mathematics

MDPI

Review

Finite Element Method-Based Elastic Analysis of Multibody Systems: A Review

Sorin Vlase [1,2,*], Marin Marin [3,*] and Negrean Iuliu [2,4]

[1] Department of Mechanical Engineering, Transilvania University of Brașov, 500036 Brașov, Romania
[2] Romanian Academy of Technical Sciences, B-dul Dacia 26, 030167 Bucharest, Romania; iuliu.negrean@mep.utcluj.ro
[3] Department of Mathematics and Informatics, Transilvania University of Brașov, B-dul Eroilor 20, 500036 Brașov, Romania
[4] Department of Mechanical Systems Engineering, Faculty of Machine Building, Technical University of Cluj-Napoca, 400641 Cluj-Napoca, Romania
* Correspondence: svlase@unitbv.ro (S.V.); m.marin@unitbv.ro (M.M.)

Abstract: This paper presents the main analytical methods, in the context of current developments in the study of complex multibody systems, to obtain evolution equations for a multibody system with deformable elements. The method used for analysis is the finite element method. To write the equations of motion, the most used methods are presented, namely the Lagrange equations method, the Gibbs–Appell equations, Maggi's formalism and Hamilton's equations. While the method of Lagrange's equations is well documented, other methods have only begun to show their potential in recent times, when complex technical applications have revealed some of their advantages. This paper aims to present, in parallel, all these methods, which are more often used together with some of their engineering applications. The main advantages and disadvantages are comparatively presented. For a mechanical system that has certain peculiarities, it is possible that the alternative methods offered by analytical mechanics such as Lagrange's equations have some advantages. These advantages can lead to computer time savings for concrete engineering applications. All these methods are alternative ways to obtain the equations of motion and response time of the studied systems. The difference between them consists only in the way of describing the systems and the application of the fundamental theorems of mechanics. However, this difference can be used to save time in modeling and analyzing systems, which is important in designing current engineering complex systems. The specifics of the analyzed mechanical system can guide us to use one of the methods presented in order to benefit from the advantages offered.

Keywords: Maggi's equations; Lagrange method; Gibbs–Appell equations; Hamilton formalism; analytical mechanics

Citation: Vlase, S.; Marin, M.; Iuliu, N. Finite Element Method-Based Elastic Analysis of Multibody Systems: A Review. *Mathematics* **2022**, *10*, 257. https://doi.org/10.3390/math10020257

Academic Editor: Athanasios Tzavaras

Received: 22 December 2021
Accepted: 12 January 2022
Published: 15 January 2022

1. Introduction

In recent years, research, led by practical applications in engineering, and which uses increasingly complex equipment, operates at higher speeds in difficult environmental conditions, and which is subjected to intense loads, has driven the development of methods for analyzing large deformable mechanical systems. Studies determined by these developments in technology have led to the analysis of multibody systems with deformable elements. In order to perform this analysis, the researchers developed and reinvented the existing methods to apply them to the new situations that has arisen. The development of this field is based on the numerical techniques related to the finite element method (FEM) as well as on the application of classical methods used in analytical mechanics (the latter representing the best way to approach such systems). Analytical mechanics has the advantage of general methods in which the procedures that apply follow a certain order, the same in which all cases may occur. These methods can be easily algorithmized. The

methods of analytical mechanics fit well with such an analysis because of the problems of complexity and the possibility to study a mechanical system with a large number of degrees of freedom (DOF). Such an analysis avoids the study of each element and the assembly of the obtained systems of equations, and it allows for the obtainment of the evolution equations, starting from the fundamental notions written for the whole system (kinetic energy, potential energy, work, acceleration energy, momentum, Hamiltonian, etc.). This is possible due to the tools used by analytical mechanics, which generalize the way of presentation of motion. The constraints that occur in engineering systems (due to the connections between the elements and the connection to the fixed space) that reduce the DOF number of the system are best described in the analytical mechanics, regardless of the method applied. Analytical mechanics has proven to be a powerful tool for analyzing complex systems, providing a set of equivalent formulations from which the most appropriate formulation can be selected.

The basic notions from the analytical descriptions stated above allow for a complete description of the mechanical system and can easily obtain the solution.

The main advantage of using Lagrange's equations in engineering is that it allows for a unitary solution of the problems of the dynamics of mechanical systems, according to a well-established itinerary and by moving through the same stages each time. Thus, the method is well suited for algorithms. Another advantage is that only scalar quantities are used instead of the vector quantities used in Newton's equations. Lagrange's equations can also highlight the existence of motion constants, which in some cases, can simplify solving the problem [1].

However, analytical mechanics offers other alternatives for writing the equations of motion of a dynamic system that, in some situations, may have advantages over Lagrange's classical approach. For this reason, during the development of the elastic MBS study, different methods were applied, identifying the advantages and disadvantages. The engineering requirements for the study of high speed and high load systems have driven the development of research for the application of MEF in the case of multibody systems (MBS), and there are many papers that address such a problem in order to obtain immediate practical results [2–9].

The practical applications for solving concrete problems, immediate in the industry, were supported by theoretical works that developed the mathematical bases of the numerical modeling of such problems. The aim of these works is to develop wide possibilities of computer simulations for the analysis of more accurate models, which will capture significant details in the operation of mechanical systems. The development of such appropriate algorithms allows for the modeling and analysis of systems that cannot be solved with existing computer codes. The main problem is that the procedures used in MEF differ from the procedures used in flexible multibody system codes.

Three ways of approaching such problems are used. The first of these approaches uses algorithms that involve successful simulations by establishing an interface between existing codes. The second way is to implement algorithms aimed at studying MBS in existing finite element algorithms. The third way used is the finite element formulation of the behavior of MBS with elastic elements, which requires a great effort for implementation in computer algorithms. A number of papers presents these methods [10–12].

Works dealing with the effective integration of equations of motion show results obtained to facilitate this step in [13]. Precise modeling of such systems is an important contribution to the development of engineering, but the potential is low due to the numerical calculation involved. The integration of the equations of motion of a complex multibody system is a time consuming computation. For example, simulating the behavior of a crankshaft that interacts with the surrounding elements requires a processing time for the CPU of several hours and determines the total time required for such a calculation. Methods for improving this step are presented in [13].

In addition, the use of symbolic calculus allows for a faster simulation code. A general integration procedure using the symbolic calculation was presented in [14]. A comparison with other models used in the literature highlighted the advantages of this approach.

In this theoretical development, multibody codes proved to be powerful tools for studying the nonlinear response of MBS with elastic elements. These systems might have had rigid movements over which small elastic deformations overlapped. For example, Refs. [15,16] presented a unitary approach and an overview of how to prepare the necessary data for dynamic models. The determination and writing of the matrix coefficients that intervened in the equations of motion were performed outside the code of the finite element. These data were then stored in a standardized object-oriented structure, and thus the dataset became independent of the wording of the MBS code.

Interesting methods of analysis and calculation simplification and some engineering applications are presented in the literature. A computational system framework was introduced in [17] using the finite element absolute nodal coordinate formulation. In that paper, they created the geometry of the system that made the analysis of the system possible. These two methods of analysis, MBS and FEM, were used to analyze a rocket sled, and the main problem was to establish the boundary conditions [18]. Using, in conjunction, these two methods, the simulation became more effective than those previously made using classical analyses.

