mathematics

Article

Geometric Compatibility Indexes in a Local AHP-Group Decision Making Context: A Framework for Reducing Incompatibility

Juan Aguarón , **María Teresa Escobar *** , **José María Moreno-Jiménez** and **Alberto Turón**

Grupo Decisión Multicriterio Zaragoza (GDMZ), Facultad de Economía y Empresa, Universidad de Zaragoza, Gran Vía 2, 50005 Zaragoza, Spain; aguaron@unizar.es (J.A.); moreno@unizar.es (J.M.M.-J.); turon@unizar.es (A.T.)

* Correspondence: mescobar@unizar.es

Abstract: This paper deals with the measurement of the compatibility in a local AHP-Group Decision Making context. Compatibility between two individuals or decision makers is understood as the property that reflects the proximity between their positions or preferences, usually measured by a distance function. An acceptable level of incompatibility between the individual and the group positions will favour the acceptance of the collective position by the individuals. To facilitate the compatibility measurement, the paper utilises four indicators based on log quadratic distances between matrices or vectors which can be employed in accordance with the information that is available from the individual decision makers and from the group. The indicators make it possible to measure compatibility in decision problems, regardless of how the collective position and the priorities are obtained. The paper also presents a theoretical framework and a general, semi-automatic procedure for reducing the incompatibility measured by the four indicators. Using relative variations, the procedure identifies and slightly modifies the judgement of the collective matrix that further improves the indicator. This process is undertaken without modifying the initial information provided by the individuals. A numerical example illustrates the application of the theoretical framework and the procedure.

Keywords: multiple criteria analysis; AHP-group decision making; incompatibility improvement; row geometric mean; GCOMPI

Citation: Aguarón, J.; Escobar, M.T.; Moreno-Jiménez, J.M.; Turón, A. Geometric Compatibility Indexes in a Local AHP-Group Decision Making Context: A Framework for Reducing Incompatibility. *Mathematics* **2022**, *10*, 278. https://doi.org/10.3390/math10020278

Academic Editor: Pascale Zaraté

Received: 27 December 2021
Accepted: 13 January 2022
Published: 17 January 2022

1. Introduction

One of the most outstanding characteristics of the Analytic Hierarchy Process (AHP) [1,2] is its suitability for addressing multiactor decision making in the three scenarios detailed by [3,4]: Group Decision Making (GDM); Negotiated Decision Making (NDM) and Systemic Decision Making (SDM).

In the first (GDM), under the principle of consensus, the individuals work together searching for a common goal. In this case, consensus refers to the approach, model, tools, or procedures (voting, aggregation, agreement, etc.) used for deriving the collective position. For the collective position to be accepted by all the actors or decision makers involved in solving the problem, or at least a significant number of them, it is necessary to guarantee an acceptable level of the incompatibility between the individual and the collective positions.

In general, the confirmation of the acceptance could be determined by means of personal intervention or by measuring the degree of incompatibility using any appropriate index. Personal intervention entails differentiated behaviour of the actors that is more typical of NDM than GDM. In GDM, as the individuals work together, the same rules are assumed for all decision makers. It is therefore necessary to measure the level of incompatibility and establish procedures for its reduction. This question is especially relevant in Consensus Reaching Processes (CRPs) where the participation of the decision makers is limited to the incorporation of their preferences at the beginning of the process.

This paper describes the measurement of the compatibility of the actors' individual positions with regards to the group collective position in a local (single criterion) AHP-GDM context. Based on the log quadratic distance between matrices or vectors, the study presents four geometric compatibility (GCOMPI) indexes (three of which are new) and a framework for reducing incompatibility.

The framework uses similar ideas to those of the procedure proposed in [5]. In particular, it assumes relative rather than absolute changes because they better reflect the perceived importance of the modifications. In addition, to improving incompatibility, it modifies the judgements of the collective matrix, without modifying the initial information provided by the individuals. Finally, following Saaty's proposal [6], only slight modifications are included, i.e., the relative changes allowed will be limited by a permissibility parameter set by the decision makers or the facilitator (if there is one).

In [5], the authors, propose a semi-automatic procedure (AEM-COM) to reduce the incompatibility measured by an indicator (GCOMPI) that combines the individual judgement matrices (input of the model) and the collective priority vector (output of the model). It assumes that the individual and collective judgement matrices are known, and that the collective priority vector is derived by the RGM. The current work introduces three new indicators which combine: (i) the individual and collective matrices; (ii) the individual priority vectors and the collective matrix; and (iii) the individual and collective priority vectors. These original indicators and their respective incompatibility improvement procedures will be employed in line with the information available and the context (local or global). This is particularly relevant for adapting compatibility measurement and incompatibility improvement to any situation, irrespective of the procedures employed to obtain the collective matrix and to derive the priority vectors.

The paper is structured as follows: Section 2 outlines the background of AHP-GDM and the measurement of compatibility in AHP; Section 3 presents four compatibility measures based on log quadratic distances; Section 4 sets out the theoretical results necessary for the reduction of incompatibility using these measures; Section 5 includes a general, semi-automatic procedure for reducing the incompatibility measured with the different GCOMPIs by revising judgements of the collective matrix; Section 6 highlights the most important conclusions of the study.

2. Background

2.1. AHP in a Multiactor Decision Making Context

One of the most widely extended and applied multi-criteria decision making techniques is AHP. The AHP methodology consists of three stages [2]: (i) Modelling; (ii) Valuation; and (iii) Prioritisation and Synthesis. In the first stage, a hierarchical structure including all the relevant factors (goal, criteria, sub-criteria of different levels, and alternatives) of the decision problem is constructed; the second stage incorporates the preferences of the actors by means of eliciting the judgements of the pairwise comparison matrices (PCMs); the third stage calculates the local priorities using a prioritisation method, the global priorities by means of the composition principle and the total priorities of the alternatives with an aggregation procedure. The most common prioritisation methods are the Eigenvector (EV) [2] and the Row Geometric Mean (RGM) [2,7].

To validate the local priorities derived from a PCM, the internal coherence (consistency) of decision makers when eliciting their judgements must be measured and its value must be below a given threshold. The Consistency Ratio (CR) [8] is generally used for the EV and the Geometric Consistency Index (GCI) [7,9] is used with the RGM.

AHP is a multicriteria decision making technique that has great potential in multiactor contexts. The two aggregation procedures traditionally employed in local AHP-Group Decision Making (AHP-GDM) situations [10–12] are the Aggregation of Individual Judgements (AIJ) and the Aggregation of Individual Priorities (AIP). A review of these and other AHP-GDM approaches can be seen in [5]. Both AIJ and AIP use weighted geometric averages, of judgments (AIJ) and priorities (AIP), to obtain the collective priority vector.

As is well known, the average is representative of the position of the group when the elements are sufficiently homogeneous. Unfortunately, this issue is rarely taken into account in AHP-GDM when using the two procedures or any other approach proposed in the scientific literature [5].

In this study, the measurement of homogeneity in the case of a local AHP-GDM context is carried out by means of a compatibility indicator suitable for addressing the reciprocity property of AHP [13]. In what follows, compatibility between two individuals or decision makers is understood as the property that reflects the proximity between their positions or preferences, usually measured by a distance function. In an analogous way, compatibility in a GDM context is defined as the distance (objective measure) between the individuals and the collective positions.

2.2. Compatibility in AHP-GDM

The evaluation of compatibility between individual and collective positions, requires the establishment of compatibility measures, procedures for their improvement (agreement and consensus searching processes) and thresholds that allow validation of the use of collective priorities that represent the individual priorities. Compatibility also refers to the internal coherence of the group when selecting its collective priority vector, that is to say, its representativeness in relation to the individual positions. In terms of decision making, ref. [14] defines compatibility as the sharing of similar value systems. Incompatibility improvement, or consensus searching, has been studied from different perspectives that depend on the level of intervention of the actors (automatic, semi-automatic, personal), and the approximation.

A number of ordinal and cardinal measures have been proposed for the evaluation of compatibility [15–17]. Ordinal indicators use rankings of the alternatives. As pointed out by [18], this is a myopic view of reality in the context of AHP (weighted spaces) so cardinal indicators are preferred. Some of the cardinal compatibility measures for the evaluation of the group coherence are: S-compatibility [19], G-compatibility [20], the coefficient of multiple determination R^2 [21], and the Geometric Compatibility Index (GCOMPI) [16,17]. This last indicator is particularly relevant for the measurement of collective coherence due to its analytical properties and the suitability of log quadratic distances for PCMs [13].

The expression of this GCOMPI indicator (presented in Section 3.2 as $GCOMPI_2$, see Definition 6), measures the compatibility between the individual PCMs (input of decision makers) and the collective priority vector (output of the group). The paper examines this compatibility measure and defines similar indicators whose expressions differ with regards to the elements being compared.

