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Bounds for Statistical Curvatures of Submanifolds in Kenmotsu-like Statistical Manifolds

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Abstract: In this article, we obtain certain bounds for statistical curvatures of submanifolds with any codimension of Kenmotsu-like statistical manifolds. In this context, we construct a class of optimum inequalities for submanifolds in Kenmotsu-like statistical manifolds containing the normalized scalar curvature and the generalized normalized Casorati curvatures. We also define the second fundamental form of those submanifolds that satisfy the equality condition. On Legendrian submanifolds of Kenmotsu-like statistical manifolds, we discuss a conjecture for Wintgen inequality. At the end, some immediate geometric consequences are stated.

Keywords: statistical manifolds; kenmotsu-like statistical manifolds; casorati curvatures; generalized wintgen inequality

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1. Introduction

A statistical manifold is an extension of a statistical model that is abstraction. The statistical manifold characterization is based on a statistical model in which the density functions are swapped by any Riemannian manifold $\bar{M}$, the Riemannian metric $\bar{d}$ substitutes the Fisher information matrix of the manifold $\bar{M}$, the dual connections $\bar{D}^{(-1)}$ and $\bar{D}^{(1)}$ are exchanged by a couple of dual connections $\bar{D}$ and $\bar{D}^*$, and the skewness tensor is modified by a three-covariant skewness tensor, which counts the cummulants of the third order. Amari [1] first developed statistical manifolds in 1985. He looked at it from the standpoint of information geometry, and such manifolds include the concept of dual connections, also known as conjugate connections in affine geometry, which is strongly linked to affine differential geometry and has applications in numerous fields of scientific disciplines.

In its most basic form, information geometry is a part of mathematics that uses differential geometry concepts to the topic of probability theory. A model is a statistical manifold, and the amount of the parameters affects the point of statistical manifold and its transformation are well known from thermodynamics. The natural Riemannian manifold for thermodynamics is the statistical manifold. The geometrical representation in the framework of Gibbs's statistical mechanical representation for reversible and irreversible fluctuations in the value of the critical variable. Entropy mechanics, which can give significant mechanical techniques in the analysis of thermodynamics, remains at the center of Gibbs' work. Gibbs statistical manifolds frameworks have been extended to represent statistical manifold kinematics. Furthermore, thermodynamic equilibrium exists in the form of statistical groups across independent states, which serve as a link between statistical thermodynamics and information geometry theories. Fisher information matrix, for example, is useful to neural networks and Langevin kinetics. Furthermore, Newtonian dynamics can be recast in the language of Riemannian geometry applied to probability theory, namely, information geometry, where the link is made using the probability distribution's average. As a result, the dynamics take place on a statistical manifold that is naturally endowed with a metric structure furnished by information geometry and the curvature of the statistical manifolds has a significant influence. For example, an entropy dynamics (ED) statistical model has been constructed on a $6n$-dimensional statistical manifold M. The micro-coordinated on the manifolds are represented by the expectation values of micro variables associated with Gaussian distribution.

The normalized square of the length of the second basic form of a submanifold of a Riemannian manifold was defined by Casorati [2] as Casorati curvature. This concept went beyond the primary direction of Riemannian manifold hypersurfaces. Geometry and other domains, such as computer visual information, have examined the Casorati curvature. Some findings were achieved in terms of isotropical Casorati curvature of production surfaces. In Riemannian manifolds, a geometrical explanation of the Casorati curvature of submanifolds was also investigated. Recently, a geometric analysis of the Cauchy–Schwarz inequality in terms of Casorati curvature has been considered. In this reason, the geometric study of Casorati curvatures for submanifolds is new and has many research problems. A couple of optimal Casorati inequalities had been obtained by many distinguished geometers in different ambient space forms (for example, [3,4]). Decu et al. have built certain inequalities for statistical submanifolds of Kenmotsu statistical manifolds with constant ϕ-sectional curvature involving normalized δ-Casorati curvatures and scalar curvature in [5]. Inequalities for statistical warped product submanifolds were explored by Aliya et al. in [6].

Wintgen [7], on the other hand, proposed a crucial relationship between the Gauss curvature, normal curvature, and squared mean curvature of any surface N in a four-dimensional Euclidean space $\mathbb{E}^4$, as well as the necessary and sufficient conditions under which the equality case holds. Guadalupe and Rodriguez generalized Wintgen's inequality to a real-space surface of arbitrary codimension in the form $\mathbb{R}^{m+2}(c)$, $m \geq 2$. After that, Chen extended this inequality to surfaces in a 4-dimensional pseudo-Euclidean space $\mathbb{E}^4_2$ with a neutral metric. In [8], DeSmet, Dillen, Verstraelen, and Vrancken found the DDVV conjecture (called the generalized Wintgen inequality in general) for an isometric immersion of a Riemannian manifold into a real space form. Furthermore, in [8], they conjectured this inequality for a submanifold with codimension 2 in a real space form $\mathbb{R}^{m+2}(c)$. The solution of this conjecture was independently proven by Lu [9] and Ge and Tang [10] for general case. Since then, many remarkable articles were published and several inequalities of this type were obtained for other kinds of submanifolds in different ambient spaces (see [11–15]).

The derivation of inequality in terms of Casorati curvatures for various submanifolds in various ambient spaces is focused on an optimization approach that establishes that the polynomial of quadratic type in the components of the second fundamental form is parabolic. However, in the present paper, the proof of the inequality involving Casorati curvatures (of submanifolds in Kenmotsu-like statistical manifolds) in Theorem 3 is emphasised on a constrained extremum problem on the submanifold given in Lemma 1. Equality case is also examined. On the other hand, we extend the classical DDVV inequality to a Legendrian submanifold in Kenmotsu-like statistical manifolds. The main ingredient in proving Theorem 6 is given by Theorem 5, which actually translates the DDVV-conjecture to an algebraic problem involving some traceless symmetric matrices.

2. Preliminaries

A semi-Riemannian manifold $\bar{M}$ and non-degenerate metric $\bar{d}$, and a torsion-free affine connection by $\bar{D}$. The triplet $(\bar{M}, \bar{D}, \bar{d})$ is said to be a statistical manifold [16] with symmetric $\bar{D}\bar{d}$. It is usually denoted by $(\bar{M}, \bar{D}, \bar{d})$.

