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Article

Application of Mixed Generalized Quasi-Einstein Spacetimes in General Relativity

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Abstract: In the present article, some geometric and physical properties of $MG(QE)_n$ were investigated. Moreover, general relativistic viscous fluid $MG(QE)_4$ spacetimes with some physical applications were studied. Finally, through a non-trivial example of $MG(QE)_4$ spacetime, we proved its existence.

Keywords: Einstein manifold; mixed generalized quasi-Einstein manifold; Einstein's field equation; energy–momentum tensor; general relativistic viscous fluid

MSC: 53C25; 53Z05

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1. Introduction

A Riemannian or a semi-Riemannian manifold (M^n, g) of dimension $n(>2)$ is termed as an Einstein manifold if its $(0,2)$-type Ricci tensor $Ric(\neq 0)$ satisfies $Ric = \frac{r}{n}$, where r stands for the scalar curvature [1]. In addition to Riemannian geometry, Einstein manifolds also have a vital contribution to the general theory of relativity (GTR).

Approximately two decades ago, Chaki and Maity introduced and studied quasi-Einstein manifolds [2]. An (M^n, g), $(n > 2)$ is said to be a quasi-Einstein manifold $(QE)_n$ if its Ric $(\neq 0)$ realizes the following condition:

$$Ric(U_1, U_2) = ag(U_1, U_2) + bA(U_1)A(U_2), \tag{1}$$

where $a, b \in \mathbb{R}$ such that $b \neq 0$ and $A(\neq 0)$ is the 1-form such that

$$g(U_1, \rho) = A(U_1), \quad g(\rho, \rho) = A(\rho) = 1, \tag{2}$$

for any vector field U_1, and a unit vector field ρ called the generator of (M^n, g). In addition, A is named the associated 1-form. Einstein manifolds form a natural subclass of the class of $(QE)_n$.

Under the study of exact solutions of the Einstein field equations, as well as under the consideration of quasi-umbilical hypersurfaces of semi-Euclidean spaces, $(QE)_n$ came into existence. For instance, the Robertson–Walker spacetimes are $(QE)_n$. Thus, $(QE)_n$ have great importance in GTR.

An (M^n, g), $(n \geq 2)$ is said to be a generalized quasi-Einstein manifold $G(QE)_n$ [3] if its $Ric(\neq 0)$ realizes the following condition:

$$Ric(U_1, U_2) = ag(U_1, U_2) + bA(U_1)A(U_2) + cB(U_1)B(U_2), \tag{3}$$

where a, b, c are non-zero scalars and A, B are two non-zero 1-forms such that

$$g(U_1, \rho) = A(U_1), \quad g(U_1, \sigma) = B(U_1), \tag{4}$$

where ρ and σ are mutually orthogonal unit vector fields, i. e., $g(\rho, \sigma) = 0$. The vector fields ρ and σ are called the generators of the manifold. If $c = 0$, then the manifold reduces to a quasi-Einstein manifold.

In 2007, Bhattacharya, De and Debnath [4] introduced the notion of a mixed generalized quasi-Einstein manifold. A non-flat Riemannian manifold is said to be a mixed generalized quasi-Einstein manifold and is denoted by $MG(QE)_n$, if its $Ric(\neq 0)$ satisfies the following condition:

$$\begin{aligned} Ric(U_1, U_2) = ag(U_1, U_2) + bA(U_1)A(U_2) + cB(U_1)B(U_2) \\ + d[A(U_1)B(U_2) + B(U_1)A(U_2)], \end{aligned} \tag{5}$$

where a, b, c, d are non-zero scalars and A, B are two non-zero 1-forms such that

$$g(U_1, \rho) = A(U_1), \quad g(U_1, \sigma) = B(U_1), \tag{6}$$

where ρ and σ are mutually orthogonal unit vector fields and are called the generators of the manifold. Recently, $MG(QE)_n$ have been studied by various geometers in several ways to a different extent, such as [5–8] and many others.