Some issues regarding the harmonization of the two MBS and FEM analysis methods are presented in [19,20]. After developing a formalism that allows for the simultaneous analysis of systems using the two methods, the authors present engineering applications confirming the validity of the analysis and the models used. A successful application for a complex mechanical system is made in [21], where railway vehicles were analyzed using flexible tracks models. The study proposed a model that has flexible railway tracks. Classical theory of Timoshenko curved beam was used for modeling. The paper presented new aspects concerning the computation procedures applied in this analysis. Applications for the study of composites were developed in [22,23], and mathematical methods of solving with practical applications justifying the applicability and advantages were studied in [24–26].

Modeling a flexible multibody system used in parallel with MEF requires significant computing resources. One strategy used by researchers is the use of reduced-order models (ROMs). For plane systems, an approach using classical Lagrange equations is presented in [27]. The method can be extended to three-dimensional systems. This method helps to significantly reduce the effort and computation time required.

Topological representations and models have been used to simplify the symbolic writing of equations of motion within these systems in order to reduce the time required for modeling. An example showing this mode of analysis is presented in [28].

Mathematical methods for solving MBS problems with elastic elements have been continuously developed in order to find the best way to deal with this type of problem, which involves considerable modeling and calculation effort. In this sense, the reduced transfer matrix method for a multibody system (MSRTMM) was developed. An application of this method, along with the Riccati transformation, is presented in [29]. Three case studies were analyzed: a thin rectangular plane plate, a parallelogram thin plane plate and a multibody system with two-dimensional elements. MSRTMM has the advantage of high computing speed, ease of writing algorithms and numerical stability.

Specific methods have been developed for systems with certain features. An exact calculation method using dynamic stiffness was used to analyze vibrations of multibody systems with flexible beams connecting rigid bodies [30]. Rigid bodies can have any geometry and can have connections between them by means of elastic beams. The results were compared with other published results. This method is a powerful tool for optimizing such systems or for identifying modal parameters. A mathematical framework for calculating the mass matrix of a rigid–flexible multibody system with parameters is presented in [31]. The proposed method of analysis was applied to the calculation of a parallel Delta robot. In

the field of mathematics involved in solving these complex problems, different models and ways of approaching systems have been proposed. For example, in [32], the well-known Ritz method was applied, which is especially known for its computational efficiency and is used extensively by engineers. It was proposed to use a generalized version of the MBS study method with a general topology. The numerical examples developed illustrate the advantages of the method.

The evaluation of inertial forces is a central and complicated task for the dynamic analysis of flexible multibody systems (FMS). A high-precision formulation for a 3D problem of flexible multibody systems is presented in [33]. The novelty is that the equations of motion were obtained with the principles of virtual power, without having to use the differentiation of the rotation matrix. Some numerical examples support the method proposed in that paper.

A practical method for numerically solving problems related to eigenvalues is presented in [34].

This paper presents, critically, the main methods of analytical mechanics used for the analysis of MBS with linear elastic components. To achieve this, a brief review of the use of MEF in this type of analysis is made.

2. FEA of Elastic MBS

To analyze an MBS having elastic components and to use FEM to consider the elastic behavior, the most used method thus far is the method of Lagrange equations, which will be highlighted in the future presentation. The purpose of an analysis of this type is to obtain, in a first step, the evolution equations for a single finite element from the studied system, if shape functions are known (thus, the type of finite element). These shape functions will determine the matrix coefficients of differential equations obtained. The equations are expressed in a local reference frame related to the finite element analyzed in rigid motion, together with the whole body that it discretizes. They must be reported, as a whole, to a global reference system against which the movement of the whole system will be analyzed. After this transition is made, it is necessary to assemble the obtained systems of equations. In this way, we finally obtain the system of evolution equations. All these presented procedures are performed according to the classical and well-known methods applied in FEM. By introducing boundary conditions and loads, one can then proceed to solve the system of equations and determine the answer to the system. In this analysis, the deformations are considered small enough such that the general movement of the system (rigid movement) is not influenced in any way by these deformations. The main problem remains regarding the harmonization of the two methods, which both use different procedures.

The analyzed works present the evolution in the study of these problems, starting from simple, one-dimensional elements and gradually moving to more and more complex finite elements. In the first studies of such problems, one-dimensional finite elements were studied, and the movement of the system was considered plane [35–37]. Complex, bi- and three-dimensional finite elements have been studied and applied in [38]. Recently, methods of analysis have been developed, and more sophisticated models have been studied. For example, the damping issues in such systems are presented in [39–41]. The contributions in these papers refer to the development of different types of finite elements that serve the purpose. Iterative methods and the Newton–Raphson algorithm can be used to solve the equations of motion. The analysis of the effects of temperature in the study of MBS with flexible elements is presented in [42]. The paper proposes a sandwich beam element that is convenient for describing large displacements and rotations. As in the cases mentioned above, an incremental–iterative method is used to solve the evolution equations together with the Grünwald approximation and the Newton–Raphson algorithm. The use of composite materials in flexible multibody systems is presented in [43], and a systematic presentation of the results obtained in FEM applied to the study of elastic MBS is made in [44].

Lagrange's equations are the main tools to obtain motion equations for a finite element that discretizes an MBS system, regardless of whether one-, two-, or three-dimensional finite elements or the type of motion of the elements of the MBS system are used. This method has proven to be, over time, useful and has been relatively convenient in the application and verification of the countless applications studied within it. The major advantage of this method is the use of notions with which researchers are familiar. At this time, in the studies, Lagrange's equations are the most widely used method for studying such problems.

However, analytical mechanics offers alternative methods of analysis equivalent to the method of Lagrange's equations. These methods use less commonly used concepts, which distances researchers from using these alternative methods. At the moment, the diversity of the approached problems and the needs for analysis imposed by the development of technology make it necessary to re-evaluate these methods, as they can show their advantages in certain situations. This paper will show an analysis of the main methods of analysis in analytical mechanics and will try to point out the advantages and disadvantages involved. The methods presented and analyzed in the paper are Lagrange's equations, Gibbs–Appell equations, Maggi's formalism and Hamilton's equations. These methods are the most used methods in application. There are, of course, other equivalent or alternative methods, but they are much less used and we do not present them here, especially since there are only a few papers that use them, and they do not seem to have obvious advantages.

We observe here that the Lagrangian has, in its component, physical quantities with which we are well acquainted (kinetic energy, potential energy, work)—a strong reason is that it is frequently used. Another reason can be represented by the fact that the generalized coordinates allow for the unitary treatment of such a system and the representations used allow for an easy application of numerical methods. Another advantage is represented by the fact that the liaison forces (or Lagrange's multipliers) are eliminated in writing these equations such that the number of unknowns is reduced and limited to the generalized coordinates in the first instance. FEM, where the number of DOFs used is high, can lead to a significant decrease in working time, and it is a major advantage for the user.

The energy of accelerations is a notion little used by researchers. A disadvantage is that the expression for velocity contains four matrix terms, while the acceleration contains five such terms [45,46]. For this reason, the number of operations for determining the energy of accelerations is slightly higher than for obtaining kinetic energy. This disadvantage is offset by the significantly lower number of differentiation operations required compared to Lagrange's equations. The method is little used, although in recent years, the need for calculations has led to reconsideration of the method [47–51]. The main advantage is the lower number of differentiation operations required to obtain the final equations of motion.