In case of a statistical manifold, we have a second connection $\bar{D}^*$ as:

$$G\bar{d}(x, y) = \bar{g}(\bar{D}_z x, y) + \bar{d}(x, \bar{D}^*_z y), \tag{1}$$

for any $x, y, z \in T_r\bar{M}$, $r \in \bar{M}$. The torsion-free affine connection $\bar{D}^*$ is called dual (or conjugate) of the connection $\bar{D}$ with respect to the $\bar{d}$ and obeys

$$(\bar{D}^*)^* = \bar{D}.$$

$$2\bar{D}^0 = \bar{D} + \bar{D}^*, \tag{2}$$

where $\bar{D}^0$ is indicates the Levi-Civita connection on $\bar{M}$.

A semi-Riemannian manifold $(\bar{M}, \bar{d})$ is said to be an almost contact metric manifold with almost contact structure (ψ, ξ, η, d) of certain kind [17] if it admits the almost contact structure (ψ, ξ, η) which satisfies the following equations:

$$\eta(\xi) = 1, \quad \psi(\xi) = 0 \quad \text{and} \quad \eta \circ \psi = 0, \tag{3}$$

and also has another tensor field ψ^* of type $(1,1)$ which obeys

$$\bar{d}(\psi x, y) = -\bar{g}(x, \psi^* y), \tag{4}$$

for any $x, y \in T_r\bar{M}$. It is easy to see that

$$\psi^{*2}x = -x + \eta(x)\xi \quad \text{and} \quad \bar{d}(\psi x, \psi^* y) = \bar{d}(x, y) - \eta(x)\eta(y), \tag{5}$$

$$\psi^*\xi = 0 \quad \text{and} \quad \eta(\psi^*(x)) = 0. \tag{6}$$

As a tensor field ψ is not symmetric, it shows that $\psi + \psi^* \neq 0$ everywhere.

Kenmotsu [18] initiated the study of Kenmotsu geometry, which is a crucial class of contact geometry. The almost contact metric manifold of certain class $(\bar{M}, \bar{D}, \phi, \xi, \eta, \bar{d})$ is said to be Kenmotsu-like statistical manifold [19], if the following axioms hold

$$\bar{D}_x\xi = x - \eta(x)\xi, \quad \bar{D}^*_x\xi = x - \eta(x)\xi, \tag{7}$$

$$(\bar{D}_x\psi)y = d(\psi x, y)\xi - \eta(z)\psi x \quad \text{and} \quad (\bar{D}^*_x\psi^*)y = \bar{d}(\psi^* x, y)\xi - \eta(z)\psi^*. \tag{8}$$

The curvature tensor $\bar{R}$ with respect to $\bar{D}$ on a Kenmotsu-like statistical manifold is given as:

$$\begin{aligned} \bar{R}(x,y)z \quad = \quad & \frac{c-3}{4}\{\bar{d}(y,z)x - \bar{d}(x,z)y\} \\ & + \frac{c+1}{4}\{\bar{d}(\psi y, z)\psi x - \bar{d}(\psi x, z)\psi y \\ & - 2\bar{d}(\psi x, y)\psi z - \bar{d}(y,\xi)\bar{d}(z,\xi)x \\ & + \bar{d}(x,\xi)\bar{d}(z,\xi)y + \bar{d}(y,\xi)\bar{d}(z,x)\xi \\ & - \bar{d}(x,\xi)\bar{d}(z,y)\xi\}, \end{aligned} \tag{9}$$

where $c \in \mathbb{R}$.

After, shifting ψ to ψ^* in (9), we turn up the expression of the curvature tensor $\bar{R}^*$ for $\bar{D}^*$.

Certainly, $(\bar{M}, \bar{D}^*, \bar{d})$ is a statistical manifold. For example, every semi-Riemannian manifold $(\bar{M}, \bar{D}, \bar{d})$ endowed with a Riemannian connection $\bar{D}$ is a trivial statistical manifold. In this case, one turn up

$$\bar{d}(\bar{R}(x,y)z, z') = -\bar{d}(z, \bar{R}^*(x,y)z'), \tag{10}$$

for any $x, y, z, z' \in T_r\bar{M}$ [16].

Let M be a submanifold of $\bar{M}$ on a statistical manifold $(\bar{M}, \bar{D}, \bar{d})$. Then $((M, D, d)$ is also said to be a statistical manifold with the induced statistical structure (D, d) on M from $(\bar{D}, \bar{d})$ and we say (M, D, d) is a statistical submanifold in $(\bar{M}, \bar{D}, \bar{d})$.

The Gauss and Weingarten equations are used in statistical settings, respectively, defined by [20]:

$$\left.\begin{array}{ll}\bar{D}_x y = D_x y + h(x,y), & \bar{D}^*_x y = D^*_x y + h^*(x,y), \\ \bar{D}_x \mathcal{U} = -A_{\mathcal{U}}(x) + D^{\perp}_x \mathcal{U}, & \bar{D}^*_x \mathcal{U} = -A^*_{\mathcal{U}}(x) + D^{\perp *}_x \mathcal{U},\end{array}\right\} \tag{11}$$

for any $x, y \in T_r M$ and $\mathcal{U} \in T_r^{\perp} M$, where $\bar{D}$ and $\bar{D}^*$ are the dual connections on $\bar{M}$. Similarly, on M, we denote them by D and D^*. For $\bar{D}$ and $\bar{D}^*$, the bilinear and symmetric imbedding curvature tensor of M in $\bar{M}$ are, respectively, indicated by h and h^*.

The finest relation between h (respectively, h^*) and A (respectively, A^*) can be seen as [20]

$$\bar{d}(h(x,y), \mathcal{U}) = d(A^*_{\mathcal{U}} x, y) \quad \text{and} \quad \bar{d}(h^*(x,y), \mathcal{U}) = d(A_{\mathcal{U}} x, y), \tag{12}$$

for any $x, y \in T_r M$ and $\mathcal{U} \in T_r^{\perp} M$.

It is also noted that the relations $2h^0 = h + h^*$ and $2A^0_{\mathcal{U}} = A_{\mathcal{U}} + A^*_{\mathcal{U}}$ exist by using (2).

