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On Some Characterizations for Uniform Dichotomy of Evolution Operators in Banach Spaces

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Abstract: The present paper deals with two of the most significant behaviors in the theory of dynamical systems: the uniform exponential dichotomy and the uniform polynomial dichotomy for evolution operators in Banach spaces. Assuming that the evolution operator has uniform exponential growth, respectively uniform polynomial growth, we give some characterizations for the uniform exponential dichotomy, respectively for the uniform polynomial dichotomy. The proof techniques that we use for the polynomial case are new. In addition, connections between the concepts approached are established.

Keywords: evolution operators; uniform exponential dichotomy; uniform polynomial dichotomy

MSC: 34D05; 34D09

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1. Introduction

The qualitative theory of evolution equations in Banach spaces represents a topic of great interest in the last few years. One of the most important behaviors in the theory of dynamical systems is the exponential dichotomy. For an evolution equation on a Banach space, the exponential dichotomy refers to the existence of a projections family denoted by $\{P(t)\}$ that leads to a decomposition of the space, at every moment, into a direct sum of a stable subspace where the norms decay exponentially as $t \to \infty$ and an unstable subspace where the norms grow exponentially as $t \to \infty$.

The exponential dichotomy was studied by Perron [1] and which has gained prominence since the appearance of two fundamental monographs of J. L. Massera and J. J. Schäffer [2] , J. L. Daleckii and M. G. Krein [3]. These were followed by the important books of Coppel [4], Chicone and Latushkin [5], who obtained significant results in the infinite dimensional spaces.

In recent years, according to Sacker and Sell [6] research, the theory of exponential dichotomy has proven to be a useful method for studying the stable, unstable and center manifolds, perturbation theories, bifurcation theory, linearization theories, homoclinic behavior and many other domains. This asymptotic property was intensively studied in both finite and infinite dimensional cases, and it was generalized in many papers (see [7–13] and the references therein).

Another direction of studying the dichotomic behavior refers to the situation when the asymptotic behaviors are of a polynomial type. In this case, we stress the concepts of nonuniform polynomial dichotomy, which were introduced independently by Barreira and Valls in [14] for the continuous case of evolution operators and respectively by Bento and Silva in [15] for discrete time systems.

Another line of research regarding the topics on dichotomies is represented by the relationship between dichotomy and admissibility. There are many papers which present different input—output techniques used in order to characterize both exponential and polynomial dichotomy. The most recent results in this direction were obtained by Dragicevic, Sasu and Sasu ([16,17]) who gave some new admissibility criteria for polynomial dichotomies of discrete nonautonomous systems on the half-line.

Our work is motivated by the great number of domains that are based on the theory of exponential and polynomial dichotomy: impulsive equations [18], delay evolution equations [19], discrete dynamical systems [20], and dynamical equations on time scales [21].

The main aim of this paper is to give some necessary and sufficient conditions for the uniform exponential dichotomy and for the uniform polynomial dichotomy of evolution operators in Banach spaces. More precisely, considering an evolution operator with uniform exponential growth respectively uniform polynomial growth and a family of projections invariant to the evolution operator, we obtain different characterizations of Datko type for both concepts, as well as characterizations that use Lyapunov function in order to describe a dichotomic behavior. Moreover, we give a new method of proving the polynomial behavior, and we establish connections between concepts.

2. Notations and Definitions

Let X be a real or complex Banach space and $\mathcal{B}(X)$ the Banach algebra of all bounded linear operators acting on X. The norms on X and on $\mathcal{B}(X)$ will be denoted by $\|.\|$. The identity operator on X is denoted by I. We also denote by

$$\Delta = \{(t,s) \in \mathbb{R}_+^2 : t \geq s\} \text{ and } T = \{(t,s,t_0) \in \mathbb{R}_+^3 : t \geq s \geq t_0\}.$$

Definition 1. *An application $U : \Delta \to \mathcal{B}(X)$ is said to be an evolution operator on X if*

(e_1) $U(t,t) = I$ *for every* $t \geq 0$

(e_2) $U(t,s)U(s,t_0) = U(t,t_0)$ *for all* $(t,s,t_0) \in T$.

Definition 2. *An evolution operator $U : \Delta \to \mathcal{B}(X)$ is said to be* strongly measurable *if, for all $(s,x) \in \mathbb{R}_+ \times X$, the mapping $t \mapsto \|U(t,s)x\|$ is measurable on $[s,\infty)$.*

Definition 3. *An application $P : \mathbb{R}_+ \to \mathcal{B}(X)$ is said to be a projection family on X if $P^2(t) = P(t)$, for all $t \geq 0$.*

Remark 1. *If $P : \mathbb{R}_+ \to \mathcal{B}(X)$ is a projection family on X, then the mapping $Q : \mathbb{R}_+ \to \mathcal{B}(X), Q(t) = I - P(t)$ is also a projection family on X, which is called the complementary projection of P.*

Definition 4. *A projection family $P : \mathbb{R}_+ \to \mathcal{B}(X)$ is said to be invariant to the evolution operator $U : \Delta \to \mathcal{B}(X)$ if*

$$U(t,s)P(s) = P(t)U(t,s),$$

for all $(t,s) \in \Delta$.

In what follows, if $P : \mathbb{R}_+ \to \mathcal{B}(X)$ is an invariant projection family to the evolution operator $U : \Delta \to \mathcal{B}(X)$, we will say that (U,P) is a dichotomic pair.

Definition 5. *The pair (U,P) is called uniformly exponentially dichotomic (u.e.d.) if there are $N \geq 1$ and $\nu > 0$ such that:*

(ued_1) $\|U(t,s)P(s)x\| \leq Ne^{-\nu(t-s)}\|P(s)x\|$,

(ued_2) $e^{\nu(t-s)}\|Q(s)x\| \leq N\|U(t,s)Q(s)x\|$,

for all $(t,s,x) \in \Delta \times X$.

Remark 2. *Let (U,P) be a dichotomic pair. Then, (U,P) is uniformly exponentially dichotomic if and only if there are $N \geq 1$ and $\nu > 0$ such that:*

(ued'_1) $\|U(t,t_0)P(t_0)x_0\| \leq Ne^{-\nu(t-s)}\|U(s,t_0)P(t_0)x_0\|$

(ued'_2) $e^{\nu(t-s)}\|U(s,t_0)Q(t_0)x_0\| \leq N\|U(t,t_0)Q(t_0)x_0\|$,

for all $(t,s,t_0,x_0) \in T \times X$.

Definition 6. *We say that the dichotomic pair* (U,P) *has uniform exponential growth (u.e.g.) if there are* $M \geq 1$ *and* $\omega > 0$ *such that:*

(ueg_1) $\|U(t,s)P(s)x\| \leq Me^{\omega(t-s)}\|P(s)x\|$,

(ueg_2) $e^{-\omega(t-s)}\|Q(s)x\| \leq M\|U(t,s)Q(s)x\|$,

for all $(t,s,x) \in \Delta \times X$.

Remark 3. *The dichotomic pair* (U,P) *has uniform exponential growth if and only if there are* $M \geq 1$ *and* $\omega > 0$ *such that:*

(ueg'_1) $\|U(t,t_0)P(t_0)x_0\| \leq Me^{\omega(t-s)}\|U(s,t_0)P(t_0)x_0\|$

(ueg'_2) $e^{-\omega(t-s)}\|U(s,t_0)Q(t_0)x_0\| \leq M\|U(t,t_0)Q(t_0)x_0\|$,

for all $(t,s,t_0,x_0) \in T \times X$.

Definition 7. *The pair* (U,P) *is called uniformly polynomially dichotomic (u.p.d.) if there are* $N \geq 1$ *and* $\nu > 0$ *such that:*

(upd_1) $(t+1)^{\nu}\|U(t,s)P(s)x\| \leq N(s+1)^{\nu}\|P(s)x\|$

(upd_2) $(t+1)^{\nu}\|Q(s)x\| \leq N(s+1)^{\nu}\|U(t,s)Q(s)x\|$,

for all $(t,s,x) \in \Delta \times X$.

Remark 4. *Let* (U,P) *be a dichotomic pair. Then,* (U,P) *is uniformly polynomially dichotomic if and only if there are* $N \geq 1$ *and* $\nu > 0$ *such that:*

(upd'_1) $(t+1)^{\nu}\|U(t,t_0)P(t_0)x_0\| \leq N(s+1)^{\nu}\|U(s,t_0)P(t_0)x_0\|$

(upd'_2) $(t+1)^{\nu}\|U(s,t_0)Q(t_0)x_0\| \leq N(s+1)^{\nu}\|U(t,t_0)Q(t_0)x_0\|$,

for all $(t,s,t_0,x_0) \in T \times X$.

Definition 8. *We say that the dichotomic pair* (U,P) *has uniform polynomial growth (u.p.g.) if there are* $M \geq 1$ *and* $\omega > 0$ *such that:*

(upg_1) $(s+1)^{\omega}\|U(t,s)P(s)x\| \leq M(t+1)^{\omega}\|P(s)x\|$

(upg_2) $(s+1)^{\omega}\|Q(s)x\| \leq M(t+1)^{\omega}\|U(t,s)Q(s)x\|$,

for all $(t,s,x) \in \Delta \times X$.

Remark 5. *The dichotomic pair* (U,P) *has uniform polynomial growth if and only if there are* $M \geq 1$ *and* $\omega > 0$ *such that:*

(upg'_1) $(s+1)^{\omega}\|U(t,t_0)P(t_0)x_0\| \leq M(t+1)^{\omega}\|U(s,t_0)P(t_0)x_0\|$

(upg'_2) $(s+1)^{\omega}\|U(s,t_0)Q(t_0)x_0\| \leq M(t+1)^{\omega}\|U(t,t_0)Q(t_0)x_0\|$,

for all $(t,s,t_0,x_0) \in T \times X$.

