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A Note on the Abelian Complexity of the Rudin-Shapiro Sequence

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Abstract: Let $\{r(n)\}_{n\geq 0}$ be the Rudin-Shapiro sequence, and let $\rho(n) := \max\{\sum_{j=i}^{i+n-1} r(j) \mid i \geq 0\} + 1$ be the abelian complexity function of the Rudin-Shapiro sequence. In this note, we show that the function $\rho(n)$ has many similarities with the classical summatory function $S_{\mathbf{r}}(n) := \sum_{i=0}^{n} r(i)$. In particular, we prove that for every positive integer n, $\sqrt{3} \leq \frac{\rho(n)}{\sqrt{n}} \leq 3$. Moreover, the point set $\{\frac{\rho(n)}{\sqrt{n}} : n \geq 1\}$ is dense in $[\sqrt{3}, 3]$.

Keywords: Rudin-Shapiro sequence; abelian complexity; growth order; dense property

1. Introduction

In this note, we are concerned with the abelian complexity $\rho(n) = \max\{\sum_{j=i}^{i+n-1} r(j) \mid i \geq 0\} + 1$ of the Rudin-Shapiro sequence $\mathbf{r}$. The Rudin-Shapiro sequence $\mathbf{r} = r(0)r(1)\cdots \in \{-1,1\}^{\mathbb{N}}$ is given by the following recurrence relations:

$$r(0) = 1,\ r(2n) = r(n),\ r(2n+1) = (-1)^n r(n)\ (n \geq 0).$$

The Rudin-Shapiro sequence $\mathbf{r}$ is a typical 2-automatic sequence [1]. It has been proved in [2] that the sequence $\rho(n)$ satisfies $\rho(1) = 2$, $\rho(2) = 3$, $\rho(3) = 4$ and for every $n \geq 1$,

$$\rho(4n) = 2\rho(n)+1,\ \rho(4n+1) = 2\rho(n),\ \rho(4n+2) = \rho(n)+\rho(n+1),\ \rho(4n+3) = 2\rho(n+1).$$

Let $\mathbf{w} = w(0)w(1)w(2)\cdots$ be an infinite sequence with $w(i) \in \mathbb{Z}$ for every $i \geq 0$. There are many papers focusing on the summatory function $S_{\mathbf{w}}(n) := \sum_{j=0}^{n} w(j)$. In [3–5], Brillhart and Morton studied the summatory function $S_{\mathbf{r}}(n) := \sum_{i=0}^{n} r(i)$ of the Rudin-Shapiro sequence. The sequence $S_{\mathbf{r}}(n)$ satisfies $S_{\mathbf{r}}(0) = 1$, $S_{\mathbf{r}}(1) = 2$, $S_{\mathbf{r}}(2) = 3$, $S_{\mathbf{r}}(3) = 2$ and for every $n \geq 1$,

$$S_{\mathbf{r}}(4n) = 2S_{\mathbf{r}}(n) + r(n),\ S_{\mathbf{r}}(4n+1) = S_{\mathbf{r}}(4n+3) = 2S_{\mathbf{r}}(n),\ S_{\mathbf{r}}(4n+2) = 2S_{\mathbf{r}}(n) + (-1)^n r(n).$$

In detail, Brillhart and Morton proved that for every $n \geq 1$,

$$\sqrt{3/5} \leq \frac{S_{\mathbf{r}}(n)}{\sqrt{n}} \leq \sqrt{6},$$

and $\{\frac{S_{\mathbf{r}}(n)}{\sqrt{n}} : n \geq 1\}$ is dense in $[\sqrt{3/5}, \sqrt{6}]$. In [6], Lafrance, Rampersad and Yee introduced a Rudin-Shapiro-like sequence $(l(n))_{n\geq 0}$ which satisfies $l(0) = 1$ and for every $n \geq 0$,

$$l(4n) = l(n),\ l(4n+1) = l(2n),\ l(4n+2) = -l(2n) \text{ and } l(4n+3) = l(n).$$

They studied the properties of the summatory function $S_{\mathbf{l}}(N) := \sum_{n=0}^{N} l(n)$. The sequence $S_{\mathbf{l}}(N)$ satisfies $S_{\mathbf{l}}(0) = 1$ and for every $m \geq 0$,

