_{t}$ $\sin(2\theta, 2) = 0, 2 > 0$ -dimensional boundary conditions are
 $\frac{d^2}{2}$ $u(0,\mathfrak{J}) = \cos(\omega \mathfrak{J})$, $\lim_{\mathfrak{J} \to \infty} \varpi(\mathfrak{J}, \mathfrak{J}) = 0$; $\mathfrak{J} > 0$, (38) *t t t t s* ω αυτό το *τ* -dimensional boundary conditions are
 $\frac{1}{2}$ $u(0, 3) = \cos(\omega 3)$. $\lim_{\infty} \varphi(Y, 3) = 0$: $3 > 0$. (38) (20) $M_{\alpha} = \frac{N_{\alpha}}{2} \left(\frac{\mu(0,0)}{2} \right) = \cos((\omega_0)(\omega_0))$ $\frac{1}{1}$ \cdots $\frac{1}{1}$ *Mathematics* **2022**, *10*, 3681 4 of 11 *Mathematics* **2022**, *10*, 3681 4 of 11

or α

or
\n
$$
\left(1 + \alpha \frac{\partial}{\partial \mathfrak{J}} + \beta \frac{\partial^2}{\partial \mathfrak{J}^2}\right) \varpi(0, \mathfrak{J}) = \sin(\omega \mathfrak{J}), \quad \lim_{\Upsilon \to \infty} \varpi(\Upsilon, \mathfrak{J}) = 0; \mathfrak{J} > 0.
$$
\n(39)
\nBearing in mind the results of the previous section, it is clear that the dimensionless

bearing in finite the results of the previous section, it is clear that the differentiality
velocity fields $\omega_{cp}(\Upsilon, \mathfrak{J})$ and $\omega_{sp}(\Upsilon, \mathfrak{J})$ corresponding to these motions are given by the
following relations (se velocity fields $\omega_{cp}(\Upsilon, \mathfrak{J})$ and $\omega_{sp}(\Upsilon, \mathfrak{J})$ corresponding to these motions are given by the following relations (see Equations (23)–(25)) 11 12 13 14 15 16 17 17 17 17 17 2 Bearing in mind the results of the previous section, it is clear that the dimensionless
velocity fields $\varpi_{cp}(\Upsilon, \mathfrak{J})$ and $\varpi_{sp}(\Upsilon, \mathfrak{J})$ corresponding to these motions are given by the
following relations (see *Bearing in mind the results of the previous section, it is clear that the dimensionless*
Sity fields $Q_2(X, \tilde{Z})$ *and* $Q_3(X, \tilde{Z})$ *corresponding to these mations are given by the* \mathbf{u} **3. Exact Steady-State Solutions** $\mathcal{B} \in \mathcal{D}_{\text{cp}}(\Upsilon, \mathfrak{J})$ and $\mathcal{D}_{\text{sp}}(\Upsilon, \mathfrak{J})$ corresponding to these motions are given by the **3. Exact Steady-State Solutions** corresponding to similar motions of incompressible Maxwell and Oldroyd-B fluids. If velocity fields $\omega_{cp}(\Upsilon, \overline{3})$ and $\omega_{sp}(\Upsilon, \overline{3})$ corresponding to these motions are given by the following relations (see Equations (23)–(25)) the previous section, it is clear that the unnensionle $\frac{1}{2}$ is different to $\frac{1}{2}$ or to $\frac{1}{2}$. both a and general to zero, the previous section, it is clear that the dimensional stress and $\mathcal{L}(2, \infty)$ *bearing in mind the results of the previous section, it is clear that the dimensionless* $\frac{1}{2}$, $\frac{1}{2}$, $\frac{1}{2}$, $\frac{1}{2}$, $\frac{1}{2}$ (20) (20) = +− + + (12) $\cos(\sec E)$ and $\cos(23)$ $-(25)$) $\frac{1}{2}$ (11) $\frac{1}{2}$ ($\frac{1}{\sqrt{2}}$ *tt t t S* ^ω αω ^ω ⁺ = −− + + (11)

following relations (see Equations (23)–(25))
\n
$$
\omega_{cp}(\Upsilon, \mathfrak{J}) = \omega \Re e \left\{ \frac{i}{(1 + i\omega \gamma)\delta^2} e^{-\delta \Upsilon + i\omega \mathfrak{J}} \right\}, \ \omega_{sp}(\Upsilon, \mathfrak{J}) = \omega \text{Im} \left\{ \frac{i}{(1 + i\omega \gamma)\delta^2} e^{-\delta \Upsilon + i\omega \mathfrak{J}} \right\}, \tag{40}
$$
\nor equivalently