The curvature tensor of $\bar{D}$ and D are given by $\bar{R}$ and R, respectively. Then, for any $x, y, z, z' \in T_r M$. Now, the corresponding Gauss formula are [20]

$$\begin{aligned}\bar{d}(\bar{R}(x,y)z, z') &= d(R(x,y)z, z') + \bar{d}(h(x,z), h^*(y,z')) \\ &\quad - \bar{d}(h^*(x,z'), h(y,z)),\end{aligned} \tag{13}$$

and

$$\begin{aligned}\bar{d}(\bar{R}^*(x,y)z, z') &= d(R^*(x,y)z, z') + \bar{d}(h^*(x,z), h(y,z')) \\ &\quad - \bar{d}(h^*(x,z'), h(y,z)).\end{aligned} \tag{14}$$

The Ricci equations for $D^{\perp}$ and $D^{\perp *}$ are, respectively, given below [20]

$$\bar{d}(R^{\perp}(x,y)\mathcal{U}, \mathcal{V}) = \bar{d}(\bar{R}(x,y)\mathcal{U}, \mathcal{V}) + d([A^*_{\mathcal{U}}, A_{\mathcal{V}}]x, y), \tag{15}$$

and

$$\bar{d}(R^{*\perp}(x,y)\mathcal{U}, \mathcal{V}) = \bar{d}(\bar{R}^*(x,y)\mathcal{U}, \mathcal{V}) + d([A_{\mathcal{U}}, A^*_{\mathcal{V}}]x, y), \tag{16}$$

for any $x, y \in T_r M$ and $\mathcal{U}, \mathcal{V} \in T_r^{\perp} M$. Here $R^{\perp}$ and $R^{*\perp}$ are the normal curvature tensors for $D^{\perp}$ and $D^{\perp *}$ on $T_r^{\perp} M$, respectively.

The usual definitions, in general, cannot produce a sectional curvature with regard to dual connections (which, of course, are not metric). In [21,22], Opozda presented a statistical sectional curvature on a statistical manifold. As a result, the statistical curvature tensor fields of $\bar{M}$ and M are defined as:

$$2\bar{\mathcal{S}} = (\bar{R} + \bar{R}^*), \quad \text{and} \quad 2\mathcal{S} = (R + R^*).$$

Additionally, the normal statistical curvature tensor $\mathcal{S}^{\perp}$ is defined as:

$$2\mathcal{S}^{\perp} = (R^{\perp} + R^{*\perp}).$$

For $x \in T_r M$, we put

$$\begin{aligned} x &= tan(\psi x) + nor(\psi x) \\ &= \mathbf{P}x + \mathbf{F}x,\end{aligned}$$

where $\mathbf{P}x$ is the tangential and $\mathbf{F}x$ is the normal component of ψx. Likewise, we can write

$$\begin{aligned} \psi^* x &= tan(\psi^* x) + nor(\psi^* x) \\ &= \mathbf{P}^* x + \mathbf{F}^* x, \end{aligned}$$

where $\mathbf{P}^* x$ is the tangential and $\mathbf{F}^* x$ is the normal component of $\psi^* x$.

Similar to the classical definition of *C-totally real and Legendrian submanifolds* of a Kenmotsu manifold (see [23]), we provide the following definition:

Definition 1. *Let M be an n-dimensional submanifold of an $(2m+1)$-dimensional Kenmotsu-like statistical manifold $(\bar{M}, d, \psi, \eta, \xi)$. If $\psi(T_r M) \subset T_r^{\perp} M$, then M is called C-totally real. However, if $n = m$, then a C-totally real submanifold turns to a* Legendrian submanifold. *Thus, it is easy to say that Legendrian submanifold is a C-totally real submanifold with the smallest possible codimension.*

In the following sections, we prove several sharp inequalities on statistical submanifold immersed into a Kenmotsu-like statistical manifold with a curvature tensor of the kind (9).

3. Bounds for Normalized Scalar Curvature

In this section, we derive an inequality on the normalized scalar curvature of a statistical submanifold immersed into a Kenmotsu-like statistical manifold with a curvature tensor of the kind (9).

Let a statistical submanifold M of dimension m in a Kenmotsu-like statistical manifold $\bar{M}^{2n+1}$. We assume a local orthonormal tangent (respectively, normal) frame $\{v_1, \ldots, v_m\}$ (respectively, $\{v_{m+1}, \ldots, v_{2n+1}\}$) of $T_r M$ (respectively, $T_r^{\perp} M$), $r \in M$. Then, $\sigma(r)$ is the scalar curvature of M and hence the normalized scalar curvature ρ of M are express as:

$$\begin{aligned} \sigma(r) &= \sum_{\substack{i,j=1 \\ i<j}}^{m} d(\mathcal{S}(v_i, v_j)v_j, v_i), \\ \rho &= \frac{2\sigma}{m(m-1)}. \end{aligned}$$

The mean curvature vectors are given by:

$$H = \frac{1}{m}\sum_{i=1}^{m} h(v_i, v_i), \ \text{ and } \ H^* = \frac{1}{m}\sum_{i=1}^{m} h^*(v_i, v_i).$$

We set

$$h_{ij}^k = \bar{d}(h(v_i, v_j), v_k), \ \text{ and } \ h_{ij}^* = \bar{d}(h^{*k}(v_i, v_j), v_k),$$

for $i, j = \{1, \ldots, m\}$ and $k = \{m+1, \ldots, 2n+1\}$.

We show the following.

Theorem 1. *Let a statistical submanifold M^m in a Kenmotsu-like statistical manifold $\bar{M}^{2n+1}$ with curvature tensor is of the form (9). Then*

$$\begin{aligned} \rho \geq \ & \frac{c-3}{4} + \frac{c+1}{4m(m-1)}\left\{2\|\mathbf{P}\|^2 - (trace(\mathbf{P}))^2 - trace(\mathbf{P}^{*2})\right\} \\ & + \frac{m}{m-1}\bar{d}(H, H^*) - \frac{1}{m(m-1)}\|h\|\|h^*\|. \end{aligned} \quad (17)$$

Proof. Let $\{v_1, \ldots, v_m\}$ and $\{v_{m+1}, \ldots, v_{2n+1}\}$ be orthonormal frames of T_rM and $T_r^{\perp}M$, $r \in M$, respectively. From Equations (9) and (13), we get:

$$\begin{aligned} 2\sigma &= \frac{c-3}{4}(m^2-m) + \frac{c+1}{4}\Big\{2\|\mathbf{P}\|^2 - (trace(\mathbf{P}))^2 - trace(\mathbf{P}^{*2})\Big\} \\ &\quad + m^2\bar{d}(H, H^*) - \sum_{k=m+1}^{2n+1}\sum_{i,j=1}^{m} h_{ij}^k h_{ij}^{*k} \\ &\geq \frac{c-3}{4}(m^2-m) + \frac{c+1}{4}\Big\{2\|\mathbf{P}\|^2 - (trace(\mathbf{P}))^2 - trace(\mathbf{P}^{*2})\Big\} \\ &\quad + m^2\bar{d}(H, H^*) - \|h\|\|h^*\|. \end{aligned}$$

By adopting the definition of the normalized scalar curvature ρ of M, we turn up the desired inequality. □

The characterization of equality cases in Theorem 1.