Putting $U_1 = U_2 = e_i$ in (5), where $\{e_i\}$ is an orthonormal basis of the tangent space at each point of the manifold, and taking summation over $i(1 \leq i \leq n)$, we obtain

$$r = na + b + c. \tag{7}$$

A Lorentzian four-dimensional manifold is said to be a mixed generalized quasi-Einstein spacetime with the generator ρ as the unit timelike vector field if its $Ric(\neq 0)$ satisfies (5). Here, A and B are non-zero 1-forms such that σ is the heat flux vector field perpendicular to the velocity vector field ρ. Therefore, for any vector field U_1, we have

$$\begin{aligned} g(U_1, \rho) = A(U_1), \quad g(U_1, \sigma) = B(U_1), \\ g(\rho, \rho) = A(\rho) = -1, \quad g(\sigma, \sigma) = B(\sigma) = 1. \end{aligned} \tag{8}$$

Further, we know that if the Riemannian curvature tensor $\overline{K}$ of type $(0, 4)$ has the form

$$\overline{K}(U_1, U_2, U_3, U_4) = k[g(U_2, U_3)g(U_1, U_4) - g(U_1, U_3)g(U_2, U_4)], \tag{9}$$

then the manifold is said to be of constant curvature k. The generalization of this manifold is the manifold of quasi-constant curvature and, in this case, the curvature tensor has the following form:

$$\begin{aligned} \overline{K}(U_1, U_2, U_3, U_4) = f_1[g(U_2, U_3)g(U_1, U_4) - g(U_1, U_3)g(U_2, U_4)] \\ + f_2[g(U_2, U_3)A(U_1)A(U_4) - g(U_2, U_4)A(U_1)A(U_3) \\ + g(U_1, U_4)A(U_2)A(U_3) - g(U_1, U_3)A(U_2)A(U_4)], \end{aligned} \tag{10}$$

where $g(K(U_1, U_2)U_3, U_4) = \overline{K}(U_1, U_2, U_3, U_4)$, K is the curvature tensor of type $(1, 3)$ and f_1, f_2 are scalars, and ρ is a unit vector field defined by

$$g(U_1, \rho) = A(U_1),$$

It can be easily seen that, if the curvature tensor $\overline{K}$ is of the form (10), then the manifold is conformally flat [3]. Thus, a Riemannian or semi-Riemannian manifold is said to be of quasi-constant curvature if the curvature tensor $\overline{K}$ satisfies the relation (10); we denote such a manifold of dimension n by $(QC)_n$.

A non-flat Riemannian or semi-Riemannian manifold (M^n, g) $(n \geq 3)$ is said to be a manifold of generalized quasi-constant curvature if the curvature tensor $\overline{K}$ of type $(0,4)$ satisfies the condition [3]

$$\begin{aligned}\overline{K}(U_1,U_2,U_3,U_4) = {} & f_1[g(U_2,U_3)g(U_1,U_4) - g(U_1,U_3)g(U_2,U_4)] \\ & + f_2[g(U_1,U_4)A(U_2)A(U_3) - g(U_2,U_4)A(U_1)A(U_3) \\ & + g(U_2,U_3)A(U_1)A(U_4) - g(U_1,U_3)A(U_2)A(U_4)] \\ & + f_3[g(U_1,U_4)B(U_2)B(U_3) - g(U_2,U_4)B(U_1)B(U_3) \\ & + g(U_2,U_3)B(U_1)B(U_4) - g(U_1,U_3)B(U_2)B(U_4)],\end{aligned} \tag{11}$$

where f_1, f_2, f_3 are scalars and A, B are two non-zero 1-forms. ρ and σ are orthonormal unit vectors corresponding to A and B such that $g(U_1,\rho) = A(X)$, $g(U_1,\sigma) = B(X)$ and $g(\rho,\sigma) = 0$. Such a manifold is denoted by $G(QC)_n$.