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$$S_{\mathbf{1}}(4m) = 2S_{\mathbf{1}}(m) - l(m) = S_{\mathbf{1}}(4m+2),\ S_{\mathbf{1}}(4m+1) = 2S_{\mathbf{1}}(m) - l(m) + l(2m),\ S_{\mathbf{1}}(4m+3) = 2S_{\mathbf{1}}(m).$$

Moreover, Lafrance, Rampersad and Yee showed that

$$\limsup_{n\to+\infty} \frac{S_{\mathbf{1}}(n)}{\sqrt{n}} = \sqrt{2},\ \liminf_{n\to+\infty} \frac{S_{\mathbf{1}}(n)}{\sqrt{n}} = \frac{\sqrt{3}}{3}.$$

The sequences $S_{\mathbf{r}}(n)$ and $S_{\mathbf{1}}(n)$ are both 2-regular sequences (in the sense of Allouche and Shallit [1]). For the definition and properties of k-regular sequences, one can refer to [1]. Let $(s(n))_{n\geq 0}$ be a k-regular sequence over $\mathbb{Z}$. It was proved in [1] that there exists a constant c such that $s(n) = O(n^c)$. In general, it is a difficult task to compute the exact growth order of sequences satisfying certain recursive relations such as k-regular sequences.

In [7], Gawron and Ulas obtained the sequence $\{a(n) \mid n \in \mathbb{N}\} := \{m \in \mathbb{N} \mid c(m) = 1\}$ where $(c(n))_{n\geq 0}$ is the sequence of coefficients of the compositional inverse of the generating function of the Thue-Morse sequence. The sequence $(a(n))_{n\geq 0}$ satisfies that $a(0) = 0, a(1) = 1, a(2) = 2, a(3) = 7$ and for all $n \geq 1$, $a(4n+i) = a(4n-1) + i + 1$ with $i \in [0,2]$, $a(8n+3) = a(8n) + 7$ and $a(8n+7) = 4a(4n+3) + 3$. They proved that

$$\liminf_{n\to+\infty} \frac{a(n)}{n^2} = \frac{1}{6},\ \limsup_{n\to+\infty} \frac{a(n)}{n^2} = \frac{1}{2}$$

and $\{\frac{a(n)}{n^2} : n \geq 1\}$ is dense in $[\frac{1}{6}, \frac{1}{2}]$. In [2], Chen, Wen, Wu and the first author studied the maximal digit sum sequence $M_{\mathbf{r}}(n) := \max\{\sum_{j=i}^{i+n-1} r(j) \mid i \geq 0\}$ and proved that the abelian complexity $\rho(n)$ of the Rudin-Shapiro sequence satisfies $\rho(n) = M_{\mathbf{r}}(n) + 1$ for every $n \geq 1$. It is remarkable that the authors in [2] just gave the recursive formulas for the sequence $M_{\mathbf{r}}(n)$ and proved the 2-regularity of the sequence $\rho(n)$. It is natural to ask whether the function $\rho(n)$ has similar properties as the summatory function $S_{\mathbf{r}}(n)$. In fact, it is of great interest to study the properties of sequences which satisfy certain recursive formulas.

This note focuses on the growth order of the abelian complexity $(\rho(n))_{n\geq 1}$ of the Rudin-Shapiro sequence $\mathbf{r}$. Firstly, by studying the maximal and minimal values of the function $\rho(n)$ in the interval $I_k := [4^k, 4^{k+1} - 1]$ with $k \geq 0$, we got $\rho(n) = \Theta(\sqrt{n})$. Then, we investigated two functions $s(k) := \min\{n \mid \rho(n) = k\}$ and $\ell(k) := \max\{n \mid \rho(n) = k\}$, and obtained the optimal lower and upper bound of the sequence $(\frac{\rho(n)}{\sqrt{n}})_{n\geq 1}$. Finally, we showed that $\rho(n)$ is a quasi-linear function for 4. As a consequence, the set $\{\frac{\rho(n)}{\sqrt{n}} : n \geq 1\}$ is dense between its optimal lower bound and upper bound. In detail, we proved the following theorems.

Theorem 1. *For every integer $n \geq 1$, we have*

$$\sqrt{3} \leq \frac{\rho(n)}{\sqrt{n}} \leq 3.$$

Theorem 2. *The set $\{\rho(n)/\sqrt{n} : n \geq 1\}$ is dense in $[\sqrt{3}, 3]$.*

The outline of this note is as follows. In Section 2, we compute the maximal and minimal values of the function $\rho(n)$ in the interval $I_k := [4^k, 4^{k+1} - 1]$ for every $k \geq 0$. In Section 3, we give the proofs of Theorem 1 and Theorem 2.

