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Asymptotic Behavior of Solutions of Integral Equations with Homogeneous Kernels

Oleg Avsyankin

Institute of Mathematics, Mechanics and Computer Sciences, Regional Mathematical Center, Southern Federal University, Mil'chakova St., 8a, 344090 Rostov-on-Don, Russia; ogavsyankin@sfedu.ru

Abstract: The multidimensional integral equation of second kind with a homogeneous of degree $(-n)$ kernel is considered. The special class of continuous functions with a given asymptotic behavior in the neighborhood of zero is defined. It is proved that, if the free term of the integral equation belongs to this class and the equation itself is solvable, then its solution also belongs to this class. To solve this problem, a special research technique is used. The above-mentioned technique is based on the decomposition of both the solution and the free term in spherical harmonics.

Keywords: integral equation; homogeneous kernel; solution of equation; asymptotics; spherical harmonics

1. Introduction

Nowadays, there are numerous papers devoted to the integral operators with homogeneous kernels of degree $(-n)$. The investigation of such operators was started by L. G. Mikhailov in connection with studying elliptic differential equations (e.g., see [1,2]). It was shown that such operators naturally arise when the method of potentials is applied to equations of the form

$$|x|^2\Delta u + |x|\sum_{k=1}^{n} a_k(x)\frac{\partial u}{\partial x_k} + b(x)u = 0$$

in the domain $G \subset \mathbb{R}^n$ that contains the point $x = 0$ (here, $a_k(x)$ and $b(x)$ are bounded functions). Operators with homogeneous kernels are also used in mechanics (see [3]). Over the past two decades, the theory of integral operators with homogeneous kernels has made significant progress. For such operators, criteria for invertibility and the Fredholm property were obtained, the Banach algebras generated by these operators were studied and the conditions for the projection method to be applied were found (e.g., see [4–10] and the bibliography therein). However, despite considerable advances, a lot of problems remain yet unsolved. This paper is devoted to one of such problems, i.e., studying the asymptotic behavior of the solution of an integral equation with a homogeneous kernel.

The object of research of this paper is the integral equation of second kind with a kernel which is homogeneous of degree $(-n)$ and invariant with respect to the rotation group $SO(n)$. This equation is considered in the space of continuous functions. The aim of this paper is to obtain the asymptotic behavior of the solution from the known asymptotic behavior of the free term of the equation. More precisely, it is assumed that the free term belongs to the class $A^{\alpha}_{s,\delta}(\mathbb{B}_n)$, which consist of all functions that are continuous in the unit ball $\mathbb{B}_n$, except for the point $x = 0$, and have a given asymptotic behavior in the neighborhood of zero. It is proved that, if the equation is solvable, then its solution also belongs to the class $A^{\alpha}_{s,\delta}(\mathbb{B}_n)$.

In conclusion, it should be noted that analogous results for the operators with difference kernels can be found in [11,12]. It is also worth noting that asymptotics similar to the class $A^{\alpha}_{s,\delta}(\mathbb{B}_n)$ appeared in M. V. Korovina's studies [13–15] in connection with investigations of the differential operators with degeneracy. We also note papers [16–18], in which

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questions about the dimension of space and the asymptotics of solutions for the elasticity system and the biharmonic (polyharmonic) equation are considered, provided that the weighted energy (or Dirichlet) integral is bounded at infinity.

2. Preliminaries and Problem Statement

2.1. Notation

We use the following notation:
$\mathbb{R}^n$—n-dimensional Euclidean space; $x = (x_1, \dots, x_n) \in \mathbb{R}^n$.
$|x| = \sqrt{x_1^2 + \dots + x_n^2}$; $x' = x/|x|$; $x \cdot y = x_1 y_1 + \dots + x_n y_n$.
$\mathbb{B}_n(a) = \{x \in \mathbb{R}^n : \ |x| \leqslant a\}$. $\mathbb{B}_n = \mathbb{B}_n(1)$. $\mathbb{S}_{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}$.
$|\mathbb{S}_{n-1}| = 2\pi^{n/2}/\Gamma(n/2)$ is the area of the sphere $\mathbb{S}_{n-1}$.
$\mathbb{Z}_+$ is the set of non-negative integers.
I is the identity operator (below, it is clear from the context in what space this operator is considered).
$d_n(m)$ is the dimension of the space of spherical harmonics of order m, i.e.,

$$d_n(m) = (n + 2m - 2)\frac{(n + m - 3)!}{m!(n - 2)!}.$$

In addition, we need to deal with the following spaces of functions:
$C(\mathbb{B}_n(a))$ is the space of all complex-valued continuous functions on $\mathbb{B}_n(a) \setminus \{0\}$, having a finite limit as $x \to 0$.
$C_0(\mathbb{B}_n(a)) = \{g \in C(\mathbb{B}_n(a)) \colon \lim\limits_{x \to 0} g(x) = 0\}$.
$C[0,1]$ is the space of all complex-valued continuous functions on $(0,1]$, having a finite limit as $x \to 0$.
$C_0[0,1] = \{g \in C[0,1] \colon \lim\limits_{r \to 0} g(r) = 0\}$.

