mathematics

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Dynamic Analysis of Software Systems with Aperiodic Impulse Rejuvenation

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Abstract: This paper aims to obtain the dynamical solution and instantaneous availability of software systems with aperiodic impulse rejuvenation. Firstly, we formulate the generic system with a group of coupled impulsive differential equations and transform it into an abstract Cauchy problem. Then we adopt a difference scheme and establish the convergence of this scheme by applying the Trotter–Kato theorem to obtain the system's dynamical solution. Moreover, the instantaneous availability as an important evaluation index for software systems is derived, and its range is also estimated. At last, numerical examples are shown to illustrate the validity of theoretical results.

Keywords: software systems; dynamical solution; instantaneous availability; Trotter–Kato theorem

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Citation: Huo, H.; Xu, H.; Chen, Z. Dynamic Analysis of Software Systems with Aperiodic Impulse Rejuvenation. *Mathematics* **2022**, *10*, 197. https://doi.org/10.3390/math10020197

Academic Editor: Simeon Reich

Received: 29 November 2021
Accepted: 6 January 2022
Published: 9 January 2022

1. Introduction

With the rapid development of science and technology, computers have become a necessary part of people's lives. However, in the process of using computers, it is inevitable that there will be failures, mainly software failures [1]. Due to the accumulation of error conditions and the exhaustion of system resources over long periods of execution, a software system can have faults appear, which are called aging-related bugs, and they manifest as increases in the failure rate and drops in performance. In addition, there may be crash-related bugs, which cause unexpected interruptions of software systems in users' environments [2,3]. The phenomenon caused by aging-related bugs and crash-related bugs is known as software aging [4,5], which can result in economic loss and endanger human life [6].

Software rejuvenation is an active fault management policy to combat software aging. The concept of software rejuvenation was proposed by Huang et al. [7] for the first time, in which the state process can be modeled as a continuous time Markov chain. Dohi et al. [8] illustrated that the model of Huang et al. [7] can be generalized to semi-Markov processes and developed a non-parametric algorithm to estimate the optimal software rejuvenation strategy. Xu [9] proved the well-posedness of software rejuvenation systems using the theory of the strong continuous semigroup, and presented the instantaneous availability by the method of finite differential scheme. Dohi and Okamura [10] obtained the optimal dynamic software rejuvenation strategy for maximizing the steady-state availability of the operational software system with multiple degradation levels. Koutras and Platis [11] proposed a multi-objective optimization strategy to optimize a system's overall performance capability. Zheng et al. [12] presented a composite stochastic Petri reward network and its resulting non-Markovian availability model for operational software systems.

Impulse systems refer to dynamic systems that change their states by jumping at a given time, which have become a research hotspot in the fields of control theory and engineering [13–17]. There has been rapid development in the literature on periodic impulse systems. Kember and Babitsky [18] investigated an exact steady state solution of vibro-impact systems with periodic impulse excitation via the periodic Green function method. Using Horn's fixed point theorem, Shen et al. [19] discussed the bounded and periodic properties of solutions of impulsive ordinary differential equations and functional differential equations. Zeng [20] proved the existence of multiple periodic solutions for n-dimensional functional differential equations with impulses. Huo et al. [21] analyzed the relationships between system availability and impulsive indexes to improve the availability of a software rejuvenation system with periodic impulses.

For the aperiodic impulse system, the research focuses on the stability of the system. Sofiyev [22] exhibited the stability of conical shells made of functionally graded materials subject to aperiodic impulsive loading. Naghshtabrizi et al. [23] established the exponential stability of nonlinear time-varying impulse systems by utilizing the Lyapunov function with discontinuous impulse time. Lu et al. [24] proposed the concept of the average impulse interval and studied the globally exponential synchronization of impulsive dynamical networks. Zhao et al. [25] investigated the global exponential stability of impulsive systems with infinite distributed delay. Shao and Yuan [26] researched sampling dependent stability for aperiodic sampled-data systems by employing a Lyapunov-like functional.

Although the dynamical solutions and instantaneous availability for periodic impulse systems have been studied by several researches, little work has been carried out for the dynamical solutions and instantaneous availability of aperiodic impulse systems. As one of the branches of functional analysis, operator semigroup theory is a useful tool with which to study the well-posedness of the abstract equation [27–29]. The Trotter–Kato theorem provides a very useful framework for studying the numerical approximation convergence of solutions of partial differential equations [30]. Xu and Hu in [31,32] studied the dynamical solutions of a kind of repairable system with preventive maintenance and a system consisting of two machines separated by finite storage buffers, respectively. Huo et al. [33] analyzed the dynamic behavior of a computer integrated manufacturing system. However, these papers [31–33] mainly involved dynamical systems without impulses. In this paper, we investigate the dynamical solutions and instantaneous availability of software systems with aperiodic impulse rejuvenation by applying the Trotter–Kato theorem. Note that aperiodic (periodic) impulse rejuvenation refers to a rejuvenation policy that changes the state of a software system by jumping when the time intervals of two successive rejuvenations are not equal (must equal), which can be widely used in the field of information, such as telecommunication switching and billing systems [34], cloud computing infrastructure [35] and Android operating systems [36].