 T_{max}

or equivalently
\n
$$
\varpi_{cp}(\Upsilon, \mathfrak{J}) = \omega \sqrt{\frac{p^2 + q^2}{\wp^2 + \eta^2}} e^{-\wp \Upsilon} \cos(\omega \mathfrak{J} - \eta \Upsilon + \phi + \psi), \tag{41}
$$

$$
\omega_{sp}(\Upsilon, \mathfrak{J}) = \omega \sqrt{\frac{p^2 + q^2}{\wp^2 + \eta^2}} e^{-\wp \Upsilon} \sin(\omega \mathfrak{J} - \eta \Upsilon + \phi + \psi),
$$
\n(42)
\nwhere ϕ and ψ have already been defined.

where ϕ and ψ have already been defined. Γ he corresponding shear stresses namely (,) oeen defined.
n stresses, parroly where ϕ and ψ have already been defined.
The corresponding shear stresses, namely

 $\frac{1}{1}$ or steady-state components of state components of state components of starting solutions. Unfortunately, there is the state components of state components of state components of starting $\frac{1}{1}$ \mathcal{Y} been defined.
r stresses, namely resses, namely ∂ ∂ ∂ ∂ ϒℑ + + ℑ = + = ℑ ϒ = ℑ> ∂ℑ ∂ℑ ∂ℑ ∂ϒ (17) _n \mathbf{v} α 2 α 2 α Where φ and ψ have already been defined.
The corresponding shear stresses, namely $\overline{2}$ 2 $\overline{2}$ 2 where φ and ψ have already been defined.
The corresponding shear stresses, namely $\frac{1}{\sqrt{2}}$ or $\frac{1}{\sqrt{2}}$, one obtains the next government governing equations the next government governing equations the next government governing equations (15), one obtains the next government governing equations (\mathbf{f} the dimensionless velocity field \mathbf{f} $\frac{1}{2}$ ding shear stresses, namely $\overline{0}$ $\overline{0}$ $\overline{0}$ $\overline{1}$, $\overline{0}$ $T_{\rm eff}$ forms of the relations \sim the relations (4) and (5), namely \sim γ and γ have already been defined.
The corresponding shear stresses, namely $\frac{1}{2}$ is different to $\frac{1}{2}$ is different to $\frac{1}{2}$ and $\frac{1}{2}$. $\epsilon_{\rm s}$, matricry

where
$$
\varphi
$$
 and ψ have already been defined.
\nThe corresponding shear stresses, namely
\n
$$
\tau_{cp}(\Upsilon, \mathfrak{J}) = \omega^2 \Re e \left\{ \frac{1}{(1 + i\omega \gamma)\delta^3} e^{-\delta \Upsilon + i\omega \mathfrak{J}} \right\}, \tau_{sp}(\Upsilon, \mathfrak{J}) = \omega^2 \text{Im} \left\{ \frac{1}{(1 + i\omega \gamma)\delta^3} e^{-\delta \Upsilon + i\omega \mathfrak{J}} \right\},
$$
\nor equivalently