2.2. Problem Statement

In the space $C(\mathbb{B}_n)$, we consider the integral equation

$$\varphi(x) = \int\limits_{\mathbb{B}_n} k(x,y)\varphi(y)\,dy + f(x), \quad x \in \mathbb{B}_n, \tag{1}$$

where the function $k(x,y)$ is defined on $\mathbb{R}^n \times \mathbb{R}^n$ (here and below, it is assumed that $n \geqslant 2$) and satisfies the following conditions:

1° homogeneity of degree $(-n)$, i.e.,

$$k(\alpha x, \alpha y) = \alpha^{-n} k(x,y), \quad \forall\, \alpha > 0;$$

2° invariance with respect to the rotation group $SO(n)$, i.e.,

$$k(\omega(x), \omega(y)) = k(x,y), \quad \forall\, \omega \in SO(n);$$

3° integrability, i.e.,

$$\int\limits_{\mathbb{R}^n} |k(e_1, y)|(1 + |\ln|y||)^{\nu}\,dy < \infty, \quad e_1 = (1, 0, \dots, 0),$$

where ν is some positive number.

The function

$$k(x,y) = \frac{|x|}{|y|^{3/2}|x - y|^{n-1/2}}\left(1 + \left|\ln\frac{|y|}{|x|}\right|\right)^{-\nu}$$

is an example of the function that satisfies conditions 1°–3°.

By K, we denote the integral operator that forms the right-hand side of Equation (1), i.e.,

$$(K\varphi)(x) = \int\limits_{\mathbb{B}_n} k(x,y)\varphi(y)\,dy, \quad x \in \mathbb{B}_n. \tag{2}$$

It is known (see [1,2]) that the operator K is a bounded operator from $C(\mathbb{B}_n)$ to $C(\mathbb{B}_n)$ and from $C_0(\mathbb{B}_n)$ to $C_0(\mathbb{B}_n)$.

In the space $C(\mathbb{B}_n)$, let us determine a special class of functions with a given asymptotic behavior at zero.

Definition 1 ([9]). *Let $0 < \alpha \leqslant 1$, $0 < \delta < 1$ and $s \in \mathbb{Z}_+$. The class $\boldsymbol{A}^{\alpha}_{s,\delta}(\mathbb{B}_n)$ is a set of all functions $g \in C(\mathbb{B}_n)$ such that, for $|x| < \delta$, the following representation is valid:*

$$g(x) = b + \sum_{j=0}^{s} \frac{b_j}{(1 - \ln|x|)^{j+\alpha}} + \frac{v(x)}{(1 - \ln|x|)^{s+\alpha}}, \quad v \in C_0(\mathbb{B}_n(\delta)).$$

It should be noted that the structure of the functions from the class $\boldsymbol{A}^{\alpha}_{s,\delta}(\mathbb{B}_n)$ corresponds to the operators of form (2) that have a singularity at zero. The class $\boldsymbol{A}^{\alpha}_{s,\delta}(\mathbb{B}_n)$ plays the same role in the theory of operators of form (2) as the asymptotics with respect to powers $\frac{1}{t+1}$ play in the theory of convolution-type operators.

Proposition 1 ([9]). *Let the numbers s and ν be such that $s \leqslant [\nu] - 1$. Then, the following apply:*

(1) the class $\boldsymbol{A}^{\alpha}_{s,\delta}(\mathbb{B}_n)$ is invariant with respect to the operator K, i.e.,

$$K(\boldsymbol{A}^{\alpha}_{s,\delta}(\mathbb{B}_n)) \subset \boldsymbol{A}^{\alpha}_{s,\delta}(\mathbb{B}_n);$$

(2) if $\varphi \in \boldsymbol{A}^{\alpha}_{s,\delta}(\mathbb{B}_n)$ and, for $|x| < \delta$, the representation

$$\varphi(x) = \frac{v(x)}{(1 - \ln|x|)^{s+\alpha}}, \quad v \in C_0(\mathbb{B}_n(\delta))$$

is valid, then $K\varphi \in \boldsymbol{A}^{\alpha}_{s,\delta}(\mathbb{B}_n)$ and, for $|x| < \delta$, the representation

$$(K\varphi)(x) = \frac{w(x)}{(1 - \ln|x|)^{s+\alpha}}, \quad w \in C_0(\mathbb{B}_n(\delta))$$

is valid.