An issue associated with the measurement of compatibility concerns the facilitation of procedures for the reduction of incompatibility. In [5] a semi-automatic procedure (named AEM-COM) for reducing incompatibility when using the $GCOMPI_2$ as the compatibility measure is presented. The AEM-COM procedure identifies and slightly modifies the judgements of the collective PCM that further improve the indicator. Among the most outstanding characteristics of this procedure are: (i) it considers relative changes; (ii) as the initial individual matrices are not modified, it does not require the continuous intervention of decision makers; (iii) it provides closed (optimal) results in terms of the judgements that most rapidly reduce incompatibility; and, (iv) it applies slight modifications to the judgements that result in slight modifications of the associated priority vector. Section 4 summarises the theoretical results on which that proposal was based and proves similar results for the other indicators that are defined. The semi-automatic procedure put forward in [5] can also be adapted for the other indicators. An outline of the general procedure is presented in Section 5. As with the AEM-COM, it considers modifications made in relative terms (as recommended by Kahneman and Tversky) [22] which are bounded to guarantee slight modifications in the collective matrix (Saaty) [6].

3. Compatibility Indexes in AHP

This section presents the definitions of the Geometric Compatibility Indexes based on log quadratic distances, in which the elements being compared in the expressions are different. This is followed by ideas for using each of the indicators; they are all continuous and derivable functions that capture the idea of reciprocity that is characteristic of AHP. All the PCMs and the priority vectors are of order n.

3.1. Geometric Compatibility Indexes. Basic Definitions

The basic definitions of the Geometric Compatibility Indexes correspond to the log quadratic distances between two matrices, a matrix and a vector, and two vectors.

Definition 1. *Let* $A = (a_{ij})$ *and* $B = (b_{ij})$ *be two PCMs. The* Geometric Compatibility Index *between* A *and* B *is defined as*

$$\text{GCOMPI}(A, B) = \frac{1}{(n-1)(n-2)} \sum_{i,j} \log^2 \frac{a_{ij}}{b_{ij}} \tag{1}$$

Definition 2. *Let* $A = (a_{ij})$ *be a PCM and* $v = (v_i)$ *be a priority vector. The* Geometric Compatibility Index *between* A *and* v *is defined as*

$$\text{GCOMPI}(A, v) = \frac{1}{(n-1)(n-2)} \sum_{i,j} \log^2 \frac{a_{ij}}{v_i/v_j} \tag{2}$$

Definition 3. *Let* $u = (u_i)$ *be a priority vector and* $B = (b_{ij})$ *be a PCM. The* Geometric Compatibility Index *between* u *and* B *is defined as*

$$\text{GCOMPI}(u, B) = \frac{1}{(n-1)(n-2)} \sum_{i,j} \log^2 \frac{u_i/u_j}{b_{ij}} \tag{3}$$

Definition 4. *Let* $u = (u_i)$ *and* $v = (v_i)$ *be two priority vectors. The* Geometric Compatibility Index *between* u *and* v *is defined as*

$$\text{GCOMPI}(u, v) = \frac{1}{(n-1)(n-2)} \sum_{i,j} \log^2 \frac{u_i/u_j}{v_i/v_j} \tag{4}$$

Remark 1. *It is obvious that* $\text{GCOMPI}(A, B) = \text{GCOMPI}(B, A)$, $\text{GCOMPI}(A, v) = \text{GCOMPI}(v, A)$ *and* $\text{GCOMPI}(u, v) = \text{GCOMPI}(v, u)$

3.2. Geometric Compatibility Indexes for Families of Matrices and Vectors

Based on the aforementioned distances, the compatibility indicators for AHP-GDM are defined below. Their expressions differ on the elements being compared and four possible combinations are analysed (see Figure 1): (i) the family of the individual judgement matrices with respect to the collective matrix; (ii) the family of the individual judgement matrices with respect to the collective priority vector; (iii) the family of the individual priority vectors with respect to the collective judgement matrix; and, (iv) the family of the individual priority vectors with respect to the collective priority vector.

The notation is as follows:

Let $\mathscr{A} = \left\{ A^{(k)} = \left(a_{ij}^{(k)}\right), k = 1, \ldots, d \right\}$ be a family of PCMs provided by d decision makers with weights α_k $\left(\sum_{k=1}^{d} \alpha_k = 1\right)$; and $\mathscr{w} = \left\{ w^{(k)}, k = 1, \ldots, d \right\}$ be a family of priority vectors provided by these decision makers.

Let $A^G = \left(a_{ij}^G\right)$ be the collective matrix obtained using the weighted geometric mean of the individual judgements of the matrices of family $\mathscr{A}$, i.e., $a_{ij}^G = \prod_{k=1}^{d} \left(a_{ij}^{(k)}\right)^{\alpha_k}$;

and $w^{G|J}$ be the priority vector of A^G obtained following a prioritisation method (AIJ aggregation procedure).

Let $w^{G|P}$ be the priority vector obtained applying the AIP aggregation procedure to the matrices of family $\mathscr{A}$. Its elements are given by the weighted geometric mean of the individual priorities, i.e., $w_i^{G|P} = \prod_{k=1}^{d} \left(w_i^{(k)}\right)^{\alpha_k}$.

When the RGM is used as the prioritisation method, both AIJ and AIP priority vectors coincide ($w^{G|J} = w^{G|P}$) [23,24].

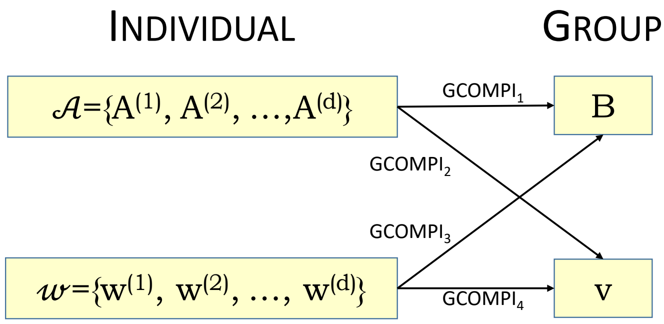

Figure 1. Outline of the GCOMPI measures.

Definition 5. *Let* $\mathscr{A} = \left\{A^{(k)}\right\}$ *be a family of PCMs and* $B = (b_{ij})$ *be a collective PCM. The* Geometric Compatibility Index *between family* $\mathscr{A}$ *and matrix B is defined as*

$$\begin{aligned} \mathrm{GCOMPI}_1(\mathscr{A}, B) &= \sum_{k=1}^{d} \alpha_k \mathrm{GCOMPI}(A^{(k)}, B) \\ &= \frac{1}{(n-1)(n-2)} \sum_{k=1}^{d} \left(\alpha_k \sum_{i,j} \log^2 \frac{a_{ij}^{(k)}}{b_{ij}} \right) \end{aligned} \tag{5}$$

In our context (local AHP-GDM), this expression measures the compatibility between the input of the decision makers (individual PCMs) and the collective PCM obtained by an aggregation procedure. This expression does not use information from the priority vectors, so it is an appropriate compatibility indicator for situations where it is not necessary to force the use of a particular prioritisation procedure. It can be applied irrespective of the prioritisation procedure each decision maker uses individually, as well as the one they apply collectively, as a group.

Definition 6. *Let* $\mathscr{A} = \left\{A^{(k)}\right\}$ *be a family of PCMs and* $v = (v_i)$ *be a collective priority vector. The* Geometric Compatibility Index *between family* $\mathscr{A}$ *and vector v is defined as*

$$\begin{aligned} \mathrm{GCOMPI}_2(\mathscr{A}, v) &= \sum_{k=1}^{d} \alpha_k \mathrm{GCOMPI}(A^{(k)}, v) \\ &= \frac{1}{(n-1)(n-2)} \sum_{k=1}^{d} \left(\alpha_k \sum_{i,j} \log^2 \frac{a_{ij}^{(k)}}{v_i / v_j} \right) \end{aligned} \tag{6}$$

This expression is already defined in the AHP-GDM literature [16,17], it measures the compatibility between the input of the decision makers (individual PCMs) and the output of the group (collective priority vector) used to rank the alternatives and make decisions. This indicator does not make use of individual priority vectors. As with the previous indicator, it may be applied irrespective of the prioritisation procedure each decision maker may use, or when none is applied. However, the use of a particular prioritisation procedure for the derivation of the collective priority vector from the collective PCM is implicit.

Definition 7. *Let* $\mathscr{w} = \left\{ w^{(k)} \right\}$ *be a family of priority vectors and* $B = (b_{ij})$ *be a collective PCM. The* Geometric Compatibility Index *between family* $\mathscr{w}$ *and matrix* B *is defined as*

$$\begin{aligned} \text{GCOMPI}_3(\mathscr{w}, B) &= \sum_{k=1}^{d} \alpha_k \text{GCOMPI}(w^{(k)}, B) \\ &= \frac{1}{(n-1)(n-2)} \sum_{k=1}^{d} \left(\alpha_k \sum_{i,j} \log^2 \frac{w_i^{(k)} / w_j^{(k)}}{b_{ij}} \right) \end{aligned} \quad (7)$$

In a local AHP-GDM context, this expression measures the compatibility between the priority vectors associated with each one of the decision makers (obtained using a prioritisation method or provided directly by them) and the collective PCM obtained by an aggregation procedure. This compatibility measure is independent of the prioritisation procedure used to obtain the collective priority vector.