 $\frac{1}{\sqrt{2}}$

permanent or steady state. To determine this time, exact expressions have to be known for the transient or steady-state components of starting solutions. Unfortunately, there is initial moment ℑ = 0 , tell us that the two unsteady motions become permanent or steady in time. An important problem for such motions is to know the need time to reach the permanent or steady state. To determine this time, exact expressions have to be known in time. An important problem for such motions is to know the need time to reach the permanent or steady state. To determine this time, exact expressions have to be known (,) 1 (0,) 1 sin(), lim (,) 0; 0. ^ϖ αβτ ^γ ^ω ^ϖ ϒ→∞ ∂ ∂ ∂ ∂ ϒℑ + + ℑ = + = ℑ ϒ ℑ = ℑ> ∂ℑ ∂ℑ ∂ℑ ∂ϒ (18) permanent or steady state. To determine this time, exact expressions have to be known for the transient or steady-state components of starting solutions. Unfortunately, there is or (,) 1 (0,) 1 cos(), lim (,) 0; 0, *^t* ^ϖ αβτ ^γ ^ω ^ϖ ϒ→∞ ϒ= ∂ ∂ ∂ ∂ ϒℑ + + ℑ = + = ℑ ϒ = ℑ> ∂ℑ ∂ℑ ∂ℑ ∂ϒ (17) 2 (,) 1 (0,) 1 cos(), lim (,) 0; 0, *^t* ^ϖ αβτ ^γ ^ω ^ϖ ϒ→∞ ∂ ∂ ∂ ∂ ϒℑ + + ℑ = + = ℑ ϒ = ℑ> ∂ℑ ∂ℑ ∂ℑ ∂ϒ (17) for the dimensionless velocity field ^ϖ (,) ϒ ℑ . The corresponding dimensionless boundary conditions are (,) 1 (0,) 1 sin(), lim (,) 0; 0. ^ϖ αβτ ^γ ^ω ^ϖ ϒ→∞ ∂ ∂ ∂ ∂ ϒℑ + + ℑ = + = ℑ ϒ ℑ = ℑ> ∂ℑ ∂ℑ ∂ℑ ∂ϒ (18) for the dimensionless velocity field ^ϖ (,) ϒ ℑ . The corresponding dimensionless boundary conditions are 2 (,) 1 (0,) 1 sin(), lim (,) 0; 0. ^ϖ αβτ ^γ ^ω ^ϖ ϒ→∞ ∂ ∂ ∂ ∂ ϒℑ + + ℑ = + = ℑ ϒ ℑ = ℑ> ∂ℑ ∂ℑ ∂ℑ ∂ϒ (18) nating ^τ (,) ϒ ℑ between the two relations (15), one obtains the next governing equation 2 2 (,) (,) 1 1 ; 0, 0, ^ϖ ^ϖ αβ ^γ ∂ ∂ ∂ ϒℑ ∂ ∂ ϒℑ + + = + ϒ> ℑ> ∂ℑ ∂ℑ ∂ℑ ∂ℑ ∂ϒ (16) The corresponding dimensionless boundary conditions are 2 (,) 1 (0,) 1 cos(), lim (,) 0; 0, *^t* ^ϖ αβτ ^γ ^ω ^ϖ ϒ→∞ ∂ ∂ ∂ ∂ ϒℑ + + ℑ= + = ℑ ϒ = ℑ> ∂ℑ ∂ℑ ∂ℑ ∂ϒ (17) 2 2 (,) (,) 1 1 ; 0, 0, ^ϖ ^ϖ αβ ^γ ∂ ∂ ∂ ϒℑ ∂ ∂ ϒℑ + + = + ϒ> ℑ> ∂ℑ ∂ℑ ∂ℑ ∂ℑ ∂ϒ (16) (,) (,) (,) 1 (,) 1 ; , ϖτϖ αβτ ^γ ∂ ∂ ∂ ∂ ϒℑ ∂ ϒℑ ∂ ϒℑ + + ϒℑ = + = ∂ℑ ∂ℑ ∂ℑ ∂ϒ ∂ϒ ∂ℑ (15) are immediately obtained using Equation (14) and dropping out the star notation. Elimi-2 22 2 (,) (,) 1 1 ;0,0, ^ϖ ^ϖ αβ ^γ ∂ ∂ ∂ ϒℑ ∂ ∂ ϒℑ + + = + ϒ> ℑ> ∂ℑ ∂ℑ ∂ℑ ∂ℑ ∂ϒ (16) (,) (,) (,) 1 (,) 1 ; , ϖτϖ αβτ ^γ ∂ ∂ ∂ ∂ ϒℑ ∂ ϒℑ ∂ ϒℑ + + ϒℑ = + = ∂ℑ ∂ℑ ∂ℑ ∂ϒ ∂ϒ ∂ℑ (15) μμ μ The dimensionless forms of the relations (4) and (5), namely 2 *τcp*((,) (,) (,) 1 (,)1 ; , ϖτϖ αβτ ^γ ∂ ∂ ∂ ∂ ϒℑ ∂ ϒℑ ∂ ϒℑ + + ϒℑ= + = ∂ℑ ∂ℑ ∂ℑ ∂ϒ ∂ϒ ∂ℑ (15) ^ˆ ˆ ˆ , ,. ^α αβ βγ γ μμ μ == = The dimensionless forms of the relations (4) and (5), namely 2 (,) (,) (,) 1 (,)1 ; , ϖτϖ αβτ ^γ ∂ ∂ ∂ ∂ ϒℑ ∂ ϒℑ ∂ ϒℑ + + ϒℑ= + = ∂ℑ ∂ℑ ∂ℑ ∂ϒ ∂ϒ ∂ℑ (15) 11 1 1 11 ˆ ˆ ^ˆ ^ˆ , , ,, , ^ˆ ^ˆ ^ˆ ˆ ˆ *S S y tu SS S* ρ μ τ ϖ τ ωω μν μ ∗ ∗ ϒ= ℑ= = = = , J) = −*ω* 2 p *p* ² + *q* 2 ℘² + *η* 2 e −℘ 11 1 1 11 ˆ ˆ ^ˆ ^ˆ , , ,, , ^ˆ ^ˆ ^ˆ ˆ ˆ *y tu SS S* ρ μ τ ϖ τ ωω μν μ ϒ= ℑ= = = = cos(*ω* J − *η* 11 1 1 11 ˆ ˆ ^ˆ ^ˆ , , ,, , ^ˆ ^ˆ ^ˆ ˆ ˆ *S S y tu SS S* ϖ τ ωω ϒ= ℑ= = = = + *φ* + 2*ψ*), (44) Let us introduce the following non-dimensional variables, functions and parameters in order to get exact results that are independent of the flow geometry. **3. Exact Steady-State Solutions** Let us introduce the following non-dimensional variables, functions and parameters in order to get exact results that are independent of the flow geometry. Let us introduce the following non-dimensional variables, functions and parameters in order to get exact results that are independent of the flow geometry.