The rest of this paper is organized as follows. Section 2 formulates a software system with aperiodic impulse rejuvenation, which contains the comprehensive characteristics of a continuous system and a discrete system. Section 3 analyzes the system's dynamical solution, presents the expression of the instantaneous availability as an important evaluation index of the system and estimates its range. In Section 4, numerical examples are shown to illustrate the validity of the theoretical results. Section 5 concludes the paper.

2. System Formulation

In this section, we formulate a model for software systems with aperiodic impulse rejuvenation as a group of coupled impulsive differential equations.

As shown in Figure 1 [37], robust state, failure probable state and failure state, denoted as 0, 1 and 2, are three states of software systems with aperiodic impulse rejuvenation.

Suppose that the system starts at initial state 0 when time $t = 0$, impulse time sequence $\{t_k\},\ k = 1, 2, \cdots$ satisfies $0 = t_0 < t_1 < t_2 < \cdots$ and $\lim_{k\to\infty} t_k = \infty$. The sojourn times of states 0 to 1, 1 to 2 and 0 to 2 follow an exponential distribution with parameters α_0, β and α_1, respectively. The repair time of state 2 to 0 follows a general distribution with repair rate $\nu(x)$, which satisfies the following reasonable assumptions:

$$\nu(x) > 0 \; a.e. \; x \in [0,\infty); \quad \sup_{x\in[0,\infty)} \nu(x) < \infty;$$
$$\int_0^l \nu(x)\mathrm{d}x < \infty, \; \forall l \in [0,\infty); \; \int_0^\infty \nu(x)\mathrm{d}x = \infty. \tag{1}$$

The rejuvenation time of state 1 to 0 follows an exponential distribution with parameter θ.

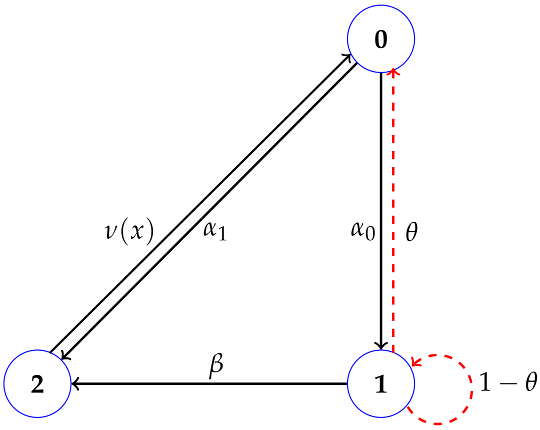

Figure 1. State transition diagram of software systems with aperiodic impulse rejuvenation.

By the supplementary variable method [37], software systems with aperiodic impulse rejuvenation can be formulated as

$$\begin{cases} \dfrac{\mathrm{d}p_0(t)}{\mathrm{d}t} = -(\alpha_0+\alpha_1)p_0(t) + \displaystyle\int_0^\infty \nu(x)p_2(t,x)\mathrm{d}x, & t \neq t_k, \\ \dfrac{\mathrm{d}p_1(t)}{\mathrm{d}t} = \alpha_0 p_0(t) - \beta p_1(t), & t \neq t_k, \\ \dfrac{\partial p_2(t,x)}{\partial t} + \dfrac{\partial p_2(t,x)}{\partial x} = -\nu(x)p_2(t,x), & t \neq t_k, \\ p_2(t,0) = \alpha_1 p_0(t) + \beta p_1(t), & t \neq t_k, \\ \Delta p_0(t_k) = \theta p_1(t_k), & \\ \Delta p_1(t_k) = -\theta p_1(t_k), & \\ \Delta p_2(t_k,x) = 0, & \\ p_0(0) = 1, \; p_1(0) = p_2(0,x) = 0, & x > 0, \end{cases} \tag{2}$$

where $p_0(t)$ and $p_1(t)$ stand for the probabilities of the system being in the robust state and in the failure probable state at time t, respectively; $p_2(t,x)$ stands for the probability density that the failed system is in failure state and has an elapsed repair time of x at time t; $\forall \varepsilon > 0$, $p_0(t_k)$, $p_1(t_k)$ and $p_2(t_k,x)$ are left continuous for $t \in (t_k-\varepsilon, t_k]$, $k = 1,2,\cdots$.