Hamilton's equations were less frequently used in the dynamic analysis of mechanical systems. There is little literature to present the advantages or disadvantages of this method. However, if we take into account that the system of second-order equations is replaced by a system of first-order equations, the use of this method may show its advantages for suitable applications [52–55].

Recent contributions to the development of this field are presented in [56–59].

3. Kinematics

A brief recapitulation of basic notions in analytical mechanics is necessary [60]. In the following, the element will relate to a local (mobile) reference frame. The mobile reference system participates in the general movement of the MBS system. This element is known via the angular velocity $\overline{\omega}$, angular acceleration $\bar{\varepsilon}$, velocity $\overline{v}_o$ and acceleration $\overline{a}_o$ of the origin of the local reference system. We use two indices L (from local) and G (from global) to denote the sizes corresponding to the local and global coordinate systems. The orthonormal operator $[ROT]$ makes the transformation of the components from the local system to the global one, $\{a\}_G = [ROT]\{a\}_L$.

By differentiating the transformation operator $[ROT]$, it is now possible to obtain the angular velocity and acceleration operator [61].

An arbitrary point M becomes, after deformation, M'. In this case, the deformation process is expressed by:

$$\{r_{M\prime}\}_G = \{r_O\}_G + [ROT](\{r\}_L + \{u\}_L). \tag{1}$$

The linear dependence between nodal displacement and vector displacement of the current point of the element is expressed in FEA through a linear relation:

$$\{u\}_L = [N]\{\delta\}_L. \tag{2}$$

Here, $\{\delta\}_L$ is denoted by the vector of the independent coordinates. With this assumption, the velocity vector of M' becomes:

$$\{v_{M'}\}_G = \{\dot{r}_{M'}\}_G = \{\dot{r}_O\}_G + \left[R\dot{O}T\right]\{r\}_L + \left[R\dot{O}T\right][N]\{\delta\}_L + [ROT][N]\left\{\dot{\delta}\right\}_L. \tag{3}$$

and the acceleration vector:

$$\{a_{M'}\}_G = \{\ddot{r}_O\}_G + \left[R\ddot{O}T\right]\{r\}_L + \left[R\ddot{O}T\right][N]\{\delta\}_L + 2\left[R\dot{O}T\right][N]\left\{\dot{\delta}\right\}_L + [ROT][N]\left\{\ddot{\delta}\right\}_L. \tag{4}$$

We can observe that some of these sizes are expressed in the local system coordinate and others in a global system coordinate. Passing to the local frame, we have:

$$\begin{aligned}\{v_{M'}\}_L = [ROT]^T\{v_{M'}\}_G &= \{\dot{r}_O\}_L + [ROT]^T\left[R\dot{O}T\right]\{r\}_L + [ROT]^T\left[R\dot{O}T\right][N]\{\delta\}_L + [N]\left\{\dot{\delta}\right\}_L \\ &= \begin{bmatrix}[E] & [ROT]^T\left[R\dot{O}T\right] & [ROT]^T\left[R\dot{O}T\right][N] & [N]\end{bmatrix}\begin{Bmatrix}\{\dot{r}_O\}_L \\ \{r\}_L \\ \{\delta\}_L \\ \left\{\dot{\delta}\right\}_L\end{Bmatrix}.\end{aligned} \tag{5}$$

$$\begin{aligned}\{a_{M'}\}_L = [ROT]^T\{a_{M'}\}_G &= \{\ddot{r}_O\}_L + [ROT]^T\left[R\ddot{O}T\right]\{r\}_L + [ROT]^T\left[R\ddot{O}T\right][N]\{\delta\}_L + 2[ROT]^T\left[R\dot{O}T\right][N]\left\{\dot{\delta}\right\}_L + [N]\left\{\ddot{\delta}\right\}_L \\ &= \begin{bmatrix}[E] & [ROT]^T\left[R\ddot{O}T\right] & [ROT]^T\left[R\ddot{O}T\right][N] & 2[ROT]^T\left[R\dot{O}T\right][N] & [N]\end{bmatrix}\begin{Bmatrix}\{\ddot{r}_O\}_L \\ \{r\}_L \\ \{\delta\}_L \\ \left\{\dot{\delta}\right\}_L \\ \left\{\ddot{\delta}\right\}_L\end{Bmatrix}.\end{aligned} \tag{6}$$

4. Fundamental Notions in Dynamics of FEA of MBS

4.1. Kinetic Energy

The kinetic energy plays an important role in an analytical description. Its expression is given by the equation:

$$E_c = \frac{1}{2}\int_V \rho\{v_{M\prime}\}_G^T\{v_{M\prime}\}_G dV. \tag{7}$$

Considering Equation (5), it obtains, for kinetic energy, the complete expression:

$$E_c = \begin{bmatrix} \{\dot{r}_O\}_L & \{r\}_L & \{\delta\}_L & \left\{\dot{\delta}\right\}_L \end{bmatrix}$$

$$\begin{bmatrix} [E] & [ROT]^T\left[\dot{ROT}\right] & [ROT]^T\left[\dot{ROT}\right][N] & [N] \\ \left[\dot{ROT}\right]^T[ROT] & \left[\dot{ROT}\right]^T\left[\dot{ROT}\right] & \left[\dot{ROT}\right]^T\left[\dot{ROT}\right] & \left[\dot{ROT}\right]^T[ROT][N] \\ [N]^T\left[\dot{ROT}\right]^T[ROT] & [N]^T\left[\dot{ROT}\right]^T\left[\dot{ROT}\right] & [N]^T\left[\dot{ROT}\right]^T\left[\dot{ROT}\right] & [N]^T\left[\dot{ROT}\right]^T[ROT][N] \\ [N]^T & [N]^T[ROT]^T\left[\dot{ROT}\right] & [N]^T[ROT]^T\left[\dot{ROT}\right][N] & [N]^T[N] \end{bmatrix} \begin{Bmatrix} \{\dot{r}_O\}_L \\ \{r\}_L \\ \{\delta\}_L \\ \left\{\dot{\delta}\right\}_L \end{Bmatrix} \quad (8)$$

We denote:

$$[m] = \int_V \rho[N]^T[N]\,dV; \quad (9)$$

$$\left[m_O^i\right] = \int_V \rho[N]^T dV; \ \left\{q^i(\varepsilon)\right\}_L = \int_V \rho[N]^T[\varepsilon]_L\{r\}_L dV; \quad (10)$$

$$\left\{q^i(\omega)\right\}_L = \int_V \rho[N]^T[\omega]_L[\omega]_L\{r\}_L dV; \quad (11)$$

$$[k(\varepsilon)] = \int_V \rho[N]^T[\varepsilon][N]dV; \quad (12)$$

$$[k(\omega)] = \int_V \rho[N]^T[\omega]_L[\omega]_L[N]dV; \quad (13)$$

$$[c] = \int_V \rho[N]^T[\omega]_L[N]dV; \quad (14)$$

$$\{m_{ix}\} = \int_V \rho\left[S_{(i)}\right]^T x dV \quad ; \quad \{m_{iy}\} = \int_V \rho\left[N_{(i)}\right]^T y dV \quad ; \quad \{m_{iz}\} = \int_V \rho\left[N_{(i)}\right]^T z dV. \quad (15)$$