Theorem 2. *Let a statistical submanifold M^m in a Kenmotsu-like statistical manifold $\bar{M}^{2n+1}$ with curvature tensor is of the form (9). Equalities hold in (17) if and only if either $h = 0$ or $h^* = 0$ holds.*

4. Optimizations on a Statistical Submanifold with Casorati Curvatures

In this section, first we study Casorati curvatures (in short CC) for a statistical submanifold, with respect to D and D^*, in a Kenmotsu-like statistical manifold.

Let a statistical submanifold M of dimension m in a Kenmotsu-like statistical manifold $\bar{M}^{2n+1}$. We assume a local orthonormal tangent (respectively, normal) frame $\{v_1, \ldots, v_m\}$ (respectively, $\{v_{m+1}, \ldots, v_{2n+1}\}$) of T_rM (respectively, $T_r^{\perp}M$), $r \in M$. Then, the squared norm of second fundamental forms h and h^* ($\|h\|^2$ and $\|h\|^{*2}$) are, respectively, indicated by $\mathcal{C}$ and $\mathcal{C}^*$, known by the *Casorati curvatures* (CC) of M in $\bar{M}$. Therefore, we have:

$$m\mathcal{C} = \|h\|^2 \quad \text{and} \quad m\mathcal{C}^* = \|h^*\|^2,$$

here

$$\begin{aligned} \|h\|^2 &= \sum_{k=m+1}^{2n+1}\sum_{i,j=1}^{m} (h_{ij}^k)^2, \\ \|h^*\|^2 &= \sum_{k=m+1}^{2n+1}\sum_{i,j=1}^{m} (h_{ij}^{*k})^2. \end{aligned}$$

Next, we define an orthonormal basis $\{v_1, \ldots, v_t\}$ of a t-dimensional subspace L of TM, $d \geq 2$. Then, the scalar curvature of the d-plane section L and the CC of subspace L are, respectively, given below:

$$\sigma(L) = \sum_{\substack{i,j=1 \\ i<j}}^{d} \mathcal{S}(v_i, v_j, v_j, v_i)$$

and

$$\begin{aligned} t\mathcal{C}(L) &= \sum_{k=m+1}^{2n+1}\sum_{i,j=1}^{d} (h_{ij}^k)^2, \\ t\mathcal{C}^*(L) &= \sum_{k=m+1}^{2n+1}\sum_{i,j=1}^{d} (h_{ij}^{*k})^2). \end{aligned}$$

The normalized *CC* $\delta_{\mathcal{C}}(m-1)$ and $\widehat{\delta}_{\mathcal{C}}(m-1)$ are, respectively, defined as:

$$[\delta_{\mathcal{C}}(m-1)]_r = \frac{1}{2}\mathcal{C}_r + \left(\frac{m+1}{2m}\right)\inf\{\mathcal{C}(L)|L : \text{a hyperplane of } T_r M\}$$

and

$$[\widehat{\delta}_{\mathcal{C}}(m-1)]_r = 2\mathcal{C}_r - \left(\frac{2m-1}{2m}\right)\sup\{\mathcal{C}(L)|L : \text{a hyperplane of } T_r M\}.$$

Further, we, respectively, define the generalized normalized *CC* $\delta_{\mathcal{C}}(p; m-1)$ and $\widehat{\delta}_{\mathcal{C}}(p; m-1)$ as follows:

(1) For $0 < p < m^2 - m$

$$[\delta_{\mathcal{C}}(p; m-1)]_r = p\mathcal{C}_r + \zeta(p)\ \inf\{\mathcal{C}(L)|L : \text{a hyperplane of } T_r M\}.$$

(2) For $p > m^2 - m$

$$[\widehat{\delta}_{\mathcal{C}}(p; m-1)]_r = p\mathcal{C}_r - \zeta(p)\ \sup\{\mathcal{C}(L)|L : \text{a hyperplane of } T_r M\},$$

where

$$\zeta(p) = \frac{1}{pm}(m-1)(m+p)(m^2-m-p), \quad p \neq m(m-1).$$

In a similar way, the normalized *CCs* $\delta^*_{\mathcal{C}}(m-1)$ and $\widehat{\delta}^*_{\mathcal{C}}(m-1)$ and the generalized normalized *CC* $\delta^*_{\mathcal{C}}(p; m-1)$ and $\widehat{\delta}^*_{\mathcal{C}}(p; m-1)$ can be defined. We also notice that

$$\begin{aligned} 2\mathcal{C}^0 &= \mathcal{C} + \mathcal{C}^*, \\ 2\delta^0_{\mathcal{C}}(p; m-1) &= \delta_{\mathcal{C}}(p; m-1) + \delta^*_{\mathcal{C}}(p; m-1). \end{aligned}$$

To derive the section's optimum inequalities, we require the following key lemma.

Lemma 1 ([24]). *If* $\vartheta = \{(u_1, u_2, \ldots, u_m) \in \mathbb{R}^m : u_1 + u_2 + \cdots + u_m = \mathcal{L}\}$ *be a hyperplane of* $\mathbb{R}^m$ *and* $f : \mathbb{R}^m \to \mathbb{R}$ *is a quadratic form, then*

$$f(u_1, u_2, \ldots, u_m) = \mu_1 \sum_{i=1}^{m-1}(u_i)^2 + \mu_2(u_m)^2 - 2\sum_{\substack{i,j=1 \\ i<j}}^{m} u_i u_j, \quad \mu_1, \mu_2 > 0. \tag{18}$$

The restricted extremum problem therefore gives f the following global solution:

$$\begin{aligned} u_1 = u_2 = \cdots = u_{m-1} &= \frac{\mathcal{L}}{\mu_1+1}, \\ u_m = \frac{\mathcal{L}}{\mu_2+1} &= \frac{\mathcal{L}(m-1)}{(\mu_1+1)\mu_2} = (\mu_1 - m + 2)\frac{\mathcal{L}}{\mu_1+1}, \end{aligned}$$

provided that

$$\mu_2 = \frac{m-1}{\mu_1 - m + 2}. \tag{19}$$

Let $\{v_1, \ldots, v_m\}$ and $\{v_{m+1}, \ldots, v_{2n+1}\}$ be orthonormal frames of T_rM and $T_r^{\perp}M$, $r \in M$, respectively. From Equations (9) and (13), we arrive at

$$\begin{aligned} 2\sigma &= \frac{c-3}{4}(m^2-m) + \frac{c+1}{4}\left\{2\|\mathbf{P}\|^2 - (trace(\mathbf{P}))^2 - trace(\mathbf{P}^{*2})\right\} \\ &\quad +2m^2\|H^0\|^2 - 2m\mathcal{C}^0 - \frac{m^2}{2}\left(\|H\|^2 + \|H^*\|^2\right) + \frac{m}{2}(\mathcal{C}+\mathcal{C}^*). \end{aligned}$$