Define the state space $X = R^2 \times L^1[0,\infty)$ with $\|\cdot\|_X = |\cdot| + |\cdot| + \|\cdot\|_{L^1[0,\infty)} < \infty$ and the system operator A in X with its domain $D(A)$. The system impulse operator B from $X \to X$ is as follows:

$$A\begin{pmatrix} p_0 \\ p_1 \\ p_2(x) \end{pmatrix} = \begin{pmatrix} -(\alpha_0+\alpha_1)p_0 + \displaystyle\int_0^\infty \nu(x)p_2(x)\mathrm{d}x \\ \alpha_0 p_0 - \beta p_1 \\ -\dfrac{\mathrm{d}p_2(x)}{\mathrm{d}x} - \mu(x)p_2(x) \end{pmatrix}, B = \begin{pmatrix} 0 & \theta & 0 \\ 0 & -\theta & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$D(A)=\left\{(p_0,p_1,p_2(x))^{\mathrm{T}}\in X\left|\begin{array}{l}p_2(x)\text{ is an absolutely continuous function,}\\ p_2(x)\in W^{1,1}[0,\infty),\ p_2(0)=\alpha_1p_0+\beta p_1\end{array}\right.\right\},$$

where $W^{1,1}[0,\infty)$ is a Sobolev space; $p_2(x)\in W^{1,1}[0,\infty)$ refers to $p_2(x)\in L^1[0,\infty)$, $\frac{dp_2(x)}{dx}\in L^1[0,\infty)$.

Let $\vec{P}(t)=(p_0(t),p_1(t),p_2(t,\cdot))^{\mathrm{T}}$. The system (3) can be formulated as an abstract Cauchy problem in X:

$$\begin{cases}\dfrac{\mathrm{d}\vec{P}(t)}{\mathrm{d}t}=A\vec{P}(t), & t\neq t_k,\ k=1,2,\cdots\\ \Delta\vec{P}(t)=B\vec{P}(t), & t=t_k,\ k=1,2,\cdots\\ \vec{P}(0)=(1,\ 0,\ 0)^{\mathrm{T}}.\end{cases}\tag{3}$$

In this paper, we aim to analyze the dynamic behavior of system (3), which means that we only need to consider the system behavior in the time interval $[0,T]$. Without loss of generality, we assume that there are N impulse points (the number of times that an impulse occurs) satisfying $0=t_0<t_1<t_2<\cdots<t_N<T\leq t_{N+1}$.

3. Dynamic Analysis

In this section, we study the dynamical solution and instantaneous availability of the system (3).

3.1. Well-Posedness

To investigate the dynamical solution of the system (3), we need to prove the well-posedness, which is shown in Theorem 1.

Theorem 1. *The software system with aperiodic impulse rejuvenation (3) has a unique nonnegative dynamical solution $\vec{P}(t)$ which satisfies*

$$\vec{P}(t)=\begin{cases}S(t)\vec{P}(0), & t\in[0,\ t_1]\\ S(t-t_k)(\prod_{i=1}^{k}(I+B)S(t_i-t_{i-1}))\vec{P}(0), & t\in(t_k,\ t_{k+1}],\ k=1,2,\cdots,N-1,\\ S(t-t_N)(\prod_{i=1}^{N}(I+B)S(t_i-t_{i-1}))\vec{P}(0), & t\in(t_N,\ T].\end{cases}\tag{4}$$

and

$$\|\ \vec{P}(t)\ \|=1,\forall t\in[0,\ T].\tag{5}$$

Proof. For $\theta=0$, we define a positive cone [38] of system (3) as

$$X_+=\{(p_0,\ p_1,\ p_2(x))^{\mathrm{T}}\in X\mid p_0\geq 0,\ p_1\geq 0,\ p_2(x)\geq 0,\ \forall x\in[0,\infty)\};$$

then the system operator A can generate a positive C_0-semigroup $S(t)$, and there exists a unique nonnegative dynamical solution [39–41]

$$\vec{P}(t)=S(t)\vec{P}(0)\tag{6}$$

satisfying $\|\ \vec{P}(t)\ \|=1,\forall t\in[0,+\infty)$.

For $t\in[0,\ t_1]$, we can directly obtain that the dynamical solution of system (3) is obviously (6).

For $t \in (t_1,\ t_2]$, it follows from (3) and (6) that

$$\vec{P}(t_1^+) = (I+B)\vec{P}(t_1) = (I+B)S(t_1)\vec{P}(0), \tag{7}$$

$$\vec{P}(t) = S(t-t_1)\vec{P}(t_1^+) = S(t-t_1)(I+B)S(t_1)\vec{P}(0), \tag{8}$$

$$\vec{P}(t_2) = S(t_2-t_1)(I+B)S(t_1)\vec{P}(0). \tag{9}$$

For $t \in (t_2,\ t_3]$, it can be derived from (3), (6) and (9) that

$$\begin{aligned}
\vec{P}(t_2^+) &= (I+B)\vec{P}(t_2) = (I+B)S(t_2-t_1)(I+B)S(t_1)\vec{P}(0),\\
\vec{P}(t) &= S(t-t_2)\vec{P}(t_2^+) = S(t-t_2)(I+B)S(t_2-t_1)(I+B)S(t_1)\vec{P}(0),\\
\vec{P}(t_3) &= S(t_3-t_2)(I+B)S(t_2-t_1)(I+B)S(t_1)\vec{P}(0).
\end{aligned}$$