Definition 9. *The pair* (U,P) *is called uniformly dichotomic (u.d.) if there exists* $N \geq 1$ *such that:*

(ud_1) $\|U(t,s)P(s)x\| \leq N\|P(s)x\|$

(ud_2) $\|Q(s)x\| \leq N\|U(t,s)Q(s)x\|$, *for all* $(t,s,x) \in \Delta \times X$.

Remark 6. *The connections between the concepts defined above are given by the following diagram:*

$$\begin{array}{ccc} u.p.g. & \Rightarrow & u.e.g \\ \Uparrow & & \Uparrow \\ u.p.d. & \Leftarrow & u.e.d. \end{array}$$

In general, the converse implications are not true, as it can be seen in the examples described below. We consider $X = \mathbb{R}^2$ *and the projections families* $P : \mathbb{R}_+ \to \mathcal{B}(X)$ *defined by* $P(t)x = (x_1, 0)$ *and* $Q(t)x = (0, x_2)$, *for all* $t \geq 0$ *and* $(x_1, x_2) \in X$.

Example 1. *Let us consider the evolution operator*

$$U : \Delta \to \mathcal{B}(X),\ U(t,s)x = \left(\frac{u(s)}{u(t)}x_1, \frac{u(t)}{u(s)}x_2\right),$$

where $u : \mathbb{R}_+ \to [1, \infty),\ u(t) = t + 1$.
Then, (U, P) *has u.p.g., but it is not u.p.d.*

Proof. It is easy to see that the pair (U, P) has u.p.g. for $M = \omega = 1$. In addition, if we suppose that (U, P) is u.p.d., it results that there exist $N \geq 1$ and $\nu > 0$ such that (upd_1) and (upd_2) are satisfied for all $t \geq s \geq 0$. In particular, for $s = 0$ and $t \to \infty$, we obtain $\infty \leq N$, absurd. $\square$

Example 2. *We consider the application* $u : \mathbb{R}_+ \to [1, \infty), u(t) = e^t$ *and the evolution operator*

$$U : \Delta \to \mathcal{B}(X),\ U(t,s)x = \left(\frac{u(t)}{u(s)}x_1, \frac{u(s)}{u(t)}x_2\right).$$

Then, the pair (U, P) *has u.e.g., but it does not have u.p.g., and it is not u.e.d.*

Proof. It is similar to the proof of Example 1. $\square$

Remark 7. *An example of a dichotomic pair which is u.p.d., but it is not u.e.d. can be found in [22].*

3. Uniform Exponential Dichotomy

In this section, we give some characterizations for the uniform exponential dichotomy behavior. The first theorem includes a logarithmic criterion, a majorization criterion and a criterion of Hai ([23]) type. Then, we give three integral characterizations of the Datko ([24]) type and three necessary and sufficient conditions which use Lyapunov functions.

Theorem 1. *Let* (U, P) *be a dichotomic pair with uniform exponential growth. Then, the following assertions are equivalent:*

(1) (U, P) *is uniformly exponentially dichotomic.*

(2) *there exists* $L > 1$ *such that:*

$(uled_1)$ $(t-s)\|U(t,s)P(s)x\| \leq L\|P(s)x\|$
$(uled_2)$ $(t-s)\|Q(s)x\| \leq L\|U(t,s)Q(s)x\|$,

for all $(t,s,x) \in \Delta \times X$.

(3) *there are* $L > 1$ *and* $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ *a nondecreasing application, with* $\lim_{t\to\infty} \varphi(t) = \infty$ *and* $\varphi(1) = 1$ *such that:*

$(umed_1)$ $\varphi(t-s)\|U(t,s)P(s)x\| \leq L\|P(s)x\|$

$(umed_2)$ $\varphi(t-s)\|Q(s)x\| \leq L\|U(t,s)Q(s)x\|$,

for all $(t,s,x) \in \Delta \times X$.

(4) *there are* $r > 1$ *and* $c \in (0,1)$ *such that:*

$(uHed_1)$ $\|U(r+s,s)P(s)x\| \leq c\|P(s)x\|$
$(uHed_2)$ $\|Q(s)x\| \leq c\|U(r+s,s)Q(s)x\|$,

for all $(s,x) \in \mathbb{R}_+ \times X$.

Proof. (1) $\Rightarrow$ (2) For $(uled_1)$ see [8].
Moreover,

$$(t-s)\|Q(s)x\| \leq N(t-s)e^{-\nu(t-s)}\|U(t,s)Q(s)x\| = N \cdot \frac{t-s}{e^{\nu(t-s)}}\|U(t,s)Q(s)x\| \leq$$
$$\leq \frac{N}{\nu e}\|U(t,s)Q(s)x\| \leq L\|U(t,s)Q(s)x\|.$$

We obtain
$$(t-s)\|Q(s)x\| \leq L\|U(t,s)Q(s)x\|, \text{ so } (uled_2) \text{ also holds .}$$

(2) $\Rightarrow$ (3) It is immediate by taking $\varphi(t) = t$.
(3) $\Rightarrow$ (4) We suppose that there are $L > 1$ and $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$, a strictly nondecreasing and bijective application with $\varphi(1) = 1$ such that the relations $(umed_1)$ and $(umed_2)$ are satisfied for all $(t,s,x) \in \Delta \times X$.
Let $r > 1$ with $\varphi(r) > L$ and $c = \dfrac{L}{\varphi(r)} < 1$.
Then, we have

$$\|U(r+s,s)P(s)x\| \leq \frac{L}{\varphi(r)}\|P(s)x\| = c\|P(s)x\|, \text{ for all } (s,x) \in \mathbb{R}_+ \times X,$$

so $(uHed_1)$ is satisfied, respectively

$$\|Q(s)x\| \leq \frac{L}{\varphi(r)}\|U(r+s,s)Q(s)x\| = c\|U(r+s,s)Q(s)x\|, \text{ for all } (s,x) \in \mathbb{R}_+ \times X,$$

so $(uHed_2)$ is also proved.
(4) $\Rightarrow$ (1) We suppose that there are $r > 1$ and $c \in (0,1)$ such that $(uHed_1)$ and $(uHed_2)$ are satisfied for all $(s,x) \in \mathbb{R}_+ \times X$.
For (ued_1), see [8].
Now, let $(t,s) \in \Delta$. Then, there are $n \in \mathbb{N}$ and $\delta \in [0,r)$ with $t - s = nr + \delta$.
We obtain

$$\|U(t,s)Q(s)x\| = \|U(t,s+nr)Q(s+nr)U(s+nr,s)Q(s)x\| \geq$$
$$\geq \frac{1}{M}e^{-\omega\delta}\|Q(s+nr)U(s+nr,s)Q(s)x\| \geq$$
$$\geq \frac{1}{M} \cdot e^{-\omega r}\|U(s+nr,s)Q(s)x\| =$$
$$= \frac{1}{M} \cdot e^{-\omega r}\|U(s+nr,s+(n-1)r)Q(s+(n-1)r)U(s+(n-1)r,s)Q(s)x\| \geq$$
$$\geq \frac{1}{M \cdot c \cdot e^{\omega r}}\|Q(s+(n-1)r)U(s+(n-1)r,s)Q(s)x\| \geq \cdots \geq \frac{1}{M \cdot c^n \cdot e^{\omega r}}\|Q(s)x\| =$$
$$= \frac{1}{M} \cdot \frac{1}{e^{\omega r}e^{n\ln c}}\|Q(s)x\| = \frac{1}{M} \cdot \frac{1}{e^{\omega r} \cdot e^{\frac{t-s-\delta}{r}\ln c}}\|Q(s)x\| =$$
$$= \frac{1}{M} \cdot \frac{1}{e^{\omega r - \frac{\delta \ln c}{r}}} \cdot \frac{1}{e^{(t-s)\frac{\ln c}{r}}}\|Q(s)x\| = \frac{1}{M} \cdot \frac{1}{\frac{e^{\omega r}}{c^{\frac{\delta}{r}}}} \cdot e^{\nu(t-s)}\|Q(s)x\| = \frac{1}{N}e^{\nu(t-s)}\|Q(s)x\|,$$

where $N = \dfrac{Me^{\omega r}}{c^{\frac{\delta}{r}}} > 1$ and $\nu = -\dfrac{\ln c}{r} > 0$, so (ued_2) is also proved.
It follows that (U, P) is uniformly exponentially dichotomic, so the proof is complete. □

Remark 8. *Another version of a majorizarion criterion for the uniform exponential dichotomy can be found in [25].*

Theorem 2. *Let $U : \Delta \to \mathcal{B}(X)$ be a strongly measurable evolution operator and (U, P) a dichotomic pair with uniform exponential growth. Then, the following statements are equivalent:*

(1) *(U, P) is uniformly exponentially dichotomic.*

(2) *there are $D > 1$ and $d > 0$ with*

$$(ueD_1) \quad \int_s^\infty e^{d(t-s)} \|U(t, t_0)P(t_0)x_0\| dt \le D\|U(s, t_0)P(t_0)x_0\|,$$
for all $(s, t_0, x_0) \in \Delta \times X$.

$$(ueD_2) \quad \int_{t_0}^t e^{d(t-s)} \|U(s, t_0)Q(t_0)x_0\| ds \le D\|U(t, t_0)Q(t_0)x_0\|,$$
for all $(t, t_0, x_0) \in \Delta \times X$.

(3) *there exists $D > 1$ with*

$$(ueD_1') \quad \int_s^\infty \|U(t, t_0)P(t_0)x_0\| dt \le D\|U(s, t_0)P(t_0)x_0\|,$$
for all $(s, t_0, x_0) \in \Delta \times X$.

$$(ueD_2') \quad \int_{t_0}^t \|U(s, t_0)Q(t_0)x_0\| ds \le D\|U(t, t_0)Q(t_0)x_0\|,$$
for all $(t, t_0, x_0) \in \Delta \times X$.