The aim of this paper is to study the asymptotic behavior of the solution of Equation (1) on the assumption that the function $f(x)$ belongs to the class $\boldsymbol{A}^{\alpha}_{s,\delta}(\mathbb{B}_n)$.

2.3. One-Dimensional Operator

A one-dimensional analog of the operator K of form (2) is the operator

$$(H\psi)(r) = \int\limits_0^1 h(r,\rho)\psi(\rho)\,d\rho, \quad r \in [0,1], \tag{3}$$

where the function $h(r,\rho)$ is defined on $\mathbb{R}_+ \times \mathbb{R}_+$, is homogeneous of degree (-1) and satisfies the condition

$$\int\limits_0^\infty |h(1,\rho)|(1 + |\ln\rho|)^{\nu}\,d\rho < \infty \tag{4}$$

for some $\nu > 0$. The operator H is bounded in the space $C[0,1]$ and in the space $C_0[0,1]$.

We denote, by $\boldsymbol{A}^{\alpha}_{s,\delta}[0,1]$, the set of all functions $g \in C[0,1]$ such that, for $r < \delta$, the following representation is valid:

$$g(r) = b + \sum_{j=0}^{s} \frac{b_j}{(1-\ln r)^{j+\alpha}} + \frac{u(r)}{(1-\ln r)^{s+\alpha}}, \quad u(r) \in C_0[0,\delta].$$

For the convenience of readers, let us formulate a one-dimensional analog of Proposition 1.

Proposition 2. *Let $s \leqslant [\nu]-1$. Then, the class $\boldsymbol{A}^{\alpha}_{s,\delta}[0,1]$ is invariant with respect to the operator H. Moreover, if, for $r < \delta$, the function $\psi \in \boldsymbol{A}^{\alpha}_{s,\delta}[0,1]$ has the form*

$$\psi(r) = \frac{u(r)}{(1-\ln r)^{s+\alpha}}, \quad u(r) \in C_0[0,\delta],$$

then, the function $H\psi \in \boldsymbol{A}^{\alpha}_{s,\delta}[0,1]$ has the form

$$(H\psi)(r) = \frac{w(r)}{(1-\ln r)^{s+\alpha}}, \quad w(r) \in C_0[0,\delta]$$

for $r < \delta$.

3. Auxiliary Statements

This section deals with one-dimensional equations. The results of this section are used in Section 4 to prove the main theorem.

In the space $C[0,1]$, we consider the integral equation

$$\psi(r) = \int_0^1 h(r,\rho)\psi(\rho)\,d\rho + g(r), \quad r \in [0,1], \tag{5}$$

where the function $h(r,\rho)$ is defined on $\mathbb{R}_+ \times \mathbb{R}_+$, is homogeneous of degree (-1) and satisfies the condition (4). Following ([4], § 5), let us name, by the symbol of Equation (5), the function

$$\sigma(\xi) = 1 - \int_0^\infty h(1,\rho)\rho^{i\xi}\,d\rho, \quad \xi \in \mathbb{R}.$$

Let the condition

$$\sigma(\xi) \neq 0, \quad \forall \xi \in \mathbb{R} \tag{6}$$

be satisfied. It is the necessary and sufficient condition for the Fredholm property of the operator $I - H$. (The operator $A : X \to Y$ is called the Fredholm operator, if its image $\operatorname{Im} A$ is closed, $\dim \operatorname{Ker} A < \infty$, and $\dim \operatorname{Coker} A < \infty$.) Let $\varkappa = -\operatorname{ind}\sigma(\xi)$. Then, Equation (5) is solvable in two cases, (a) $\varkappa \geqslant 0$ and (b) $\varkappa < 0$, and the following orthogonality conditions are satisfied:

$$\int_0^1 g(r)\overline{\chi_s(r)}\,dr = 0, \quad s = 0, 1, \ldots, |\varkappa| - 1,$$

where the functions $\chi_s(r)$ form the basis of the space of solutions of the conjugate homogeneous equation (the line denotes complex conjugation).