By induction, we deduce the dynamical solution of system (3) as

$$\vec{P}(t) = \begin{cases} S(t)\vec{P}(0), & t \in [0,\ t_1] \\ S(t-t_k)(\prod_{i=1}^{k}(I+B)S(t_i-t_{i-1}))\vec{P}(0), & t \in (t_k,\ t_{k+1}],\ k = 1,2,\cdots,N-1, \\ S(t-t_N)(\prod_{i=1}^{N}(I+B)S(t_i-t_{i-1}))\vec{P}(0), & t \in (t_N,\ T]. \end{cases}$$

In addition, it is obvious that $\vec{P}(t) \in X$ and $\parallel \vec{P}(t) \parallel = 1, \forall t \in [0,\ T]$. □

3.2. Approximation System

In this subsection, we derive a numerical approximation for solution (4) and (5) of system (3) by applying the Trotter–Kato theorem [30].

Using the method of characteristics, a common method for solving partial differential equations, $p_2(t,x) = 0$ for $x \geq t$ is obtained. To derive a numerical approximation for solution (4) and (5) of system (3) in time interval $[0,T]$, we only need to divide the finite interval $[0,T]$ into n equal subintervals $[x_j, x_{j+1}]$, $x_j = (j-1)\Delta x$, $j = 1,2,\cdots,n$ and $\Delta x = \dfrac{T}{n}$. Let $p_{2,j}(t) = p_2(t,x_j)$, $\nu_j = \nu(x_j)$. Then, the system (3) can be described as the following approximation system.

$$\begin{cases} \dfrac{\mathrm{d}p_0(t)}{\mathrm{d}t} = -(\alpha_0+\alpha_1)p_0(t) + \Delta x \sum_{j=1}^{n} \nu_j p_{2,j}(t), & t \neq t_k, \\ \dfrac{\mathrm{d}p_1(t)}{\mathrm{d}t} = \alpha_0 p_0(t) - \beta p_1(t), & t \neq t_k, \\ \dfrac{\mathrm{d}p_{2,j}(t)}{\mathrm{d}t} = \dfrac{p_{2,j-1}(t) - p_{2,j}(t)}{\Delta x} - \nu_j p_{2,j}(t), & t \neq t_k, \\ p_{2,0}(t) = \alpha_1 p_0(t) + \beta p_1(t), & t \neq t_k, \\ \Delta p_0(t_k) = \theta p_1(t_k), \\ \Delta p_1(t_k) = -\theta p_1(t_k), \\ \Delta p_{2,j}(t_k) = 0, \\ p_0(0) = 1,\ p_1(0) = p_{2,j}(0) = 0, \end{cases} \tag{10}$$

where $k = 1,2,\cdots N$; $j = 1,2,\cdots,n$. By defining

$$\vec{P}_n(t) = (p_0(t),\ p_1(t),\ p_{2,1}(t),\ p_{2,2}(t),\ \cdots, p_{2,n}(t))^{\mathrm{T}}, \tag{11}$$

and the asymptotic generator

$$A_n = \begin{pmatrix} -(\alpha_0+\alpha_1) & 0 & \Delta x\nu_1 & \Delta x\nu_2 & \cdots & \Delta x\nu_n \\ \alpha_0 & -\beta & 0 & 0 & \cdots & 0 \\ \frac{\alpha_1}{\Delta x} & \frac{\beta}{\Delta x} & \wedge_1 & 0 & \cdots & 0 \\ 0 & 0 & \frac{1}{\Delta x} & \wedge_2 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \\ 0 & 0 & 0 & 0 & \frac{1}{\Delta x} & \wedge_n \end{pmatrix}, \; B_n = \begin{pmatrix} 0 & \theta & 0 & \cdots & 0 \\ 0 & -\theta & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

on $X_n = R^{n+2}$, where $\wedge_j = -(\frac{1}{\Delta x} + \nu_j)$, $j = 1, 2, \cdots, n$, the above approximation system (10) can be transformed into an abstract Cauchy problem of the ODE system

$$\begin{cases} \frac{\mathrm{d}\vec{P}_n(t)}{\mathrm{d}t} = A_n \vec{P}_n(t), & t \neq t_k,\ k = 1, 2, \cdots, N, \\ \Delta \vec{P}_n(t) = B_n \vec{P}_n(t), & t = t_k,\ k = 1, 2, \cdots, N, \\ \vec{P}_n(0) = (1, 0, 0, \cdots, 0)^{\mathrm{T}}_{1\times(n+2)}, \end{cases} \tag{12}$$

and then we have the following important theorem.