2. Basic Properties of the Function $\rho(n)$

In this section, we exhibit some basic properties of the abelian complexity function $\rho(n)$ of the Rudin-Shapiro sequence $\mathbf{r}$.

Following from ([2] Theorem 1 and Lemma 3), the abelian complexity function $\rho(n)$ of the Rudin-Shapiro sequence is given by the following formulas: $\rho(1) = 2$, $\rho(2) = 3$, $\rho(3) = 4$ and for every integer $n \geq 1$,

$$\rho(4n) = 2\rho(n) + 1, \qquad \rho(4n+1) = 2\rho(n),$$
$$\rho(4n+2) = \rho(n) + \rho(n+1), \qquad \rho(4n+3) = 2\rho(n+1). \tag{1}$$

Set $\rho(0) := 1$. For every integer $n \geq 0$, let $\Delta\rho(n) := \rho(n+1) - \rho(n)$. Then $\Delta\rho(0) = \Delta\rho(1) = \Delta\rho(2) = \Delta\rho(3) = 1$, and for every integer $n \geq 1$,

$$\Delta\rho(4n) = -1, \ \ \Delta\rho(4n+3) = 1,$$
$$\Delta\rho(4n+1) = \Delta\rho(4n+2) = \Delta\rho(n). \tag{2}$$

This implies that $\Delta\rho(n) \in \{-1, 1\}$ for every integer $n \geq 0$. The first 16 terms of $\rho(n)$, starting with $n = 1$, are listed in Table 1.

Table 1. The first 16 terms of the sequence $\rho(n)$.

n	$\rho(n)$	n	$\rho(n)$
1	2	9	6
2	3	10	7
3	4	11	8
4	5	12	9
5	4	13	8
6	5	14	9
7	6	15	10
8	7	16	11

For simplicity of notation, for every integer $k \geq 0$, put $m_k := \frac{4^{k+1}-1}{3}$, $M_k := 4^{k+1} - 1$ and $I_k := [4^k, 4^{k+1} - 1]$. Then we have the following two lemmas which give the minimal and maximal values of the function $\rho(n)$ in the interval I_k for every $k \geq 0$.

Lemma 1. *For every integer $k \geq 0$, the minimum value of $\rho(n)$ in $I_k = [4^k, 4^{k+1} - 1]$ is 2^{k+1}. Moreover,*

$$\max\left\{n \in I_k : \rho(n) = 2^{k+1}\right\} = m_k.$$

Proof. We will prove this by induction on the variable k. For $k = 0$, it follows from Table 1 that this assertion is true. Assume the assertion holds for the interval I_k.

We first show that 2^{k+2} is the lower bound for $\rho(n)$ in I_{k+1}. If n lies in $I_{k+1} = [4^{k+1}, 4^{k+2} - 1]$, then we can write $n = 4m + d$ for some $m \in I_k$ and some $d \in \{0, 1, 2, 3\}$. There are two cases to be considered.

1. When $4^k \leq m \leq 4^{k+1} - 2$, (1) yields that for every $d \in \{0, 1, 2, 3\}$

$$\rho(n) = \rho(4m + d) \geq 2\min\{\rho(m) : m \in I_k\} = 2^{k+2}.$$

   The last equality is true under the inductive assumption.

2. When $m = 4^{k+1} - 1$, it follows from (2) that $\Delta(m) = 1$, which implies

$$\rho(m + 1) = \rho(4^{k+1} - 1) + 1 = \rho(m) + 1.$$

   Hence, for every $d \in \{0, 1, 2, 3\}$, using (1) again, we have

$$\rho(n) = \rho(4m + d) \geq 2\rho(m) > 2\min\{\rho(m) : m \in I_k\} = 2^{k+2}. \tag{3}$$

At the same time, using the fact $m_{k+1} = 4m_k + 1$, it is easy to check that

$$\rho(m_{k+1}) = \rho(4m_k + 1) = 2\rho(m_k) = 2^{k+2}.$$

Now it suffices to show that

$$m_{k+1} = \max\left\{n \in I_{k+1} : \rho(n) = 2^{k+2}\right\}.$$

Following from the inductive assumption, for every $m \in I_k$ satisfying $m > m_k$, we have $\rho(m) \geq \rho(m_k) + 1$. By (1), we can get

$$\begin{aligned}\rho(m_{k+1} + 1) &= \rho(4m_k + 2) = \rho(m_k) + \rho(m_k + 1) \geq 2^{k+2} + 1, \\ \rho(m_{k+1} + 2) &= \rho(4m_k + 3) = 2\rho(m_k + 1) \geq 2^{k+2} + 2.\end{aligned}$$