$$
\tau_{sp}(\Upsilon,\mathfrak{J}) = -\omega^2 \frac{\sqrt{p^2 + q^2}}{\wp^2 + \eta^2} e^{-\wp \Upsilon} \sin(\omega \mathfrak{J} - \eta \Upsilon + \phi + 2\psi), \tag{45}
$$

 \mathcal{L} is introduce the following non-dimensional variables, functions and parameters and parameter

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have been obtained using the relations (8) and (15)₂. The equivalence of the expressions $\frac{1}{2}$ π ($\frac{1}{2}$) and π ($\frac{1}{2}$) is it can be Equivalence in the state from the have been obtained using the relations (8) and (15)₂. The equivalence of the expressions of $\tau_{cp}(\Upsilon, \mathfrak{J})$ and $\tau_{sp}(\Upsilon, \mathfrak{J})$ given by Equation (43)₁, respectively (43)₂ to those from the equalities (44) and 11 1 μν μ μν μ equalities (44) and (45) is graphically proved by Figure 3. equalities (44) and (45) is graphically proved by Figur[e 3](#page-7-0). have been obtained using the relations (8) and (15)₂. The equivalence of the expressions
of $\tau_{\rm cr}(\Upsilon,3)$ and $\tau_{\rm cr}(\Upsilon,3)$ given by Equation (43), respectively (43), to those from the

Let us introduce the following non-dimensional variables, functions and parameters and parameters and parameters

Let us introduce the following non-dimensional variables, functions and parameters

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Figure 3. Profiles of $\tau_{cp}(\Upsilon, \mathfrak{J})$ and $\tau_{sp}(\Upsilon, \mathfrak{J})$ given by Equations (43)₁ and (44), respectively (43)₂ and (45) for $\alpha = 0.8$, $\beta = 0.7$, $\gamma = 0.6$, $\omega = \pi/12$ and $\mathfrak{J} = 5$. (43)² and (45) for $\alpha = 0.8$, $\beta = 0.7$, $\gamma = 0.6$, $\omega = \pi/12$ and $\gamma = 3$. (43)₂ and (45) for $\alpha = 0.8$, $\beta = 0.7$, $\gamma = 0.6$, $\omega = \pi/12$ and $\mathfrak{J} = 5$. $\mathfrak{J}=5.$