Proof. $(1) \Rightarrow (2)$. It is a simple verification.

$(2) \Rightarrow (3)$. It is immediate.

$(3) \Rightarrow (1)$. See [11]. □

Corollary 1. *Let $U : \Delta \to \mathcal{B}(X)$ be a strongly measurable evolution operator and (U, P) a dichotomic pair with uniform exponential growth. Then, (U, P) is uniformly exponentially dichotomic if and only if there are $D > 1$ and $d \ge 0$ such that:*

$$(ueD_1) \quad \int_s^\infty e^{d(t-s)} \|U(t, t_0)P(t_0)x_0\| dt \le D\|U(s, t_0)P(t_0)x_0\|,$$
for all $(s, t_0, x_0) \in \Delta \times X$.

$$(ueD_2) \quad \int_{t_0}^t e^{d(t-s)} \|U(s, t_0)Q(t_0)x_0\| ds \le D\|U(t, t_0)Q(t_0)x_0\|,$$
for all $(t, t_0, x_0) \in \Delta \times X$.

Proof. It follows immediately from Theorem 2. □

Theorem 3. *Let $U : \Delta \to \mathcal{B}(X)$ be a strongly measurable evolution operator and (U, P) a dichotomic pair with uniform exponential growth. Then, (U, P) is uniformly exponentially dichotomic if and only if there are $D > 1$ and $L : \Delta \times X \to \mathbb{R}_+$ with the following properties:*

(ueL_1) $L(t, t_0, x_0) \leq D(\|U(s, t_0)P(t_0)x_0\| + \|U(t, t_0)Q(t_0)x_0\|)$, *for all* $(t, t_0, x_0) \in \Delta \times X$.

(ueL_2) $L(t, t_0, P(t_0)x_0) + \int_s^t e^{d(\tau - s)}\|U(\tau, t_0)P(t_0)x_0\|d\tau = L(s, t_0, P(t_0)x_0)$, *for all* $(t, s, t_0, x_0) \in T \times X$.

(ueL_3) $L(s, t_0, Q(t_0)x_0) + \int_s^t e^{d(t-s)}\|U(s, t_0)Q(t_0)x_0\|ds = L(t, t_0, Q(t_0)x_0)$, *for all* $(t, s, t_0, x_0) \in T \times X$.

Proof. Necessity. It follows from Theorem 2 by taking the function $L : \Delta \times X \to \mathbb{R}_+$ defined by

$$L(t, t_0, x_0) = \int_t^\infty e^{d(\tau - s)}\|U(\tau, t_0)P(t_0)x_0\|d\tau + \int_{t_0}^t e^{d(t-s)}\|U(s, t_0)Q(t_0)x_0\|ds.$$

Sufficiency. If there exists a function $L : \Delta \times X \to \mathbb{R}_+$ with the properties $(ueL_1) - (ueL_3)$, then

$$\int_s^t e^{d(\tau - s)}\|U(\tau, t_0)P(t_0)x_0\|d\tau = L(s, t_0, P(t_0)x_0) - L(t, t_0, P(t_0)x_0) \leq$$
$$\leq L(s, t_0, P(t_0)x_0) \leq D\|U(s, t_0)P(t_0)x_0\|,$$

for all $(t, s, t_0, x_0) \in T \times X$. For $t \to \infty$, we obtain

$$\int_s^\infty e^{d(\tau - s)}\|U(\tau, t_0)P(t_0)x_0\|d\tau \leq D\|U(s, t_0)P(t_0)x_0\|,$$

for all $(s, t_0, x_0) \in \Delta \times X$. In addition,

$$\int_s^t e^{d(t-s)}\|U(s, t_0)Q(t_0)x_0\|ds = L(t, t_0, Q(t_0)x_0) - L(s, t_0, Q(t_0)x_0) \leq$$
$$\leq L(t, t_0, Q(t_0)x_0) \leq D\|U(t, t_0)Q(t_0)x_0\|$$

for all $(t, s, t_0, x_0) \in T \times X$. For $s \to t_0$, we obtain

$$\int_{t_0}^t e^{d(t-s)}\|U(s, t_0)Q(t_0)x_0\|ds \leq D\|U(t, t_0)Q(t_0)x_0\|$$

for all $(t, t_0, x_0) \in \Delta \times X$. □

Another characterization of Datko type is given by

Theorem 4. *Let $U : \Delta \to \mathcal{B}(X)$ be a strongly measurable evolution operator and (U, P) a dichotomic pair with uniform exponential growth. The following assertions are equivalent:*

(1) *(U, P) is uniformly exponentially dichotomic;*

(2) *there are $D > 1$ and $d > 0$ with*

$$(ueD_1^1) \int_{t_0}^{t} \frac{e^{d(t-s)}}{\|U(s,t_0)P(t_0)x_0\|} ds \leq \frac{D}{\|U(t,t_0)P(t_0)x_0\|},$$

for all $(t, t_0, x_0) \in \Delta \times X,\ U(t,t_0)P(t_0)x_0 \neq 0.$

$$(ueD_2^1) \int_{s}^{\infty} \frac{e^{d(t-s)}}{\|U(t,t_0)Q(t_0)x_0\|} dt \leq \frac{D}{\|U(s,t_0)Q(t_0)x_0\|},$$

for all $(s, t_0, x_0) \in \Delta \times X,\ Q(t_0)x_0 \neq 0.$

(3) *there exists* $D > 1$ *with*

$$(ueD_1^2) \int_{t_0}^{t} \frac{ds}{\|U(s,t_0)P(t_0)x_0\|} \leq \frac{D}{\|U(t,t_0)P(t_0)x_0\|},$$

for all $(t, t_0, x_0) \in \Delta \times X,\ U(t,t_0)P(t_0)x_0 \neq 0.$

$$(ueD_2^2) \int_{s}^{\infty} \frac{dt}{\|U(t,t_0)Q(t_0)x_0\|} \leq \frac{D}{\|U(s,t_0)Q(t_0)x_0\|},$$

for all $(s, t_0, x_0) \in \Delta \times X,\ Q(t_0)x_0 \neq 0.$

Proof. $(1) \Rightarrow (2)$. It follows after a simple computation, by taking $D = 1 + \dfrac{N}{\nu - d}$.
$(2) \Rightarrow (3)$. It is trivial.
$(3) \Rightarrow (1)$. *Step 1.* We prove that (U, P) is uniformly dichotomic.
Let $(t, s, t_0) \in T$ with $t \geq s + 1$. Then,

$$\begin{aligned}
\frac{1}{\|U(s,t_0)P(t_0)x_0\|} &= \int_{s}^{s+1} \frac{1}{\|U(s,t_0)P(t_0)x_0\|} d\tau \leq M \int_{s}^{s+1} \frac{e^{\omega(\tau-s)}}{\|U(\tau,t_0)P(t_0)x_0\|} d\tau \leq \\
&\leq Me^{\omega} \int_{s}^{s+1} \frac{d\tau}{\|U(\tau,t_0)P(t_0)x_0\|} \leq Me^{\omega} \int_{t_0}^{t} \frac{d\tau}{\|U(\tau,t_0)P(t_0)x_0\|} \leq \\
&\leq MDe^{\omega} \cdot \frac{1}{\|U(t,t_0)P(t_0)x_0\|} \leq \frac{N_1}{\|U(t,t_0)P(t_0)x_0\|},
\end{aligned}$$

where $N_1 = MDe^{\omega} > 1$.
We obtain

$$\|U(t,t_0)P(t_0)x_0\| \leq N_1 \|U(s,t_0)P(t_0)x_0\|, \forall\, (t, s, t_0, x_0) \in T \times X,\ t \geq s + 1. \tag{1}$$

Let $(t, s, t_0) \in T, t \in [s, s+1)$. Then,

$$\begin{aligned}
\|U(t,t_0)P(t_0)x_0\| &\leq Me^{\omega(t-s)} \|U(s,t_0)P(t_0)x_0\| \leq Me^{\omega} \|U(s,t_0)P(t_0)x_0\| \\
&\leq N_1 \|U(s,t_0)P(t_0)x_0\|.
\end{aligned}$$

Thus, we have

$$\|U(t,t_0)P(t_0)x_0\| \leq N_1 \|U(s,t_0)P(t_0)x_0\|, \forall\, (t, s, t_0, x_0) \in T \times X,\ t \in [s, s+1). \tag{2}$$

From (1) and (2), it follows that (ud_1) is satisfied for all $(t, s, t_0, x_0) \in T \times X$.
Now, we prove that (ud_2) holds.

