

Article

Automorphisms and Definability (of Reducts) for Upward Complete Structures

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Abstract: The Svenonius theorem establishes the correspondence between definability of relations in a countable structure and automorphism groups of these relations in extensions of the structure. This may help in finding a description of the lattice constituted by all definability spaces (reducts) of the original structure. Results on definability lattices were previously obtained only for ω -categorical structures with finite signature. In our work, we introduce the concept of an upward complete structure and define the upward completion of a structure. For upward complete structures, the Galois correspondence between definability lattice and the lattice of closed supergroups of the automorphism group of the structure is an anti-isomorphism. We describe the natural class of structures which have upward completion, we call them discretely homogeneous graphs, present the explicit construction of their completion and automorphism groups of completions. We establish the general localness property of discretely homogeneous graphs and present examples of completable structures and their completions.



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1. Introduction

The definitions of concepts of a definability space and definability lattice (reduct lattice) can be found in [1].

The original idea of Alessandro Padoa [2] as well as general philosophy of Alfred Tarski [3] assuming “Erlangen Program” of “geometrization” and “algebraization” of logic have found their refinement in the Svenonius Theorem (“Completeness Theorem for definability”) published in 1959 [4]. This remarkable result by Lars Svenonius, however, has not found applications for a long time. One of the reasons, perhaps, is historical and subjective—the paper was published in a Swedish philosophical journal little known to mathematicians. Another, and likely main reason is that the Svenonius theorem reduces the problem of description of a definability lattice of a structure to the consideration of supergroups of automorphism groups of all countable elementary extensions of that structure; and these extensions generally appear to be obscure and hardly amenable to classification.

Starting from 1965 (see [5]), definability lattices have been constructed for several homogeneous structures with finite relational signature. Since such structures are ω -categorical, the study of closed supergroups required only the structures themselves. Furthermore, homogeneity of the structures, which is essentially a transformational and geometric property, did help to simplify greatly the problem. All found lattices proved to be finite. The hypothesis of finiteness of definability lattices of all homogeneous structures with finite basis (finite-relational signature) was proposed by Simon Thomas (see [6]) in 1991 and remains open.

Continuing the direction of [1] in the present work, we consider the structures for which the Svenonius theorem “works” by restricting the class of elementary extensions where the study of automorphism groups is required. We believe that it may become an important step in the description of corresponding definability lattices. Such an approach was used in [1], where a complete description of the infinite definability lattice for the successor structure on integers was obtained.

All undefined standard notations and terms may be found in [7].
The Svenonius Theorem is formulated as follows [7]:

Svenonius Theorem. *Let $M = \langle A, \Sigma \rangle$ be a countable structure, and $\Sigma' \subset \Sigma$. Then, for any relation $P \in \Sigma$, the following two conditions are equivalent:*

- (i) *P is not definable in Σ' ;*
- (ii) *There exists an elementary extension $M' = \langle A', \Sigma \rangle$ of the structure M and a permutation φ of the domain of A' , preserving all relations in Σ' and not preserving P .*

In the present paper, we discuss a class of structures for which the construction of their definability lattices is reduced to the analysis of closed supergroups for one structure and its automorphism group, that is, the class of *upward complete structures*. We present several examples of structures from this class.

In [1], the description of definability lattice of $\langle \mathbb{Z}, +1 \rangle$ upward completeness of an extension of this structure was used. (We called the upward complete structure “maximal” there).

For other structures, we consider in the present work (excluding $\langle \mathbb{Z}, +1 \rangle$) that full descriptions of their definability lattices are unknown. We hope, however, that the existence of upward complete elementary extensions will help to obtain such descriptions.