Now we only need to consider the case $n = 4m + d \geq 4m_k + 4$ with $d \in \{0, 1, 2, 3\}$. In fact, for every $m_k + 1 \leq m \leq 4^{k+1} - 2$ and $d \in \{0, 1, 2, 3\}$, it follows from (1) that

$$\rho(4m + d) \geq 2 \min\{\rho(m) : m_k + 1 \leq m \leq 4^{k+1} - 1\} \geq 2^{k+2} + 2.$$

By (3), the case $n = 4m + d$ for $m = 4^{k+1} - 1$ holds, which completes the proof. □

Lemma 2. *Let k be a non-negative integer. The maximum value of $\rho(n)$ in $I_k = [4^k, 4^{k+1} - 1]$ is $3 \cdot 2^{k+1} - 2$ and this value occurs only at the point $n = M_k = 4^{k+1} - 1$.*

Proof. We will prove this by induction on the variable k. For $k = 0$, this assertion holds following from Table 1. Assume the assertion is true for the interval I_k. When n lies in I_{k+1}, let $n = 4m + d$ for some $m \in I_k$ and $d \in \{0, 1, 2, 3\}$. Similarly with the proof of Lemma 1, we divide it into two cases.

1. When $4^k \leq m < 4^{k+1} - 1$. By (1) and the inductive assumption, we have

$$\rho(n) = \rho(4m + d) \leq 2 \max\{\rho(m) : m \in I_k\} + 1 < 3 \cdot 2^{k+2} - 2. \quad (4)$$

2. When $m = M_k = 4^{k+1} - 1$. Following from (1) and (2), we have

$$\begin{aligned}\rho(4M_k) &= 2\rho(M_k) + 1 = 3 \cdot 2^{k+2} - 3, \\ \rho(4M_k + 1) &= 2\rho(M_k) = 3 \cdot 2^{k+2} - 4, \\ \rho(4M_k + 2) &= \rho(M_k) + \rho(M_k + 1) = 3 \cdot 2^{k+2} - 3, \\ \rho(4M_k + 3) &= 2\rho(M_k + 1) = 3 \cdot 2^{k+2} - 2.\end{aligned} \quad (5)$$

This implies that $\rho(n) \leq \rho(4M_k + 3) = \rho(4^{k+2} - 1) = 3 \cdot 2^{k+2} - 2$.

Following from (4) and (5), we can obtain that $M_{k+1} = 4M_k + 3$ is the unique point in I_{k+1} which attains the maximal value of ρ in the interval I_{k+1}. This completes the proof. □

Remark 1. *From Lemma 1, we have that for every $n \geq 1$,*

$$\rho(n) \geq 2.$$

Remark 2. *If $n \in I_k = [4^k, 4^{k+1} - 1]$, Lemma 1 gives us*

$$\frac{\rho(n)}{\sqrt{n}} > \frac{2^{k+1}}{\sqrt{4^{k+1}}} = 1,$$

while Lemma 2 implies that

$$\frac{\rho(n)}{\sqrt{n}} < \frac{3 \cdot 2^{k+1} - 2}{\sqrt{4^k}} < 6.$$

Thus, for every integer $n \geq 1$, $1 \leq \frac{\rho(n)}{\sqrt{n}} \leq 6$, and so $\rho(n)$ is roughly a constant times $\sqrt{n}$. However, these bounds are not optimal. Note that $\rho(n) \geq 2$ for every $n \geq 1$. It is easy to verify that

$$\lim_{k\to\infty} \frac{\rho(M_k)}{\sqrt{M_k}} = \lim_{k\to\infty} \frac{3 \cdot 2^{k+1} - 2}{\sqrt{4^{k+1} - 1}} = 3$$

and

$$\lim_{k\to\infty} \frac{\rho(m_k)}{\sqrt{m_k}} = \lim_{k\to\infty} \frac{2^{k+1}}{\sqrt{\frac{4^{k+1}-1}{3}}} = \sqrt{3}.$$

In other words, 3 and $\sqrt{3}$ are two accumulation points of the set $\{\frac{\rho(n)}{\sqrt{n}} : n \geq 1\}$. In the following section, we will prove that 3 and $\sqrt{3}$ are the optimal upper and lower bound for the sequence $(\frac{\rho(n)}{\sqrt{n}})_{n\geq 1}$ respectively.