Let $(t,s,t_0) \in T, t \geq s+1$. We compute

$$\frac{1}{\|U(t,t_0)Q(t_0)x_0\|} = \int_s^{s+1} \frac{d\tau}{\|U(t,t_0)Q(t_0)x_0\|} \leq M \int_s^{s+1} \frac{e^{\omega(t-\tau)}}{\|U(\tau,t_0)Q(t_0)x_0\|} d\tau \leq$$
$$\leq Me^{\omega} \int_s^{\infty} \frac{d\tau}{\|U(\tau,t_0)Q(t_0)x_0\|} \leq \frac{MDe^{\omega}}{\|U(s,t_0)Q(t_0)x_0\|} \leq$$
$$\leq \frac{N_1}{\|U(s,t_0)Q(t_0)x_0\|}.$$

We obtain

$$\|U(s,t_0)Q(t_0)x_0\| \leq N_1\|U(t,t_0)Q(t_0)x_0\|, \forall\, (t,s,t_0,x_0) \in T \times X,\ t \geq s+1. \tag{3}$$

Let $(t,s,t_0) \in T, t \in [s,s+1)$. Then,

$$\|U(s,t_0)Q(t_0)x_0\| \leq Me^{\omega(t-s)}\|U(t,t_0)Q(t_0)x_0\| \leq Me^{\omega}\|U(t,t_0)Q(t_0)x_0\|$$
$$\leq N_1\|U(t,t_0)Q(t_0)x_0\|.$$

Thus, we have

$$\|U(s,t_0)Q(t_0)x_0\| \leq N_1\|U(t,t_0)Q(t_0)x_0\|, \forall\, (t,s,t_0,x_0) \in T \times X,\ t \in [s,s+1). \tag{4}$$

From (3) and (4), it follows that (ud_2) holds for all $(t,s,t_0,x_0) \in T \times X$, which means that (U,P) is uniformly dichotomic.
Step 2. We prove that (U,P) is uniformly exponentially dichotomic:

$$\frac{(t-s)}{\|U(t,t_0)P(t_0)x_0\|} = \int_s^{t} \frac{d\tau}{\|U(s,t_0)P(t_0)x_0} \leq N_1 \int_s^{t} \frac{d\tau}{\|U(\tau,t_0)P(t_0)x_0\|} \leq$$
$$\leq N_1 \int_{t_0}^{t} \frac{d\tau}{\|U(\tau,t_0)P(t_0)x_0\|} \leq \frac{N_1 D}{\|U(t,t_0)P(t_0)x_0\|}.$$

We obtain

$$(t-s)\|U(t,t_0)P(t_0)x_0\| \leq N_1 D\|U(s,t_0)P(t_0)x_0\|,\ \forall (t,s,t_0,x_0) \in T \times X. \tag{5}$$

$$\frac{(t-s)}{\|U(t,t_0)Q(t_0)x_0\|} = \int_s^{t} \frac{d\tau}{\|U(t,t_0)Q(t_0)x_0\|} \leq N_1 \int_s^{t} \frac{d\tau}{\|U(\tau,t_0)Q(t_0)x_0\|} \leq$$
$$\leq N_1 \int_s^{\infty} \frac{d\tau}{\|U(\tau,t_0)Q(t_0)x_0\|} \leq \frac{N_1 D}{\|U(s,t_0)Q(t_0)x_0\|}.$$

We obtain In my opinion, it is not redundant.

$$(t-s)\|U(s,t_0)Q(t_0)x_0\| \leq N_1 D\|U(t,t_0)Q(t_0)x_0\|,\ \forall (t,s,t_0,x_0) \in T \times X. \tag{6}$$

From (5) and (6) using the logarithmic criterion from Theorem 1, it follows that (U,P) is uniformly exponentially dichotomic. □

Corollary 2. *Let $U : \Delta \to \mathcal{B}(X)$ be a strongly measurable evolution operator and (U,P) a dichotomic pair with uniform exponential growth. Then, (U,P) is uniformly exponentially dichotomic if and only if there are $D > 1$ and $d \geq 0$ such that:*

$$(ueD_1^1)\quad \int_{t_0}^{t} \frac{e^{d(t-s)}}{\|U(s,t_0)P(t_0)x_0\|}ds \leq \frac{D}{\|U(t,t_0)P(t_0)x_0\|},$$
for all $(t,t_0,x_0) \in \Delta \times X,\ U(t,t_0)P(t_0)x_0 \neq 0.$

$$(ueD_2^1)\quad \int_{s}^{\infty} \frac{e^{d(t-s)}}{\|U(t,t_0)Q(t_0)x_0\|}dt \leq \frac{D}{\|U(s,t_0)Q(t_0)x_0\|},$$
for all $(s,t_0,x_0) \in \Delta \times X,\ Q(t_0)x_0 \neq 0.$

Proof. It is immediate using Theorem 4. □

Theorem 5. *Let $U : \Delta \to \mathcal{B}(X)$ be a strongly measurable evolution operator and (U,P) a dichotomic pair with uniform exponential growth. Then, (U,P) is uniformly exponentially dichotomic if and only if there are $D > 1$ and $L : \Delta \times X \to \mathbb{R}_+$ with the following properties:*

$$(ueL_1')\quad L(t,t_0,x_0) \leq D\left(\frac{1}{\|U(t,t_0)P(t_0)x_0\|} + \frac{1}{\|U(s,t_0)Q(t_0)x_0\|}\right),$$
for all $(t,t_0,x_0) \in \Delta \times X,\ U(t,t_0)P(t_0)x_0 \neq 0,\ Q(t_0)x_0 \neq 0.$

$$(ueL_2')\quad L(s,t_0,P(t_0)x_0) + \int_{s}^{t} \frac{e^{d(t-\tau)}}{\|U(\tau,t_0)P(t_0)x_0\|}d\tau = L(t,t_0,P(t_0)x_0),$$
for all $(t,s,t_0,x_0) \in T \times X,\ U(t,t_0)P(t_0)x_0 \neq 0.$

$$(ueL_3')\quad L(t,t_0,Q(t_0)x_0) + \int_{s}^{t} \frac{e^{d(t-\tau)}}{\|U(\tau,t_0)Q(t_0)x_0\|}d\tau = L(s,t_0,Q(t_0)x_0),$$
for all $(t,s,t_0,x_0) \in T \times X,\ Q(t_0)x_0 \neq 0.$

Proof. Necessity. It follows from Theorem 4 by taking the function $L : \Delta \times X \to \mathbb{R}_+$ defined by

$$L(s,t_0,x_0) = \int_{t_0}^{s} \frac{e^{d(t-\tau)}}{\|U(\tau,t_0)P(t_0)x_0\|}d\tau + \int_{s}^{\infty} \frac{e^{d(\tau-t)}}{\|U(\tau,t_0)Q(t_0)x_0\|}d\tau.$$

Sufficiency. It follows in a similar manner as the sufficiency proved in Theorem 3. □

Theorem 6. *Let $U : \Delta \to \mathcal{B}(X)$ be a strongly measurable evolution operator and (U,P) a dichotomic pair with uniform exponential growth. Then, (U,P) is uniformly exponentially dichotomic if and only if there are $D > 1$ and $d \in (0,1)$ with*

$$(ueD_1^3)\quad \int_{s}^{\infty} e^{(d-1)t}\|U(t,t_0)P(t_0)x_0\|dt \leq De^{ds}\|U(s,t_0)P(t_0)x_0\|,$$
for all $(s,t_0,x_0) \in \Delta \times X.$

$$(ueD_2^3)\quad \int_{s}^{\infty} \frac{e^{(d-1)t}}{\|U(t,t_0)Q(t_0)x_0\|}dt \leq \frac{De^{ds}}{\|U(s,t_0)Q(t_0)x_0\|},$$
for all $(s,t_0,x_0) \in \Delta \times X, Q(t_0)x_0 \neq 0.$

Proof. Necessity. It is a simple verification.

Sufficiency. We suppose that there are $D > 1$ and $d \in (0,1)$ such that (ueD_1^3) and (ueD_2^3) are satisfied. We have to prove that (U,P) is uniformly exponentially dichotomic, which means, according to Remark 2, that (ued_1') and (ued_2') hold. For (ueD_1^3) implies (ued_1')—see [7].

In order to prove the second relation, we firstly consider $(t,s,t_0) \in T,\ e^t \geq 2s$. Then,

$$\frac{e^{dt}}{\|U(t,t_0)Q(t_0)x_0\|} = \frac{2}{e^t}\int_{\frac{e^t}{2}}^{e^t} \frac{e^{dt}}{\|U(t,t_0)Q(t_0)x_0\|}d\tau = \frac{2}{e^t}\int_{\frac{e^t}{2}}^{e^t} \frac{e^{dt}}{\|U(t,s)U(s,t_0)Q(t_0)x_0\|}d\tau \leq$$

$$\leq \frac{2M}{e^t}\int_{\frac{e^t}{2}}^{e^t} \frac{e^{dt}e^{\omega(t-\tau)}}{\|U(\tau,s)U(s,t_0)Q(t_0)x_0\|}d\tau = \frac{2M}{e^t}\int_{\frac{e^t}{2}}^{e^t} \frac{e^{dt}e^{\omega(t-\tau)}}{\|U(\tau,t_0)Q(t_0)x_0\|}d\tau =$$

$$= \frac{2M}{e^t}\int_{\frac{e^t}{2}}^{e^t} \frac{e^{dt}e^{\omega(t-\tau)}e^{(d-1)(t-\tau)}e^{(1-d)(t-\tau)}}{\|U(\tau,t_0)Q(t_0)x_0\|}d\tau \leq 2M\int_{\frac{e^t}{2}}^{e^t} \frac{e^{(d-1)\tau}}{\|U(\tau,t_0)Q(t_0)x_0\|}d\tau \leq$$

$$\leq 2M\int_{s}^{\infty} \frac{e^{(d-1)\tau}}{\|U(\tau,t_0)Q(t_0)x_0\|}d\tau \leq \frac{2MDe^{ds}}{\|U(s,t_0)Q(t_0)x_0\|} \leq \frac{Ne^{ds}}{\|U(s,t_0)Q(t_0)x_0\|},$$

where $N = 2MD > 1$.