We now define a quadratic polynomial in terms of the components of the second fundamental form Q as

$$\begin{aligned} Q &= p\mathcal{C}^0 + \zeta(p)\mathcal{C}^0(L) - 2\sigma + \frac{c-3}{4}(m^2-m) + \frac{c+1}{4}\Big\{2\|\mathbf{P}\|^2 - (trace(\mathbf{P}))^2 \\ &\quad -trace(\mathbf{P}^{*2})\Big\} - \frac{m^2}{2}\left(\|H\|^2 + \|H^*\|^2\right) + \frac{m}{2}(\mathcal{C}+\mathcal{C}^*). \end{aligned} \tag{20}$$

We suppose that L is spanned by $v_1, \ldots, v_m$ (without loss of generality), combining (20), it follows that:

$$\begin{aligned} Q &= \frac{m+p}{m}\sum_{k=m+1}^{2n+1}\sum_{i,j=1}^{m}(h_{ij}^{0k})^2 + \frac{\zeta(p)}{m-1}\sum_{k=m+1}^{2n+1}\sum_{i,j=1}^{m-1}(h_{ij}^{0k})^2 - \sum_{k=m+1}^{2n+1}\left(\sum_{i=1}^{m}h_{ii}^{0k}\right)^2 \\ &= \sum_{k=m+1}^{2n+1}\sum_{i=1}^{m-1}\left[2(\lambda+1)(h_{ii}^{0k})^2 + \frac{2(m+p)}{m}(h_{im}^{0k})^2\right] \\ &\quad +\left[\lambda\sum_{1\le i\ne j\le m-1}(h_{ij}^{0k})^2 - 2\sum_{1\le i\ne j\le m}(h_{ii}^{0k}h_{jj}^{0k}) + \frac{p}{m}(h_{mm}^{0k})^2\right] \\ &\ge \sum_{k=m+1}^{2n+1}\left[\sum_{i=1}^{m-1}\lambda(h_{ii}^{0k})^2 - 2\sum_{1\le i\ne j\le m}h_{ii}^{0k}h_{jj}^{0k} + \frac{p}{m}(h_{mm}^{0})^2\right], \end{aligned} \tag{21}$$

where

$$\lambda = \left(\frac{p}{m} + \frac{\zeta(p)}{m-1}\right).$$

Let us assume the quadratic form $f^k : \mathbb{R}^m \to \mathbb{R}$, for $k = m+1, \ldots 2n+1$, defined by

$$f^k(h_{11}^{0k}, \ldots, h_{mm}^{0k}) = \sum_{i=1}^{m-1}\lambda(h_{ii}^{0k})^2 - 2\sum_{1\le i\ne j\le m}h_{ii}^{0k}h_{jj}^{0k} + \frac{p}{m}(h_{mm}^{0k})^2 \tag{22}$$

and the problem as follows:

$$\min\{f^k(h_{11}^{0k}, \ldots, h_{mm}^{0k}) : h_{11}^{0k} + \cdots + h_{mm}^{0k} = \mathcal{L}^k, \mathcal{L}^k \in \mathbb{R}\}.$$

On Comparing the functions (18) and (22), we find that

$$\mu_1 = \frac{p}{m} + \frac{\zeta(p)}{m-1} \quad \text{and} \quad \mu_2 = \frac{p}{m},$$

which verify the relation the relation (19). Thus, by Lemma 1, entails that the critical point $(h_{11}^{0k}, \ldots, h_{mm}^{0k})$ is given by:

$$h_{11}^{0k} = \cdots = h_{m-1m-1}^{0k} = \frac{\mathcal{L}^k}{\mu_1+1} = \frac{\mathcal{L}}{\lambda+1}, h_{mm}^{0k} = \frac{\mathcal{L}^k}{\mu_2+1} = \frac{m\mathcal{L}^k}{m+p}, \tag{23}$$

and a result, it is the global lowest point.

Next, plugging (23) into (22), we arrive at

$$f^k(h_{11}^{0k}, \ldots, h_{mm}^{0k}) = 0. \tag{24}$$

From (21) and (24), we get $Q \geq 0$ and hence we have the following:

$$\begin{aligned} 2\sigma \leq\ & p\mathcal{C}^0 + \zeta(p)\mathcal{C}^0(L) + \frac{m}{2}\left(\mathcal{C} + \mathcal{C}^*\right) + \frac{c-3}{4}(m^2 - m) + \frac{c+1}{4} \\ & \left\{2\|\mathbf{P}\|^2 - (trace(\mathbf{P}))^2 - trace(\mathbf{P}^{*2})\right\} \\ & -\frac{m^2}{2}\left(\|H\|^2 + \|H^*\|^2\right). \end{aligned}$$

Further, we find that:

$$\begin{aligned} \rho \leq\ & \frac{p}{m(m-1)}\mathcal{C}^0 + \frac{\zeta(p)}{m(m-1)}\mathcal{C}^0(L) + \frac{1}{2(m-1)}\left(\mathcal{C} + \mathcal{C}^* + \frac{c-3}{4}\right) \\ & + \frac{c+1}{4m(m-1)}\left\{2\|\mathbf{P}\|^2 - (trace(\mathbf{P}))^2 - trace(\mathbf{P}^{*2})\right\} \\ & - \frac{m}{2(m-1)}\left(\|H\|^2 + \|H^*\|^2\right). \end{aligned}$$

Hence, we get the following inequality.

Theorem 3. *Let a statistical submanifold M^m in a Kenmotsu-like statistical manifold $\bar{M}^{2n+1}$ with curvature tensor is of the form (9). Then, the generalized normalized CC $\delta_{\mathcal{C}}(p; m-1)$ and $\delta^*_{\mathcal{C}}(p; m-1)$ holds*

$$\begin{aligned} \rho \leq\ & \frac{2\delta^0_{\mathcal{C}}(p; m-1)}{m(m-1)} + \frac{1}{m-1}\mathcal{C}^0 + \frac{c-3}{4} + \frac{c+1}{4m(m-1)}\left\{2\|\mathbf{P}\|^2 - (trace(\mathbf{P}))^2\right. \\ & \left. - trace(\mathbf{P}^{*2})\right\} - \frac{m}{2(m-1)}\left(\|H\|^2 + \|H^*\|^2\right). \end{aligned} \tag{25}$$

Remark 1. *In a similar way ones can obtain inequality for the generalized normalized CC $\widehat{\delta}_{\mathcal{C}}(p; m-1)$ and $\widehat{\delta}^*_{\mathcal{C}}(p; m-1)$, that is,*

$$\begin{aligned} \rho \leq\ & \frac{2\widehat{\delta}^0_{\mathcal{C}}(p; m-1)}{m(m-1)} + \frac{1}{m-1}\mathcal{C}^0 + \frac{c-3}{4} + \frac{c+1}{4m(m-1)}\left\{2\|\mathbf{P}\|^2\right. \\ & \left. - (trace(\mathbf{P}))^2 - trace(\mathbf{P}^{*2})\right\} - \frac{m}{2(m-1)}\left(\|H\|^2 + \|H^*\|^2\right). \end{aligned} \tag{26}$$

The characterization of equality cases in Theorem 3.