4.2. Potential Energy

The classic expression for the internal work (potential energy) is:

$$E_p = \frac{1}{2}\int_V \{\sigma\}^T\{\varepsilon\}dV, \quad (16)$$

where $\{\varepsilon\}$ is the strain vector and $\{\sigma\}$ is the stress vector;
The generalized Hooke law has the well-known form:

$$\{\sigma\} = [H]\{\varepsilon\}. \quad (17)$$

The strains can be expressed as [6]:

$$\{\varepsilon\} = [b]\{u\} = [b][N]\{\delta\}_L. \quad (18)$$

Using Equations (17) and (18), we obtain:

$$E_p = \frac{1}{2}\{\delta\}_L^T\left(\int_V [N]^T[b]^T[H]^T[b][N]dV\right)\{\delta\}_L. \quad (19)$$

Matrix $[k]$ is the stiffness matrix:

$$[k] = \int_V [N]^T[b]^T[H]^T[b][N]dV. \quad (20)$$

Equation (19) has the traditional form:

$$E_p = \frac{1}{2}\int_V \{\delta\}_L^T [k]\{\delta\}_L dV. \tag{21}$$

4.3. Work

The concentrated forces $\{q\}_L$ and volume forces $\{p\} = \{p(x,y,z)\}$ produce mechanical work:

$$W^c = \{q\}_L^T \{\delta\}_L, \tag{22}$$

and:

$$W = \int\limits_V \{p\}_L^T \{f\}_L dV = \left(\int\limits_V \{p\}_L^T [N] dV\right)\{\delta\}_L = \{q^*\}_L^T \{\delta\}_L, \tag{23}$$

4.4. Lagrangian

The expression of the Lagrangian is [60]:

$$L = E_c - E_p + W + W^c, \tag{24}$$

Using Equations (21)–(23), the Lagrangian takes the form:

$$L = E_c - \frac{1}{2}\int_V \{\delta\}_L^T [k]\{\delta\}_L dV + \{q^*\}_L^T \{\delta\}_L + \{q\}_L^T \{\delta\}_L. \tag{25}$$

4.5. Momentum

The momentum for a finite element is:

$$\begin{aligned} \{p\}_G &= \textstyle\int_V \rho \{v_{M\prime}\}_G dV \\ &= \textstyle\int_V \rho\Big(\{\dot{r}_O\}_G + \left[R\dot{O}T\right]\{r\}_L + \left[R\dot{O}T\right][N]\{\delta\}_L + [ROT][N]\left\{\dot{\delta}\right\}_L\Big) dV \\ &= m\{\dot{r}_O\}_G + \left[R\dot{O}T\right]\{\overline{S}\}_L + \left[R\dot{O}T\right]\left(\textstyle\int_V \rho[N]dV\right)\{\delta\}_L + [ROT]\left(\textstyle\int_V \rho[N]dV\right)\left\{\dot{\delta}\right\}_L \\ &= m\{\dot{r}_O\}_G + m\left[R\dot{O}T\right]\{\overline{r}_C\}_L + \left[R\dot{O}T\right]\left[m_O^i\right]\{\delta\}_L + [ROT]\left[m_O^i\right]\left\{\dot{\delta}\right\}_L \end{aligned} \tag{26}$$

The notation $m = \int_V \rho dV$ represents the total mass of the finite element, $\{\overline{S}\}_L$ is the static moment and $\left[m_O^i\right] = \int_V \rho[N]dV$ is the matrix of the inertia of the element (see Equation (11)).

In the local system, there is the relation:

$$\begin{aligned} \{p\}_L &= [ROT]^T \{p\}_G = \\ &= m\{\dot{r}_O\}_L + m[ROT]^T\left[R\dot{O}T\right]\{\overline{r}_C\}_L + [ROT]^T\left[R\dot{O}T\right]\left[m_O^i\right]\{\delta\}_L + \left[m_O^i\right]\left\{\dot{\delta}\right\}_L. \end{aligned} \tag{27}$$

In an alternative way, the momentum can be calculated with the relation:

$$\{p\}_L = \left\{\frac{\partial L}{\partial\left\{\dot{d}\right\}_L}\right\}. \tag{28}$$

From (27), the vector of velocities $\left\{\dot{\delta}\right\}_L$ can be obtained:

$$\left\{\dot{\delta}\right\}_L = \left[m_O^i\right]^{-1}\left(\{p\}_L - m\{\dot{r}_O\}_L - m[ROT]^T\left[R\dot{O}T\right]\{\overline{r}_C\}_L - [ROT]^T\left[R\dot{O}T\right]\left[m_O^i\right]\{\delta\}_L\right). \tag{29}$$

4.6. Hamiltonian

Using the previous notations, the Hamiltonian becomes:

$$H = \left\{ \frac{\partial L}{\partial \left\{ \dot{\delta} \right\}_L} \right\}^T \left\{ \dot{\delta} \right\}_L - L = \{p\}^T \left[m_O^i \right]^{-1} \left(\{p\}_L - m\{\dot{r}_O\}_L - m[ROT]^T \left[\dot{ROT} \right] \{\bar{r}_C\}_L - [ROT]^T \left[\dot{ROT} \right] \left[m_O^i \right] \{\delta\}_L \right) - L. \quad (30)$$

where, for the Lagrangian, Equation (25) is used.

4.7. Energy of Accelerations

We introduce the notion of energy of acceleration. The expression of this is, for N material points, [61]:

$$E_a = \frac{1}{2} \sum_{i=1}^{N} m_i a_i^2. \quad (31)$$

For a solid body, the expression becomes:

$$E_a = \frac{1}{2} \int_V \rho a^2 dV. \quad (32)$$

Using Equation (4) for acceleration, Equation (32) becomes:

$$E_a = \tfrac{1}{2} \int_V \rho a_{M'}^2 dV = \tfrac{1}{2} \int_V \rho \{a_{M'}\}^T \{a_{M'}\} dV$$

$$= \tfrac{1}{2} \int_V \rho \left(\{\ddot{r}_O\}_G^T + \{r\}_L^T \left[\ddot{ROT} \right]^T + \{\delta\}_L^T [N]^T \left[\ddot{ROT} \right]^T + 2 \left\{ \dot{\delta} \right\}_L^T [N]^T \left[\dot{ROT} \right]^T + \left\{ \ddot{\delta} \right\}_L^T [N]^T [ROT]^T \right) x \quad (33)$$

$$x \left(\{\ddot{r}_O\}_G + \left[\ddot{ROT} \right] \{r\}_L + \left[\ddot{ROT} \right] [N] \{\delta\}_L + 2 \left[\dot{ROT} \right] [N] \left\{ \dot{\delta} \right\}_L + [ROT][N] \left\{ \ddot{\delta} \right\}_L \right) dV$$

More comments concerning this notion are presented in [62].