 $p = \gamma = 0$ in Equations (40)–(43)
ns corresponding to isothermal mo Finally, taking $\alpha = \beta = \gamma = 0$ in Equations (40)–(45), we recover the di (E)₂ and (E) 101 $a = 0.0$, $p = 0.7$, $\gamma = 0.0$, $\omega = \pi/12$ and $y = 0.5$.
Finally, taking $\alpha = \beta = \gamma = 0$ in Equations (40)–(45), we recover the dimension-Newtonian fluids generated by the flat plate that moves in its plane with the oscillatory velocity fields, namely $\beta = \gamma = 0$ in Equations (40)–(45), $\gamma = 0$ in Equations (40)–(45), we recover
responding to isothermal motions of the velocities $\hat{V}_1 \cos(\omega \hat{t}_1)$ or $\hat{V}_1 \sin(\omega \hat{t}_1)$ (see [17], Equations (27)–(33)). The corresponding \overline{a} Finally, taking $\alpha = \beta = \gamma = 0$ in Equations (40)–(45), we recover the dimension-
star du at the solutions convey an ding to isothermal mations of the incompressible less steady-state solutions corresponding to isothermal motions of the incompressible Γ , Γ , **3. Exact Steady-State Solutions** div

velocity fields, namely
\n
$$
\varpi_{Ncp}(\Upsilon, \mathfrak{J}) = \exp\left(-\Upsilon \sqrt{\frac{\omega}{2}}\right) \cos\left(\omega \mathfrak{J} - \Upsilon \sqrt{\frac{\omega}{2}}\right), \varpi_{Nsp}(\Upsilon, \mathfrak{J}) = \exp\left(-\Upsilon \sqrt{\frac{\omega}{2}}\right) \sin\left(\omega \mathfrak{J} - \Upsilon \sqrt{\frac{\omega}{2}}\right), \tag{46}
$$

are the dimensionless forms of the solutions (12) and (17) obtained by Erdogan $\overline{1}$ are the anneholdeness forms of the solutions (12) and (17) obtained by Erdogan β mensionless forms of the solutions (12) and (17) obtained by Erdogan [18] $\sum_{i=1}^{\infty}$, the two units of the solutions $\sum_{i=1}^{\infty}$ and $\sum_{i=1}^{\infty}$ solutions becomes imensionless forms of the solutions (12) and (17) obtained by Erdogan [18] $\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \sum_{j$ forms of the solutions (12) and (17) obtained by Erdogan [18] are the dimensionless forms of the solutions (12) and (17) obtained by Erdogan [18].
5 Conclusions *SS S* are the dimensionless forms of the solutions (12) and (17) obtained by Erdogan [\[18\]](#page-9-1).
5 Conclusions \mathfrak{m} \mathfrak{p} . *SS S*

5. Conclusions is to know the need time to reach the need to $\bf n$ clusions is to know the need time to the need time to the need time to the need time to $\bf n$ θ . Concrusions **5. Conclusions** \mathbf{u}_{H}

The exact steady-state solutions to initial-boundary value problems often explain the motions or deformations of various fluids or solids. Additionally, they may be used as tests motions or deformations of various fluids or solids. Additionally, they may be used as tests calculate the amount of time needed to reach the steady or permanent state. For two mixed to vandate numerical schemes created to research more difficult problems as well as
calculate the amount of time needed to reach the steady or permanent state. For two mix-
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calculate the amount of time needed to reach the steady or permanent state. For two mixed
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blems describing isothermal unsteady motions of IBF over an we provided equivalent closed-form expressions for the dimensionless steady-state velocity $\frac{1}{\sqrt{2}}$ for vandate numerical schemes created to research more unificant problems as wen as to
calculate the amount of time needed to reach the steady or permanent state. For two mixed
initial-boundary value problems describing is \mathcal{S} $\mathcal{$ infinite flat plate when differential expressions of the shear stress are given on the boundary,