2. Upward Complete Structures and Completions

Definition 1. *A countable structure \mathcal{M} is called upward complete, if $\mathcal{M} \cong \mathcal{M}'$ for every countable elementary extension of $\mathcal{M}' \succ \mathcal{M}$. An upward complete structure elementary equivalent to a structure is called a completion of the latter.*

It immediately follows from the Svenonius theorem that the definability lattice of every upward complete structure \mathcal{M} (and of each structure elementary equivalent to that) is determined completely by the lattice of all closed supergroups of its automorphism group $Aut(\mathcal{M}) \subset Sym(\mathcal{M})$ (the group of all permutations of the domain of \mathcal{M}).

It is obvious that any ω -categorical structure is upward complete. For example, the structure of non-negative rational numbers with the standard order is ω -categorical. It provides an example of upward complete but non-homogeneous structure. The simplest example of a non- ω -categorical upward complete structure is \mathbb{Z}^ω , the union of countably many disjoint copies of $\langle \mathbb{Z}, +1 \rangle$. Following tradition, we call such copies (and similar components in other cases) *galaxies*. It is easy to see that \mathbb{Z}^ω is a completion of $\langle \mathbb{Z}, +1 \rangle$.

For every natural number n , we denote:

The relation $|x_1 - x_2| = n$ by $A_{0,n}$;

The relation $x_1 - x_2 = x_3 - x_4 = n \vee x_1 - x_2 = x_3 - x_4 = -n$ by $A_{1,n}$;

The relation $x_1 - x_2 = n$ by $A_{2,n}$.

It is shown in [1] that the lattice of closed supergroups of the group $Aut(\mathbb{Z}^\omega)$ consists of groups $\Gamma_{A_{0,d}}, \Gamma_{A_{1,d}}, \Gamma_{A_{2,d}}$ for all $d \in \mathbb{N}$ corresponding to the relations above.

For any d , we denote by $C(d)$ the set of all residue classes modulo d taken from all galaxies.

Then:

- The group $\Gamma_{A_{2,d}}$ consists of all permutations γ , such that $\gamma(a + d) = \gamma(a) + d$ for all a . It is possible to have an explicit description of the elements of $\Gamma_{A_{2,d}}$ —these are mappings arbitrarily permuting the elements of $C(d)$ and preserving the order on each element.

- The group $\Gamma_{A_{1,d}}$ is a proper supergroup of $\Gamma_{A_{2,d}}$, which consists of all permutations γ such that either $\gamma(a + d) = \gamma(a) + d$ for all a , or $\gamma(a + d) = \gamma(a) - d$ for all a . The explicit description of the elements of $\Gamma_{A_{1,d}}$ is similar to the previous case of $\Gamma_{A_{2,d}}$. The difference here is that two variants are possible: the residue classes in $C(d)$ are permuted either all-preserving or all-reversing the order.
- The group $\Gamma_{A_{0,d}}$ is a proper supergroup of $\Gamma_{A_{1,d}}$, which consists of all permutations γ such that $|\gamma(a + d) - \gamma(a)| = d$ for all $a \in \mathbb{Z}^\omega$. The explicit description of the elements of the group is similar to the previous cases. A permutation from $\Gamma_{A_{0,d}}$ sends each residue class from $C(d)$ to another residue class with either preserving or reversing the order. The choice of the variant for each class is done independently.

In [1], it is proved that these groups exhaust the lattice of closed supergroups for the successor structure on the domain of integers, and generators for the corresponding definability spaces are presented as given above. Notice that all these groups with the exception of $\Gamma_{A_{0,1}}$ are realizable as supergroups of the automorphism groups of the successor structure over integers. So, we need an extension for $\Gamma_{A_{0,1}}$ only.

Here are some more examples. Every countable infinite vector space over a finite field is an upward complete structure. The countably dimensional vector space over the field of rationals is a completion of the additive group of rationals.

Further, we present more examples of upward complete structures.

The set of all integers with successor operation can be naturally generalized to a countable set with several successor operations (the definitions will be given later).

In this paper, we consider two examples of such structures and show that for each of them, an upward complete elementary extension does exist.

3. Graphs

Consider a countable infinite-oriented graph $\langle G, R \rangle$ with its vertices and edges labeled by the elements of a finite set and denote the resulting signature by Σ .