3. Proofs of Theorems 1 and 2

Following that M_k and m_k both go to infinity with k tending to infinity (by Lemmas 1 and 2), we can see that there are only finite number of places n such that $\rho(n)$ has a fixed value k. When $\rho(n) = k$, for a fixed k, the ratio $\rho(n)/\sqrt{n}$ will be the smallest when n is largest while it will be largest if n is smallest. This leads us to the following idea: for a fixed $k \in \mathbb{N}$ with $k \geq 1$, let us focus on the smallest and largest values of n such that $\rho(n) = k$. For this purpose, we introduced two auxiliary functions $s(k)$ and $\ell(k)$.

Definition 1. *Given an integer $k \geq 1$, let $s(k)$ and $\ell(k)$ be the smallest and largest values of n such that $\rho(n) = k$ respectively, i.e.,*

$$s(k) := \min\{n : \rho(n) = k\},$$
$$\ell(k) := \max\{n : \rho(n) = k\}.$$

Following from (1), Table 1 and $\rho(0) = 1$, the initial 8 terms of the sequences $s(k)$ and $\ell(k)$ are given in Table 2.

Table 2. The initial values for the sequences $s(k)$ and $\ell(k)$.

k	1	2	3	4	5	6	7	8
$s(k)$	0	1	2	3	4	7	8	11
$\ell(k)$	0	1	2	5	6	9	10	21

For the sequences $s(k)$ and $\ell(k)$, we have the following results.

Lemma 3. *The sequence $\ell(k)$ satisfies $\ell(1) = 0$ and for every integer $k \geq 1$,*

$$\ell(2k) = 4\ell(k) + 1, \quad \ell(2k+1) = 4\ell(k) + 2.$$

Proof. The assertion holds for $k = 1$ by Table 2. Fix some integer $k \geq 2$, assume $\ell(k) = n$. Then $\rho(n) = k$. By the definition of $\ell(n)$ and (2), for every integer $m > n$, $\rho(m) \geq \rho(n) + 1 = k + 1$ and

$$\rho(n+1) = \rho(n) + 1 = k + 1.$$

Therefore, for every $m > n$ and $d \in \{0, 1, 2, 3\}$,

$$\rho(4m + d) \geq 2\min\{\rho(m) : m > n\} = 2k + 2.$$

At the same time, when $m = n$, it is obvious that

$$\begin{aligned} \rho(4n) &= 2\rho(n) + 1 = 2k + 1, \\ \rho(4n+1) &= 2\rho(n) = 2k, \\ \rho(4n+2) &= \rho(n) + \rho(n+1) = 2k + 1, \\ \rho(4n+3) &= 2\rho(n+1) = 2k + 2. \end{aligned}$$

This implies that $\ell(2k) = 4n + 1 = 4\ell(k) + 1$ and $\ell(2k+1) = 4n + 2 = 4\ell(k) + 2$. □

Lemma 4. *The sequence $s(k)$ satisfies $s(1) = 0$, $s(2) = 1$, $s(3) = 2$ and for every $k \geq 2$,*

$$s(2k) = 4s(k) - 1, \quad s(2k+1) = 4s(k).$$

Proof. The initial values $s(1)$, $s(2)$ and $s(3)$ can be easily verified by Table 2. For an integer $k \geq 2$, suppose $s(k) = n$ for some integer n. Then $\rho(n) = k$. By the definition of $s(k)$ and (2), we have that $\rho(m) < \rho(n)$ whenever $m < n$ and

$$\rho(n-1) = \rho(n) - 1 = k - 1.$$

Therefore, following from (1), for every $0 \leq m < n - 1$ and $d \in \{0, 1, 2, 3\}$, we have

$$\rho(4m + d) \leq 2 \max\{\rho(m) : \ 0 \leq m \leq n - 1\} + 1 = 2k - 1.$$

Note that the value of $4m + d$ ranges from 0 to $4n - 5$. At the same time, it follows from (1) that

$$\begin{aligned} \rho(4n-4) &= 2\rho(n-1) + 1 = 2k - 1, \\ \rho(4n-3) &= 2\rho(n-1) = 2k - 2, \\ \rho(4n-2) &= \rho(n-1) + \rho(n) = 2k - 1, \\ \rho(4n-1) &= 2\rho(n) = 2k. \end{aligned}$$