Theorem 2. *Define U_n, E_n, $\|\cdot\|_n$ as*

$$U_n\vec{\psi} = \left(p_0,\ p_1,\ \frac{1}{\Delta x}\int_{x_0}^{x_1} p_2(x)\mathrm{d}x, \ldots, \frac{1}{\Delta x}\int_{x_{n-1}}^{x_n} p_2(x)\mathrm{d}x\right)^{\mathrm{T}}, \tag{13}$$

$$E_n\vec{\varphi}^n = \left(p_0,\ p_1,\ \sum_{j=1}^{n} p_{2,j}\chi_{(x_{j-1},x_j]}\right)^{\mathrm{T}}, \tag{14}$$

$$\|\vec{\varphi}^n\|_n = |p_0| + |p_1| + \Delta x \sum_{j=1}^{n} |p_{2,j}|, \tag{15}$$

where

$$\vec{\psi} = (p_0,\ p_1,\ p_2(x))^{\mathrm{T}} \in X,$$
$$\vec{\varphi}^n = (p_0,\ p_1,\ p_{2,1},\ p_{2,2},\ \cdots, p_{2,n})^{\mathrm{T}} \in X_n,$$

and let $S(t)$ and $\mathrm{e}^{A_n t}$ be the semigroup generated by A and A_n on X and X_n, respectively, then for every $x \in X$ and $t \geq 0$, $\left\|E_n \mathrm{e}^{A_n t} U_n x - S(t)x\right\| \to 0$ as $n \to \infty$ uniformly on bounded t-intervals. Moreover, the dynamical solution $\vec{P}(t) = (p_0(t), p_1(t), p_2(t,x))^{\mathrm{T}}$ of system (3) is

$$p_0(t) = \begin{cases} \lim\limits_{n\to\infty} \vec{H}_0 \mathrm{e}^{A_n t} \vec{P}_n(0), & t \in [0,\ t_1], \\ \lim\limits_{n\to\infty} \vec{H}_0 \mathrm{e}^{A_n(t-t_k)} \prod\limits_{i=1}^{k} (I + B_n)\mathrm{e}^{A_n(t_i - t_{i-1})} \vec{P}_n(0), & t \in (t_k,\ t_{k+1}], \\ \lim\limits_{n\to\infty} \vec{H}_0 \mathrm{e}^{A_n(t-t_N)} \prod\limits_{i=1}^{N} (I + B_n)\mathrm{e}^{A_n(t_i - t_{i-1})} \vec{P}_n(0), & t \in (t_N,\ T], \end{cases} \tag{16}$$

$$p_1(t) = \begin{cases} \lim\limits_{n\to\infty} \vec{H}_1 \mathrm{e}^{A_n t} \vec{P}_n(0), & t \in [0,\ t_1], \\ \lim\limits_{n\to\infty} \vec{H}_1 \mathrm{e}^{A_n (t-t_k)} \prod\limits_{i=1}^{k} (I+B_n) \mathrm{e}^{A_n(t_i - t_{i-1})} \vec{P}_n(0), & t \in (t_k,\ t_{k+1}], \\ \lim\limits_{n\to\infty} \vec{H}_1 \mathrm{e}^{A_n (t-t_N)} \prod\limits_{i=1}^{N} (I+B_n) \mathrm{e}^{A_n(t_i - t_{i-1})} \vec{P}_n(0), & t \in (t_N,\ T], \end{cases} \tag{17}$$

and

$$\begin{aligned} p_2(t,x) =& \lim_{n\to\infty} \sum_{j=1}^{n} p_{2,j}(t) \chi_{(x_{j-1},x_j]} \\ =& \begin{cases} \lim\limits_{n\to\infty} \vec{H}_2 \mathrm{e}^{A_n t} \vec{P}_n(0), & t \in [0,\ t_1], \\ \lim\limits_{n\to\infty} \vec{H}_2 \mathrm{e}^{A_n (t-t_k)} \prod\limits_{i=1}^{k} (I+B_n) \mathrm{e}^{A_n(t_i - t_{i-1})} \vec{P}_n(0), & t \in (t_k,\ t_{k+1}], \\ \lim\limits_{n\to\infty} \vec{H}_2 \mathrm{e}^{A_n (t-t_N)} \prod\limits_{i=1}^{N} (I+B_n) \mathrm{e}^{A_n(t_i - t_{i-1})} \vec{P}_n(0), & t \in (t_N,\ T], \end{cases} \end{aligned} \tag{18}$$

in which $k = 1,\ 2,\ \cdots,\ N-1$, *by setting* $\vec{H}_0 = (1,\ 0,\ 0,\ 0,\ \cdots,\ 0)_{1\times(n+2)}$, $\vec{H}_1 = (0,\ 1,\ 0,\ 0,\ \cdots,\ 0)_{1\times(n+2)}$, $\vec{H}_2 = (0,\ 0,\ 1,\ 1,\ \cdots,\ 1)_{1\times(n+2)}$.

Proof. The first part of Theorem 2 can be proved by [30]. On this basis, the dynamical solution $\vec{P}(t) = (p_0(t), p_1(t), p_2(t,\cdot))^{\mathrm{T}}$ of system (3) can be expressed as

$$\begin{aligned} \vec{P}(t) =& \lim_{n\to\infty} E_n \vec{P}_n(t) \\ =& \lim_{n\to\infty} \left(p_0(t),\ p_1(t),\ \sum_{j=1}^{n} p_{2,j}(t) \chi_{(x_{j-1},x_j]} \right)^{\mathrm{T}} \\ =& \left(p_0(t),\ p_1(t),\ \lim_{n\to\infty} \sum_{j=1}^{n} p_{2,j}(t) \chi_{(x_{j-1},x_j]} \right)^{\mathrm{T}}, \end{aligned} \tag{19}$$

by Equations (5) and (14), which yield (16)–(18). □

3.3. Instantaneous Availability

In this subsection, we present the expression of the instantaneous availability of system (3) and estimate its range.