We will use the concepts of *connectedness, distances between vertices, diameter, neighborhoods, neighborhood center, radius*, etc., ignoring the orientations and labels.

Further on, we assume that our graphs have infinite diameter. It is true if the degrees of all vertices are finite.

3.1. Non-Standard Extension of Graphs

Let us enrich the signature by adding the relation $d_i(x, y)$ for every $i \in \mathbb{N}$, meaning that the distance between vertices x, y is greater than i , and the constant symbol \odot denoting an arbitrary fixed element of G . We denote the resulting structure by G^+ .

Let us now add to the theory of G^+ the statements on the existence of infinitely many non-standard and disconnected-to-each-other elements, that is, elements c_j , such that for all k, i, j both $d_k(c_i, \odot)$ and $d_k(c_i, c_j), i \neq j$ hold true. By the compactness argument, there is a countable model of the theory. Let us denote it by G^{+*} , and let G^* be a reduct of the structure G^{+*} to the structure in the original signature Σ .

Proposition 1. *Every elementary extension of the structure G^* contains:*

- *The original structure G ;*
- *Countably many components not connected to each other and to G .*

Proof. It follows from the existence of the countably many non-standard disconnected elements. \square

We call the components of every structure obtained by the procedure above and all its elementary extensions *galaxies* of the structure (graph) G .

Theorem 1. *Let all galaxies of G be isomorphic. Then, the structure G^* is an upward complete extension of G .*

Proof. We need to demonstrate that

- (i) G^* is an elementary extension of G ;
- (ii) Any countable elementary extension G^* is isomorphic to G^* .

The condition (i) is obvious since G^* is just obtained by signature reduction.

As we have noticed in Proposition 1, G^* is the union of the original structure G and countably many disconnected galaxies. The same is true for any countable elementary extension of G . Then, (ii) follows from Proposition 1. \square

3.2. Discretely Homogeneous Graphs

We call a graph G *discretely homogeneous* if it is connected and for each radius r , all neighborhoods, with the exception of finitely many, of radius r are isomorphic (as oriented labeled graphs with a pointed neighborhood center of it). We call these (non-exceptional) neighborhoods *typical*.

Theorem 2. *Let a graph G be discretely homogeneous. Then, each of its galaxies is isomorphic to a countable union of increasing typical neighborhoods with a common center.*

Proof. For every natural n and vertex $a \in G$, we denote by $D_n(a)$ the neighborhood of radius n with the center a . Thus, $D_n(a) = \{x \mid \neg d_n(a, x)\}$. Define the equivalence relation $E_n(x, y)$ on the vertices of the graph G as

$$E_n(a, b) \iff D_n(a) \cong D_n(b)$$

The relation $E_n(x, y)$ partitions the set of graph vertices into finitely many equivalence classes. The condition of discrete homogeneity means that for each fixed n , only one of the equivalence classes is infinite and it is definable. We denote such a class by A_n .

Let us consider now an elementary extension of G and an element a of a galaxy of it. The element a equals to none of the elements of the original graph G . So, the condition $a \in A_n$ is satisfied for every n . Furthermore, since each connected component G' of a graph coincides with the union $\bigcup_{i \in \mathbb{N}} D_i(a)$ for some vertex $a \in G'$, every galaxy coincides with $\bigcup_{i \in \mathbb{N}} D_i(a)$ for every (non-standard) element a from that galaxy. \square

3.3. Automorphisms

For every discretely homogeneous G , the automorphism group of its upward complete extension $Aut(G^*)$ has a very simple structure.

Proposition 2. *Let $\Gamma(G)$ consist of all galaxies of G plus G itself, if G is isomorphic to a galaxy. All automorphisms from the $Aut(G^*)$ can be constructed as follows:*

- Take an arbitrary permutation α of $\Gamma(G)$;
- For every g from $\Gamma(G)$, take an arbitrary isomorphism β_g from g to $\alpha(g)$;
- Make the union of all β_g .