5. Analytical Method in FEA of MBS

5.1. Lagrange's Equations

The classic Lagrange's equations are:

$$\frac{d}{dt} \left\{ \frac{\partial L}{\partial \dot{\delta}} \right\}_L - \left\{ \frac{\partial L}{\partial \delta} \right\}_L = 0. \quad (34)$$

By $\left\{ \frac{\partial E}{\partial X} \right\}$, it is denoted as:

$$\left\{ \frac{\partial E}{\partial X} \right\} = \begin{Bmatrix} \frac{\partial E}{\partial x_1} \\ \frac{\partial E}{\partial x_2} \\ \vdots \\ \frac{\partial E}{\partial x_n} \end{Bmatrix} \text{ and: } \{X\} = \begin{Bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{Bmatrix}. \quad (35)$$

Using the Lagrangian previously obtained in Equation (25) results in:

$$[m] \left\{ \ddot{\delta} \right\}_L + [c] \left\{ \dot{\delta} \right\}_L + ([k] + [k(\varepsilon)] + [k(\omega)]) \{\delta\}_L = \{q\}_L + \{q^*\}_L - \left\{ q^i(\varepsilon) \right\}_L - \left\{ q^i(\omega) \right\}_L - \left[m_O^i \right] \{\ddot{r}_O\}_L. \quad (36)$$

We mention here the difference between the application of Lagrange's equations (three differentiations $\left\{\frac{\partial L}{\partial \dot{d}}\right\}$, $\frac{d}{dt}\left\{\frac{\partial L}{\partial \dot{d}}\right\}$, $\left\{\frac{\partial L}{\partial d}\right\}$) and the application of the Gibbs–Appell equations (when it is necessary to make only a single differentiation $\left\{\frac{\partial E_a}{\partial \ddot{d}}\right\}$) [61].

5.2. Gibbs–Appell Formalism

The Gibbs–Appell equation represents an alternative to Lagrange's equations. To use these, it is necessary to know the energy of acceleration, obtained in Equation (33). The Gibbs–Appell equations are [62]:

$$\frac{\partial E_a}{\partial \ddot{q}_j} = Q_j \quad j = \overline{1,n}. \tag{37}$$

Equation (33) has, in its component, the following terms [22]:

- E_{a2} containing the quadratic terms in accelerations:

$$E_{a2} = \frac{1}{2}\int_V \rho\left(\left\{\ddot{\delta}\right\}_L^T [N]^T [N]\left\{\ddot{\delta}\right\}_L\right) dV; \tag{38}$$

- E_{a1} containing the linear terms in accelerations:

$$E_{a1} = \int_V \rho\left(\left\{\ddot{\delta}\right\}_L^T [N]^T [ROT]^T \{\ddot{r}_O\} + \left\{\ddot{\delta}\right\}_L^T [N]^T [ROT]^T \left[R\ddot{O}T\right]\{r\}_L + \left\{\ddot{\delta}\right\}_L^T [N]^T [ROT]^T \left[R\ddot{O}T\right][N]\{d\}_L + 2\left\{\ddot{\delta}\right\}_L^T [N]^T [ROT]^T \left[R\dot{O}T\right][N]\left\{\dot{\delta}\right\}_L\right) dV \tag{39}$$

- The terms E_{a0} without any term with accelerations that play no role in obtaining the equations.

Equation (37) can be written if we take into account our notations as:

$$\left\{\frac{\partial E_a}{\partial \ddot{d}}\right\}_L - \{Q\}_L = 0; \tag{40}$$

The term E_a is:

$$E_a = E_{ao}(\dot{q}) + E_{a1}(\dot{q}, \ddot{q}) + E_{a2}(\ddot{q}); \tag{41}$$

and:

$$\{Q\}_L = [k]\{\delta\}_L + \{q\}_L + \{q^*\}_L; \tag{42}$$

If we differentiate it, we obtain:

$$\frac{\partial E_{a2}}{\partial\left\{\ddot{d}\right\}_L} = \left(\int_V \rho[N]^T [S]\, dV\right)\left\{\ddot{d}\right\}_L = [m]\left\{\ddot{d}\right\}_L; \tag{43}$$

$$\frac{\partial E_{a1}}{\partial\left\{\ddot{d}\right\}_L} = -\left[m_O^i\right]\{\ddot{r}_O\}_L - \left\{q^i(\omega)\right\} - \left\{q^i(\varepsilon)\right\} + ([k(\omega)] + [k(\varepsilon)])\{d\}_L + [c]\left\{\dot{d}\right\}_L; \tag{44}$$

$$\frac{\partial E_{a0}}{\partial\left\{\ddot{d}\right\}_L} = 0. \tag{45}$$

Performing the calculations in the end, we obtain Equation (36).

Compared to Lagrange's method, this method requires a smaller number of differentiations. In this way, the number of calculations decreases and thus the time required to solve such problems. If we take into account that finite element models involve a large number of DOFs and thus a large number of calculations, reducing the number of operations offered by this method can lead to significant savings in computer time.

5.3. Hamilton's Method

The use of the Lagrange formalism (or Gibbs-Appell) leads to the obtaining of a system of second-order differential equations. Technically, solving this system of second-order equations is achieved by transforming it into a system of first-order differential equations of double dimension. Hamiltonian mechanics use unknown $2n$, and the system of differential equations obtained is from the beginning a system of differential equations of the first order of size $2n$. The unknowns are the generalized coordinates $\{\delta\}_L$ and canonically conjugated moment:

$$\{p\}_L = -\left\{\frac{\partial L}{\partial\{\delta\}_L}\right\}. \tag{46}$$

Thus, the main difference between Lagrange's and Hamilton's method is the use of the canonical conjugated moment instead of the generalized velocities. The major advantage of applying this method could precisely be the direct obtainment of a system of first-order equations, which can be solved directly using the usual commercial software.

Hamilton's equations are a first-order system of differential equations [62]. They are:

$$\left\{\dot{\delta}\right\}_L = \left\{\frac{\partial H}{\partial\{p\}_L}\right\};\ \{\dot{p}\}_L = -\left\{\frac{\partial H}{\partial\{\delta\}_L}\right\}. \tag{47}$$

From Equations (27)–(29), we obtain:

$$\begin{aligned}
\left\{\dot{\delta}\right\}_L &= [m_O^i]^{-1}\left(\{p\}_L - m\{\dot{r}_O\}_L - m[ROT]^T\left[\dot{ROT}\right]\{\bar{r}_C\}_L - [ROT]^T\left[\dot{ROT}\right][m_O^i]\{\delta\}_L\right);\\
\{\dot{p}\}_L &= \{p\}^T[m_O^i]^{-1}[ROT]^T\left[\dot{ROT}\right][m_O^i] + \int_V \rho\left(\{\dot{r}_O\}_L^T[ROT]^T\left[\dot{ROT}\right][N]\right)d\ V\\
&+\int_V \rho\left(\{\ r\}_L^T\left[\dot{ROT}\right]^T\left[\dot{ROT}\right][S]\right)dV + \int_V \rho\left([N]^T\left[\dot{ROT}\right]^T\left[\dot{ROT}\right][N]\{\delta\}_L + [N]^T\left[\dot{ROT}\right]^T[ROT][N]\{\delta\}_L\right)dV\\
&-\int_V\ [k]\{d\}_L dV + \{q^*\}_L^T + \{q\}_L^T
\end{aligned} \tag{48}$$

These represent the equations of motion sought.

The main advantage of Hamilton's method is that it provides us with a system of first-order differential equations. However, the number of unknowns to be found is doubled. In the case of using other methods, the differential equations obtained are of the second order. Solving techniques require for their transformation into first-order differential systems by introducing new variables. In the case of Hamilton's method, these new variables are obtained directly and have physical significance.

5.4. Maggi's Equations

The form of these equations are [63]:

$$\sum_{k=1}^{n} a_{kj}\left[\left(\frac{d}{dt}\left(\frac{\partial E_c}{\partial\dot{q}_k}\right) - \frac{\partial E_c}{\partial q_k}\right) - Q_k\right] = 0 \quad ; \quad j = \overline{1, n-m}, \tag{49}$$

representing a number of $n - m$ independent equations called Maggi's equations.