Theorem 4. *Let a statistical submanifold M^m in a Kenmotsu-like statistical manifold $\bar{M}^{2n+1}$ with curvature tensor is of the form (9). Equalities occur identically, at all points of M, in the Equations (25) and (26) if and only if the totally geodesic submanifolds endowed with respect to the Levi–Civita connection.*

5. Bounds for the Normalized Normal Scalar Curvature

In this segment, we use the most important ingredient, derived by Lu for the symmetric and trace-free operators in [9], in the proof of our desired DDVV inequality. This is as follows.

Theorem 5. *Let a Riemannian submanifold of dimension n immersed into $(n+m)$-dimensional Riemannian manifold. For every set $\{B_1,\dots,B_m\}$ of symmetric $(m\times m)$-matrices with trace zero the following inequality holds:*

$$\sum_{r,s=1}^{m} ||B_r, B_s||^2 \leq \left(\sum_{r=1}^{m} ||B_r||^2\right)^2.$$

Let an m-dimensional Legendrian submanifold M of a Kenmotsu-like statistical manifold $\bar{M}^{2m+1}$ with curvature tensor is of the form (9) and $\{v_1,\dots,v_m\}$ and $\{v_{m+1},\dots,v_{2m+1}\}$ be orthonormal frames of T_rM and $T_r^{\perp}M$, $r\in M$, respectively. From Equations (9) and (13), we arrive at:

$$\begin{aligned} 2\sigma &= \frac{c-3}{4}(m^2-m) + 2m^2\|H^0\|^2 - \frac{m^2}{2}\left(\|H\|^2+\|H^*\|^2\right) \\ &\quad -2\|h^0\|^2 + \frac{1}{2}(\|h\|^2+\|h^*\|^2). \end{aligned}$$

Then, the normalized scalar curvature of of M is ρ

$$\begin{aligned} \rho &= \frac{c-3}{4} + \frac{2m}{m-1}\|H^0\|^2 - \frac{m}{2(m-1)}\left(\|H\|^2+\|H^*\|^2\right) \\ &\quad -\frac{2}{m(m-1)}\|h^0\|^2 + \frac{1}{2m(m-1)}(\|h\|^2+\|h^*\|^2). \end{aligned} \tag{27}$$

However, if we fix $h - Hd = \tau$, $h^* - H^*d = \tau^*$ and $h^0 - H^0 d = \tau^0$, the traceless part of the second fundamental forms, then, respectively, we get:

$$\begin{aligned} &\|\tau\|^2 = \|h\|^2 - m\|H\|^2, \ \ \|\tau^*\|^2 = \|h^*\|^2 - m\|H^*\|^2, \\ &\|\tau^0\|^2 = \|h^0\|^2 - m\|H^0\|^2. \end{aligned}$$

Thus, the expression (28) becomes:

$$\begin{aligned} \rho &= \frac{c-3}{4} + 2\|H^0\|^2 - \frac{1}{2}\left(\|H\|^2+\|H^*\|^2\right) - \frac{2}{m(m-1)}\|\tau^0\|^2 \\ &\quad + \frac{1}{2m(m-1)}(\|\tau\|^2+\|\tau^*\|^2). \end{aligned} \tag{28}$$

Now, we compute our main extrinsic curvature and the normalized normal scalar curvature of M as below.

$$\begin{aligned} \rho^{\perp} &= \frac{1}{m(m-1)}\left\{\sum_{1\leq a<b\leq m}\sum_{1\leq i<j\leq m}\left[d(R^{\perp}(v_i,v_j)v_{m+a},v_{m+b})\right.\right. \\ &\quad \left.\left. + d(R^{*\perp}(v_i,v_j)v_{m+a},v_{m+b})\right]^2\right\}^{1/2} \\ &= \frac{1}{m(m-1)}\left\{\sum_{1\leq a<b\leq m}\sum_{1\leq i<j\leq m}\left[\bar{d}(\bar{R}(v_i,v_j)v_{m+a},v_{m+b})\right.\right. \\ &\quad + d([A^*_{v_{m+a}}, A_{v_{m+b}}]v_i,v_j) \\ &\quad + \bar{d}(\bar{R}^*(v_i,v_j)v_{m+a},v_{m+b}) \\ &\quad \left.\left. + d([A_{v_{m+a}}, A^*_{v_{m+b}}]v_i,v_j)\right]^2\right\}^{1/2}, \end{aligned}$$

which refers to the following:

$$\begin{aligned}\rho^{\perp} &= \frac{1}{m(m-1)}\Bigg\{\sum_{1\le a<b\le n}\sum_{1\le i<j\le m}\Bigg[\frac{c+1}{4}(\delta_{ib}\delta_{ja}-\delta_{ia}\delta_{jb})\\ &+4d([A^0_{v_{m+a}},A^0_{v_{m+b}}]v_i,v_j)+g([A_{v_{m+a}},A_{v_{m+b}}]v_i,v_j)\\ &+d([A^*_{v_{m+a}},A^*_{v_{m+b}}]v_i,v_j)\Bigg]^2\Bigg\}^{1/2}. \end{aligned} \tag{29}$$

On simplifying (29) and applying the Cauchy-Schwarz inequality, $(\alpha+\beta+\gamma+\pi)^2-4(\alpha^2+\beta^2+\gamma^2+\pi^2)\le 0$, for $\alpha,\beta,\gamma,\pi\in\mathbb{R}$, we introduce an inequality in (29) for $\rho^{\perp}$ as follows:

$$\begin{aligned}\rho^{\perp} &\le \frac{2}{m(m-1)}\Bigg\{\sum_{1\le a<b\le n}\sum_{1\le i<j\le m}\Bigg[\left(\frac{c+1}{4}\right)^2(\delta_{ib}\delta_{ja}-\delta_{ia}\delta_{jb})^2\\ &+16d([A^0_{v_{m+a}},A^0_{v_{m+b}}]v_i,v_j)^2+d([A_{v_{m+a}},A_{v_{m+b}}]v_i,v_j)^2\\ &+d([A^*_{v_{m+a}},A^*_{v_{m+b}}]v_i,v_j)^2\Bigg]\Bigg\}^{1/2}\\ &\le \frac{2}{m(m-1)}\Bigg\{\frac{m^2(m+1)^2(c+1)^2}{64}+\frac{1}{4}\sum_{a,b=1}^{n}\Big[16||A^0_a,A^0_b||^2\\ &+||A_a,A_b||^2+||A^*_a,A^*_b||^2\Big]\Bigg\}^{1/2}.\end{aligned}$$