If G is not isomorphic to a galaxy, then multiply directly the union by $Aut(G)$.

The proposition does not give us an immediate description for the lattice of closed supergroups for the automorphism group. However, it provides a possible base for further investigation.

The situation becomes more interesting and productive when the galaxies are not isomorphic to the original structure. They can have richer groups of automorphisms representing “imaginary” elements of the geometry of the original space G . The group $Aut(G^*)$ in this case is the direct product of $Aut(G)$ and the group acting on galaxies, as described above.

3.4. Localness

From an intuitive viewpoint, discretely homogeneous structures should satisfy the *localness* property: the value of any formula on them should not depend on the “distant” values of its variables. This is made precise by the following proposition.

Proposition 3. *For every formula $F(x_1, \dots, x_n, y)$ in the signature of a discretely homogeneous graph G , there exists a natural number w such that for every tuple $\bar{a} \in G$ and for every $b_1, b_2 \in G$ if $d(\odot, b_j) > w$ and $d(a_i, b_j) > w, j = 1, 2; 1 \leq i \leq n$, then $G \models (F(\bar{a}, b_1) \equiv F(\bar{a}, b_2))$.*

Proof. Consider a structure GN with a domain which is the union of the graph G and the set of natural numbers \mathbb{N} . The signature of GN consists of the signature of G applied to elements of G only (false outside), as well as:

- Two unary relations $G(x), N(x)$, defining the set of vertices of G and the set of natural numbers, respectively;
- A ternary relation $D(a, b, k)$, meaning that a, b are vertices of the graph G with the distance between them greater than k and $NG \models N(k)$.

Let $F(x_1, \dots, x_n, y)$ be a formula in the signature of graph G . Let

$$P(x_1, \dots, x_n, y, w) \Leftrightarrow (G(y) \wedge N(w) \wedge D(\odot, y, w) \wedge \bigwedge_{i=1}^n (G(x_i) \wedge D(x_i, y, w)))$$

The statement to prove is

$$NG \models (\exists w)(\forall \bar{a}, b_1, b_2)(P(\bar{a}, b_1, w) \wedge P(\bar{a}, b_2, w)) \rightarrow (F(\bar{a}, b_1) \equiv F(\bar{a}, b_2)))$$

Consider a structure NG^* being an arbitrary non-standard extension of NG , in which the set $\mathbb{N}^* = \{x | NG^* \models N(x)\}$ contains non-standard elements and the structure $G^* = \{x | NG^* \models G(x)\}$ is a “non-standard” elementary extension of G .

Let $w_0 \in \mathbb{N}^* \setminus \mathbb{N}$ be an arbitrary non-standard element. If $P(\bar{a}, b_1, w_0)$ holds true, then

- $b \in G^* \setminus G$ is a non-standard element;
- The galaxies containing the elements a_i and b , respectively, do not intersect.

Furthermore, if

$$(P(\bar{a}, b_1, w_0) \wedge P(\bar{a}, b_2, w_0))$$

holds true, then there exists an automorphism of G^* mapping b_1 to b_2 which is an identity on G and an identity on all galaxies not containing b_1, b_2 . It means that the formula

$$(\forall \bar{a}, b_1, b_2)((P(\bar{a}, b_1, w_0) \wedge P(\bar{a}, b_2, w_0)) \rightarrow (F(\bar{a}, b_1) \equiv F(\bar{a}, b_2)))$$

is satisfied in the structure NG^* , that is,

$$NG^* \models (\forall \bar{a}, b_1, b_2)((P(\bar{a}, b_1, w) \wedge P(\bar{a}, b_2, w)) \rightarrow (F(\bar{a}, b_1) \equiv F(\bar{a}, b_2)))$$

for some $w \in \mathbb{N}$, and hence, it is satisfied in NG too. \square

4. Examples of Discretely Homogeneous Structures

The important examples of homogeneous structures (alongside of rational numbers) are provided by random graphs. Further homogeneous structures were studied, whose homogeneity resembles homogeneity (universality) of random graphs [8]. The graphs we studied are also homogeneous, but in a different sense—they are the same “almost everywhere” (and, in particular, could be the same “everywhere”).