Lemma 1. *Let the condition (6) be satisfied and Equation (5) be solvable in the space $C[0,1]$. If $g \in \boldsymbol{A}^{\alpha}_{s,\delta}[0,1]$, where $s \leqslant [\nu]-1$, then the solution $\psi(r)$ also belongs to the class $\boldsymbol{A}^{\alpha}_{s,\delta}[0,1]$.*

Proof. Since the symbol $\sigma(\xi)$ satisfies condition (6), it can be factorized as follows:

$$\sigma(\xi) = \sigma_-(\xi)\left(\frac{\xi - i}{\xi + i}\right)^{-\varkappa}\sigma_+(\xi),$$

where $\sigma_+(\xi)$ and $\sigma_-(\xi)$ are the functions that are analytical inside and continuous, including the boundary, in the half-planes $\mathbb{C}_+ = \{\xi \colon \Im\xi \geqslant 0\}$ and $\mathbb{C}_- = \{\xi \colon \Im\xi \leqslant 0\}$, respectively. Moreover,

$$\sigma_+(\xi) \neq 0,\ \forall\xi \in \mathbb{C}_+ \quad \text{and} \quad \sigma_-(\xi) \neq 0,\ \forall\xi \in \mathbb{C}_- \tag{7}$$

(e.g., see [4], pp. 34–35). Thus, the operator $N = I - H$ can be factorized in the form

$$N = N_- V^{-\varkappa} N_+, \tag{8}$$

where $N_\pm = I - H_\pm$ are the operators with the symbols $\sigma_\pm(\xi)$, respectively. Here, $H_\pm$ are the integral operators of form (3) with kernels that are homogeneous of degree (-1) and satisfy condition (4). The operator $V^{-\varkappa}$ is the operator whose symbol is the function $\left(\frac{\xi-i}{\xi+i}\right)^{-\varkappa}$. The operators $V^{\pm 1}$ are defined by the formulas

$$(V\psi)(r) = \psi(r) - 2\int\limits_r^1 \frac{r}{\rho^2}\psi(\rho)\,d\rho, \quad (V^{-1}\psi)(r) = \psi(r) - 2\int\limits_0^r \frac{1}{r}\psi(\rho)\,d\rho,$$

where $r \in [0,1]$. It should be emphasized that the operator V^{-1} is the *left*-inverse operator for the operator V. It is easy to see that the kernels of operators $V^{\pm 1}$ satisfy condition (4) for any $\nu > 0$.

Since condition (7) is satisfied, the operators $N_\pm$ are invertible. Moreover, the inverse operators $N_\pm^{-1}$ have the same structure, i.e., their kernels are homogeneous of degree (-1) and satisfy condition (4). Then, it follows, from Formula (8), that the solution of Equation (5) has the form

$$\psi(r) = \begin{cases} \left(N_+^{-1}N_-^{-1}g\right)(r), & \varkappa = 0, \\ \left(N_+^{-1}V^{\varkappa}N_-^{-1}g\right)(r) + \sum\limits_{\ell=0}^{\varkappa-1} b_\ell N_+^{-1}\left(r\left(\ln\dfrac{1}{r}\right)^\ell\right), & \varkappa > 0, \\ \left(N_+^{-1}V^{\varkappa}N_-^{-1}g\right)(r), & \varkappa < 0, \end{cases} \tag{9}$$

where b_ℓ are arbitrary constants and the functions $r\left(\ln\frac{1}{r}\right)^\ell$, $\ell = 0, 1, \ldots, \varkappa - 1$, form the basis of the space of solutions of equation $V^{-\varkappa}\psi = 0$.

Let $g \in \boldsymbol{A}^\alpha_{s,\delta}[0,1]$. Then, by virtue of Proposition 2, $N_+^{-1}V^{\varkappa}N_-^{-1}g \in \boldsymbol{A}^\alpha_{s,\delta}[0,1]$ for any $\varkappa$. This implies that Lemma 1 is valid for $\varkappa \leqslant 0$. Next, for any $\ell \in \mathbb{Z}_+$ the function $r\left(\ln\frac{1}{r}\right)^\ell$ belongs to the class $\boldsymbol{A}^\alpha_{s,\delta}[0,1]$, because the representation

$$r\left(\ln\frac{1}{r}\right)^\ell = \frac{v(r)}{(1 - \ln r)^{s+\alpha}},$$

where $v(r) = r\left(\ln\frac{1}{r}\right)^\ell(1 - \ln r)^{s+\alpha} \in C_0[0,1]$, is valid. Then, the function $N_+^{-1}\left(r\left(\ln\frac{1}{r}\right)^\ell\right)$ belongs to the class $\boldsymbol{A}^\alpha_{s,\delta}[0,1]$ for any $\ell \in \mathbb{Z}_+$. It follows, from the above, that Lemma 1 is valid for $\varkappa > 0$. $\square$