Definition 8. *Let* $\mathscr{w} = \left\{ w^{(k)} \right\}$ *be a family of priority vectors and* $v = (v_i)$ *be a collective priority vector. The* Geometric Compatibility Index *between family* $\mathscr{w}$ *and vector* v *is defined as*

$$\begin{aligned} \text{GCOMPI}_4(\mathscr{w}, v) &= \sum_{k=1}^{d} \alpha_k \text{GCOMPI}(w^{(k)}, v) \\ &= \frac{1}{(n-1)(n-2)} \sum_{k=1}^{d} \left(\alpha_k \sum_{i,j} \log^2 \frac{w_i^{(k)} / w_j^{(k)}}{v_i / v_j} \right) \end{aligned} \quad (8)$$

This expression measures the compatibility between the priority vectors associated with each one of the decision makers (obtained using a prioritisation method or provided directly by them) and the output of the group (collective priority vector) used to rank the alternatives and make decisions. The use of a particular prioritisation procedure for the derivation of the collective priority vector from the collective PCM is implicit.

The last two indicators, GCOMPI_3 and GCOMPI_4, may be used in a local context (one criterion) and in a more general context (hierarchy) with several criteria. In the latter, the individual priority vectors would correspond to those associated with the hierarchy. They may also be appropriate in NDM situations, but these two extensions are not dealt with in this paper.

The following result shows the relationships between different expressions of the GCOMPIs indicators when the consistent matrices associated with the priority vectors are taken into account.

Remark 2. *Let* $\mathscr{W} = \left\{ W^{(k)}, k = 1, \ldots, d \right\}$*, be the family of consistent matrices associated with priority vectors* $w^{(k)}$*,* $W^{(k)} = (w_{ij}^{(k)}) = (w_i^{(k)} / w_j^{(k)})$*, and* $V = (v_{ij}) = (v_i / v_j)$ *be the consistent matrix associated with a collective priority vector* v*. It holds that:*

$$\text{GCOMPI}_1(\mathscr{A}, V) = \text{GCOMPI}_2(\mathscr{A}, v) \quad (9)$$

$$\text{GCOMPI}_1(\mathscr{W}, B) = \text{GCOMPI}_3(\mathscr{w}, B) \quad (10)$$

$$\text{GCOMPI}_1(\mathscr{W}, V) = \text{GCOMPI}_4(\mathscr{w}, v) \quad (11)$$

$$\text{GCOMPI}_2(\mathscr{W}, v) = \text{GCOMPI}_4(\mathscr{w}, v) \quad (12)$$

$$\text{GCOMPI}_3(\mathscr{w}, V) = \text{GCOMPI}_4(\mathscr{w}, v) \quad (13)$$

4. A Theoretical Framework for Reducing Geometric Compatibility Measures in a Local Context

This section gives the theoretical results necessary for reducing, in a local AHP-GDM context, the incompatibility measured by each one of the four GCOMPI_i ($i = 1, \ldots, 4$) indicators. The results associated with the GCOMPI_1, GCOMPI_3 and GCOMPI_4 are original and were obtained by following developments parallel to those used for the GCOMPI_2 in [5]. To maintain an homogeneous notation with that paper, in what follows, the collective matrix is denoted as $P = (p_{ij})$. The theoretical results allow the identification of the judgement of the collective matrix that produces the greater reduction in the corresponding GCOMPI_i. In all cases, it is assumed that the collective matrix P is known.

The sequence followed to identify the judgement of the collective PCM, $P = (p_{ij})$, that should be slightly modified to reduce the incompatibility is analogous for the four indicators. It consists of five stages or results: (i) a theorem that identifies the judgement p_{rs} which, in absolute terms, most rapidly reduces the GCOMPI_i; (ii) a theorem that establishes the change produced in the value of the GCOMPI_i when the judgement p_{rs} is modified in relative terms; (iii) a theorem that provides the judgement p_{rs} which, in relative terms, most rapidly reduces the GCOMPI_i; (iv) a corollary that applies the previous theorem when considering small variations; and, (v) a corollary that provides the relative variation of judgement p_{rs} which produces the greatest decrease of the GCOMPI_i and the maximum reduction of that measure.

It is worth noting that the use of changes in relative terms (third theorem for each indicator) is justified by the fact that relative changes capture the human perception of the relevance of modifications better than absolute changes. Moreover [5], the use of relative changes follows the suggestions of [22]: "the preferences, associated with the same physical magnitude, are relative rather than absolute, depending on the situation of gain or loss, and also on the point of departure" and [25]: "small errors (in terms of absolute values) may significantly change the final rankings if they are big in relation to the true value". The importance given to the modification of a unit in the value of a judgement in the pairwise comparison matrix depends on the initial value; increasing a unit in a small judgement, such as 2 (an increase of 50%) is not the same as in a medium judgement such as 5 (an increase of 20%).

The first two indicators measure the compatibility between the individual pairwise comparison matrices and the collective position (the collective matrix with GCOMPI_1 and the collective priority vector with GCOMPI_2).

4.1. GCOMPI_1

The theoretical results that support the procedure to reduce incompatibility measured by the GCOMPI_1 through modifying the judgements of the collective matrix are as follows.

The first theorem identifies the judgement p_{rs} that most rapidly reduces the GCOMPI_1.

Theorem 1. *Let $\mathscr{A} = \left\{A^{(k)}\right\}$ be a family of PCMs and $P = (p_{ij})$, $i, j = 1, \ldots, n$, be a collective PCM. It holds that*

$$\frac{\partial \text{GCOMPI}_1(\mathscr{A}, P)}{\partial p_{rs}} = \frac{4}{(n-1)(n-2)} \frac{1}{p_{rs}} \log \frac{p_{rs}}{a^G_{rs}} \tag{14}$$

Proof. It is obvious that

$$\text{GCOMPI}_1(\mathscr{A}, P) = \frac{1}{(n-1)(n-2)} \sum_{k=1}^{d} \left(\alpha_k \sum_{i,j} \log^2 \frac{a^{(k)}_{ij}}{p_{ij}} \right) = \frac{2}{(n-1)(n-2)} \sum_{k=1}^{d} \left(\alpha_k \sum_{i<j} \log^2 \frac{a^{(k)}_{ij}}{p_{ij}} \right)$$

and then

$$
\begin{aligned}
\frac{\partial \mathrm{GCOMPI}_1(\mathscr{A},P)}{\partial p_{rs}} &= \frac{2}{(n-1)(n-2)}\sum_{k=1}^{d}\frac{\partial}{\partial p_{rs}}\left(\alpha_k\sum_{i<j}\log^2\frac{a_{ij}^{(k)}}{p_{ij}}\right)\\
&= \frac{2}{(n-1)(n-2)}\sum_{k=1}^{d}\left(\alpha_k\frac{-2}{p_{rs}}\log\frac{a_{rs}^{(k)}}{p_{rs}}\right) = \frac{4}{(n-1)(n-2)}\frac{1}{p_{rs}}\sum_{k=1}^{d}\left(\alpha_k\log\frac{p_{rs}}{a_{rs}^{(k)}}\right)\\
&= \frac{4}{(n-1)(n-2)}\frac{1}{p_{rs}}\log\frac{p_{rs}}{\prod_{k=1}^{d}\left(a_{rs}^{(k)}\right)^{\alpha_k}} = \frac{4}{(n-1)(n-2)}\frac{1}{p_{rs}}\log\frac{p_{rs}}{a_{rs}^{G}}
\end{aligned}
$$

□

From previous theorem, if $p_{rs} < a_{rs}^G$, the derivative is negative and as the value of p_{rs} increases, the value of $\mathrm{GCOMPI}_1(\mathscr{A},P)$ decreases. If $p_{rs} > a_{rs}^G$, the reasoning is analogous. The optimum (greatest reduction) is reached for $p_{rs} = a_{rs}^G$.

The previous result considers absolute variations in the judgements. As previously explained, the modification of a unit in the value of a judgement depends on the initial value being small or large. Relative variations are now looked at: if judgement p_{rs} is modified with p'_{rs} as its new value, $t_{rs} = p'_{rs}/p_{rs}$ denotes the relative variation of this judgement and $P' = \left(p'_{ij}\right)$ the modified collective PCM. The modified value of the GCOMPI_1 is given by the following theorem.

Theorem 2. *Let $\mathscr{A} = \left\{A^{(k)}\right\}$ be a family of PCMs and $P = \left(p_{ij}\right)$, $i,j = 1,\ldots,n$, be a collective PCM. It holds that*

$$
\mathrm{GCOMPI}_1(\mathscr{A},P') = \mathrm{GCOMPI}_1(\mathscr{A},P) + \frac{2}{(n-1)(n-2)}\log t_{rs}\left(2\log\frac{p_{rs}}{a_{rs}^G} + \log t_{rs}\right) \tag{15}
$$