Using these equations makes it simpler to analyze such a system from the point of view of a formal description. In this case, only the kinetic energy is necessary to compute. The liaisons between elements offer us the possibility to eliminate the liaison forces and thus to simplify the calculus.

6. Conclusions and Discussion

The most important step in the dynamic analysis of an elastic MBS is to write the equations of evolution. The next steps that follow, namely, the assembly of the equations of motion and their solution, will be performed according to the classical methods used in the commercial software of FEM. To obtain the equations is the most difficult problem to

solve, given the multitude of terms that appear in such a description. As a result, finding a formalism that would make it possible to write these equations as easily as possible is an important step in this analysis. The method used almost exclusively in this type of analysis, until now, was the Lagrange's equations. This is primarily due to the fact that researchers are familiar with this method, and they use fundamental notions currently used by researchers (kinetic energy, potential, work, ...). However, analytical mechanics offer several formulations that are equivalent to each other and to Lagrange's equations. Gibbs–Appell equations, Hamilton equations, Maggi equations, Jacobs equations and other equivalent forms can be used in this way. With such a multitude of methods that can be used that are equivalent to each other, the question arises as to which of these methods can be applied more easily than the method of Lagrange's equations. The paper analyzes several analytical forms used to determine the equations of motion of MBS systems with elastic elements to identify and analyze the advantages and disadvantages of these methods, which could allow for a more economical result. Lagrange's equations have the advantage of being a method widely used by researchers due to a familiarity of researchers with it. The Gibbs–Appell equations prove to be easy to write by skipping some steps related to the derivation of the equations. This turns out to be a more economical method in terms of the time required to write the equations. In this method, the number of differentiations of terms decreases, and as a result, the total number of calculations required decreases. However, we mention that Lagrange's method has the advantage of using kinetic energy, a well-known notion with which we operate easily. The Gibbs–Appell equations use the energy of accelerations, a notion that most engineers are less familiar with. Not many papers contain applications of the Gibbs–Appell equations. Generally, the papers present the Gibbs–Appell formalism as a secondary method to solve a problem (useful but not necessary) [62].

These equations are formally more elegant and simpler, and the necessary number of differentiations is smaller. Using this method, a system with a holonomic constraint can be handled in an economical manner, as with Lagrange's equations.

Maggi's method also has the advantages of simplicity in approaching problems, being essentially equivalent to the Gibbs–Appell method [64]. It is proven that this formulation is a simple and stable method for determining the dynamic response of constrained multibody systems [65].

Hamilton's method of equations proves to be the least profitable for the type of problems studied; in general, the time required to obtain the equations is not economical, and the complexity of the intermediate calculations is high. However, we do not deny that this approach could prove useful in certain applications because the system of differential equations obtained is first order and thus avoids a computational step, used in the classical solution of these systems, where systems of equations obtained are second-order differentials.

If we take into account all these considerations, we can reasonably assume that the alternative and equivalent methods developed in analytical mechanics (and which, for the moment, do not seem to have practical applicability) will be reevaluated and developed due to more faithful modeling requirements imposed by technology.

Funding: This research received no external funding. The APC was funded by Transilvania University of Brasov.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Conflicts of Interest: The authors declare no conflict of interest.

Notations

$\{r_{M'}\}_G$	position vector of point M′;
$\{r_M\}_G$	position vectors of point M;
$\{u\}_L$	displacement vector;
$\{r_O\}_G$	position vector of origin O (of the mobile reference frame)
index G	vector with components express in the global reference frame;
index L	vector with components express in the local reference frame;
$[ROT]$	rotation matrix;
$[N]$	shape functions matrix;
$\{v_{M'}\}$	velocity of point M′;
$\{a_{M'}\}$	acceleration of point M′;
$\{\delta\}$	nodal displacement vector;
E_c	kinetic energy;
E_p	potential energy;
L	Lagrangian;
H	Hamiltonian;
$\{\sigma\}$	stress vector;
$\{\varepsilon\}$	strain vector;
$\{v\}^T\{w\}$	dot product between the vectors $\{v\}$ and $\{w\}$;
$\{p\}$	conjugated moment;
E_a	energy of acceleration;
W^c	work of the concentrated forces;
W	work of the volume forces.

References

1. Gans, F.R. *Engineering Dynamics: From the Lagrangian to Simulation*; Springer: New York, NY, USA, 2013; ISBN 978-1-4614-3929-5.
2. Shi, Z.; Meacci, M.; Meli, E.; Wang, K.Y.; Rindi, A. Validation of a Finite Element Multibody System Model for Vehicle-Slab Track Application. In *Advances in Dynamics of Vehicles on Roads and Tracks, Iavsd 2019, Proceedings of the 26th Symposium of the International Association of Vehicle System Dynamics (IAVSD), Gothenburg, Sweden, 12–16 August 2019*; Proceedings Paper in Book Series: Lecture Notes in Mechanical Engineering; Springer: Cham, Switzerland, 2020; pp. 407–414. [CrossRef]
3. Tokarczyk, J. Migration of Computational Models in Virtual Prototyping of Complex Mechanical Systems. In *Book Group Author IAENG, Proceedings of the World Congress on Engineering and Computer Science, WCECS 2012, San Francisco, CA, USA, 24–26 October 2012*; Proceedings Paper, Book Series: Lecture Notes in Engineering and Computer Science; Newswood Limited: Hong Kong, China, 2012; Volume II, pp. 1334–1337.
4. Marce-Nogue, J.; Klodowski, A.; Sanchez, M.; Gil, L. Coupling finite element analysis and multibody system dynamics for biological research. *Palaeontol. Electron.* **2015**, *18*, 5T.
5. Miao, B.R.; Zhang, W.H.; Huang, G.H.; Wu, S.C.; Zhao, Y.X. Research of High Speed Train Carbody Structure Vibration Behaviors and Structure Fatigue Strength Characteristic Technology. In *Advanced Materials Research*; Advanced in Product Development and Reliability III 2012; Proceedings Paper; Trans Tech Publications Ltd.: Freienbach, Switzerland, 2012; Volume 544, pp. 256–261. [CrossRef]
6. Ding, J.G.; Dai, Y.W.; Qiao, Z.; Huang, H.J.; Zhuang, W. Analysis of the Response of a Frame Structure during an Earthquake Using the Transfer Matrix Method of a Multibody System. *J. Eng. Mech.* **2015**, *141*, 04015020. [CrossRef]
7. Wallrapp, O.; Sachau, D. Space Flight Dynamic Simulations using Finite Element Results in Multibody System Codes. In *Advances in Computational Mechanics, Proceedings of the 2nd International Conference on Computational Structures Technology, CST' 94, Athens, Greece, 30 August–1 September 1994*; Proceedings Paper; Greece; 1994; pp. 149–158. Available online: https://elib.dlr.de/28194/ (accessed on 10 October 2021).
8. Zhang, J.H.; Jiang, S.S. Definition of Boundary Conditions and Dynamic Analysis of Rocket Sled and Turntable. In *Applied Mechanics and Materials, Proceedings of the 1st International Conference on Mechanical Engineering, Phuket, Thailand, 2 March 2011*; Proceedings Paper; Trans Tech Publications Ltd.: Freienbach, Switzerland, 2011; Volume 52–54, pp. 261–268. [CrossRef]
9. Scutaru, M.L.; Chircan, E.; Marin, M.; Grif, H.S. Liaison Forces Eliminating and Assembling of the Motion Equation in the Study of Multibody System with Elastic Elements. In *Procedia Manufacturing, Proceedings of the 13th International Conference Interdisciplinarity in Engineering (Inter-Eng. 2019), Targu Mures, Romania, 3–4 October 2019*; Proceedings Paper; Elservier B.V.: Amsterdam, The Netherlands, 2020; Volume 46, pp. 78–86. [CrossRef]
10. Shabana, A.A. *Dynamics of Multibody Systems*, 4th ed.; Cambridge University Press: Cambridge, UK, 2013.
11. Shabana, A.A.; Bauchau, O.A.; Hulbert, G.M. Integration of Large Deformation Finite Element and Multibody System Algorithms. *J. Comput. Nonlinear Dyn.* **2007**, *2*, 351–359. [CrossRef]