Lemma 2. *Let the condition (6) be fulfilled and Equation (5) be solvable in the space* $C[0,1]$*. If* $s \leqslant [\nu] - 1$ *and, for* $r < \delta$*, the function* $g \in \boldsymbol{A}^{\alpha}_{s,\delta}[0,1]$ *has the form*

$$g(r) = \frac{u(r)}{(1 - \ln r)^{s+\alpha}}, \quad u \in C_0[0,\delta],$$

then, for $r < \delta$*, the solution* $\psi(r)$ *has the form*

$$\psi(r) = \frac{w(r)}{(1 - \ln r)^{s+\alpha}}, \quad w \in C_0[0,\delta].$$

Proof. Proof follows from Formula (9) with Proposition 2 taken into account. □

4. The Main Result

In the space $C(\mathbb{B}_n)$, we consider Equation (1). We introduce the symbol of Equation (1) as a set of functions

$$\sigma_m(\xi) = 1 - \int\limits_{\mathbb{R}^n} k(e_1, y) P_m(e_1 \cdot y') |y|^{i\xi}\, dy, \quad \xi \in \mathbb{R}, \tag{10}$$

where $m \in \mathbb{Z}_+$ and $P_m(t)$ are the Legendre polynomials. The condition

$$\sigma_m(\xi) \neq 0, \quad \forall \xi \in \mathbb{R}, \quad \forall m \in \mathbb{Z}_+ \tag{11}$$

is the necessary and sufficient condition for the Fredholm property of the operator $I - K$ (e.g., see [4], p. 78, and [5]). Let us assume that (11) is satisfied. Then, we put

$$\varkappa_m = -\operatorname{ind} \sigma_m(\xi), \quad \widetilde{\varkappa} = -\sum_{\varkappa_m < 0} d_n(m) \varkappa_m.$$

Equation (1) is solvable in the space $C(\mathbb{B}_n)$ if and only if the following orthogonality conditions are satisfied:

$$\int\limits_{\mathbb{B}_n} f(x) \overline{\tau_s(x)}\, dx = 0, \quad s = 0, 1, \ldots, \widetilde{\varkappa} - 1,$$

where the functions $\tau_s(x)$ form the basis of the space of solutions of the conjugate homogeneous equation.

Below, it is assumed that condition (11) is satisfied and Equation (1) is solvable. Using condition 1°, we can rewrite Equation (1) in the form

$$\varphi(x) = \int\limits_{\mathbb{B}_n} \frac{1}{|x|^n} k\left(x', \frac{y}{|x|}\right) \varphi(y)\, dy + f(x).$$

Since the function $k(x,y)$ satisfies condition 2°, there is a function $k_0(r,\rho,t)$ such that $k(x,y) = k_0(|x|, |y|, x' \cdot y')$ ([4], p. 68). Taking this into account and passing to the spherical coordinates $x = r\sigma$ and $y = \rho\theta$ in the last equation, we obtain

$$\varphi(r\sigma) = \int\limits_0^1 \int\limits_{\mathbb{S}_{n-1}} \frac{1}{r} D\left(\frac{\rho}{r}, \sigma \cdot \theta\right) \varphi(\rho\theta)\, d\rho d\theta + f(r\sigma), \tag{12}$$

where

$$D(\rho, t) = k_0(1, \rho, t) \rho^{n-1}. \tag{13}$$

Using condition 3°, it is easy to verify that

$$\int_0^\infty \int_{-1}^1 |D(\rho,t)|(1+|\ln\rho|)^\nu (1-t^2)^{(n-3)/2} d\rho\, dt < \infty. \tag{14}$$

Let us fix the orthonormal basis $\{Y_{m\mu}\}_{m\in\mathbb{Z}_+,\,\mu=1,2,\ldots,d_n(m)}$ that consists of real spherical harmonics. In the space $C(\mathbb{B}_n)$, we define the projector P_M by the formula

$$(P_M\varphi)(r\sigma) = \sum_{m=0}^{M}\sum_{\mu=1}^{d_n(m)} \varphi_{m\mu}(r)Y_{m\mu}(\sigma),$$

where $\varphi_{m\mu}(r)$ are the Fourier–Laplace coefficients of the function $\varphi(r\sigma)$, i.e.,

$$\varphi_{m\mu}(r) = \int_{\mathbb{S}_{n-1}} \varphi(r\sigma)Y_{m\mu}(\sigma)\, d\sigma.$$

Let $Q_M = I - P_M$. We use the following notation:

$$\varphi_M(r\sigma) = (P_M\varphi)(r\sigma), \quad \varphi^M(r\sigma) = (Q_M\varphi)(r\sigma).$$