Proof. When judgement p_{rs} changes to p'_{rs}, the only terms of expression (5) that differs are those corresponding to indexes (r,s) and (s,r). Moreover, $\log^2 a_{sr}^{(k)}/p_{sr} = \log^2 a_{rs}^{(k)}/p_{rs}$ by the reciprocal property, and then:

$$
\begin{aligned}
\Delta\mathrm{GCOMPI}_1 &= \mathrm{GCOMPI}_1(\mathscr{A},P') - \mathrm{GCOMPI}_1(\mathscr{A},P)\\
&= \frac{2}{(n-1)(n-2)}\sum_{k=1}^{d}\alpha_k\left(\log^2\frac{a_{rs}^{(k)}}{p'_{rs}} - \log^2\frac{a_{rs}^{(k)}}{p_{rs}}\right)\\
&= \frac{2}{(n-1)(n-2)}\sum_{k=1}^{d}\alpha_k\left(\log\frac{a_{rs}^{(k)}}{p'_{rs}} + \log\frac{a_{rs}^{(k)}}{p_{rs}}\right)\left(\log\frac{a_{rs}^{(k)}}{p'_{rs}} - \log\frac{a_{rs}^{(k)}}{p_{rs}}\right)\\
&= \frac{2}{(n-1)(n-2)}\sum_{k=1}^{d}\alpha_k\left(2\log a_{rs}^{(k)} - \log p'_{rs}p_{rs}\right)\log\frac{p_{rs}}{p'_{rs}}\\
&= \frac{2}{(n-1)(n-2)}\log\frac{1}{t_{rs}}\left(2\sum_{k=1}^{d}\alpha_k\log a_{rs}^{(k)} - \log t_{rs}p_{rs}^2\right)\\
&= \frac{2}{(n-1)(n-2)}\log\frac{1}{t_{rs}}\left(2\log\prod_{k=1}^{d}\left(a_{rs}^{(k)}\right)^{\alpha_k} - 2\log p_{rs} - \log t_{rs}\right)\\
&= \frac{2}{(n-1)(n-2)}\log\frac{1}{t_{rs}}\left(2\log a_{rs}^G - 2\log p_{rs} - \log t_{rs}\right)\\
&= \frac{2}{(n-1)(n-2)}\log t_{rs}\left(2\log\frac{p_{rs}}{a_{rs}^G} + \log t_{rs}\right)
\end{aligned}
$$

□

The following theorem identifies judgement p_{rs} that most rapidly reduces the GCOMPI_1 when considering relative changes.

Corollary 1. *Let $\mathscr{A} = \left\{A^{(k)}\right\}$ be a family of PCMs and $P = (p_{ij})$, $i,j = 1,\dots,n$, be a collective PCM. It holds that*

$$\frac{\partial \mathrm{GCOMPI}_1(\mathscr{A}, P)}{\partial t_{rs}} = \frac{4}{(n-1)(n-2)} \frac{1}{t_{rs}} \log \frac{p_{rs} t_{rs}}{a^G_{rs}} \tag{16}$$

Proof. Immediate from Theorem 2. □

When considering small variations, the value of t_{rs} moves around 1, so the expression of the previous partial derivative, (16), is given by next corollary.

Corollary 2. *Let $\mathscr{A} = \left\{A^{(k)}\right\}$ be a family of PCMs and $P = (p_{ij})$, $i,j = 1,\dots,n$, be a collective PCM. It holds that*

$$\left.\frac{\partial \mathrm{GCOMPI}_1(\mathscr{A}, P)}{\partial t_{rs}}\right|_{t_{rs}=1} = \frac{4}{(n-1)(n-2)} \log \frac{p_{rs}}{a^G_{rs}} \tag{17}$$

Proof. Immediate from Corollary 1. □

Finally, the following corollary establishes the optimal relative variation of a judgement, and the maximum reduction that can be achieved.

Corollary 3. *Let $\mathscr{A} = \left\{A^{(k)}\right\}$ be a family of PCMs and $P = (p_{ij})$, $i,j = 1,\dots,n$, be a collective PCM. The relative variation of judgement p_{rs} that produces the greatest decrease of $\mathrm{GCOMPI}_1(\mathscr{A}, P)$ is*

$$t^*_{rs} = \frac{p^*_{rs}}{p_{rs}} = \frac{a^G_{rs}}{p_{rs}} \tag{18}$$

and the maximum reduction of the $\mathrm{GCOMPI}_1(\mathscr{A}, v)$ is

$$\nabla^* \mathrm{GCOMPI}_1 = \frac{2}{(n-1)(n-2)} \log^2 \frac{p_{rs}}{a^G_{rs}} \tag{19}$$

Proof. Immediate from Theorem 2 and Corollary 1. □

The judgement p_{rs} that most rapidly decreases the value of the GCOMPI_1 is the one for which there is a greater relative difference between the ratios p_{rs} and a^G_{rs} (Corollary 2). This judgement is also the one that allows the greatest reduction of the GCOMPI_1 in absolute terms (Corollary 3).

From Corollary 1 it is obvious that the matrix that minimises the value $\mathrm{GCOMPI}_1(\mathscr{A}, P)$ is given by matrix A^G. This is coherent with the previous results which indicate that the modifications that produce the largest reduction in GCOMPI_1 cause the P matrix to move closer to A^G.

4.2. GCOMPI_2

This section is based on [5]. The main results are included here in order to be able to appreciate the parallelism with the other three indicators and to establish a global framework.

Theorem 3. *Let $\mathscr{A} = \left\{A^{(k)}\right\}$ be a family of PCMs, $P = (p_{ij})$ be a collective PCM and $v = (v_i)$ be the corresponding priority vector associated with P obtained with the RGM method. It holds that*

$$\frac{\partial \mathrm{GCOMPI}_2(\mathscr{A}, v)}{\partial p_{rs}} = \frac{4}{(n-1)(n-2)} \frac{1}{p_{rs}} \log \frac{v_r / v_s}{w^{G|J}_r / w^{G|J}_s} \tag{20}$$

Proof. See [5]. □

Theorem 4. *Let $\mathscr{A} = \left\{ A^{(k)} \right\}$ be a family of PCMs, $P = (p_{ij})$ be a collective PCM and $v = (v_i)$ be the corresponding priority vector associated with P obtained with the RGM method. It holds that*

$$\mathrm{GCOMPI}_2(\mathscr{A}, v') = \mathrm{GCOMPI}_2(\mathscr{A}, v) + \frac{4}{(n-1)(n-2)} \log t_{rs} \left(\frac{\log t_{rs}}{n} + \log \frac{v_r / v_s}{w_r^{G|J} / w_s^{G|J}} \right) \tag{21}$$

Proof. See [5]. □

Corollary 4. *Let $\mathscr{A} = \left\{ A^{(k)} \right\}$ be a family of PCMs, $P = (p_{ij})$ be a collective PCM and $v = (v_i)$ be the corresponding priority vector associated with P obtained with the RGM method. It holds that*

$$\frac{\partial \mathrm{GCOMPI}_2(\mathscr{A}, v)}{\partial t_{rs}} = \frac{4}{(n-1)(n-2)} \frac{1}{t_{rs}} \left(\frac{2 \log t_{rs}}{n} + \log \frac{v_r / v_s}{w_r^{G|J} / w_s^{G|J}} \right) \tag{22}$$

Proof. See [5]. □

Corollary 5. *Let $\mathscr{A} = \left\{ A^{(k)} \right\}$ be a family of PCMs, $P = (p_{ij})$ be a collective PCM and $v = (v_i)$ be the corresponding priority vector associated with P obtained with the RGM method. It holds that*

$$\left. \frac{\partial \mathrm{GCOMPI}_2(\mathscr{A}, v)}{\partial t_{rs}} \right|_{t_{rs}=1} = \frac{4}{(n-1)(n-2)} \log \frac{v_r / v_s}{w_r^{G|J} / w_s^{G|J}} \tag{23}$$

Proof. See [5]. □

Corollary 6. *Let $\mathscr{A} = \left\{ A^{(k)} \right\}$ be a family of PCMs, $P = (p_{ij})$ be a collective PCM and $v = (v_i)$ be the corresponding priority vector associated with P obtained with the RGM method. The relative variation of judgement p_{rs} that produces the greatest decrease of* $\mathrm{GCOMPI}_2(\mathscr{A}, v)$ *is*

$$t_{rs}^* = p_{rs}^* / p_{rs} = \left(\frac{w_r^{G|J} / w_s^{G|J}}{v_r / v_s} \right)^{n/2} \tag{24}$$

and the maximum reduction of the $\mathrm{GCOMPI}_2(\mathscr{A}, v)$ *is*

$$\nabla^* \mathrm{GCOMPI}_2 = \frac{n}{(n-1)(n-2)} \log^2 \frac{v_r / v_s}{w_r^{G|J} / w_s^{G|J}} \tag{25}$$

Proof. See [5]. □

Corollaries 5 and 6 show that the judgement p_{rs} for which there is a greater relative difference between the ratios v_r / v_s and $w_r^{G|J} / w_s^{G|J}$ is the one that most rapidly decreases the value of the GCOMPI_2 and it is also the one that allows the greatest reduction of this indicator in absolute terms.

For a family of PCMs $\mathscr{A}$, ref. [5] proved that

$$\min_u \mathrm{GCOMPI}_2(\mathscr{A}, u) = \mathrm{GCOMPI}_2(\mathscr{A}, w^{G|J}) \tag{26}$$

This is coherent with the previous results which indicate that the modifications leading to the further reduction of GCOMPI_2 cause the priority vector derived from the P matrix to approach $w^{G|J}$.

Figure 2 shows an outline of the reduction process when using the two previously described indicators. The input for the incompatibility reduction in cases of Sections 4.1 and 4.2 comprises the individual comparison matrices $\mathscr{A}$ and the initial collective matrix (P). The output in both cases is the collective matrix P'. When using the GCOMPI$_2$, the associated priority vector v' is derived using the RGM.