We obtain

$$\|U(s,t_0)Q(t_0)x_0\| \leq Ne^{-d(t-s)}\|U(t,t_0)Q(t_0)x_0\|,\ \forall(t,s,t_0,x_0) \in T \times X, e^t \geq 2s. \tag{7}$$

Now, let $(t,s,t_0) \in T,\ e^t < 2s$. Then,

$$\frac{e^{dt}}{\|U(t,t_0)Q(t_0)x_0\|} \leq \frac{Me^{dt}e^{\omega(t-s)}}{\|U(s,t_0)Q(t_0)x_0\|} = \frac{Me^{(d+\omega)(t-s)}e^{ds}}{\|U(s,t_0)Q(t_0)x_0\|} \leq$$

$$\leq M \cdot \left(\frac{2}{e}\right)^{d+\omega} \cdot \frac{e^{ds}}{\|U(s,t_0)Q(t_0)x_0\|} \leq \frac{Ne^{ds}}{\|U(s,t_0)Q(t_0)x_0\|}.$$

We obtain

$$\|U(s,t_0)Q(t_0)x_0\| \leq Ne^{-d(t-s)}\|U(t,t_0)Q(t_0)x_0\|,\ \forall(t,s,t_0,x_0) \in T \times X, e^t < 2s. \tag{8}$$

From (7) and (8), it follows that (ued_2') is satisfied for all $(t,s,t_0,x_0) \in T \times X$.

In conclusion, from Remark 2, it follows that (U,P) is uniformly exponentially dichotomic, so the proof is completed. □

Theorem 7. *Let $U : \Delta \to \mathcal{B}(X)$ be a strongly measurable evolution operator and (U,P) a dichotomic pair with uniform exponential growth. Then, (U,P) is uniformly exponentially dichotomic if and only if there are $D > 1$ and $L : \Delta \times X \to \mathbb{R}_+$ with the following properties:*

(ueL_1'') $L(t,t_0,x_0) \leq D\left(\|U(s,t_0)P(t_0)x_0\| + \dfrac{1}{\|U(s,t_0)Q(t_0)x_0\|}\right),$
for all $(t,t_0,x_0) \in \Delta \times X,\ Q(t_0)x_0 \neq 0$.

(ueL_2'') $L(t,t_0,P(t_0)x_0) + \displaystyle\int_s^t \frac{e^{(d-1)\tau}}{e^{ds}}\|U(\tau,t_0)P(t_0)x_0\|d\tau = L(s,t_0,P(t_0)x_0),$
for all $(t,s,t_0,x_0) \in T \times X$.

(ueL_3'') $L(t,t_0,Q(t_0)x_0) + \displaystyle\int_s^t \frac{e^{(d-1)\tau}}{e^{ds}} \cdot \frac{1}{\|U(\tau,t_0)Q(t_0)x_0\|} = L(s,t_0,Q(t_0)x_0),$
for all $(t,s,t_0,x_0) \in T \times X,\ Q(t_0)x_0 \neq 0$.

Proof. Necessity. It follows from Theorem 6 by taking the function $L : \Delta \times X \to \mathbb{R}_+$ defined by

$$L(t,t_0,x_0) = \int_t^\infty \frac{e^{(d-1)\tau}}{e^{ds}} \|U(\tau,t_0)P(t_0)x_0\| d\tau + \int_t^\infty \frac{e^{(d-1)t}}{e^{ds}} \cdot \frac{d\tau}{\|U(\tau,t_0)Q(t_0)x_0\|} d\tau.$$

Sufficiency. It follows by using similar arguments as in the sufficiency of Theorem 3. □

4. Uniform Polynomial Dichotomy

In this section, we focus on the uniform polynomial dichotomy. In fact, we obtain similar results as in the exponential case and we use the characterizations obtained in the exponential behavior in order to prove the theorems for the polynomial behavior. All the proofs from this section are based on the connection between the exponential and the polynomial case, a connection which is established through two evolution operators defined as follows:

$$U_1 : \Delta \to \mathcal{B}(X),\ U_1(t,s) = U(e^t - 1, e^s - 1)$$

and

$$U_2 : \Delta \to \mathcal{B}(X),\ U_2(t,s) = U(\ln(t+1), \ln(s+1)).$$

In addition, we define the projections' families associated with this operators

$$P_1 : \mathbb{R}_+ \to \mathcal{B}(X),\ P_1(t) = P(e^t - 1)$$

and

$$P_2 : \mathbb{R}_+ \to \mathcal{B}(X),\ P_2(t) = P(\ln(t+1)).$$

Proposition 1. *The pair (U,P) is uniformly polynomially dichotomic if and only if the pair (U_1,P_1) is uniformly exponentially dichotomic.*

Proof. Necessity. We suppose that (U,P) is u.p.d., which means that the relations (upd_1) and (upd_2) are satisfied. A simple computation shows us that (ued_1) and (ued_2) are true for the pair (U_1,P_1).
Sufficiency. We suppose that the pair (U_1,P_1) is u.e.d. Then,

$$\|U_1(t,s)P_1(s)x\| = \|U(e^t - 1, e^s - 1)P_1(e^s - 1)x\|.$$

If we denote by $e^t - 1 = u$ and $e^s - 1 = v$, we obtain

$$\begin{aligned}
&\|U(u,v)P(v)x\| = \|U(e^t - 1, e^s - 1)P(e^s - 1)x\| = \|U_1(t,s)P_1(s)x\| \leq \\
&\leq Ne^{-\nu(t-s)}\|P_1(s)x\| = Ne^{-\nu(\ln(1+u)-\ln(1+v))}\|P(e^s - 1)x\| = \\
&= N\left(\frac{v+1}{u+1}\right)^\nu \|P(v)x\|.
\end{aligned}$$

In addition,

$$\begin{aligned}
&N\|U(u,v)Q(v)x\| = N\|U(e^t - 1, e^s - 1)Q(e^s - 1)x\| = \\
&= N\|U_1(t,s)Q_1(s)x\| \geq e^{\nu(t-s)}\|Q_1(s)x\| = e^{\nu(\ln(1+u)-\ln(1+v))}\|Q_1(s)x\| = \\
&= \left(\frac{1+u}{1+v}\right)^\nu \|Q(e^s - 1)x\| = \left(\frac{1+u}{1+v}\right)^\nu \|Q(v)x\|.
\end{aligned}$$

□

Proposition 2. *The dichotomic pair (U_2, P_2) is uniformly polynomially dichotomic if and only if the dichotomic pair (U, P) is uniformly exponentially dichotomic.*

Proof. The relation (ued_1) is equivalent to:

$$\begin{aligned}\|U_2(t,s)P_2(s)x\| &= \|U(\ln(t+1),\ln(s+1))P(\ln(s+1))x\| \leq \\ &\leq Ne^{-\nu(\ln(t+1)-\ln(s+1))}\|P(\ln(s+1))x\| \\ &= N\left(\frac{t+1}{s+1}\right)^{-\nu}\|P(\ln(s+1))x\| = N\left(\frac{t+1}{s+1}\right)^{-\nu}\|P_2(s)x\|.\end{aligned}$$

The relation (ued_2) is equivalent to:

$$\begin{aligned}N\|U_2(t,s)Q_2(s)x\| &= N\|U(\ln(t+1),\ln(s+1))Q(\ln(s+1))x\| \geq \\ &\geq e^{\nu(\ln(t+1)-\ln(s+1))}\|Q(\ln(s+1))x\| \\ &= \left(\frac{t+1}{s+1}\right)^{\nu}\|Q(\ln(s+1))x\| = \left(\frac{t+1}{s+1}\right)^{\nu}\|Q_2(s)x\|.\end{aligned}$$

□

Next, we give a majorization criterion for the uniform polynomial dichotomy.

Theorem 8. *Let $U : \Delta \to \mathcal{B}(X)$ be a strongly measurable evolution operator and (U, P) a dichotomic pair with uniform polynomial growth. Then, (U, P) is uniformly polynomially dichotomic if and only if there exists a nondecreasing function*

$$\varphi : [1,\infty) \to \mathbb{R}_+ \textit{ with } \lim_{t\to\infty}\varphi(t) = \infty \textit{ such that}$$

(cm_1) $\varphi\left(\frac{t+1}{s+1}\right)\|U(t,t_0)P(t_0)x_0\| \leq \|U(s,t_0)P(t_0)x_0\|$

(cm_2) $\varphi\left(\frac{t+1}{s+1}\right)\|U(s,t_0)Q(t_0)x_0\| \leq \|U(t,t_0)Q(t_0)x_0\|,$

for all $(t, s, t_0, x_0) \in T \times X$.

Proof. Necessity. We suppose that (U, P) is u.p.d., which implies from Remark 6 and from Proposition 1 that (U, P) has u.p.g. and (U_1, P_1) is u.e.d., so (U_1, P_1) has u.e.g. Then, from the majorization criterion for the exponential dichotomy, we have that there exists a nondecreasing function $\varphi_1 : \mathbb{R}_+ \to \mathbb{R}_+$, with $\lim_{t\to\infty}\varphi_1(t) = \infty$ and

(i) $\varphi_1(u-v)\|U_1(u,w)P_1(w)x_0\| \leq \|U_1(v,w)P_1(w)x_0\| \qquad (*)$

(ii) $\varphi_1(u-v)\|U_1(v,w)Q_1(w)x_0\| \leq \|U_1(u,w)Q_1(w)x_0\| \qquad (**)$

for all $(u, v, w, x_0) \in T \times X$.
Moreover, for all $(u, v, w) \in \mathbb{R}^3_+$, there are $(t, s, t_0) \in \mathbb{R}^3_+$ such that $u = \ln(t+1)$, $v = \ln(s+1)$, $w = \ln(t_0+1)$.
Since $u \geq v \geq w$, then $t \geq s \geq t_0$. We compute the left side of the inequalities $(*)$ and $(**)$, and the necessity is proved. Sufficiency. We suppose that there exists a nondecreasing function $\varphi : [1,\infty) \to \mathbb{R}_+$ with $\lim_{t\to\infty}\varphi(t) = \infty$ such that the relations (cm_1) and (cm_2) hold.
We have to prove that (U, P) is u.p.d, which is equivalent from Proposition 1 with (U_1, P_1) being u.e.d.
Let $(t, s, t_0) \in T$, which implies that there are $(u, v, w) \in T$ with $u = \ln(t+1)$, $v = \ln(s+1)$, $w = \ln(t_0+1)$.