Now, by following the same steps done in [13], the sets $\{B^0_1,B^0_2,\ldots,B^0_m\}$, $\{B_1,B_2,\ldots,B_m\}$ and $\{B^*_1,B^*_2,\ldots,B^*_m\}$ of symmetric with trace zero operators on T_rM are defined as:

$$< B^0_aE,F >=< \tau^0(E,F),\mathcal{U}_a >,\quad < B_aE,F >=< \tau(E,F),\mathcal{U}_a >, \tag{30}$$

$$< B^*_aE,F >=< \tau^*(E,F),\mathcal{U}_a > . \tag{31}$$

Clearly, we have the relations

$$B^0_a = A^0_a- < H^0,\mathcal{U}_a > Id,\quad B_a = A_a- < H,\mathcal{U}_a > Id,\quad B^*_a = A^*_a- < H^*,\mathcal{U}_a > Id$$

and

$$[A^0_a,A^0_b]=[B^0_a,B^0_b],\quad [A_a,A_b]=[B_a,B_b],\quad [A^*_a,A^*_b]=[B^*_a,B^*_b], \tag{32}$$

for any $b\in\{1,\ldots,m\}$.

Therefore, we have:

$$\begin{aligned}\rho^{\perp} &\le \frac{2}{m(m-1)}\Bigg\{\frac{m^2(m+1)^2(c+1)^2}{64}+\frac{1}{4}\sum_{a,b=1}^{n}\Big[16||B^0_a,B^0_b||^2\\ &+||B_a,B_b||^2+||B^*_a,B^*_b||^2\Big]\Bigg\}^{1/2}.\end{aligned}$$

Then, by applying Theorem 5, we can easily arrive at:

$$\begin{aligned}\rho^{\perp} &\le \frac{|c+1|}{4}+\frac{1}{m(m-1)}\sum_{a,b=1}^{n}\Big[4||B^0_a||^2+||B_a||^2+||B^*_a||^2\Big]\\ &\le \frac{|c+1|}{4}+\frac{1}{m(m-1)}\Big[4||\tau^0||^2+||\tau||^2+||\tau^*||^2\Big].\end{aligned}$$

Substituting (28) in the last relation, we get:

$$\rho^{\perp} \leq 2\rho + \frac{|c+1|}{4} - \frac{c-3}{2} + \frac{8}{m(m-1)}||\tau^0||^2 - 4\|H^0\|^2 + \|H\|^2 + \|H^*\|^2. \tag{33}$$

However, the normalized scalar curvature ρ^0 of M for Levi–Civita connection is:

$$\begin{aligned} \rho &= \frac{c-3}{4} + \frac{m}{m-1}\|H^0\|^2 - \frac{1}{m(m-1)}\|h^0\|^2 \\ &= \frac{c-3}{4} + \|H^0\|^2 - \frac{1}{m(m-1)}\|\tau^0\|^2. \end{aligned} \tag{34}$$

Putting (34) into (35), we give the main result of this segment.

Theorem 6. *Let a Legendrian submanifold M^m of a Kenmotsu-like statistical manifold $\bar{M}^{2m+1}$ with curvature tensor is of the kind (9). Then we have:*

$$\begin{aligned} \rho^{\perp} &\leq 2\rho - 8\rho^0 + \frac{|c+1| + 6c}{4} - \frac{9}{2} \\ &\quad +4\|H^0\|^2 + \|H\|^2 + \|H^*\|^2. \end{aligned} \tag{35}$$

6. Some Geometric Applications

In this part, we study some immediate applications of the results (Theorems 1 and 3) proved in the previous section.

In light of Lemma 1 and taking $q = \frac{m(m-1)}{2}$ in $\delta_C(p; m-1)$ (respectively, $\delta^*_C(p; m-1)$), we find that:

$$\begin{aligned} \left[\delta_C\left(\frac{m(m-1)}{2}; m-1\right)\right]_q &= m(m-1)[\delta_C(m-1)]_r \\ (\text{respectively,} \left[\delta^*_C\left(\frac{m(m-1)}{2}; m-1\right)\right]_q &= m(m-1)[\delta^*_C(m-1)]_r) \end{aligned}$$

at any point $r \in M$.

Thus, we turn up the following corollaries on Theorem 3.

Corollary 1. *Let a statistical submanifold M^m of a Kenmotsu-like statistical manifold $\bar{M}^{2n+1}$ with curvature tensor is of the kind (9). Then, the normalized CC $\delta_C(m-1)$ and $\delta^*_C(m-1)$ holds*

$$\begin{aligned} \rho &\leq 2\delta^0_C(m-1) + \frac{1}{m-1}\mathcal{C}^0 + \frac{c-3}{4} + \frac{c+1}{4m(m-1)}\Big\{2\|\mathbf{P}\|^2 - (trace(\mathbf{P}))^2 \\ &\quad -trace(\mathbf{P}^{*2})\Big\} - \frac{m}{2(m-1)}\Big(\|H\|^2 + \|H^*\|^2\Big). \end{aligned} \tag{36}$$

Remark 2. *In a similar way, ones can obtain inequality for the normalized CC $\widehat{\delta}_C(m-1)$ and $\widehat{\delta}^*_C(m-1)$.*

Now, from

$$\|H^0\|^2 = \frac{1}{4}\Big(\|H\|^2 + \|H^*\|^2 + 2\bar{d}(H, H^*)\Big),$$

and together with $H^0 = 0$, we have the relation

$$\|H\|^2 + \|H^*\|^2 = -2\bar{d}(H, H^*).$$

Hence, the following corollary follows directly from inequalities (25) and (26).