In [9], Bhattacharya and De introduced the notion of mixed generalized quasi-constant curvature. A non-flat Riemannian or semi-Riemannian manifold (M^n, g) $(n \geq 3)$ is said to be a manifold of mixed generalized quasi-constant curvature if the curvature tensor $\overline{K}$ of type $(0,4)$ satisfies the condition

$$\begin{aligned}\overline{K}(U_1,U_2,U_3,U_4) = {} & f_1[g(U_2,U_3)g(U_1,U_4) - g(U_1,U_3)g(U_2,U_4)] \\ & + f_2[g(U_1,U_4)A(U_2)A(U_3) - g(U_2,U_4)A(U_1)A(U_3) \\ & + g(U_2,U_3)A(U_1)A(U_4) - g(U_1,U_3)A(U_2)A(U_4)] \\ & + f_3[g(U_1,U_4)B(U_2)B(U_3) - g(U_2,U_4)B(U_1)B(U_3) \\ & + g(U_2,U_3)B(U_1)B(U_4) - g(U_1,U_3)B(U_2)B(U_4)] \\ & + f_4[\{A(U_2)B(U_3) + B(U_2)A(U_3)\}g(U_1,U_4) \\ & - \{A(U_1)B(U_3) + B(U_1)A(U_3)\}g(U_2,U_4) \\ & + \{A(U_1)B(U_4) + B(U_1)A(U_4)\}g(U_2,U_3) \\ & - \{A(U_2)B(U_4) + B(U_2)A(U_4)\}g(U_1,U_3)],\end{aligned} \tag{12}$$

where f_1, f_2, f_3, f_4 are scalars. A, B are two non-zero 1-forms. ρ and σ are orthonormal unit vectors corresponding to A and B such that $g(U_1,\rho) = A(X)$, $g(U_1,\sigma) = B(X)$ and $g(\rho,\sigma) = 0$. Such a manifold is denoted by $MG(QC)_n$.

The spacetime of general relativity and cosmology is regarded as a connected four-dimensional semi-Riemannian manifold (M^4, g) with Lorentzian metric g with signature $(-,+,+,+)$. The geometry of the Lorentz manifold begins with the study of a causal character of vectors of the manifold. Due to this causality, the Lorentz manifold becomes a convenient choice for the study of general relativity. Spacetimes have been studied by various authors in several ways, such as [10–14] and many others.

2. $MG(QE)_n$ Admitting the Generators ρ and σ as Recurrent Vector Fields

Let us consider the generators ρ and σ corresponding to the associated recurrent 1-forms A and B. Then, we have

$$(D_{U_1}A)(U_2) = \eta(U_1)A(U_2),$$

$$(D_{U_1}B)(U_2) = \varphi(U_1)B(U_2),$$

where η and φ are non-zero 1-forms.

A non-flat Riemannian or semi-Riemannian manifold (M^n, g), $(n > 2)$ is said to be Ricci-recurrent [15,16] if its $Ric(\neq 0)$ satisfies the following condition:

$$(D_{U_1}Ric)(U_2,U_3) = \alpha(U_1)Ric(U_2,U_3), \tag{13}$$

where α is in non-zero 1-form. Since we know that

$$(D_{U_1}Ric)(U_2,U_3) = U_1Ric(U_2,U_3) - Ric(D_{U_1}U_2,U_3) - Ric(U_2,D_{U_1}U_3), \quad (14)$$

using (14) in (13), it follows that

$$\alpha(U_1)Ric(U_2,U_3) = U_1Ric(U_2,U_3) - Ric(D_{U_1}U_2,U_3) - Ric(U_2,D_{U_1}U_3). \quad (15)$$

Using (5) in (15), we obtain

$$\begin{aligned} &\alpha(U_1)[ag(U_2,U_3) + bA(U_2)A(U_3) + cB(U_2)B(U_3) \\ &+ d\{A(U_2)B(U_3) + A(U_3)B(U_2)\}] = U_1[ag(U_2,U_3) + bA(U_2)A(U_3) \\ &+ cB(U_2)B(U_3) + d\{A(U_3)B(U_2) + A(U_2)B(U_3)\}] \\ &- [ag(D_{U_1}U_2,U_3) + bA(D_{U_1}U_2)A(U_3) + cB(D_{U_1}U_2)B(U_3) \\ &+ d\{A(D_{U_1}U_2)B(U_3) + A(U_3)B(D_{U_1}U_2)\}] \\ &- [ag(U_2,D_{U_1}U_3) + bA(U_2)A(D_{U_1}U_3) + cB(U_2)B(D_{U_1}U_3) \\ &+ d\{A(U_2)B(D_{U_1}U_3) + A(D_{U_1}U_3)B(U_2)\}]. \end{aligned} \quad (16)$$