Then, we have

$$\varphi\left(\frac{t+1}{s+1}\right) = \varphi\left(e^{\ln\frac{t+1}{s+1}}\right) = \varphi(e^{\ln(t+1)-\ln(s+1)}) = \varphi(e^{u-v}) = \varphi_1(u-v),$$

where $\varphi_1 = \varphi(e^x)$.
$\|U(t,t_0)P(t_0)x_0\| = \|U(e^u-1,e^w-1)P(e^w-1)x_0\| = \|U_1(u,w)P_1(w)\|$.
$\|U(s,t_0)P(t_0)x_0\| = \|U(e^v-1,e^w-1)P(e^w-1)x_0\| = \|U_1(v,w)P_1(w)\|$.
$\|U(t,t_0)Q(t_0)x_0\| = \|U(e^u-1,e^w-1)Q(e^w-1)x_0\| = \|U_1(u,w)Q_1(w)\|$.
$\|U(s,t_0)Q(t_0)x_0\| = \|U(e^v-1,e^w-1)Q(e^w-1)x_0\| = \|U_1(u,w)Q_1(w)\|$.

Using the relations (cm_1) and (cm_2), it follows that:

$\varphi_1(u-v)\|U_1(u,w)P_1(w)x_0\| \le \|U_1(v,w)P_1(w)x_0\|$.
$\varphi_1(u-v)\|U_1(v,w)Q_1(w)x_0\| \le \|U_1(u,w)Q_1(w)x_0\|$.

Using the majorization criterion for u.e.d., it follows that the pair (U_1,P_1) is u.e.d, and, from Proposition 1, we obtain that (U,P) is u.p.d., so the proof is complete. □

Theorem 9. *Let $U : \Delta \to \mathcal{B}(X)$ be a strongly measurable evolution operator and (U,P) a dichotomic pair with uniform polynomial growth. The following assertions are equivalent:*

(1) *(U,P) is uniformly polynomially dichotomic.*
(2) *there are $D > 1$ and $d > 0$ with*

$$(upD_1)\quad \int_s^\infty \left(\frac{\tau+1}{s+1}\right)^d \frac{\|U(\tau,t_0)P(t_0)x_0\|}{\tau+1}d\tau \le D\|U(s,t_0)P(t_0)x_0\|,$$

for all $(s,t_0,x_0) \in \Delta \times X$.

$$(upD_2)\quad \int_{t_0}^t \left(\frac{t+1}{\tau+1}\right)^d \frac{\|U(\tau,t_0)Q(t_0)x_0\|}{\tau+1}d\tau \le D\|U(t,t_0)Q(t_0)x_0\|,$$

for all $(t,t_0,x_0) \in \Delta \times X$.

(3) *there exists $D > 1$ with*

$$(upD_1')\quad \int_s^\infty \frac{\|U(\tau,t_0)P(t_0)x_0\|}{\tau+1}d\tau \le D\|U(s,t_0)P(t_0)x_0\|,$$

for all $(s,t_0,x_0) \in \Delta \times X$.

$$(upD_2')\quad \int_{t_0}^t \frac{\|U(\tau,t_0)Q(t_0)x_0\|}{\tau+1}d\tau \le D\|U(t,t_0)Q(t_0)x_0\|,$$

for all $(t,t_0,x_0) \in \Delta \times X$.

Proof. $(1) \Rightarrow (2)$.

We suppose that (U,P) is uniformly polynomially dichotomic, which is equivalent from Proposition 1 with (U_1,P_1) being uniformly exponentially dichotomic. Then, using Theorem 2, it results that there are $D > 1$ and $d > 0$ such that

$$(i)\quad \int_s^\infty e^{d(t-s)}\|U_1(t,t_0)P_1(t_0)x_0\|dt \le D\|U_1(s,t_0)P_1(t_0)x_0\|,$$

for all $(s,t_0,x_0) \in \Delta \times X$.

$$(ii)\quad \int_{t_0}^t e^{d(t-s)}\|U_1(s,t_0)Q_1(t_0)x_0\|ds \le D\|U_1(t,t_0)Q_1(t_0)x_0\|,$$

for all $(t,t_0,x_0) \in \Delta \times X$.

Then, we have

$$\int_s^\infty e^{d(t-s)}\|U(e^t-1,e^{t_0}-1)P(e^{t_0}-1)x_0\|dt \le D\|U(e^s-1,e^{t_0}-1)P(e^{t_0}-1)x_0\|.$$

We do the change of variable $e^t - 1 = u$, and we obtain

$$\int_{e^s-1}^{\infty} e^{d(\ln(u+1)-s)} \|U(u, e^{t_0}-1)P(e^{t_0}-1)x_0\| \frac{du}{u+1} \leq D\|U(e^s-1, e^{t_0}-1)P(e^{t_0}-1)x_0\|.$$

We denote $e^s - 1 = v$ and $e^{t_0} - 1 = u_0$, and we have

$$\int_{v}^{\infty} \left(\frac{u+1}{v+1}\right)^d \frac{\|U(u,u_0)P(u_0)x_0\|}{u+1} du \leq D\|U(v,u_0)P(u_0)x_0\|,$$

which is equivalent to (upD_1).

On the other hand, the inequality (ii) is equivalent to

$$\int_{t_0}^{t} e^{d(t-s)} \|U(e^s-1, e^{t_0}-1)Q(e^{t_0}-1)x_0\| ds \leq D\|U(e^t-1, e^{t_0}-1)Q(e^{t_0}-1)x_0\|.$$

We do the change of variable $e^s - 1 = v$, we denote by $e^t - 1 = u,\ e^{t_0} - 1 = u_0$ and we obtain

$$\int_{u_0}^{u} \left(\frac{u+1}{v+1}\right)^d \frac{\|U(v,u_0)Q(u_0)x_0\|}{v+1} dv \leq D\|U(u,u_0)Q(u_0)x_0\|,$$

which is equivalent to (upD_2).

$(2) \Rightarrow (3)$. It is immediate.

$(3) \Rightarrow (1)$. We suppose that there exists $D > 1$ such that (upD_1') şi (upD_2') hold. We have to prove that (U, P) is uniformly polynomially dichotomic. In order to do this, according to Proposition 1, it is enough to prove that the pair (U_1, P_1) is uniformly exponentially dichotomic. We have

$$\int_{s}^{\infty} \|U_1(t,t_0)P_1(t_0)x_0\| dt = \int_{s}^{\infty} \|U(e^t-1, e^{t_0}-1)P(e^{t_0}-1)x_0\| dt.$$

We do the change of variable $e^t - 1 = u$, we denote by $e^s - 1 = v,\ e^{t_0} - 1 = u_0$ and we obtain

$$\int_{v}^{\infty} \frac{\|U(u,u_0)P(u_0)x_0\|}{u+1} du \leq D\|U(v,u_0)P(u_0)x_0\| = D\|U(e^s-1, e^{t_0}-1)P(e^{t_0}-1)x_0\| =$$
$$= D\|U_1(s,t_0)P_1(t_0)x_0\|.$$

Thus,

$$\int_{s}^{\infty} \|U_1(t,t_0)P_1(t_0)x_0\| dt \leq D\|U_1(s,t_0)P_1(t_0)x_0\|,$$

which means that (ueD_1') holds for (U_1, P_1). $(*)$

In addition,

$$\int_{t_0}^{t} \|U_1(s,t_0)Q_1(t_0)x_0\| ds = \int_{t_0}^{t} \|U(e^s-1, e^{t_0}-1)Q(e^{t_0}-1)x_0\| ds.$$

We do the change of variable $e^s - 1 = v$, we denote by $e^t - 1 = u$, $e^{t_0} - 1 = u_0$, and we obtain

$$\int_{u_0}^{u} \frac{\|U(v,u_0)Q(u_0)x_0\|}{v+1} dv \le D\|U(u,u_0)Q(u_0)x_0\| = D\|U(e^t-1, e^{t_0}-1)Q(e^{t_0}-1)x_0\| =$$

$$= D\|U_1(t,t_0)Q_1(t_0)x_0\|.$$

We obtain

$$\int_{t_0}^{t} \|U_1(s,t_0)Q_1(t_0)x_0\| ds \le D\|U_1(t,t_0)Q_1(t_0)x_0\|,$$

which means that (ueD_2') holds for (U_1, P_1). $(**)$

From $(*)$ and $(**)$, it follows that the relation (3) from Theorem 2 is satisfied for the pair (U_1, P_1). Thus, (U_1, P_1) is uniformly exponentially dichotomic, which implies, using Proposition 1, that (U, P) is uniformly polynomially dichotomic, so the proof is complete. □

Corollary 3. *Let $U : \Delta \to \mathcal{B}(X)$ be a strongly measurable evolution operator and (U, P) a dichotomic pair with uniform polynomial growth. Then, (U, P) is uniformly polynomially dichotomic if and only if there are $D > 1$ and $d \ge 0$ such that*

(upD_1'') $$\int_{s}^{\infty} \left(\frac{\tau+1}{s+1}\right)^d \frac{\|U(\tau,t_0)P(t_0)x_0\|}{\tau+1} d\tau \le D\|U(s,t_0)P(t_0)x_0\|,$$
for all $(s, t_0, x_0) \in \Delta \times X$.

(upD_2'') $$\int_{t_0}^{t} \left(\frac{t+1}{\tau+1}\right)^d \frac{\|U(\tau,t_0)Q(t_0)x_0\|}{\tau+1} d\tau \le D\|U(t,t_0)Q(t_0)x_0\|,$$
for all $(t, t_0, x_0) \in \Delta \times X$.

Proof. It is follows immediately from Theorem 9. □

Remark 9. *Another variant of Theorem 9 was proved using a distinct technique by Rămneanţu and Ceauşu in [22].*

In addition, a different proof of the above theorem for the particular case $d = 0$ can be found in [26].