Corollary 2. *Let a statistical submanifold M^m of a Kenmotsu-like statistical manifold $\bar{M}^{2n+1}$ with curvature tensor is of the kind (9). If M is minimal, that is, $H^0 = 0$. Then, we have:*

$$\rho \leq \frac{2\delta^0_{\mathcal{C}}(p; m-1)}{m(m-1)} + \frac{1}{m-1}\mathcal{C}^0 + \frac{c-3}{4} + \frac{c+1}{4m(m-1)}\Big\{2\|\mathbf{P}\|^2 - (trace(\mathbf{P}))^2 - trace(\mathbf{P}^{*2})\Big\} + \frac{m}{(m-1)}\bar{d}(H, H^*). \quad (37)$$

Corollary 3. *Let a statistical submanifold M^m of a Kenmotsu-like statistical manifold $\bar{M}^{2n+1}$ with curvature tensor is of the kind (9). If M is minimal, that is, $H^0 = 0$. Then, we have:*

$$\rho \leq 2\delta^0_{\mathcal{C}}(m-1) + \frac{1}{m-1}\mathcal{C}^0 + \frac{c-3}{4} + \frac{c+1}{4m(m-1)}\Big\{2\|\mathbf{P}\|^2 - (trace(\mathbf{P}))^2 - trace(\mathbf{P}^{*2})\Big\} + \frac{m}{(m-1)}\bar{d}(H, H^*). \quad (38)$$

Some consequences of Theorem 1 are the following.

Corollary 4. *Let a statistical submanifold M^m of a Kenmotsu-like statistical manifold $\bar{M}^{2n+1}$ with curvature tensor is of the kind (9). If $c = -1$, then*

$$m(m-1)(\rho + 1) \geq m^2 \bar{d}(H, H^*) - \|h\|\|h^*\|.$$

Corollary 5. *Let a statistical submanifold M^m of a Kenmotsu-like statistical manifold $\bar{M}^{2n+1}$ with curvature tensor is of the kind (9). Suppose that*

1. *$h(x, y) = d(x, y)H$ and $h^*(x, y) = d(x, y)H^*$;*
2. *$h = 0$ and $h^* = 0$;*
3. *$c = -1$.*

Then $\rho + 1 \geq 0$.

Remark 3. *In the above Corollary 5, we have $0 = h(x, y) = d(x, y)H$, $x, y \in T_r\mathbf{M}$, which gives $H = 0$. Similarly, $0 = h^*(x, y) = g(x, y)H^*$ implies $H^* = 0$. Hence, an inequality (17) reduces to $\rho + 1 \geq 0$.*

Corollary 6. *Let a statistical submanifold M^m of a Kenmotsu-like statistical manifold $\bar{M}^{2n+1}$ with curvature tensor is of the kind (9). If the following holds*

$$\rho = \frac{c-3}{4} + \frac{c+1}{4m(m-1)}\Big\{2\|\mathbf{P}\|^2 - (trace(\mathbf{P}))^2 - trace(\mathbf{P}^{*2})\Big\}.$$

Then neither $h = 0$ nor $h^* = 0$.

Let us take a minimal submanifold M for the Levi-Civita connection, which gives $H + H^* = 0$ because of $H^0 = 0$ and thus, we have the relation $-2\bar{d}(H, H^*) = ||H||^2 + ||H^*||^2$. Then Theorem 6 gives

Corollary 7. *Let a statistical submanifold M^m of a Kenmotsu-like statistical manifold $\bar{M}^{2n+1}$ with curvature tensor is of the kind (9). Then we have*

$$\rho^{\perp} \leq 2\rho - 8\rho^0 + \frac{|c+1| + 6c}{4} - \frac{9}{2} - 2\bar{d}(H, H^*).$$

7. Related Examples

Example 1. *Let $\mathbb{R}^4$ be a Euclidean space with local coordinate system $\{x_1, x_2, y_1, y_2\}$, which admits the following almost complex structure J*

$$J = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix},$$

the metric $d_{\mathbb{R}^4} = 2\partial x_1^2 + 2\partial x_2^2 - \partial y_1^2 - \partial y_2^2$ and the flat affine connection $D^{\mathbb{R}^4}$ is a Kähler-like statistical manifold (see [17]). If $\left(\mathbb{R}, \nabla^D, \partial t^2\right)$ is a trivial statistical manifold, it is known from [13] that the product manifold $\left(\bar{M}^5 = \mathbb{R} \times_f \mathbb{R}^4, \bar{D}, \bar{d} = \partial t^2 + f^2 d_{\mathbb{R}^4}\right)$ is a Kenmotsu-like statistical manifold, where $f = e^t \in \mathbb{R}^+$.

We define ψ, ξ and η by

$$\psi = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \end{bmatrix}, \quad \xi = dt = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

and $\eta = (1, 0, -y_1, 0, -y_2)$. We also find

$$\psi^* = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & -\frac{1}{2} \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \end{bmatrix}.$$

Next, we give examples of submanifolds in $\bar{M}^5$ as follows:

Example 2. *The statistical submanifold $M^3 = \mathbb{R} \times_f \mathbb{R}^2$ of $\bar{M}^5$, where $f = e^t \in \mathbb{R}^+$, attains equality for both inequalities (25) and (26) because M^3 is a totally geodesic submanifold of $\bar{M}^5$ with respect to the Levi–Civita connection. Thus, Theorem 4 is satisfied.*

Example 3. *Let us take a 3-dimensional submanifold M^3 of $\bar{M}^5$ defined by an isometric immersion $\omega : M^3 \to \bar{M}^5$ as $\omega(u, v, t) = (0, v, u, 0, t)$. Then the vector fields at each point of M^3 are given as $E_1 = (0, 0, 1, 0, 0)$, $E_2 = (0, 1, 0, 0, 0)$ and $E_3 = (0, 0, 0, 0, 1)$. By direct calculations, we have $[E_1, E_2] = [E_2, E_3] = [E_3, E_1] = 0$ and hence by Koszula formula, we calculate the Levi-Civita connection D^0 as $D^0_{X_i} X_j = 0$ for $1 \le i, j \le 3$. This tells us that M^3 is a totally geodesic submanifold of $\bar{M}^5$ with respect to D^0. Thus, Theorem 4 is satisfied.*

Example 4. *Let ω be an isometric immersion from M^3 to $\bar{M}^5$ defined by $\omega(x_1, y_1, z_1) = (x_1, y_1, z_1, 0, 0)$. The vector fields at each point of M^3 are given by $X_1 = e^{-z_1}(x_1 \partial x_1 + y_1 \partial y_1)$, $X_2 = e^{-z_1} \partial y_1$, and $X_3 = e^{-2z_1} \partial z_1$. By direct calculations, we find that $[X_1, X2] = -e^{-z_1} X_2$, $[X_1, X_3] = e^{-2z_1} X_1$, and $[X_2, X_3] = e^{-2z_1} X_2$. It is easy to see that M^3 is a non-totally geodesic submanifold of $\bar{M}^5$ with respect to Levi-Civita connection D^0. In this case, the inequalities (25) and (26) are precisely satisfied at all points and the case of equality cannot be attained.*

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