Lemma 3. *The instantaneous availability* $A(t)$ *of system (3) is*

$$\begin{aligned} A(t) =& p_0(t) + p_1(t) \\ =& \begin{cases} \lim\limits_{n\to\infty} \vec{H} \mathrm{e}^{A_n t} \vec{P}_n(0), & t \in [0,\ t_1], \\ \lim\limits_{n\to\infty} \vec{H} \mathrm{e}^{A_n (t-t_k)} \prod\limits_{i=1}^{k} (I+B_n) \mathrm{e}^{A_n(t_i - t_{i-1})} \vec{P}_n(0), & t \in (t_k,\ t_{k+1}], \\ \lim\limits_{n\to\infty} \vec{H} \mathrm{e}^{A_n (t-t_N)} \prod\limits_{i=1}^{N} (I+B_n) \mathrm{e}^{A_n(t_i - t_{i-1})} \vec{P}_n(0), & t \in (t_N,\ T], \end{cases} \end{aligned} \tag{20}$$

where $k = 1,\ 2,\ \cdots,\ N-1$ *and* $\vec{H} = \vec{H}_0 + \vec{H}_1 = (1,\ 1,\ 0,\ 0,\ \cdots,\ 0)_{1\times(n+2)}$.

Proof. Since state 0 and state 1 are the working states as described in Section 2, the instantaneous availability $A(t)$ of system (3) can be directly obtained by Equations (16) and (17). □

Theorem 4. *Assume that the positive numbers* τ_1 *and* τ_2 *are the lower bound and upper bound of the impulse interval satisfying*

$$0 < \tau_1 = \inf_{k \in \{1,2,\cdots,N+1\}} (t_k - t_{k-1}) \le \sup_{k \in \{1,2,\cdots,N+1\}} (t_k - t_{k-1}) = \tau_2 < \infty. \tag{21}$$

Let $A_1(t)$ *and* $A_2(t)$ *are the corresponding instantaneous availability of* τ_1 *and* τ_2*, respectively. Then, the range of the instantaneous availability for system (3) can be estimated as*

$$A_2(t) \le A(t) \le A_1(t) \tag{22}$$

for fixed impulse strength θ *in* $t \in [0, T]$*.*

Proof. Based on (21) and the upper and lower bounds of impulsive intervals method [42], we derive the instantaneous availability $A_1(t)$ and $A_2(t)$ of system (3) by

$$A_m(t) = \begin{cases} \lim\limits_{n\to\infty} \vec{H} e^{A_n t} \vec{P}_n(0), & t \in [0,\ \tau_m] \\ \lim\limits_{n\to\infty} \vec{H} e^{A_n(t-k\tau_m)} ((I+B_n)e^{A_n\tau_m})^k \vec{P}_n(0), & t \in (k\tau_m,\ (k+1)\tau_m], \\ \lim\limits_{n\to\infty} \vec{H} e^{A_n(t-N\tau_m)} ((I+B_n)e^{A_n\tau_m})^N \vec{P}_n(0), & t \in (N\tau_m,\ T] \end{cases} \tag{23}$$

where $k = 1,\ 2,\ \cdots,\ N-1$ and $m = 1, 2$. It is well known that $\lim\limits_{n\to\infty} \vec{H} e^{A_n t} \vec{P}_n(0)$ of system (12) decreases as the time increases in time $t \in [0, t_1]$. As the lower bound τ_1 and upper bound τ_2 of Equation (21) satisfy $\tau_1 \le \tau_2$, we have

$$\lim_{n\to\infty} \vec{H} e^{A_n \tau_2} \vec{P}_n(0) \le \lim_{n\to\infty} \vec{H} e^{A_n \tau_1} \vec{P}_n(0).$$

Due to the positivity of the impulse strength θ, the range of the instantaneous availability for system (3) can be estimated as (22). □

Remark 5. *Proposition 4 implies that when the impulse interval is* τ_1 *or* τ_2*, system (3) is a software system with periodic impulse rejuvenation.*

4. Numerical Examples

In this section, some numerical examples are presented to illustrate the validity of the theoretical results.

To simulate the dynamic behavior of the software system with aperiodic impulse rejuvenation (3), we take a virtual machine as an example, and then set $T = 9$, $\alpha_0 = \frac{1}{7}$, $\alpha_1 = \frac{1}{120}$, $\beta = \frac{1}{3}$ and the repair time follows a Weibull distribution with repair rate $\nu(x) = 50x$, where the parameter values $\alpha_0,\ \alpha_1,\ \beta$ used are based on the experimental studies, $\nu(x)$ is set by ourselves according to the values in references [43,44].