Since $\operatorname{Im} P_M = \operatorname{Ker} Q_M$ and $\operatorname{Im} Q_M = \operatorname{Ker} P_M$, $\operatorname{Im} P_M$ and $\operatorname{Im} Q_M$ are the closed subspaces of $C(\mathbb{B}_n)$ and $C(\mathbb{B}_n) = \operatorname{Im} P_M \oplus \operatorname{Im} Q_M$. Hence, Equation (12) is equivalent to the system

$$\begin{cases} \varphi_M(r\sigma) = \displaystyle\int_0^1 \int_{\mathbb{S}_{n-1}} \frac{1}{r} D\Big(\frac{\rho}{r}, \sigma\cdot\theta\Big)\varphi_M(\rho\theta)\, d\rho\, d\theta + f_M(r\sigma), \\ \varphi^M(r\sigma) = \displaystyle\int_0^1 \int_{\mathbb{S}_{n-1}} \frac{1}{r} D\Big(\frac{\rho}{r}, \sigma\cdot\theta\Big)\varphi^M(\rho\theta)\, d\rho\, d\theta + f^M(r\sigma). \end{cases} \tag{15}$$

The equivalence is understood in the following sense: if we are given the solution of system (15), we can write down the solution of Equation (12) and vice versa.

In the subspace $\operatorname{Im} Q_M$, let us consider the operator

$$(\mathcal{K}\varphi^M)(r\sigma) = \int_0^1 \int_{\mathbb{S}_{n-1}} \frac{1}{r} D\Big(\frac{\rho}{r}, \sigma\cdot\theta\Big)\varphi^M(\rho\theta)\, d\rho d\theta.$$

It is obvious that the operator $\mathcal{K}$ is the restriction of the operator K of type (2) to the subspace $\operatorname{Im} Q_M$. It should be noted that $\mathcal{K}(\operatorname{Im} Q_M) \subset \operatorname{Im} Q_M$. This follows from the equation $P_M\mathcal{K}Q_M = 0$ (e.g., see [7], p. 1170).

It is shown, in ([4], pp. 80–81), that the number M may be chosen to be so large that the inequality $\|\mathcal{K}\|_{\mathcal{L}(\operatorname{Im} Q_M)} < 1$ is satisfied. Then, the operator $I - \mathcal{K}$ is invertible in $\mathcal{L}(\operatorname{Im} Q_M)$. Hence, the second equation of system (15) has the unique solution defined by the formula

$$\varphi^M(r\sigma) = \Big((I-\mathcal{K})^{-1} f^M\Big)(r\sigma) = \sum_{\ell=0}^{\infty} \Big(\mathcal{K}^\ell f^M\Big)(r\sigma), \tag{16}$$

where the series is sup-norm convergent.

Let us transform the first equation of system (15). Multiplying both sides of this equation by $Y_{m\mu}(\sigma)$, where $m \leqslant M$, and integrating over the unit sphere, we obtain

$$\varphi_{m\mu}(r) = \int_{\mathbb{S}_{n-1}} Y_{m\mu}(\sigma)\, d\sigma \int_0^1 \int_{\mathbb{S}_{n-1}} \frac{1}{r} D\Big(\frac{\rho}{r}, \sigma\cdot\theta\Big)\varphi_M(\rho\theta)\, d\rho\, d\theta + f_{m\mu}(r).$$

Using the Funk–Hecke formula ([4], p. 74), we write down the chain of equations

$$\int\limits_{\mathbb{S}_{n-1}} Y_{m\mu}(\sigma)\, d\sigma \int\limits_0^1 \int\limits_{\mathbb{S}_{n-1}} \frac{1}{r} D\Big(\frac{\rho}{r}, \sigma\cdot\theta\Big)\, \varphi_M(\rho\theta)\, d\rho\, d\theta$$

$$= \int\limits_0^1 \int\limits_{\mathbb{S}_{n-1}} \frac{1}{r} \varphi_M(\rho\theta)\, d\rho\, d\theta \int\limits_{\mathbb{S}_{n-1}} D\Big(\frac{\rho}{r}, \sigma\cdot\theta\Big) Y_{m\mu}(\sigma)\, d\sigma$$

$$= \int\limits_0^1 \int\limits_{\mathbb{S}_{n-1}} \frac{1}{r} \varphi_M(\rho\theta) D_m\Big(\frac{\rho}{r}\Big) Y_{m\mu}(\theta)\, d\rho\, d\theta = \int\limits_0^1 \frac{1}{r} D_m\Big(\frac{\rho}{r}\Big) \varphi_{m\mu}(\rho)\, d\rho,$$

where

$$D_m(\rho) = |\mathbb{S}_{n-2}| \int\limits_{-1}^1 D(\rho, t) P_m(t) (1-t^2)^{(n-3)/2} dt \tag{17}$$

and $P_m(t)$ are the Legendre polynomials.