In the following paragraphs, we present corresponding examples. In each case, it is not difficult to establish the property of discrete homogeneity. In some cases, the property of “strong” discrete homogeneity holds, where all neighborhoods of a given radius are isomorphic.

Acyclic (free) non-oriented n -graph (that is a graph in which the degree of every vertex is n). Discrete homogeneity holds for both graphs with labeled and non-labeled edges.

Oriented n -tree with a root, in which for every vertex, there are n outgoing edges, and for every vertex except the root, there is one incoming edge. The variants with labeled and non-labeled edges are possible. In the case of labeled edges, we get an unoid, an algebraic structure. One can also consider trees without root.

The points of the plane with integer coordinates and two successor relations: the domain is $\mathbb{Z} \times \mathbb{Z}$; the signature with two binary relations:

$$\langle\langle x, y \rangle, \langle x + 1, y \rangle\rangle; \langle\langle x, y \rangle, \langle x, y + 1 \rangle\rangle$$

The description of the automorphisms group $Aut(\mathcal{M}^*)$ for these examples is given by Proposition 2. For example, in the case of $G = \mathbb{Z} \times \mathbb{Z}$, the group $Aut(G)$ is generated by sheets along the axes and the full automorphisms group is constructed similar to the one-dimensional case.

The modifications of the proposed construction can be used to construct completions of the structure of rational numbers with binary relations for multiplication by rational constants, and plausibly, also to construct completions of the following structures:

- Positive rational numbers with binary relations for multiplication by rational constants;
- The same for non-negative rational numbers.

5. Discussion, Open Problems, and Hypotheses

In the proposed approach, the problem of construction of a definability lattice for a structure can be split into two problems:

- Finding an elementary extension of the structure, such that its lattice of closed supergroups is isomorphic to the definability lattice;
- Finding the structure of closed supergroups of automorphisms for the elementary extension.

In the present work, we made progress towards a solution of the first problem, which was trivial for previously homogeneous structures.

As we already noticed, definability lattices for all above-mentioned structures (with the exception of $\langle\mathbb{Z}, +1\rangle$) are unknown to us.

As far as the first problem is concerned, the structures considered in this work can be further extended by several unary relations. Among the classes of such relations, one can identify the simplest: *periodic* (further generalizable to *almost periodic* [9,10]) and *sparse* with respect to the successor relation—meaning that the distance between neighboring elements within the union of unary relations tends to infinity.

A further direction is to extend the class of discretely homogeneous structures to the class which have not one but a uniformly limited number of neighborhoods that occur infinitely.

We think that a version of “quantifier elimination” can be conducted for our structures and the generalizations. The infinite signature can consist of atomic formulas with several pointed vertices and “distance is bigger than”.

It looks plausible that the technique used in [1] could help to describe the lattices for both $\langle\mathbb{N}, +1\rangle$ and upward complete n -trees considered above. In particular, the hypothesis for $\langle\mathbb{N}, +1\rangle$ can be formulated as follows:

Every element of the lattice can be generated by a relation from the list $A_{0,d}, A_{1,d}, A_{2,d}$ generating an element of the lattice for $\langle\mathbb{Z}, +1\rangle$, as mentioned in the first section of the paper with a parameter d , possibly extended by several constants (unary one-element relations) $0, \dots, d - 1$.

At the moment, it is not clear to which extent this technique can be transferred to the structures of higher dimensions, such as in the example $\mathbb{Z} \times \mathbb{Z}$ mentioned above, etc. However, the hypothesis is that the lattice generators in this case will be the pairs/tuples of the elements corresponding to axes. A new situation will occur if we add, for example, a binary predicate of equality for the components of two-dimensional vectors.

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