Figure 2 depicts the dynamical approximation $\sum\limits_{j=1}^{n} p_{2,j}(t)\chi_{(x_{j-1},x_j]}$ of (19) with different segmentation patterns n. We can see that when n increases, $\sum\limits_{j=1}^{n} p_{2,j}(t)\chi_{(x_{j-1},x_j]}$ converges to the same curve, which reflects the validity of Theorem 2.

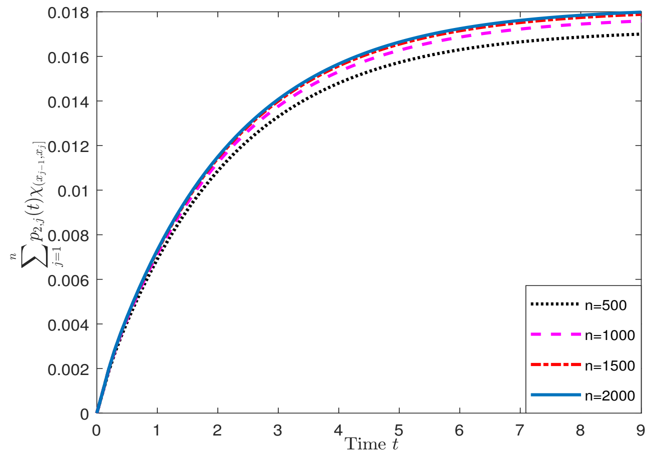

Figure 2. Dynamical approximation $\sum_{j=1}^{n} p_{2,j}(t)\chi_{(x_{j-1},x_j]}$ of (19).

Let impulse action occur in $t_1 = 1,\ t_2 = 3,\ t_3 = 6,\ t_4 = 8$, which implies that the impulse lower bound $\tau_1 = 1$ and impulse upper bound $\tau_2 = 3$, then the instantaneous availability of system (3) and its range for the impulse strength $\theta = 0.8$ are depicted in Figure 3. With Figure 3, we find that the instantaneous availability of software systems with impulse rejuvenation is significantly higher than that of software systems without impulse rejuvenation, and $\forall t \in [0,9]$, $A_2(t) \le A(t) \le A_1(t)$, which satisfies Theorem 4 and Remark 5. Therefore, the system instantaneous availability can be improved by decreasing the impulse interval.

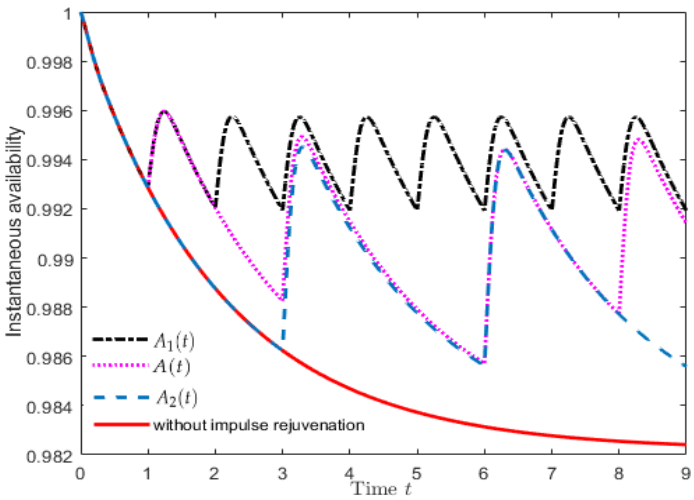

Figure 3. Instantaneous availability of the software system.

In Figure 4, with the change in impulse strength θ, the instantaneous availabilities of the system for impulse action occurred at $t_1 = 1,\ t_2 = 3,\ t_3 = 6,\ t_4 = 8$ are provided. Figure 4 shows that the instantaneous availability of software systems with impulse rejuvenation is higher than that of software systems without impulse rejuvenation, and the instantaneous availability is inversely proportional to the impulse interval in time interval $t \in (t_k, t_{k+1}]$, $k = 1,\ 2,\ 3,\ 4$. Thus, the system's instantaneous availability can be improved by increasing the system impulse strength.

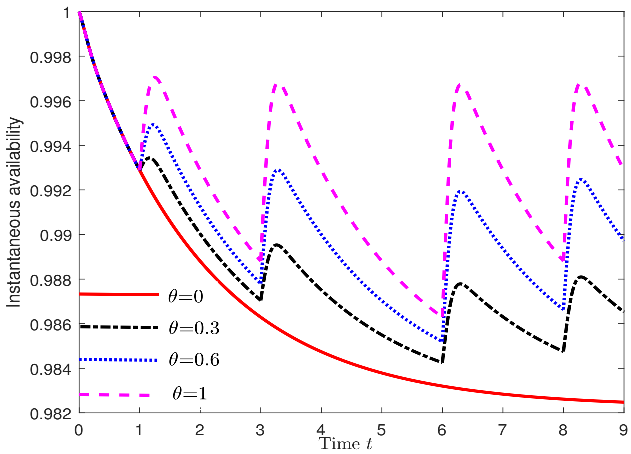

Figure 4. Instantaneous availability with different impulse strengths θ.