Thus, we obtain the following finite diagonal system of one-dimensional equations:

$$\varphi_{m\mu}(r) = \int\limits_0^1 \frac{1}{r} D_m\Big(\frac{\rho}{r}\Big) \varphi_{m\mu}(\rho)\, d\rho + f_{m\mu}(r), \tag{18}$$

where $r \in [0,1]$, $m = 0, 1, \ldots, M$, $\mu = 1, 2, \ldots, d_n(m)$. It is obvious that the kernel of Equation (18) is homogeneous of degree (-1). In addition, since $|P_m(t)| \leqslant 1$ for all $t \in [-1,1]$ and $m \in \mathbb{Z}_+$, it is follows, from condition (14), that

$$\int\limits_0^\infty |D_m(\rho)| (1 + |\ln\rho|)^\nu\, d\rho < \infty.$$

Therefore, Equation (18) is an equation of type (5). Let us show that the symbol of Equation (18) is the function $\sigma_m(\xi)$ defined by Formula (10). Indeed, applying subsequently equality (17), Catalan's formula (e.g., see [19], p. 44) and equality (13), we obtain

$$1 - \int\limits_0^\infty D_m(\rho)\rho^{i\xi}\, d\rho = 1 - |\mathbb{S}_{n-2}| \int\limits_0^\infty \rho^{i\xi}\, d\rho \int\limits_{-1}^1 D(\rho, t) P_m(t) (1-t^2)^{(n-3)/2} dt$$

$$= 1 - \int\limits_0^\infty \int\limits_{\mathbb{S}_{n-1}} D(\rho, e_1\cdot\theta) P_m(e_1\cdot\theta) \rho^{i\xi}\, d\rho\, d\theta$$

$$= 1 - \int\limits_{\mathbb{R}^n} k(e_1, y) P_m(e_1\cdot y') |y|^{i\xi}\, dy = \sigma_m(\xi).$$

It is easy to see that, if Equation (1) is solvable in the space $C(\mathbb{B}_n)$, then Equation (18) is solvable in the space $C[0,1]$ for any values of m and μ.

The main result of this paper is the following theorem.

Theorem 1. *Let condition (11) be satisfied and Equation (1) be solvable in the space $C(\mathbb{B}_n)$. If $f \in \boldsymbol{A}^\alpha_{s,\delta}(\mathbb{B}_n)$, where $s \leqslant [\nu] - 1$, then the solution $\varphi(x)$ also belongs to the class $\boldsymbol{A}^\alpha_{s,\delta}(\mathbb{B}_n)$.*

Proof. Equation (1) is equivalent to Equation (12), which, in turn, is equivalent to system (15). Let the function $f(r\sigma) \in \boldsymbol{A}^{\alpha}_{s,\delta}(\mathbb{B}_n)$ in the δ–neighborhood of zero have the form

$$f(r\sigma) = b + \sum_{j=0}^{s} \frac{b_j}{(1-\ln r)^{j+\alpha}} + \frac{v(r\sigma)}{(1-\ln r)^{s+\alpha}},$$

where $v(r\sigma) \in C_0(\mathbb{B}_n(\delta))$. Then, for $r < \delta$, the Fourier–Laplace coefficients $f_{m\mu}(r)$ have the form

$$f_{m\mu}(r) = \begin{cases} \beta + \sum\limits_{j=0}^{s} \dfrac{\beta_j}{(1-\ln r)^{j+\alpha}} + \dfrac{v_0(r)}{(1-\ln r)^{s+\alpha}}, & m = 0, \\ \dfrac{v_{m\mu}(r)}{(1-\ln r)^{s+\alpha}}, & m \geqslant 1, \end{cases} \tag{19}$$

where $\beta = b\sqrt{|\mathbb{S}_{n-1}|}$, $\beta_j = b_j\sqrt{|\mathbb{S}_{n-1}|}$ (taking into account that $Y_0(\sigma) \equiv 1/\sqrt{|\mathbb{S}_{n-1}|}$). Hence, $f_{m\mu}(r) \in \boldsymbol{A}^{\alpha}_{s,\delta}[0,1]$ for all $m \in \mathbb{Z}_+$ and $\mu = 1, 2, \ldots, d_n(m)$. It is easy to see that, for $r < \delta$,

$$f_M(r\sigma) = b + \sum_{j=0}^{s} \frac{b_j}{(1-\ln r)^{j+\alpha}} + \frac{v_M(r\sigma)}{(1-\ln r)^{s+\alpha}},$$

and, respectively,

$$f^M(r\sigma) = \frac{v^M(r\sigma)}{(1-\ln r)^{s+\alpha}}. \tag{20}$$

Therefore, the functions $f_M(r\sigma)$ and $f^M(r\sigma)$ belong to the class $\boldsymbol{A}^{\alpha}_{s,\delta}(\mathbb{B}_n)$. Let us prove that the solution $\varphi(r\sigma)$ of Equation (12) also belongs to the class $\boldsymbol{A}^{\alpha}_{s,\delta}(\mathbb{B}_n)$. With this purpose, we show that $\varphi_M(r\sigma) \in \boldsymbol{A}^{\alpha}_{s,\delta}(\mathbb{B}_n)$ and $\varphi^M(r\sigma) \in \boldsymbol{A}^{\alpha}_{s,\delta}(\mathbb{B}_n)$.