12. Rui, X.; Rong, B.; Wang, G. New Method for Dynamics Modeling and Simulation of Flexible Multibody System. In Proceedings of the Third International Conference on Mechanical Engineering and Mechanics, Beijing, China, 21–23 October 2009; Volumes 1 and 2, pp. 17–23.
13. Witteveen, W.; Stefan, P.; Pichler, F. On the Projection of a Flexible Bodies Modal Coordinates onto Another Finite Element Model with Local Modifications. *J. Comput. Nonlinear Dyn.* **2019**, *14*, 074501. [CrossRef]
14. Liang, Y.T.; McPhee, J. Symbolic integration of multibody system dynamics with the finite element method. *Multibody Syst. Dyn.* **2018**, *43*, 387–405. [CrossRef]
15. Wallrapp, O. Flexible bodies in multibody system codes. *Veh. Syst. Dyn.* **1998**, *30*, 237–256. [CrossRef]
16. Wallrapp, O. Standardization of Flexible Body Modeling in Multibody System Codes, Part I: Definition of Standard Input Data. *Mech. Struct. Mach.* **1994**, *22*, 283–304. [CrossRef]
17. Patel, M.; Orzechowski, G.; Tian, Q.; Shabana, A.A. A new multibody system approach for tire modeling using ANCF finite elements. *Proc. Inst. Mech. Engineers. Part K-J. Multibody Dyn.* **2016**, *230*, 69–84. [CrossRef]
18. Zhang, J.H.; Jiang, S.S. Rigid-Flexible Coupling Model and Dynamic Analysis of Rocket Sled. International Conference on Sustainable Construction Materials and Computer Engineering (ICSCMCE 2011). In *Advanced Materials Research, Proceedings of the Sustainable Construction Materials and Computer Engineering, Kunming, China, 24–25 September 2011*; Proceedings Paper; Trans Tech Publications Ltd.: Freienbach, Switzerland, 2012; Volume 346, pp. 447–454. [CrossRef]
19. Lu, H.J.; Rui, X.T.; Ding, Y.Y.; Chang, Y.; Chen, Y.H.; Ding, J.G.; Zhang, X.P. A hybrid numerical method for vibration analysis of linear multibody systems with flexible components. *Appl. Math. Model.* **2022**, *101*, 748–771. [CrossRef]
20. You, T.W.; Gong, D.; Zhou, J.S.; Sun, Y.; Chen, J.X. Frequency response function-based model updating of flexible vehicle body using experiment modal parameter. *Veh. Syst. Dyn.* **2021**. [CrossRef]
21. Costa, J.N.; Antunes, P.; Magalhaes, H.; Pombo, J.; Ambrosio, J. A finite element methodology to model flexible tracks with arbitrary geometry for railway dynamics applications. *Comput. Struct.* **2021**, *254*, 106519. [CrossRef]
22. Krauklis, A.E.; Gagani, A.I.; Echtermeyer, A.T. Prediction of Orthotropic Hygroscopic Swelling of Fiber-Reinforced Composites from Isotropic Swelling of Matrix Polymer. *J. Compos. Sci.* **2019**, *3*, 10. [CrossRef]
23. Krauklis, A.E.; Gagani, A.I.; Echtermeyer, A.T. Near-Infrared Spectroscopic Method for Monitoring Water Content in Epoxy Resins and Fiber-Reinforced Composites. *Materials* **2018**, *11*, 586. [CrossRef] [PubMed]
24. Gagani, A.I.; Monsås, A.B.; Krauklis, A.E.; Echtermeyer, A.T. The effect of temperature and water immersion on the interlaminar shear fatigue of glass fiber epoxy composites using the I-beam method. *Compos. Sci. Technol.* **2019**, *181*, 107703. [CrossRef]
25. Rocha, A.V.M.; Akhavan-Safar, A.; Carbas, R.; Marques, E.A.S.; Goyal, R.; El-zein, M.; da Silva, L.F.M. Numerical analysis of mixed-mode fatigue crack growth of adhesive joints using CZM. *Theor. Appl. Fract. Mechanics.* **2020**, *106*, 102493. [CrossRef]
26. Rocha, A.V.M.; Akhavan-Safar, A.; Carbas, R.; Marques, E.A.S.; Goyal, R.; El-zein, M.; da Silva, L.F.M. Fatigue crack growth analysis of different adhesive systems: Effects of mode mixity and load level. *Fatigue Fract. Eng. Mater. Struct.* **2020**, *43*, 330–341. [CrossRef]
27. Cammarata, A. Global modes for the reduction of flexible multibody systems Methodology and complexity. *Multibody Syst. Dyn.* **2021**, *53*, 59–83. [CrossRef]
28. Manca, A.G.; Pappalardo, C.M. Topology Optimization Procedure of Aircraft Mechanical Components Based on Computer-Aided Design, Multibody Dynamics, and Finite Element Analysis. In *Advances in Design, Simulation and Manufacturing III: Mechanical and Chemical Engineering, Proceedings of the 3rd International Conference on Design, Simulation, Manufacturing-(DSMIE), Kharkiv, Ukraine, 9–12 June 2020*; Book Series: Lecture Notes in Mechanical Engineering; Springer: Cham, Swizerland, 2020; Volume 2, pp. 159–168. [CrossRef]
29. Lu, H.J.; Rui, X.T.; Zhang, X.P. A computationally efficient modeling method for the vibration analyses of two-dimensional system structures using reduce transfer matrix method for multibody system. *J. Sound Vib.* **2021**, *502*, 116096. [CrossRef]
30. Liu, X.; Sun, C.L.; Banerjee, J.R.; Dan, H.C.; Chang, L. An exact dynamic stiffness method for multibody systems consisting of beams and rigid-bodies. *Mech. Syst. Signal Processing* **2021**, *150*, 107264. [CrossRef]
31. Raoofian, A.; Taghvaeipour, A.; Kamali, E.A. Elastodynamic analysis of multibody systems and parametric mass matrix derivation. *Mech. Based Des. Struct. Mach.* 2020; early access. [CrossRef]
32. Jeong, S.; Yoo, H.H. Generalized classical Ritz method for modeling geometrically nonlinear flexible multibody systems having a general topology. *Int. J. Mech. Sci.* **2020**, *181*, 105687. [CrossRef]
33. Wang, G.; Qi, Z.H.; Xu, J.S. A high-precision co-rotational formulation of 3D beam elements for dynamic analysis of flexible multibody systems. *Comput. Methods Appl. Mech. Eng.* **2020**, *360*, 112701. [CrossRef]
34. Hou, Y.S.; Liu, C.; Hu, H.Y. Component-level proper orthogonal decomposition for flexible multibody systems. *Comput. Methods Appl. Mech. Eng.* **2020**, *361*, 112690. [CrossRef]
35. Bagci, C. Elastodynamic Response of Mechanical Systems using Matrix Exponential Mode Uncoupling and Incremental Forcing Techniques with Finite Element Method. In Proceedings of the Sixth Word Congress on Theory of Machines and Mechanisms, India, Delhi, 15–20 December 1983; p. 472.
36. Bahgat, B.M.; Willmert, K.D. Finite Element Vibrational Analysis of Planar Mechanisms. *Mech. Mach. Theory* **1976**, *11*, 47. [CrossRef]
37. Cleghorn, W.L.; Fenton, E.G.; Tabarrok, K.B. Finite Element Analysis of High-Speed Flexible Mechanisms. *Mech. Mach. Theory* **1981**, *16*, 407. [CrossRef]