This implies that $s(2k) = 4n - 1 = 4s(k) - 1$. The other formula $s(2k+1) = 4s(k)$ follows from the fact that $\rho(4n) = 2\rho(n) + 1 = 2k + 1$ and $\rho(i) \leq 2k$ for every $0 \leq i \leq 4n - 1$. This ends the proof. □

The following two propositions show the upper bound for the sequence $\ell(k)$ and the lower bound for the sequence $s(k)$. For the sake of simplicity, for every integer $k \geq 2$, let

$$k = \sum_{j=0}^{m} k_j 2^j := [k_m k_{m-1} \cdots k_0]_2$$

be the binary expansion of k with $m \geq 1$ and $k_m = 1$. For every $x \geq 0$, let $\lfloor x \rfloor$ be the greatest integer which is no more than x.

Proposition 1. *For every integer $k \geq 2$, we have*

$$\ell(k) \leq \frac{k^2}{3}.$$

Proof. For every $k \geq 2$, let the binary expansion of k be $[k_m k_{m-1} \cdots k_0]_2$. Following from Lemma 3, we have

$$\ell(k) = \ell([k_m k_{m-1} \cdots k_0]_2) = 4\ell([k_m k_{m-1} \cdots k_1]_2) + k_0 + 1. \tag{6}$$

Now we apply (6) by replacing k with $[k_m k_{m-1} \cdots k_1]_2$, which yields

$$\ell([k_m k_{m-1} \cdots k_1]_2) = 4\ell([k_m k_{m-1} \cdots k_2]_2) + k_1 + 1.$$

Repeating this progress m times, by the fact that $k_j = \lfloor \frac{k}{2^j} \rfloor - 2\lfloor \frac{k}{2^{j+1}} \rfloor$ for every $0 \le j \le m-1$, we can obtain that

$$\begin{aligned}
\ell(k) &= \ell([k_m k_{m-1} \cdots k_0]_2) \\
&= 4\ell([k_m k_{m-1} \cdots k_1]_2) + k_0 + 1 \\
&= 4^2 \ell([k_m k_{m-1} \cdots k_2]_2) + 4(k_1 + 1) + k_0 + 1 \\
&= \cdots\cdots \\
&= 4^m \ell(k_m) + \sum_{j=0}^{m-1} 4^j (k_j + 1) \\
&= 4^m \ell(1) + \sum_{j=0}^{m-1} 4^j (\lfloor \frac{k}{2^j} \rfloor - 2\lfloor \frac{k}{2^{j+1}} \rfloor + 1) \\
&= \frac{4^m - 1}{3} + \sum_{j=0}^{m-1} 4^j (\lfloor \frac{k}{2^j} \rfloor - 2\lfloor \frac{k}{2^{j+1}} \rfloor) \\
&\le \frac{4^m}{3} + k - 2\lfloor \frac{k}{2} \rfloor + 4\lfloor \frac{k}{2} \rfloor - 8\lfloor \frac{k}{2^2} \rfloor + \cdots + 4^{m-1}\lfloor \frac{k}{2^{m-1}} \rfloor - 2 \cdot 4^{m-1} \lfloor \frac{k}{2^m} \rfloor \\
&= \frac{4^m}{3} - 2 \cdot 4^{m-1} + k + 2\sum_{j=1}^{m-1} 4^{j-1} \lfloor \frac{k}{2^j} \rfloor \\
&\le \frac{4^m}{3} - 2 \cdot 4^{m-1} + k + 2\sum_{j=1}^{m-1} 4^{j-1} \frac{k}{2^j} \\
&= 2^{m-1} k - \frac{2}{3} \cdot 4^{m-1}.
\end{aligned}$$