It is clear that $\varphi^M(r\sigma) \in C(\mathbb{B}_n)$. Let us find the form of the function $\varphi^M(r\sigma)$ in the δ–neighborhood of zero. It follows, from Formula (20) and Proposition 1, that, for any $\ell > 0$, the representation

$$(\mathcal{K}^{\ell} f^M)(r\sigma) = \frac{u_\ell(r\sigma)}{(1-\ln r)^{s+\alpha}}, \quad u_\ell(r\sigma) \in C_0(\mathbb{B}_n(\delta)),$$

where $r < \delta$, holds. Then, taking into account (16), we obtain

$$\varphi^M(r\sigma) = \sum_{\ell=0}^{\infty} \frac{u_\ell(r\sigma)}{(1-\ln r)^{s+\alpha}} = \frac{u(r\sigma)}{(1-\ln r)^{s+\alpha}},$$

where $u(r\sigma) = \sum\limits_{\ell=0}^{\infty} u_\ell(r\sigma) \in C_0(\mathbb{B}_n(\delta))$. Then $\varphi^M(r\sigma) \in \boldsymbol{A}^{\alpha}_{s,\delta}(\mathbb{B}_n)$.

Now, we consider the solution $\varphi_M(r\sigma)$ of the first equation of system (15). Since $f_{m\mu}(r) \in \boldsymbol{A}^{\alpha}_{s,\delta}[0,1]$ for all $m = 0, 1, \ldots, M$ and $\mu = 1, 2, \ldots, d_n(m)$, by virtue of Lemma 1, the solution $\varphi_{m\mu}(r)$ of Equation (18) also belongs to the class $\boldsymbol{A}^{\alpha}_{s,\delta}[0,1]$. Moreover, it follows, from Formula (19) and Lemma 2, that, for $r < \delta$, the functions $\varphi_{m\mu}(r)$ have the form

$$\varphi_{m\mu}(r) = \begin{cases} \gamma + \sum\limits_{j=0}^{s} \dfrac{\gamma_j}{(1-\ln r)^{j+\alpha}} + \dfrac{w_0(r)}{(1-\ln r)^{s+\alpha}}, & m = 0, \\ \dfrac{w_{m\mu}(r)}{(1-\ln r)^{s+\alpha}}, & 1 \leqslant m \leqslant M, \end{cases}$$

where $w_0(r), w_{m\mu}(r) \in C_0[0,\delta]$. Then, in the δ–neighborhood of zero, the function $\varphi_M(r\sigma)$ is defined by the formula

$$\varphi_M(r\sigma) = a + \sum_{j=0}^{s} \frac{a_j}{(1-\ln r)^{j+\alpha}} + \frac{w(r\sigma)}{(1-\ln r)^{s+\alpha}},$$

where $a = \gamma/\sqrt{|\mathbb{S}_{n-1}|}$, $a_j = \gamma_j/\sqrt{|\mathbb{S}_{n-1}|}$ and

$$w(r\sigma) = \sum_{m=0}^{M} \sum_{\mu=1}^{d_n(m)} w_{m\mu}(r) Y_{m\mu}(\sigma).$$

It is obvious that $w(r\sigma) \in C_0(\mathbb{B}_n)$. Then, $\varphi_M(r\sigma) \in \boldsymbol{A}^{\alpha}_{s,\delta}(\mathbb{B}_n)$, thus also $\varphi(r\sigma) \in \boldsymbol{A}^{\alpha}_{s,\delta}(\mathbb{B}_n)$. $\square$

5. Conclusions

As is known, there is no common method for constructing an *exact* solution of integral Equation (1). However, in many applied problems, it is sufficient to know only the asymptotic behavior of the solution in the neighborhood of zero. The main result of this paper is Theorem 1, which allows us to find the asymptotic behavior of the solution of Equation (1) from the given asymptotic behavior of the free term.

Further, the results of this paper can be used for matrix equations with homogeneous kernels. Moreover, the above method for constructing the asymptotics of the solution can be applied to the integral equations with homogeneous kernels and a multiplicative shift. It should also be noted that this method for constructing the asymptotics can be used in the problems of mechanics.

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