38. Vlase, S.; Dănăşel, C.; Scutaru, M.L.; Mihălcică, M. Finite Element Analysis of a Two-Dimensional Linear Elastic Systems with a Plane "rigid Motion. *Rom. Journ. Phys.* **2014**, *59*, 476–487.
39. Deü, J.-F.; Galucio, A.C.; Ohayon, R. Dynamic responses of flexible-link mechanisms with passive/active damping treatment. *Comput. Struct.* **2008**, *86*, 258–265. [CrossRef]
40. Zhang, X.; Erdman, A.G. Dynamic responses of flexible linkage mechanisms with viscoelastic constrained layer damping treatment. *Comput. Struct.* **2001**, *79*, 1265–1274. [CrossRef]
41. Shi, Y.M.; Li, Z.F.; Hua, H.X.; Fu, Z.F.; Liu, T.X. The Modeling and Vibration Control of Beams with Active Constrained Layer Damping. *J. Sound Vib.* **2001**, *245*, 785–800. [CrossRef]
42. Hou, W.; Zhang, X. Dynamic analysis of flexible linkage mechanisms under uniform temperature change. *J. Sound Vib.* **2009**, *319*, 570–592. [CrossRef]
43. Neto, M.A.; Ambrósio, J.A.C.; Leal, R.P. Composite materials in flexible multibody systems. *Comput. Methods Appl. Mech. Eng.* **2006**, *195*, 6860–6873. [CrossRef]
44. Piras, G.; Cleghorn, W.L.; Mills, J.K. Dynamic finite-element analysis of a planar high-speed, high-precision parallel manipulator with flexible links. *Mech. Mach. Theory* **2005**, *40*, 849–862. [CrossRef]
45. Gibbs, J.W. On the fundamental formulae of dynamics. *Am. J. Math.* **1879**, *2*, 49–64. [CrossRef]
46. Appell, P. Sur une forme générale des equations de la dynamique. *J. Reine Angew. Math.* **1900**, *121*, 310–319.
47. Mirtaheri, S.M.; Zohoor, H. The Explicit Gibbs-Appell Equations of Motion for Rigid-Body Constrained Mechanical System. In Proceedings of the RSI International Conference on Robotics and Mechatronics ICRoM, Tehran, Iran, 23–25 October 2018; pp. 304–309.
48. Korayem, M.H.; Dehkordi, S.F. Motion equations of cooperative multi flexible mobile manipulator via recursive Gibbs-Appell formulation. *Appl. Math. Model.* **2019**, *65*, 443–463. [CrossRef]
49. Shafei, A.M.; Shafei, H.R. A systematic method for the hybrid dynamic modeling of open kinematic chains confined in a closed environment. *Multibody Syst. Dyn.* **2017**, *38*, 21–42. [CrossRef]
50. Korayem, M.H.; Dehkordi, S.F. Derivation of dynamic equation of viscoelastic manipulator with revolute-prismatic joint using recursive Gibbs-Appell formulation. *Nonlinear Dyn.* **2017**, *89*, 2041–2064. [CrossRef]
51. Marin, M.; Ellahi, R.; Chirilă, A. On solutions of Saint-Venant's problem for elastic dipolar bodies with voids. *Carpathian J. Math.* **2017**, *33*, 219–232. [CrossRef]
52. Cheng, Y.D.; Wang, Z.X. A new discontinuous Galerkin finite element method for directly solving the Hamilton-Jacobi equations. *J. Comput. Phys.* **2014**, *268*, 134–153. [CrossRef]
53. Zheng, F.; Qiu, J.X. Directly solving the Hamilton-Jacobi equations by Hermite WENO Schemes. *J. Comput. Phys.* **2016**, *307*, 423–445. [CrossRef]
54. Anguelov, R.; Lubuma, J.M.S.; Minani, F. A monotone scheme for Hamilton-Jacobi equations via the nonstandard finite difference method. *Math. Methods Appl. Sci.* **2010**, *33*, 41–48. [CrossRef]
55. Liu, H.L.; Pollack, M.; Saran, H. Alternating Evolution Schemes for Hamilton-Jacobi Equations. *SIAM J. Sci. Comput.* **2013**, *35*, A122–A149. [CrossRef]
56. Rong, B.; Rui, X.T.; Tao, L.; Wang, G.P. Theoretical modeling and numerical solution methods for flexible multibody system dynamics. *Nonlinear Dyn.* **2019**, *98*, 1519–1553. [CrossRef]
57. Vlase, S.; Teodorescu, P.P. Elasto-Dynamics of a Solid with a General "RIGID" Motion using FEM Model. Part I. Theoretical Approach. *Rom. J. Phys.* **2013**, *58*, 872–881.
58. Vlase, S.; Negrean, I.; Marin, M.; Nastac, S. Kane's Method-Based Simulation and Modeling Robots with Elastic Elements, Using Finite Element Method. *Mathematics* **2020**, *8*, 805. [CrossRef]
59. Mitu, G.L.; Chircan, E.; Scutaru, M.L.; Vlase, S. Kane's Formalism Used to the Vibration Analysis of a Wind Water Pump. *Symmetry* **2020**, *12*, 1030. [CrossRef]
60. Ursu-Fisher, N. *Elements of Analytical Mechanics*; House of Science Book Press: Cluj-Napoca, Romania, 2015.
61. Vlase, S.; Negrean, I.; Marin, M.; Scutaru, M.L. Energy of Accelerations Used to Obtain the Motion Equations of a Three-Dimensional Finite Element. *Symmetry* **2020**, *12*, 321. [CrossRef]
62. Negrean, I.; Kacso, K.; Schonstein, C.; Duca, A. *Mechanics*; UTPESS: Cluj Napoca, Romania, 2012.
63. Vlase, S.; Marin, M.; Scutaru, M.L. Maggi's equations used in the finite element analysis of the multibody systems with elastic elements. *Mathematics* **2020**, *8*, 399. [CrossRef]
64. de Jalón, J.G.; Callejo, A.; Hidalgo, A. Efficient Solution of Maggi's Equations. *J. Comput. Nonlinear Dyn.* **2012**, *7*, 021003. [CrossRef]
65. Desloge, E. The Gibbs–Appell equations of motion. *Am. J. Phys. Am. Assoc. Phys. Teach.* **1988**, *56*, 841–847. [CrossRef]