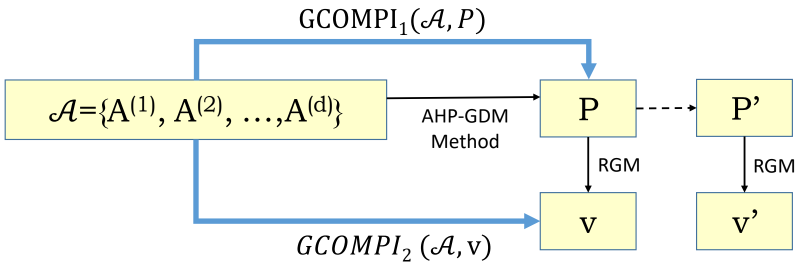

Figure 2. Outline of the incompatibility reduction of indicators GCOMPI$_1$ and GCOMPI$_2$.

The two following indicators measure the compatibility between the priority vectors of the decision makers and the collective position (the collective matrix with GCOMPI$_3$ and the collective priority vector with GCOMPI$_4$).

4.3. GCOMPI$_3$

Theorem 5. *Let $w = \left\{ w^{(k)} \right\}$ be a family of priority vectors and $P = (p_{ij})$ be a collective PCM. It holds that*

$$\frac{\partial \mathrm{GCOMPI}_3(w, P)}{\partial p_{rs}} = \frac{4}{(n-1)(n-2)} \frac{1}{p_{rs}} \log \frac{p_{rs}}{w_r^{G|P}/w_s^{G|P}} \tag{27}$$

Proof. From Remark 2 it holds that:

$$\mathrm{GCOMPI}_3(w, B) = \mathrm{GCOMPI}_1(\mathscr{W}, B)$$

using Theorem 1

$$\frac{\partial \mathrm{GCOMPI}_3(w, P)}{\partial p_{rs}} = \frac{\partial \mathrm{GCOMPI}_1(\mathscr{W}, P)}{\partial p_{rs}} = \frac{4}{(n-1)(n-2)} \frac{1}{p_{rs}} \log \frac{p_{rs}}{w_{rs}^G}$$

where

$$w_{rs}^G = \prod_{k=1}^d \left(w_{rs}^{(k)} \right)^{\alpha_k} = \prod_{k=1}^d \left(\frac{w_r^{(k)}}{w_s^{(k)}} \right)^{\alpha_k} = \frac{\prod_{k=1}^d \left(w_r^{(k)} \right)^{\alpha_k}}{\prod_{k=1}^d \left(w_s^{(k)} \right)^{\alpha_k}} = \frac{w_r^{G|P}}{w_s^{G|P}}$$

□

Theorem 6. *Let $\mathscr{A} = \left\{ A^{(k)} \right\}$ be a family of PCMs and $P = (p_{ij})$, $i, j = 1, \ldots, n$, be a collective PCM. It holds that*

$$\mathrm{GCOMPI}_3(w, P') = \mathrm{GCOMPI}_3(w, P) + \frac{2}{(n-1)(n-2)} \log t_{rs} \left(2 \log \frac{p_{rs}}{w_r^{G|P}/w_s^{G|P}} + \log t_{rs} \right) \tag{28}$$

Proof. Analogous to proof of Theorem 5. □

Corollary 7. *Let $w = \left\{ w^{(k)} \right\}$ be a family of priority vectors and $P = (p_{ij})$ be a collective PCM. It holds that*

$$\frac{\partial \mathrm{GCOMPI}_3(w, P)}{\partial t_{rs}} = \frac{4}{(n-1)(n-2)} \frac{1}{t_{rs}} \log \frac{p_{rs} t_{rs}}{w_r^{G|P}/w_s^{G|P}} \tag{29}$$

Proof. Analogous to proof of Theorem 5. □

Corollary 8. *Let $\mathscr{w} = \left\{ w^{(k)} \right\}$ be a family of priority vectors and $P = (p_{ij})$ be a collective PCM. It holds that*

$$\left.\frac{\partial \mathrm{GCOMPI}_3(\mathscr{w}, P)}{\partial t_{rs}}\right|_{t_{rs}=1} = \frac{4}{(n-1)(n-2)} \log \frac{p_{rs}}{w_r^{G|P} / w_s^{G|P}} \tag{30}$$

Proof. Immediate from Corollary 7. □

Corollary 9. *Let $\mathscr{w} = \left\{ w^{(k)} \right\}$ be a family of priority vectors and $P = (p_{ij})$ be a collective PCM. The relative variation of judgement p_{rs} that produces the greatest decrease of* $\mathrm{GCOMPI}_3(\mathscr{w}, P)$ *is*

$$t_{rs}^* = p_{rs}^* / p_{rs} = \frac{w_r^{G|P} / w_s^{G|P}}{p_{rs}} \tag{31}$$

and the maximum reduction of the $\mathrm{GCOMPI}_3(\mathscr{A}, v)$ *is*

$$\nabla^* \mathrm{GCOMPI}_3 = \frac{2}{(n-1)(n-2)} \log^2 \frac{p_{rs}}{w_r^{G|P} / w_s^{G|P}} \tag{32}$$

Proof. Immediate from Theorem 6 and Corollary 7. □

The judgement p_{rs} that most rapidly decreases the value of the GCOMPI_3 is the one for which there is a greater relative difference between the ratios p_{rs} and $w_r^{G|P} / w_s^{G|P}$ (Corollary 8). This judgement is also the one that allows the greatest reduction of the GCOMPI_3 in absolute terms (Corollary 9).

From Corollary 7 it is obvious that the matrix that minimises the value $\mathrm{GCOMPI}_3(\mathscr{w}, P)$ is given by the consistent matrix $W^{G|P}$ associated with the priority vector $w^{G|P}$. This is coherent with the previous results which suggest that the modifications leading to the further reduction of GCOMPI_3 cause the matrix P to approach the matrix $W^{G|P}$.

4.4. GCOMPI$_4$

The last indicator measures the compatibility between the individual priority vectors of a set of decision makers and the priority vector obtained from the collective matrix when using the RGM. Taking into account that $\mathrm{GCOMPI}_4(\mathscr{w}, v) = \mathrm{GCOMPI}_2(\mathscr{W}, v)$ (see Remark 2) we can prove the corresponding results for GCOMPI_4 in a similar way to Section 4.3.

Theorem 7. *Let $\mathscr{w} = \left\{ w^{(k)} \right\}$ be a family of priority vectors. Let $P = (p_{ij})$ be a collective PCM and $v = (v_i)$ be the corresponding priority vector associated with P obtained with the RGM method. It holds that*

$$\frac{\partial \mathrm{GCOMPI}_4(\mathscr{w}, v)}{\partial p_{rs}} = \frac{4}{(n-1)(n-2)} \frac{1}{p_{rs}} \log \frac{v_r / v_s}{w_r^{G|P} / w_s^{G|P}} \tag{33}$$

Theorem 8. *Let $\mathscr{w} = \left\{ w^{(k)} \right\}$ be a family of priority vectors. Let $P = (p_{ij})$ be a collective PCM and $v = (v_i)$ be the corresponding priority vector associated with P obtained with the RGM method. It holds that*

$$\mathrm{GCOMPI}_4(\mathscr{w}, v') = \mathrm{GCOMPI}_4(\mathscr{w}, v) + \frac{4}{(n-1)(n-2)} \log t_{rs} \left(\frac{\log t_{rs}}{n} + \log \frac{v_r / v_s}{w_r^{G|P} / w_s^{G|P}} \right) \tag{34}$$

Corollary 10. *Let* $\mathscr{w} = \left\{w^{(k)}\right\}$ *be a family of priority vectors. Let* $P = \left(p_{ij}\right)$ *be a collective PCM and* $v = (v_i)$ *be the corresponding priority vector associated with P obtained with the RGM method. It holds that*

$$\frac{\partial \text{GCOMPI}_4(\mathscr{A}, v)}{\partial t_{rs}} = \frac{4}{(n-1)(n-2)} \frac{1}{t_{rs}} \left(\frac{2 \log t_{rs}}{n} + \log \frac{v_r / v_s}{w_r^{G|P} / w_s^{G|P}} \right) \tag{35}$$

Corollary 11. *Let* $\mathscr{w} = \left\{w^{(k)}\right\}$ *be a family of priority vectors. Let* $P = \left(p_{ij}\right)$ *be a collective PCM and* $v = (v_i)$ *be the corresponding priority vector associated with P obtained with the RGM method. It holds that*

$$\left. \frac{\partial \text{GCOMPI}_4(\mathscr{A}, v)}{\partial t_{rs}} \right|_{t_{rs}=1} = \frac{4}{(n-1)(n-2)} \log \frac{v_r / v_s}{w_r^{G|P} / w_s^{G|P}} \tag{36}$$

Corollary 12. *Let* $\mathscr{w} = \left\{w^{(k)}\right\}$ *be a family of priority vectors. Let* $P = \left(p_{ij}\right)$ *be a collective PCM and* $v = (v_i)$ *be the corresponding priority vector associated with P obtained with the RGM method. The relative variation of judgement* p_{rs} *that produces the greatest decrease of* $\text{GCOMPI}_4(\mathscr{w}, v)$ *is*

$$t^*_{rs} = p^*_{rs} / p_{rs} = \left(\frac{w_r^{G|P} / w_s^{G|P}}{v_r / v_s} \right)^{n/2} \tag{37}$$

and the variation of the $\text{GCOMPI}_4(\mathscr{w}, v)$ *is*

$$\nabla^* \text{GCOMPI}_4 = \frac{n}{(n-1)(n-2)} \log^2 \frac{v_r / v_s}{w_r^{G|P} / w_s^{G|P}} \tag{38}$$

Corollaries 11 and 12 show that the judgement p_{rs} for which there is a greater relative difference between the ratios v_r / v_s and $w_r^{G|P} / w_s^{G|P}$ is the one that most rapidly decreases the value of the GCOMPI_4 and it is also the one that allows the greatest reduction of this indicator in absolute terms.