Theorem 10. *Let $U : \Delta \to \mathcal{B}(X)$ be a strongly measurable evolution operator and (U, P) a dichotomic pair with uniform polynomial growth. Then, (U, P) is uniformly polynomially dichotomic if and only if there are $D > 1$ and $L : \Delta \times X \to \mathbb{R}_+$ with the following properties:*

(upL_1) $L(t, t_0, x_0) \le D(\|U(s,t_0)P(t_0)x_0\| + \|U(t,t_0)Q(t_0)x_0\|)$,
for all $(t, t_0, x_0) \in \Delta \times X$.

(upL_2) $$L(t, t_0, P(t_0)x_0) + \int_{s}^{t} \frac{(\tau+1)^{d-1}}{(s+1)^d} \|U(\tau,t_0)P(t_0)x_0\| d\tau = L(s, t_0, P(t_0)x_0),$$
for all $(t, s, t_0, x_0) \in T \times X$.

(upL_3) $$L(s, t_0, Q(t_0)x_0) + \int_{s}^{t} \frac{(t+1)^d}{(\tau+1)^{d+1}} \|U(\tau,t_0)Q(t_0)x_0\| d\tau = L(t, t_0, Q(t_0)x_0),$$
for all $(t, s, t_0, x_0) \in T \times X$.

Proof. It is similar to the proof of Theorem 3.2 from [22]. □

Theorem 11. *Let $U : \Delta \to \mathcal{B}(X)$ be a strongly measurable evolution operator and (U, P) a dichotomic pair with uniform polynomial growth. The following assertions are equivalent:*

(1) *(U, P) is uniformly polynomially dichotomic.*

(2) *there are $D > 1$ and $d > 0$ with*

$$(upD_1^1) \quad \int_{t_0}^{t} \left(\frac{t+1}{\tau+1}\right)^d \frac{d\tau}{(\tau+1)\|U(\tau,t_0)P(t_0)x_0\|} \leq \frac{D}{\|U(t,t_0)P(t_0)x_0\|},$$
for all $(t, t_0, x_0) \in \Delta \times X,\ U(t,t_0)P(t_0)x_0 \neq 0.$

$$(upD_2^1) \quad \int_{s}^{\infty} \left(\frac{t+1}{s+1}\right)^d \frac{dt}{(t+1)\|U(t,t_0)Q(t_0)x_0\|} \leq \frac{D}{\|U(s,t_0)Q(t_0)x_0\|},$$
for all $(s, t_0, x_0) \in \Delta \times X,\ Q(t_0)x_0 \neq 0.$

(3) *there exists $D > 1$ with*

$$(upD_1^2) \quad \int_{t_0}^{t} \frac{d\tau}{(\tau+1)\|U(\tau,t_0)P(t_0)x_0\|} \leq \frac{D}{\|U(t,t_0)P(t_0)x_0\|},$$
for all $(t, t_0, x_0) \in \Delta \times X,\ U(t,t_0)P(t_0)x_0 \neq 0.$

$$(upD_2^2) \quad \int_{s}^{\infty} \frac{dt}{(t+1)\|U(t,t_0)Q(t_0)x_0\|} \leq \frac{D}{\|U(s,t_0)Q(t_0)x_0\|},$$
for all $(s, t_0, x_0) \in \Delta \times X,\ Q(t_0)x_0 \neq 0.$

Proof. (1) $\Rightarrow$ (2). We suppose that (U, P) is uniformly polynomially dichotomic, which is equivalent from Proposition 1 to (U_1, P_1) uniformly exponentially dichotomic. Then, using Theorem 4, it follows that there are $D > 1$ and $d > 0$ such that (ueD_1^1) and (ueD_2^1) are satisfied for the pair (U_1, P_1), i.e.,

$$(i) \quad \int_{t_0}^{t} \frac{e^{d(t-s)}}{\|U_1(s,t_0)P_1(t_0)x_0\|} ds \leq \frac{D}{\|U_1(t,t_0)P_1(t_0)x_0\|}$$
for all $(t, t_0, x_0) \in \Delta \times X,\ U_1(t,t_0)P_1(t_0)x_0 \neq 0.$

$$(ii) \quad \int_{s}^{\infty} \frac{e^{d(t-s)}}{\|U_1(t,t_0)Q_1(t_0)x_0\|} dt \leq \frac{D}{\|U_1(s,t_0)Q_1(t_0)x_0\|}$$
for all $(s, t_0, x_0) \in \Delta \times X,\ Q_1(t_0)x_0 \neq 0.$

The relation (i) is equivalent to

$$\int_{t_0}^{t} \frac{e^{d(t-s)}}{\|U(e^s-1, e^{t_0}-1)P(e^{t_0}-1)x_0\|} ds \leq \frac{D}{\|U(e^t-1, e^{t_0}-1)P(e^{t_0}-1)x_0\|}.$$

We do the change of variable $e^s - 1 = v$, and we obtain

$$\int_{e^{t_0}-1}^{e^t-1} \frac{e^{d(t-\ln(v+1))}}{\|U(v, e^{t_0}-1)P(e^{t_0}-1)x_0\|} ds \leq \frac{D}{\|U(e^t-1, e^{t_0}-1)P(e^{t_0}-1)x_0\|}.$$

We denote by $e^{t_0} - 1 = u_0$ şi $e^t - 1 = u$, and we obtain

$$\int_{u_0}^{u} \left(\frac{u+1}{v+1}\right)^d \frac{dv}{(v+1)\|U(v,u_0)P(u_0)x_0\|} \leq \frac{D}{\|U(v,u_0)P(u_0)x_0\|},$$

which is equivalent to (upD_1^1).

In addition, (ii) is equivalent to

$$\int_s^\infty \frac{e^{d(t-s)}}{\|U(e^t-1,e^{t_0}-1)Q(e^{t_0}-1)x_0\|}dt \leq \frac{D}{\|U(e^s-1,e^{t_0}-1)Q(e^{t_0}-1)x_0\|}.$$

We do the change of variable $e^t-1=u$, we denote by $e^s-1=v$, $e^{t_0}-1=u_0$ and we obtain

$$\int_v^\infty \left(\frac{u+1}{v+1}\right)^d \frac{du}{(u+1)\|U(u,u_0)Q(u_0)x_0\|} \leq \frac{D}{\|U(v,u_0)Q(u_0)x_0\|},$$

which is equivalent to (upD_2^1).
$(2) \Rightarrow (3)$. It is immediate.
$(3) \Rightarrow (1)$. We suppose that there exists $D>1$ such that (upD_1^2) and (upD_2^2) hold. We have to prove that (U,P) is uniformly polynomially dichotomic. According to Proposition 1, it is enough to prove that (U_1,P_1) is uniformly exponentially dichotomic. We compute

$$\int_{t_0}^t \frac{ds}{\|U_1(s,t_0)P_1(t_0)x_0\|} = \int_{t_0}^t \frac{ds}{\|U(e^s-1,e^{t_0}-1)P(e^{t_0}-1)x_0\|}.$$

We do the change of variable $e^s-1=v$, we denote by $e^t-1=u$, $e^{t_0}-1=u_0$ and we obtain

$$\int_{u_0}^u \frac{dv}{(v+1)\|U(v,u_0)P(u_0)x_0\|} \leq \frac{D}{\|U(u,u_0)P(u_0)x_0\|} = \frac{D}{\|U(e^t-1,e^{t_0}-1)P(e^{t_0}-1)x_0\|} = \\ = \frac{D}{\|U_1(t,t_0)P_1(t_0)x_0\|}.$$

Thus,

$$\int_{t_0}^t \frac{ds}{\|U_1(s,t_0)P_1(t_0)x_0\|} \leq \frac{D}{\|U_1(t,t_0)P_1(t_0)x_0\|},$$

which means that (ueD_1^2) holds for (U_1,P_1). $(\diamond)$
Similarly,

$$\int_s^\infty \frac{dt}{\|U_1(t,t_0)Q_1(t_0)x_0\|} = \int_s^\infty \frac{dt}{\|U(e^t-1,e^{t_0}-1)Q(e^{t_0}-1)x_0\|}.$$

We do the change of variable $e^t-1=u$, we denote by $e^{t_0}-1=u_0$, $e^s-1=v$ and we obtain

$$\int_v^\infty \frac{du}{(u+1)\|U(u,u_0)Q(u_0)x_0\|} \leq \frac{D}{\|U(v,u_0)Q(u_0)x_0\|} = \frac{D}{\|U(e^s-1,e^{t_0}-1)Q(e^{t_0}-1)x_0\|} = \\ = \frac{D}{\|U_1(s,t_0)Q_1(t_0)x_0\|}.$$

We obtain

$$\int_s^\infty \frac{dt}{\|U_1(t,t_0)Q_1(t_0)x_0\|} \leq \frac{D}{\|U_1(s,t_0)Q_1(t_0)x_0\|},$$

which means that (ueD_2^2) holds for (U_1,P_1). $(\diamond\diamond)$

From $(\diamond)$ and $(\diamond\diamond)$, it results that the inequality (3) from Theorem 2 is satisfied for the dichotomic pair (U_1,P_1). Thus, (U_1,P_1) is uniformly exponentially dichotomic, which implies from Proposition 1 that (U,P) is uniformly polynomially dichotomic and the proof is complete. □

Corollary 4. *Let $U:\Delta\to\mathcal{B}(X)$ be a strongly measurable evolution operator and (U,P) a dichotomic pair which has uniform polynomial growth. Then, (U,P) is uniformly polynomially dichotomic if and only if there are $D>1$ and $d\geq 0$ such that*

$$(upD_1^1)\quad \int_{t_0}^{t}\left(\frac{t+1}{\tau+1}\right)^d \frac{d\tau}{(\tau+1)\|U(\tau,t_0)P(t_0)x_0\|}\leq \frac{D}{\|U(t,t_0)P(t_0)x_0\|},$$
for all $(t,t_0,x_0)\in\Delta\times X,\ U(t,t_0)P(t_0)x_0\neq 0$.

$$(upD_2^1)\quad \int_{s}^{\infty}\left(\frac{t+1}{s+1}\right)^d \frac{dt}{(t+1)\|U(t,t_0)Q(t_0)x_0\|}\leq \frac{D}{\|U(s,t_0)Q(t_0)x_0\|},$$
for all $(s,t_0,x_0)\in\Delta\times X,\ Q(t_0)x_0\neq 0$.