It suffices to show that

$$2^{m-1} k - \frac{2}{3} \cdot 4^{m-1} \le \frac{k^2}{3}.$$

Note that $2^m \le k < 2^{m+1}$. This implies that

$$\frac{k}{4} < 2^{m-1} \le \frac{k}{2}.$$

Consider the function $f_k(x) = kx - \frac{2}{3}x^2$. It is not hard to check that $f_k(x)$ is strictly increasing on the interval $(\frac{k}{4}, \frac{k}{2}]$ with fixed $k \ge 1$. Hence

$$kx - \frac{2}{3}x^2 \le f_k\left(\frac{k}{2}\right) = \frac{1}{3}k^2,$$

which is the desired result. □

Proposition 2. *For every integer $k \ge 2$, we have*

$$s(k) \ge \frac{k^2}{9}.$$

Proof. The assertion holds when $k = 2$ and $k = 3$ since $s(2) = 1 > 4/9$ and $s(3) = 2 > 1$. For every integer $k \ge 4$, let the binary expansion of k be $[k_m k_{m-1} \cdots k_0]_2$ with $m \ge 2$ and $k_m = 1$. Following from Lemma 4, we have

$$s(k) = s([k_m k_{m-1} \cdots k_0]_2) = 4s([k_m k_{m-1} \cdots k_1]_2) + k_0 - 1.$$

Arguing analogously as in the proof of Proposition 1, we have

$$\begin{aligned}
s(k) &= s([k_m k_{m-1} \cdots k_0]_2) \\
&= 4^{m-1} s([k_m k_{m-1}]_2) + \sum_{j=0}^{m-2} 4^j (k_j - 1) \\
&= 4^{m-1} s(2 + k_{m-1}) + \sum_{j=0}^{m-2} 4^j (\lfloor \frac{k}{2^j} \rfloor - 2\lfloor \frac{k}{2^{j+1}} \rfloor - 1) \\
&= 4^{m-1} s(2 + k_{m-1}) - \frac{4^{m-1} - 1}{3} + \sum_{j=0}^{m-2} 4^j (\lfloor \frac{k}{2^j} \rfloor - 2\lfloor \frac{k}{2^{j+1}} \rfloor) \\
&= 4^{m-1} s(2 + k_{m-1}) - \frac{4^{m-1} - 1}{3} - 2 \cdot 4^{m-2} \lfloor \frac{k}{2^{m-1}} \rfloor + k + 2 \sum_{j=1}^{m-2} 4^{j-1} \lfloor \frac{k}{2^j} \rfloor \\
&\geq 4^{m-1} s(2 + k_{m-1}) - \frac{4^{m-1}}{3} - 2 \cdot 4^{m-2} \lfloor \frac{k}{2^{m-1}} \rfloor + k + 2 \sum_{j=1}^{m-2} 4^{j-1} (\frac{k}{2^j} - 1) \\
&\geq 4^{m-1} s(2 + k_{m-1}) - 2 \cdot 4^{m-2} (3 + k_{m-1}) + 2^{m-2} k \\
&= \begin{cases} 2^{m-2} k - 2 \cdot 4^{m-2} & \text{if } k_{m-1} = 0, \\ 2^{m-2} k & \text{if } k_{m-1} = 1. \end{cases}
\end{aligned}$$

The last equality uses the fact that $s(2) = 1$ and $s(3) = 2$.

1. If $k_{m-1} = 0$, then we have $2^m \leq k < 3 \cdot 2^{m-1}$, and it follows that

$$\frac{k}{6} < 2^{m-2} \leq \frac{k}{4}.$$

Put $g_k(x) := xk - 2x^2$. It is obvious that $g_k(x)$ is strictly increasing on the interval $(\frac{k}{6}, \frac{k}{4}]$ with fixed k. Hence

$$2^{m-2} k - 2 \cdot 4^{m-2} \geq g_k\left(\frac{k}{6}\right) = \frac{k^2}{9}.$$

2. If $k_{m-1} = 1$, then we have $3 \cdot 2^{m-1} \leq k < 2^{m+1}$, and it follows that

$$\frac{k}{8} < 2^{m-2} \leq \frac{k}{6},$$

which implies

$$2^{m-2} k \geq \frac{k^2}{8} \geq \frac{k^2}{9}.$$

This ends the proof. □

Now, combining Proposition 1 and Proposition 2, we can prove Theorem 1.

Proof of Theorem 1. For every integer $n \geq 1$, assume $\rho(n) = k$ for some integer k. It follows from Remark 1 that $k \geq 2$. By Proposition 1 and the definition of $\ell(k)$, we have

$$\frac{\rho(n)}{\sqrt{n}} \geq \frac{k}{\sqrt{\ell(k)}} \geq \sqrt{3}.$$

Following from Proposition 2 and the definition of $s(k)$, we have

$$\frac{\rho(n)}{\sqrt{n}} \leq \frac{k}{\sqrt{s(k)}} \leq 3.$$

This completes the proof. □

By Remark 2 and Theorem 1, we get the following corollary.