From Corollary 10 it is obvious that the vector that minimises the value $\text{GCOMPI}_4(\mathscr{w}, v)$ is $w^{G|P}$. So, any collective matrix P whose RGM priority vector coincides with $w^{G|P}$ provides the minimum value for this compatibility measure. This is coherent with the previous results which show that the modifications leading to the further reduction of GCOMPI_4 cause the priority vector derived from the P matrix to approach $w^{G|P}$.

Figure 3 gives an outline of the reduction process when using the last two indicators. The input for the incompatibility reduction in both cases 4.3 and 4.4 consists of the individual priority vectors $\mathscr{w}$ and the initial collective matrix (P). Again, the output is the collective matrix P'. When using the GCOMPI_4 the associated priority vector, v', is derived using the RGM.

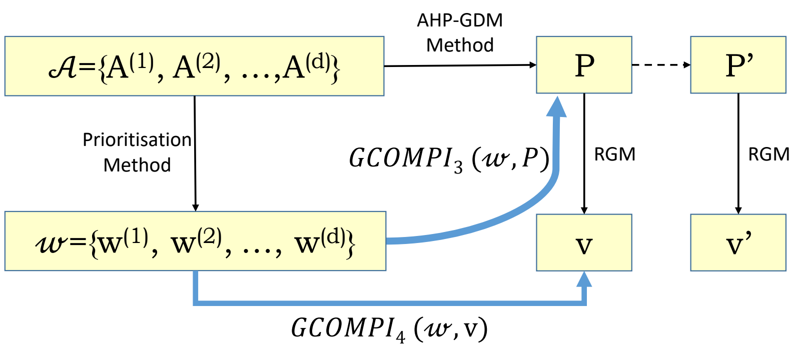

Figure 3. Outline of the incompatibility reduction of indicators GCOMPI_3 and GCOMPI_4.

5. Procedure for Incompatibility Reduction and Numerical Example

5.1. Semi-Automatic Procedure for Incompatibility Reduction

Using the theoretical results demonstrated in the previous section, a semi-automatic iterative procedure for incompatibility reduction can be designed that works in a similar way for the four compatibility indicators. Table 1 lists the values determined in Section 4 for each indicator and guides the general procedure proposed for the reduction of incompatibility.

Table 1. Values from theoretical results for the indicators.

Measure	$d_{rs} = \left.\frac{\partial \text{GCOMPI}_i}{\partial t_{rs}}\right\|_{t_{rs}=1}$	t^*_{rs}	$\nabla^*\text{GCOMPI}_i$
GCOMPI_1	$\frac{4}{(n-1)(n-2)} \log \frac{p_{rs}}{a^G_{rs}}$	$\frac{a^G_{rs}}{p_{rs}}$	$\frac{2}{(n-1)(n-2)} \log^2 \frac{p_{rs}}{a^G_{rs}}$
GCOMPI_2	$\frac{4}{(n-1)(n-2)} \log \frac{v_r/v_s}{w_r^{G\|J}/w_s^{G\|J}}$	$\left(\frac{w_r^{G\|J}/w_s^{G\|J}}{v_r/v_s}\right)^{n/2}$	$\frac{n}{(n-1)(n-2)} \log^2 \frac{v_r/v_s}{w_r^{G\|J}/w_s^{G\|J}}$
GCOMPI_3	$\frac{4}{(n-1)(n-2)} \log \frac{p_{rs}}{w_r^{G\|P}/w_s^{G\|P}}$	$\frac{w_r^{G\|P}/w_s^{G\|P}}{p_{rs}}$	$\frac{2}{(n-1)(n-2)} \log^2 \frac{p_{rs}}{w_r^{G\|P}/w_s^{G\|P}}$
GCOMPI_4	$\frac{4}{(n-1)(n-2)} \log \frac{v_r/v_s}{w_r^{G\|P}/w_s^{G\|P}}$	$\left(\frac{w_r^{G\|P}/w_s^{G\|P}}{v_r/v_s}\right)^{n/2}$	$\frac{n}{(n-1)(n-2)} \log^2 \frac{v_r/v_s}{w_r^{G\|P}/w_s^{G\|P}}$

As already mentioned, incompatibility reduction is achieved by modifying judgements in the collective matrix without modifying the initial information provided by the decision makers. Therefore, it is necessary to have the collective matrix, obtained by any aggregation procedure, as input.

Modifications in the judgements of the collective matrix are made in relative terms [22] and are limited to guarantee slight modifications in the collective matrix [6].

The results collected in Corollaries 2, 5, 8 and 11 provide the judgement from the collective matrix whose modification reduces the corresponding GCOMPI_i more rapidly. This judgement is selected from the value of derivative $d_{rs} = \left.\frac{\partial \text{GCOMPI}_i}{\partial t_{rs}}\right|_{t_{rs}=1}$ (see column 2 in Table 1). Corollaries 3, 6, 9 and 12 calculate the relative variation of these judgements, t^*_{rs} (column 3 of Table 1), that produces the largest reduction $\nabla^*\text{GCOMPI}_i$ (column 4 of Table 1).

At each iteration, the procedure selects the judgement p_{rs} from the collective matrix whose relative modification will produce the largest reduction in the corresponding GCOMPI_i. Its optimal value is determined by t^*_{rs}, but this relative variation can be excessively large, moving the judgement p_{rs} away from the initial value. To avoid major modifications of the judgments, the concept of permissibility, ρ, is introduced [26]. The parameter ρ indicates the maximum relative variation permitted for the modifications of any judgement. It is established by the decision makers or by the facilitator (if there is one) [5]. The parameter of permissibility considers the actors' attitudes or flexibility to negotiation. This allows them to adapt their initial positions, facilitating the establishment of consensus paths for reaching a more satisfactory final agreement [3]. The sensitivity analysis of the permissibility parameter may provide relevant information about the critical points and the decision opportunities of the resolution process.

With these ideas, at each iteration, the value of the relative variation, t_{rs}, of the selected judgement is determined as follows:

$$t_{rs} = \begin{cases} \min\{1+\rho, t^*_{rs}\} & \text{when } d_{rs} < 0 \\ \max\left\{\frac{1}{1+\rho}, t^*_{rs}\right\} & \text{when } d_{rs} > 0 \end{cases} \quad (39)$$

Once a judgement has been modified, it will not be examined in the subsequent iterations. The procedure finishes when all judgements in the matrix have been analysed or when an acceptable, pre-set, level of incompatibility is reached. As no thresholds for the compatibility indicators are available, this level can be set in terms of efficiency (% improvement with respect to the maximum possible reduction). From results obtained in

Section 4, we can obtain the minimum value for each GCOMPI$_i$, GCOMPI$_{i,min}$. This value allows us to calculate the efficiency achieved after t iterations:

$$E_t = \frac{\text{GCOMPI}_{i,0} - \text{GCOMPI}_{i,t}}{\text{GCOMPI}_{i,0} - \text{GCOMPI}_{i,min}} \tag{40}$$

where GCOMPI$_{i,t}$ denotes the value of GCOMPI$_i$ after t iterations. In this way, decision makers or a facilitator can set a threshold for the efficiency, E^*, and if it is reached, the procedure finishes.

An outline of the algorithm that implements these ideas is shown in Box 1. To apply this semi-automatic procedure, it is necessary to provide the permissibility value (ρ) and the efficiency threshold (E^*), in addition to the information on the individual and the collective positions (matrices or vectors, depending on the indicator). Once these values are known, the procedure works automatically to provide the final collective matrix.

Box 1. Outline of the algorithm for improving the GCOMPI$_i$ in terms of relative changes.

Step 0.	Let $J = \{(r,s), \text{with } r < s\}$.
Step 1.	Evaluate $d_{rs} = \frac{\partial \text{GCOMPI}_i}{\partial t_{rs}}\Big\vert_{t_{rs}=1}$ for all $(r,s) \in J$.
Step 2.	Choose the pair $(r', s') \in J$ for which $d_{r's'}$ has the largest absolute value.
Step 3.	If $p_{r's'} > 1$ then let $(r,s) = (r', s')$. Otherwise, let $(r,s) = (s', r')$.
Step 4.	Modify p_{rs} using expression (39).
Step 5.	Update matrix P with new values $p'_{rs} = p_{rs} t_{rs}$ and $p'_{sr} = 1/p'_{rs}$. Update $J = J \setminus (r', s')$.
Step 6.	If J is empty or $E_t \geq E^*$, stop and provide P'. Otherwise go to Step 1.

The procedure described above follows general and common guidelines for the four indicators. Some particularities of these indicators are presented below.