Proof. It follows immediately from Theorem 11. □

Theorem 12. *Let $U:\Delta\to\mathcal{B}(X)$ be a strongly measurable evolution operator and (U,P) a dichotomic pair with uniform polynomial growth. Then, (U,P) is uniformly polynomially dichotomic if and only if there are $D>1$ and $L:\Delta\times X\to\mathbb{R}_+$ with the following properties:*

$$(upL_1')\quad L(t,t_0,x_0)\leq D\left(\frac{1}{\|U(t,t_0)P(t_0)x_0\|}+\frac{1}{\|U(t,t_0)Q(t_0)x_0\|}\right),$$
for all $(t,t_0,x_0)\in\Delta\times X,\ U(t,t_0)P(t_0)x_0\neq 0,\ Q(t_0)x_0\neq 0$.

$$(upL_2')\quad L(s,t_0,P(t_0)x_0)+\int_s^t \frac{(t+1)^d}{(\tau+1)^{d+1}}\cdot\frac{d\tau}{\|U(\tau,t_0)P(t_0)x_0\|}=L(t,t_0,P(t_0)x_0),$$
for all $(t,s,t_0,x_0)\in T\times X,\ U(t,t_0)P(t_0)x_0\neq 0$.

$$(upL_3')\quad L(t,t_0,Q(t_0)x_0)+\int_s^t \frac{(\tau+1)^{d-1}}{(t+1)^{d}}\cdot\frac{d\tau}{\|U(\tau,t_0)Q(t_0)x_0\|}=L(s,t_0,Q(t_0)x_0),$$
for all $(t,s,t_0,x_0)\in T\times X,\ Q(t_0)x_0\neq 0$.

Proof. Necessity. It follows from Theorem 11 by taking the function $L:\Delta\times X\to\mathbb{R}_+$ defined by

$$L(s,t_0,x_0)=\int_{t_0}^{s}\frac{(t+1)^d}{(\tau+1)^{d+1}}\cdot\frac{d\tau}{\|U(\tau,t_0)P(t_0)x_0\|}+\int_s^\infty \frac{(\tau+1)^{d-1}}{(t+1)^d}\cdot\frac{d\tau}{\|U(\tau,t_0)Q(t_0)x_0\|d\tau}.$$

Sufficiency. It follows in a similar manner as the sufficiency proved in Theorem 10. □

5. Discussion

The results obtained in this work contribute to the development of the theory in the field of dynamical systems. More specifically, we prove some characterizations for two of the most studied asymptotic properties of evolution operators in Banach spaces, namely the uniform exponential dichotomy and the uniform polynomial dichotomy.

We give necessary and sufficient conditions that extend Datko's theorem, which has become one of the most famous theorems of the modern control theory. In addition, we characterize these concepts using Lyapunov functions, and we establish connections between the concepts mentioned in the paper. The method that we use in order to prove the polynomial part is new, and it is much simpler than the one that exists in the literature.

For the future, we intend to generalize all these results to the nonuniform case in order to study the robustness property, a notion that has a long history and was discussed for the first time in the context of the nonuniform exponential behavior by Barreira and Valls in [27].

In addition, having [28] as a start point, we would like to investigate if it is possible to analyze the behaviors described in this paper in order to obtain some numerical results.

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References

1. Perron, O. Die Stabilitätsfrage bei Differentialgleichungen. *Math. Z.* **1930**, *32*, 703–728. [CrossRef]
2. Massera, J.L.; Schäffer, J.J. *Linear Differential Equations and Function Spaces*; Academic Press: New York, NY, USA, 1966.
3. Daleckii, J.L.; Krein, M.G. *Stability of Solutions of Differential Equations in Banach Spaces*; Trans. Math. Monographs, 43; American Mathematical Society: Providence, RI, USA, 1974.
4. Coppel, W.A. *Dichotomies in Stability Theory*; Lecture Notes in Math; Springer: Berlin, Germany; New York, NY, USA, 1978.
5. Chicone, C.; Latushkin, Y. *Evolution Semigroups in Dynamical Systems and Differential Equations*; Math Surveys and Monograph, 70; American Mathematical Society: Providence, RI, USA, 1999.
6. Sacker, R.J.; Sell, G.R. Dichotomies for Linear Evolutionary Equations in Banach Spaces. *J. Differ. Equ.* **1994**, *113*, 17–67. [CrossRef]
7. Boruga (Toma), R.; Megan, M.; Toth, D.M.-M. Integral Characterizations for Uniform Stability with Growth Rates in Banach Spaces. *Axioms* **2021**, *10*, 235. [CrossRef]
8. Boruga (Toma), R.; Borlea (Pătraşcu), D.I.; Toth, D.M.-M. On uniform stability with growth rates in Banach spaces. In Proceedings of the 15th International Symposium on Applied Computational Intelligence and Informatics, Timişoara, Romania, 19–21 May 2021.
9. Megan M.; Sasu B.; Sasu, A.L. On nonuniform exponential dichotomy of evolution operators in Banach spaces. *Integral Equ. Oper. Theory* **2002**, *44*, 71–78. [CrossRef]
10. Mihiţ, C.L. On some nonuniform dichotomic behaviors of discrete skew-product semiflows. *Theory Appl. Math. Comput. Sci.* **2019**, *9*, 1–7.
11. Preda, P.; Megan, M. Exponential dichotomy of evolutionary processes in Banach spaces. *Czechoslov. Math. J.* **1985**, *35*, 312–323. [CrossRef]
12. Sasu, A.L.; Sasu, B. Integral equations, dichotomy of evolution families on the half-line and applications. *Integral Equ. Oper. Theory* **2010**, *66*, 113–140. [CrossRef]
13. Zhou, L.; Lu, K.; Zhang, W. Equivalences between nonuniform exponential dichotomy and admissibility. *J. Differ. Equ.* **2017**, *262*, 682–747. [CrossRef]
14. Barreira, L.; Valls, C. Polynomial growth rates. *Nonlinear Anal.* **2009**, *71*, 5208–5219. [CrossRef]
15. Bento, A.J.G.; Silva, C. Stable manifolds for nonuniform polynomial dichotomies. *J. Funct. Anal.* **2009**, *257*, 122–148. [CrossRef]
16. Dragičević, D.; Sasu, A.L.; Sasu, B. On polynomial dichotomies of discrete nonautonomous systems on the half-line. *Carpathian J. Math.* **2022**, *38*, 663–680. [CrossRef]
17. Dragičević, D.; Sasu, A.L.; Sasu, B. Admissibility and polynomial dichotomy of discrete nonautonomous systems. *Carpathian J. Math.* **2022**, *38*, 737–762. [CrossRef]
18. Bainov, D.D.; Kostadinov, S.I.; Zabreiko, P.P. Exponential dichotomy of linear impulsive equations in a Banach space. *Internat. J. Theoret. Phys.* **1989**, *28*, 797–814. [CrossRef]
19. Caraballo, T.; Real, J.; Shaikhet, L. Method of Lyapunov functionals construction in stability of delay evolution equations. *J. Math. Anal. Appl.* **2007**, *334*, 1130–1145. [CrossRef]
20. Pötzsche, C. *Geometric Theory of Discrete Nonautonomous Dynamical Systems*; Lecture Notes in Math; Springer: Berlin/Heidelberg, Germany, 2010; Volume 2002.
21. Zhang, J.; Fan, M.; Zhu, H. Existence and roughness of exponential dichotomies of linear dynamic equations on time scales. *Comput. Math. Appl.* **2010**, *59*, 2658–2675. [CrossRef]

22. Rămneanţu, M.L.; Ceauşu, T. Uniform polynomial dichotomy of evolution operators in Banach spaces. *Analele Univ. Timişoara Ser. Mat. Inform.* **2011**, *49*, 107–116.
23. Hai, P.V. On the polynomial stability of evolution families. *Appl. Anal.* **2015**, *95*, 1239–1255. [CrossRef]
24. Datko, R. Uniform Asymptotic Stability of Evolutionary Processes in a Banach Space. *SIAM J. Math. Anal.* **1972**, *3*, 428–445. [CrossRef]
25. Megan, M.; Stoica, C.; Buliga, L. On asymptotic behaviours of evolution operators in Banach spaces. *Prepr. Ser. Math. Univ. Vest Timişoara* **2006**, *13*, 1–22.
26. Boruga, R.; Megan, M. On uniform polynomial dichotomy in Banach spaces. *Sci. Bull. Politeh. Univ. Timişoara Trans. Math. Phys.* **2018**, *63*, 32–40.
27. Barreira, L.; Valls, C. Robustness of nonuniform exponential dichotomies in Banach spaces. *J. Differ. Equ.* **2008**, *244*, 2407–2447. [CrossRef]
28. Srivastava, H.M.; Abdel-Gawad, H.I.; Saad, K.M. Oscillatory states and patterns formation in a two-cell cubic autocatalytic reaction-diffusion model subjected to the Dirichlet conditions. *Discret. Contin. Dyn. Syst. Ser. S* **2021**, *14*, 3785–3801. [CrossRef]