Corollary 1. *The numbers* $\sqrt{3}$ *and* 3 *are the optimal lower and upper bounds for the set* $\{\frac{\rho(n)}{\sqrt{n}} : n \geq 1\}$ *respectively.*

To prove Theorem 2, we need an auxiliary notation which was firstly introduced in [8]. Let $b \geq 2$ be an integer and $f : \mathbb{N} \to \mathbb{Z}$ be a discrete function (or an integer sequence). For every $x \geq 0$, let

$$\delta_f(x) := \limsup_{n \to +\infty} \frac{|f(n)|}{n^x}.$$

Write $\Delta f(n) := f(n+1) - f(n)$. Set

$$\alpha = \alpha(f) := \inf\{x \geq 0 \mid \delta_f(x) = 0\} \text{ and } \beta = \beta(f) := \inf\{x \geq 0 \mid \delta_{\Delta f}(x) = 0\}.$$

Definition 2. *(Quasi-linear function) Let* $f : \mathbb{N} \to \mathbb{Z}$. *If* $\alpha(f) > \beta(f)$ *and there exists a constant* $C_1 > 0$ *and an integer* $b \geq 2$ *such that for all positive integers* n *and* $0 \leq i \leq b-1$,

$$|f(bn+i) - b^\alpha f(n)| \leq C_1 n^\beta, \tag{7}$$

then we call f *a quasi-linear function for* b.

For the quasi-linear functions, we have the following lemma. For more details about this, see [8].

Lemma 5 ([8])**.** *Given an integer function* $f(n)$, *set* $\alpha := \alpha(f)$. *Suppose* a_1 *and* a_2 *are two accumulation points of* $\{f(n)/n^\alpha : n \geq 1\}$. *If* f *is a quasi-linear function, then* $\{f(n)/n^\alpha : n \geq 1\}$ *is dense in* $[a_1, a_2]$.

Proof of Theorem 2. Following from the fact that $\Delta\rho(n) \in \{-1, 1\}$ for every $n \geq 0$, we have $\beta(\rho) = 0$. By Theorem 1, $\alpha(\rho) = \frac{1}{2}$. Moreover, for every $0 \leq i \leq 3$ and $n \geq 1$, (1) yields

$$|\rho(4n+i) - 2\rho(n)| \leq 2|\rho(n+1) - \rho(n)| = 2.$$

Therefore $\rho(n)$ is a quasi-linear function for $b = 4$.

It follows from Remark 2 that $\sqrt{3}$ and 3 are two accumulation points of the set $\{\rho(n)/\sqrt{n} : n \geq 1\}$. Hence we obtained the desired result by Lemma 5. □

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References

1. Allouche, J.-P.; Shallit, J. *Automatic Sequences: Theory, Applications, Generalizations*; Cambridge University Press: Cambridge, UK, 2003.
2. Lü, X.-T.; Chen, J.; Wen, Z.-X.; Wu, W. On the abelian complexity of the Rudin-Shapiro sequences. *J. Math. Anal. Appl.* **2017**, *451*, 822–838. [CrossRef]
3. Brillhart, J.; Morton, P. Uber Summen von Rudin-Shapiroschen Koeffizienten. *Ill. J. Math.* **1978**, *22*, 126–148. [CrossRef]
4. Brillhart, J.; Morton, P. A case study in mathematical research: The Golay-Rudin-Shapiro sequence. *Am. Math. Mon.* **1996**, *130*, 854–869. [CrossRef]
5. Brillhart, J. ; Erdős, P.; Morton, P. On sums of Rudin–Shapiro coefficients II. *Pac. J. Math.* **1983**, *107*, 39–69. [CrossRef]
6. Lafrance, P.; Rampersad, N.; Yee, R. Some properties of a Rudin–Shapiro-like sequence. *Adv. Appl. Math.* **2015**, *63*, 19–40. [CrossRef]
7. Gawron, M.; Ulas, M. On the formal inverse of the Prouhet-Thue-Morse sequence. *Discret. Math.* **2016**, *339*, 1459–1470. [CrossRef]
8. Lü, X.-T.; Chen, J.; Wen, Z.-X.; Wu, W. Limit behavior of the quasi-linear discrete functions. *Fractals* **2020**, *28*, 2050041. [CrossRef]