The cross-section $A(t) = 0.995$ of the three-dimensional graph for time t (impulse action occurred at $t_1 = 1,\ t_2 = 3,\ t_3 = 6,\ t_4 = 8$), impulse strength θ and $A(t)$ is depicted in Figure 5. In addition, the region of $A(t) \geq 0.995$ is also given in Figure 6, which shows that $(t,\ \theta)$ in the shaded area, whereas in the non-shaded area, the availability cannot reach 0.995 regardless of the intensity of the impulse. Therefore, we can guarantee the instantaneous availability $A(t)$ being greater than a certain value by choosing suitable t and θ.

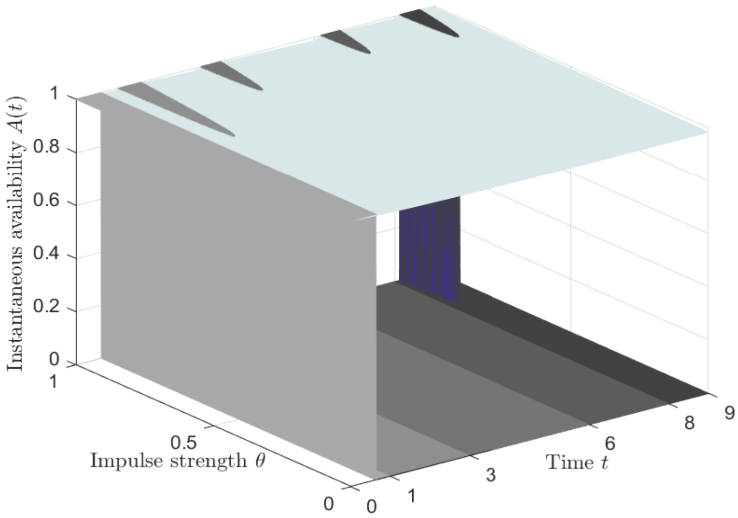

Figure 5. Cross-section of $A(t) = 0.995$.

Remark 6. *Although all the findings of the numerical examples section are based on the experimental data* $\alpha_0 = \frac{1}{7}$, $\alpha_1 = \frac{1}{120}$, $\beta = \frac{1}{3}$ *of a virtual machine, we can simulate the dynamical solution and instantaneous availability of software systems for any given transition rates* $\alpha_0,\ \alpha_1,\ \beta$, *which is consistent with Theorems 2 and 4 and Remark 5.*

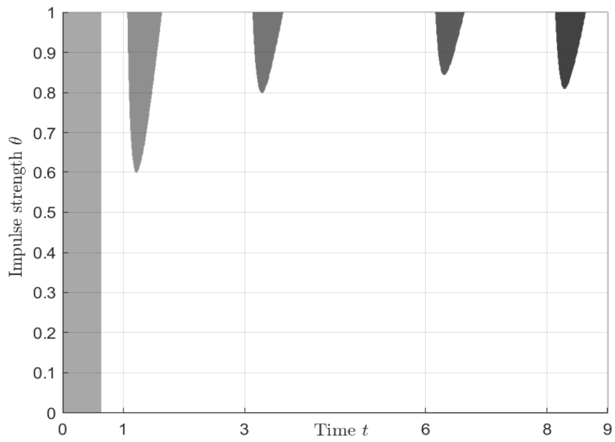

Figure 6. Projection of $A(t) \geq 0.995$.

5. Conclusions

The main contributions of this paper were obtaining the dynamical solution and instantaneous availability of software systems with aperiodic impulse rejuvenation. Firstly, the system was formulated by a group of coupled impulsive differential equations and transformed into an abstract Cauchy problem. Then we proved the existence and uniqueness of the system's dynamical solution by means of semigroup theory. To investigate the numerical approximation of the system, we constructed an approximation system and derived the dynamical solution of the software system with aperiodic impulse rejuvenation by using Trotter–Kato theorem. Moreover, the expression of the system's instantaneous availability was given, and the range of the system's instantaneous availability was estimated by applying the upper and lower bounds of the impulsive intervals method. In the numerical examples, we depicted the system dynamic behavior, estimated its instantaneous availability and then provided the region of time t and impulse strength θ that ensures instantaneous availability greater than a certain value.

Author Contributions: Conceptualization, methodology and funding acquisition, H.X.; writing—original draft, H.H.; writing—review and editing, H.H., H.X. and Z.C.; supervision and validation, H.X. and Z.C. All authors have read and agreed to the published version of the manuscript.

Funding: This work is supported by the National Natural Science Foundation of China (NSAF Grant No. U1430125).

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Acknowledgments: The authors express their thanks to the handling editor and the reviewers for their careful reading and constructive suggestions.

Conflicts of Interest: The authors declare no conflict of interest.

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