The minimum value for the GCOMPI$_1$ is obtained when $P = A^G$, and the algorithm iterates over the judgements trying to approach the value of p_{rs} to a^G_{rs} taking into account that the relative variation is limited by permissibility ρ. The problem can be expressed as

$$\begin{aligned} \min_{P'} \quad & \text{GCOMPI}_1(\mathscr{A}, P') \\ \text{s.t.} \quad & \frac{1}{1+\rho} \leq \frac{p'_{ij}}{p_{ij}} \leq 1+\rho, \text{ for all } i,j \end{aligned} \tag{41}$$

This problem is solvable because it is a separable optimisation problem and it is enough to optimise each judgement p'_{ij} separately. To find the optimal solution it is enough to apply the algorithm to each and every one of the judgements, regardless of the order.

However, the iterative algorithm above makes sense as by sequentially modifying the judgements that reduce the indicator most rapidly, it is possible to stop when a certain efficiency has been reached (E^*). Stopping before going through all the judgements allows us to move only as far as necessary from the original judgements.

The same can be said for GCOMPI$_3$: its minimum value is obtained when $P = W^{G|P}$.

The GCOMPI$_2$ is minimised by any matrix P whose priority vector, obtained applying the RGM method, coincides with $w^{G|J}$. In particular, the consistent matrix $W^{G|J}$ formed from the vector $w^{G|J}$ provides minimum distance with $GCI = 0$. This matrix can be far from the starting matrix P, and the algorithm tries, with small modifications, to move the vector v closer to the vector $w^{G|J}$.

The same is also true for GCOMPI$_4$: its minimum value is reached using any matrix P whose priority vector, obtained applying the RGM method, coincides with $w^{G|P}$.

Remark 3. *When the RGM method is used to obtain the individual priority vectors, the collective priority vector* $w^{G|P}$ *equals* $w^{G|J}$ *and the results obtained for* GCOMPI_4 *coincides with those for* GCOMPI_2*. The proposed procedure leads to the same matrix* P'*.*

The procedure for improving incompatibility can be tailored to specific interests that reflect ideas that are common in AHP-GDM [27,28]. For example, by simply customising the appropriate constraints in step 4, the algorithm may [5]: (i) delimit the judgements of the resulting collective matrix in the interval [1/9, 9]; (ii) delimit the judgements in the priority stability intervals for $P.\alpha$ or $P.\gamma$ problems; (iii) delimit the judgements in the consistency stability intervals; or (iv) allow for a greater participation of the decision makers in an interactive way.

5.2. Numerical Example

The procedure is illustrated with an example used in [17,29]. The problem has $n = 5$ alternatives and $d = 3$ decision makers with weights $\alpha_1 = 5/11$; $\alpha_2 = 4/11$; and $\alpha_3 = 2/11$. We present in more detail the procedure for incompatibility reduction when compatibility is measured with GCOMPI_1. As initial information, it is necessary to provide the individual and the collective pairwise comparison matrices. The individual pairwise comparison matrices are:

$$A^{(1)} = \begin{pmatrix} 1 & 3 & 5 & 8 & 6 \\ & 1 & 3 & 5 & 4 \\ & & 1 & 3 & 2 \\ & & & 1 & 1/3 \\ & & & & 1 \end{pmatrix} A^{(2)} = \begin{pmatrix} 1 & 3 & 7 & 9 & 5 \\ & 1 & 3 & 7 & 1 \\ & & 1 & 5 & 1/5 \\ & & & 1 & 1/5 \\ & & & & 1 \end{pmatrix} A^{(3)} = \begin{pmatrix} 1 & 5 & 7 & 7 & 5 \\ & 1 & 1 & 5 & 1 \\ & & 1 & 5 & 1/3 \\ & & & 1 & 1/5 \\ & & & & 1 \end{pmatrix}$$

The collective matrix P is provided below. It corresponds to the Precise Consistency Consensus Matrix (PCCM) determined by the method described in [17], which is based on consistency. The matrix A^G, obtained using AIJ, is also provided as its values are utilised in the reduction process of the GCOMPI_1.

$$P = \begin{pmatrix} 1 & 2.049 & 5.510 & 9.000 & 3.165 \\ & 1 & 3.000 & 6.082 & 1.739 \\ & & 1 & 2.709 & 1/1.467 \\ & & & 1 & 1/2.845 \\ & & & & 1 \end{pmatrix} A^G = \begin{pmatrix} 1 & 3.292 & 6.007 & 8.150 & 5.432 \\ & 1 & 2.457 & 5.651 & 1.878 \\ & & 1 & 3.964 & 1/1.600 \\ & & & 1 & 1/3.964 \\ & & & & 1 \end{pmatrix}$$

The value of the compatibility indicator for matrix P is $\text{GCOMPI}_1(\mathscr{A}, P) = 0.484$; for A^G it is $\text{GCOMPI}_1(\mathscr{A}, A^G) = 0.342$, which is the minimum possible value for this indicator.

The algorithm has been applied to reduce the value of GCOMPI_1 considering a permissibility of 15%. Table 2 lists the 10 iterations followed with the proposed semi-automatic procedure when there is no efficiency threshold. Each iteration shows: the selected judgement (r, s), its initial value (p_{rs}), the modified value (p'_{rs}), the new value of the compatibility indicator after the modification ($\text{GCOMPI}_{1,t}$), and the level of efficiency achieved up to that iteration (E_t). The process finishes after revising all the judgements and 56.32% efficiency is achieved. If an efficiency threshold is assumed the procedure may conclude before the tenth iteration. For example, for $E^* = 50\%$ the procedure stops at the fifth iteration achieving efficiency of 52.01%.

Table 2. Information on the iterations of the procedure with $\rho = 15\%$.

Iteration t	(r, s)	p_{rs}	p'_{rs}	GCOMPI_1, t	E_t
0				0.484	
1	(1–5)	3.165	3.640	0.462	15.57%
2	(1–2)	2.049	2.356	0.443	28.87%
3	(3–4)	2.708	3.114	0.428	39.13%
4	(4–5)	2.844	3.271	0.416	47.76%
5	(2–3)	3.000	2.608	0.410	52.01%
6	(1–4)	9.000	8.149	0.408	53.21%
7	(3–5)	1.466	1.599	0.407	54.06%
8	(1–3)	5.509	6.007	0.406	54.98%
9	(2–5)	1.738	1.877	0.405	55.68%
10	(2–4)	6.082	5.650	0.404	56.32%

After considering all the judgements (10 iterations), the final pairwise comparison matrix (P') is:

$$P' = \begin{pmatrix} 1.000 & 2.356 & 6.007 & 8.150 & 3.640 \\ 0.424 & 1.000 & 2.609 & 5.651 & 1.878 \\ 0.166 & 0.383 & 1.000 & 3.115 & 0.625 \\ 0.123 & 0.177 & 0.321 & 1.000 & 0.306 \\ 0.275 & 0.533 & 1.600 & 3.272 & 1.000 \end{pmatrix}$$

Table 3 details the resulting priorities using the RGM method for the initial and final collective matrices. It also shows the associated values of the compatibility indicator (GCOMPI_1) for both matrices. It can be observed that the differences between the two collective priority vectors are small, as recommended by Saaty [6]. In relative terms, the maximum difference is 6.6% and the average difference is 3.2%. Garuti's G value [20] is $G = 0.9612$ suggesting that the initial and final priority vectors are highly compatible ($G > 0.9$).

Table 3. Priorities and Compatibility for the collective matrices.

	v_1	v_2	v_3	v_4	v_5	GCOMPI_1
P	0.467	0.255	0.095	0.044	0.139	0.484
P'	0.486	0.238	0.096	0.042	0.138	0.404

For this example, the incompatibility reduction procedure has also been applied using the other three indicators to measure compatibility; again with a permissibility value of $\rho = 15\%$. A detailed description of the application with GCOMPI_2 can be found at [5]. In the cases where the individual priority vectors are needed (GCOMPI_3 and GCOMPI_4), we have used those obtained by applying the RGM to the individual pairwise comparison matrices. Furthermore, the final collective matrices obtained by applying the procedure using the four indicators have an acceptable level of inconsistency.

Table 4 shows, for each indicator, the efficiency achieved if the process is completed with the 10 possible judgements (E_{10}), the number of iterations necessary to achieve efficiency of 50% (#Iter), and Garuti's G value that measures the proximity between the initial and the final priority vectors. The results obtained for GCOMPI_2 and GCOMPI_4 are the same as the individual priority vectors have been calculated using the RGM (see Remark 3). It can be observed that efficiency of 73.10% can be achieved for GCOMPI_2 and GCOMPI_4, and that is not possible to achieve 50% efficiency for GCOMPI_3. Finally, in all cases, the initial and final priority vectors are highly compatible ($G > 0.9$).

Table 4. Results of the procedure for the four indicators with $\rho = 15\%$.

	E_{10}	#Iter	G
GCOMPI$_1$	56.32%	5	0.9612
GCOMPI$_2$	73.10%	5	0.9380
GCOMPI$_3$	44.11%	–	0.9498
GCOMPI$_4$	73.10%	5	0.9380

6. Conclusions

One of the strengths of AHP is its suitability for multiactor decision making. In group decision making, where individuals work together, the assessment of the representativeness of the group position has not been contemplated with the significance that it deserves. Defining compatibility between individual and collective positions as the property that reflects the proximity between them, the position of the group may be accepted if the group has an acceptable level of incompatibility.