0 ϒ= 0 ϒ= fluids over an infinite plate that move in its plane with time-dependent velocities of the 2 2 They seem to be the first exact solutions of this type in the existing literature. All of the They seem to be the first exact solutions of this type in the existing literature. All of the
results found here may be easily customized to provide exact steady-state solutions for
incompressible Oldroyd-B, Maxwell, secon incompressible Oldroyd-B, Maxwell, second grade and Newtonian fluids acting in similar the incompressible Newtonian fluids due to the infinite plate that applies a shear stress
 $\hat{\xi}_c \cos(\omega t_c)$ or $\hat{\xi}_c \sin(\omega t_c)$ to the fluid are brought to light. The results that have been the incompressible Newtonian fluids due to the infinite plate that applies a shear stress $\hat{S}_1 \cos(\omega \hat{t}_1)$ or $\hat{S}_1 \sin(\omega \hat{t}_1)$ to the fluid are brought to light. The results that have been obtained here especially t obtained here, especially the method used to find them, will be useful for establishing similar solutions for MHD motions of incompressible Burgers' fluids through a porous media. The authors also believe that present results can be used to investigate flows of seem to be the first exact solutions of this type in the existing literature. All of the
ts found here may be easily customized to provide exact steady-state solutions for
pressible Oldroyd-B, Maxwell, second grade and New The exact dimensionless steady-state solutions for unsteady motions of the same 2 2 fluids over an infinite plate that move in its plane with tire
same form as the previously applied shear stresses are de results found here may be easily customized to provide exact steady-state solutions for stratified fluids or nanofluids with Burgers' fluids as base.
Stratified fluids or nanofluids with Burgers' fluids as base. came form as the previously and fluids over an infinite plate that move in its plane with time-dependent v
same form as the previously applied shear stresses are developed using e proviously applied shear stresses fluids over an infinite plate that move in its plane with time-dependent velocities of the same form as the previously applied shear stresses are developed using these solutions. the incompressible Newtonian fluids due to the infinite plate that applies a shear stress $\hat{S}_1 \cos(\omega \hat{t}_1)$ or $\hat{S}_1 \sin(\omega \hat{t}_1)$ to the fluid are brought to light. The results that have been obtained here expecially t ilied shear stresses are developed v (infinite plate that move in its plane with time-dependent velocities of the chapter of the previously applied shear stresses are developed using these solutions. incompressible Newtonian fluids due to the infinite plate that applies a shear stress $cos(\omega t_1)$ or $\hat{S}_1 sin(\omega t_1)$ to the fluid are brought to light. The results that have been ained here, expecially the method used to fi ces are developed using these solu (order) once obtained the intertaing method of the entirely and move in its plane with time-dependent velocities of the applied shear stresses are developed using these solutions. Execution is a fluids due to the infinite plate that applies a shear stress in $(\omega \hat{t}_1)$ to the fluid are brought to light. The results that have been existly the method used to find them, will be useful for establishin (anonulas with burgers fitulas as base. nanofluids with Burgers' fluids as base. μ sing these solutions fluids over an infinite plate that move in its plane with time-dependent velocities of the
same form as the previously applied shear stresses are developed using these solutions. the incompressible Newtonian fluids due to the infinite plate that applies a shear stress $\hat{S}_1 \cos(\omega \hat{t}_1)$ or $\hat{S}_1 \sin(\omega \hat{t}_1)$ to the fluid are brought to light. The results that have been obtained here especially t $(gers)$ fitures as base. ∂ ∂ ∂ ∂ ϒℑ + + ℑ = + = ℑ ϒ ℑ = ℑ> ∂ℑ ∂ℑ ∂ℑ ∂ϒ (18) motions. For illustration, the corresponding solutions for isothermal unsteady motions of

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Nomenclature

- *T* The Cauchy stress tensor
- *S*ˆ The extra-stress tensor
- *L* The velocity gradient
- \hat{A}_1 ¹ The first Rivlin–Ericksen tensor
- *u* The velocity vector
- *p* The hydrostatic pressure
- *u* The dimensionless velocity component in the x-direction
- *α*, *β*, *γ* The material coefficients of Burgers' fluids
- *τ* The (x,y) component of the dimensionless extra-stress tensor
- *ω* The pulsation of the oscillation
- *µ* The dynamic viscosity

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