This paper presents four indicators based on log quadratic distances for the evaluation of compatibility, three of them new. The measures are continuous and derivable functions and capture the idea of reciprocity, which is essential in AHP. The elements compared by the expressions are not the same: the first indicator compares the individual and the collective pairwise comparison matrices; the second compares the individual pairwise comparison matrices and the collective priority vector; the third compares the individual priority vectors and the collective pairwise comparison matrix; and the fourth compares the individual and the collective priority vectors. The indicators make it possible to measure compatibility for different decision problems, irrespective of the aggregation and the prioritisation methods. Some recommendations on when each of the indicators could be used have been included.

Associated with each of the proposed indicators, a set of theoretical results have been proved to determine how to reduce incompatibility by modifying the judgements of the collective matrix. Due to the mathematical expressions of the indicators that measure the distance, the results are closed (optimal) in terms of the judgements that most rapidly reduce incompatibility. In all four cases, it is shown that the judgement that achieves the greatest reduction in relative terms is also the one that produces the greatest reduction in absolute terms. The values that the modified matrix approaches in each case are also observed.

Finally, the theoretical results are used for the development of a general, semi-automatic procedure for reducing the incompatibility. Adapting the proposal made in [5] to each compatibility measure, the procedure selects and modifies, at each iteration, the judgement of the collective matrix that further improves the corresponding indicator, the continuous intervention of decision makers to modify the initial information they provide is not necessary. It uses relative rather than absolute changes as they better reflect the perceived importance of the modifications and limits the variations in the judgements to guarantee slight modifications in the associated priorities. The efficiency, measured as a % of improvement with respect to the maximum possible reduction, is one of the values proposed to determine if an acceptable level of incompatibility has been reached.

The proposed procedure has been illustrated with a numerical example. With relative changes in the judgements of the collective matrix below 15%, efficiency values above 44% are achieved for the four indicators, reaching 73% for two of them. In all cases, the initial and final collective priority vectors are highly compatible, and the relative differences in their components are below the permissibility value given for the judgements.

In short, this work offers new indicators to measure compatibility that are not influenced by the method used for obtaining the collective position and the priorities. Furthermore, a general procedure for incompatibility reduction that is easy to implement and apply is also proposed. The procedure can be adapted to particular interests in a similar way as outlined in [5] for GCOMPI$_2$. Future research will include the analysis of the performance of the general algorithm; obtaining direct thresholds for the different compatibility indicators; and, extensions to deal with global contexts (hierarchy) and negotiated decision making.

Author Contributions: The paper was jointly written by the four authors. Conceptualization, J.A., M.T.E., J.M.M.-J. and A.T.; Methodology, J.A., M.T.E., J.M.M.-J. and A.T.; Software, J.A., M.T.E., J.M.M.-J. and A.T.; Supervision, J.A., M.T.E., J.M.M.-J. and A.T.; Writing—original draft, J.A., M.T.E., J.M.M.-J. and A.T. All authors have read and agreed to the published version of the manuscript.

Funding: This research is partially financed by the "Grupo Decisión Multicriterio Zaragoza" research group (S35-20R) and the project "Participación Ciudadana Cognitiva y Decisiones Públicas. Aplicaciones Sociosanitarias" (Ref. LMP35_21), both supported by grants from the Regional Government of Aragon and FEDER funds.

Acknowledgments: The authors would like to acknowledge the work of English translation professional David Jones in preparing the final text. The authors are grateful to the editor and the two anonymous referees for their comments and suggestions that have improved the quality of the article.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Saaty, T.L. A scaling method for priorities in hierarchical structures. *J. Math. Psychol.* **1977**, *15*, 234–281. [CrossRef]
2. Saaty, T.L. *Multicriteria Decision Making: The Analytic Hierarchy Process*; McGraw-Hill: New York, NY, USA, 1980.
3. Altuzarra, A.; Moreno-Jiménez, J.M.; Salvador, M. Consensus Building in AHP-Group Decision Making: A Bayesian Approach. *Oper. Res.* **2010**, *58*, 1755–1773. [CrossRef]
4. Altuzarra, A.; Gargallo, P.; Moreno-Jiménez, J.M.; Salvador, M. Homogeneous Groups of Actors in an AHP-Local Decision Making Context: A Bayesian Analysis. *Mathematics* **2019**, *7*, 294. [CrossRef]
5. Aguarón, J.; Escobar, M.T.; Moreno-Jiménez, J.M. Reducing incompatibility in a local AHP-Group Decision Making context. *Forthcoming*.
6. Saaty, T.L. Decision-making with the AHP: Why is the principal eigenvector necessary. *Eur. J. Oper. Res.* **2003**, *145*, 85–91. [CrossRef]
7. Crawford, G.; Williams, C. A note on the analysis of subjective judgment matrices. *J. Math. Psychol.* **1985**, *29*, 387–405. [CrossRef]
8. Saaty, T. *Fundamentals of Decision Making and Priority Theory with the Analytic Hierarchy Process*; AHP Series; RWS Publications: Pittsburgh, PA, USA, 1994.
9. Aguarón, J.; Moreno-Jiménez, J.M. The Geometric Consistency Index: Approximated Thresholds. *Eur. J. Oper. Res.* **2003**, *147*, 137–145. [CrossRef]
10. Saaty, T.L. Group Decision Making and the AHP. In *The Analytic Hierarchy Process: Applications and Studies*; Golden, B.L., Wasil, E.A., Harker, P.T., Eds.; Springer: Berlin/Heidelberg, Germany, 1989; pp. 59–67. [CrossRef]
11. Ramanathan, R.; Ganesh, L. Group preference aggregation methods employed in AHP: An evaluation and an intrinsic process for deriving members' weightages. *Eur. J. Oper. Res.* **1994**, *79*, 249–265. [CrossRef]
12. Forman, E.; Peniwati, K. Aggregating individual judgments and priorities with the analytic hierarchy process. *Eur. J. Oper. Res.* **1998**, *108*, 165–169. [CrossRef]
13. Aguarón, J.; Escobar, M.T.; Moreno-Jiménez, J.M.; Turón, A. The Triads Geometric Consistency Index in AHP-pairwise comparison matrices. *Mathematics* **2020**, *8*, 926. [CrossRef]
14. Garuti, C.E. A set theory justification of Garuti's compatibility index. *J. Multi-Criteria Decis. Anal.* **2020**, *27*, 50–60. [CrossRef]
15. Lipovetsky, S. Priority vector estimation: Consistency, compatibility, precision. *Int. J. Anal. Hierarchy Process* **2020**, *12*, 577–591. [CrossRef]
16. Dong, Y.; Zhang, G.; Hong, W.C.; Xu, Y. Consensus models for AHP group decision making under row geometric mean prioritization method. *Decis. Support Syst.* **2010**, *49*, 281–289. [CrossRef]
17. Escobar, M.T.; Aguarón, J.; Moreno-Jiménez, J.M. Some extensions of the precise consistency consensus matrix. *Decis. Support Syst.* **2015**, *74*, 67–77. [CrossRef]
18. Garuti, C. Consistency & compatibility (two sides of the same coin). In Proceedings of the ISAHP 2016, London, UK, 4–7 August 2016. [CrossRef]
19. Saaty, T. *The Analyutic Network Process. Decision Making with Dependence and Feedback*; AHP Series; RWS Publications: Pittsburgh, PA, USA, 1996.
20. Garuti, C. Measuring compatibility (closeness) in weighted environments. In Proceedings of the International Symposium on the AHP, Vina del Mar, Chile, 2–6 August 2007.
21. Lipovetsky, S. Global Priority Estimation in Multiperson Decision Making. *J. Optim. Theory Appl.* **2009**, *140*, 77–91. [CrossRef]
22. Kahneman, D.; Tversky, A. Prospect Theory: An Analysis of Decision under Risk. *Econometrica* **1979**, *47*, 263–291. [CrossRef]
23. Barzilai, J.; Golany, B. Ahp Rank Reversal, Normalization Furthermore, Aggregation Rules. *INFOR Inf. Syst. Oper. Res.* **1994**, *32*, 57–64. [CrossRef]
24. Escobar, M.; Aguarón, J.; Moreno-Jiménez, J. A note on AHP group consistency for the row geometric mean priorization procedure. *Eur. J. Oper. Res.* **2004**, *153*, 318–322. [CrossRef]

25. Grzybowski, A.Z. New results on inconsistency indices and their relationship with the quality of priority vector estimation. *Expert Syst. Appl.* **2016**, *43*, 197–212. [CrossRef]
26. Aguarón, J.; Escobar, M.T.; Moreno-Jiménez, J.M. Reducing inconsistency measured by the geometric consistency index in the analytic hierarchy process. *Eur. J. Oper. Res.* **2021**, *288*, 576–583. [CrossRef]
27. Aguarón, J.; Escobar, M.T.; Moreno-Jiménez, J.M.; Turón, A. AHP-Group Decision Making Based on Consistency. *Mathematics* **2019**, *7*, 242. [CrossRef]
28. Wu, Z.; Jin, B.; Fujita, H.; Xu, J. Consensus analysis for AHP multiplicative preference relations based on consistency control: A heuristic approach. *Knowl.-Based Syst.* **2020**, *191*, 105317. [CrossRef]
29. Turón, A.; Aguarón, J.; Escobar, M.T.; Moreno-Jiménez, J.M. A Decision Support System and Visualisation Tools for AHP-GDM. *Int. J. Decis Support Syst. Technol.* **2019**, *11*, 1–19. [CrossRef]