Putting $U_2 = U_3 = \rho$ in (16), we obtain

$$U_1(a+b) - \alpha(U_1)(a+b) = 2(a+b)A(D_{U_1}\rho) + 2dB(D_{U_1}\rho). \quad (17)$$

By using the fact that $A(D_{U_1}\rho) = 0$ and (6) in (17), we have

$$U_1(a+b) - \alpha(U_1)(a+b) = 2dg(D_{U_1}\rho,\sigma), \quad (18)$$

which can be written as

$$U_1(a+b) - \alpha(U_1)(a+b) = -2dA(D_{U_1}\sigma).$$

Thus, we have $A(D_{U_1}\sigma) = 0$ if and only if $U_1(a+b) - \alpha(U_1)(a+b) = 0$. This implies that either $D_{U_1}\sigma \perp \rho$ or σ is a parallel vector field.

Again, putting $U_2 = U_3 = \sigma$ in (16), we have

$$U_1(a+b) - \alpha(U_1)(a+b) = 2(a+c)B(D_{U_1}\sigma) + 2dA(D_{U_1}\sigma). \quad (19)$$

Again, using the fact that $B(D_{U_1}\sigma) = 0$ and (6) in (19), we have

$$U_1(a+b) - \alpha(U_1)(a+b) = 2dg(D_v\sigma,\rho), \quad (20)$$

$$or, \quad U_1(a+b) - \alpha(U_1)(a+b) = -2dB(D_v\rho).$$

Thus, we have $B(D_{U_1}\rho) = 0$ if and only if $U_1(a+b) - \alpha(U_1)(a+b) = 0$. This implies that either $D_{U_1}\rho \perp \sigma$ or ρ is a parallel vector field. Hence, we can state the following theorem:

Theorem 1. *Let a mixed generalized quasi-Einstein manifold $MG(QE)_n$ be Ricci-recurrent; then, the following statements are equivalent:*
(i) ρ and σ are parallel vector fields;
(ii) $U_1(a+b) - \alpha(U_1)(a+b) = 0$ if and only if $D_{U_1}\sigma \perp \rho$;
(iii) $U_1(a+b) - \alpha(U_1)(a+b) = 0$ if and only if $D_{U_1}\rho \perp \sigma$.

3. $MG(QE)_n$ Admitting the Generators ρ and σ as Concurrent Vector Fields

A vector field π is said to be concurrent if it satisfies the following condition [17,18]:

$$D_{U_1}\pi = \xi U_1, \tag{21}$$

where ξ is constant.

Let us consider the generators ρ and σ corresponding to the associated concurrent 1-forms A and B. Then, we have

$$(D_{U_1}A)(U_2) = \lambda g(U_1, U_2), \tag{22}$$

$$and \quad (D_{U_1}B)(U_2) = \mu g(U_1, U_2), \tag{23}$$

where λ and μ are non-zero constants.

Taking the covariant derivative of (5) with respect to U_3, we obtain

$$\begin{aligned}(D_{U_3}Ric)U_2 = {} & b[(D_{U_3}A)(U_1)A(U_2) + A(U_1)(D_{U_3}A)(U_2)] \\ & + c[(D_{U_3}B)(U_1)B(U_2) + B(U_1)(D_{U_3}B)(U_2)] \\ & + d[(D_{U_3}A)(U_1)B(U_2) + A(U_1)(D_{U_3}B)(U_2) \\ & + (D_{U_3}B)(U_1)A(U_2) + B(U_1)(D_{U_3}A)(U_2)].\end{aligned} \tag{24}$$

Using (22) and (23) in (24), it follows that

$$\begin{aligned}(D_{U_3}Ric)(U_1, U_2) = {} & b[\lambda g(U_1, U_3)A(U_2) + \lambda g(U_2, U_3)A(U_1)] \\ & + c[\mu g(U_1, U_3)B(U_2) + \mu g(U_2, U_3)B(U_1)] \\ & + d[\lambda g(U_1, U_3)B(U_2) + \mu g(U_1, U_3)A(U_2) \\ & + \lambda g(U_2, U_3)B(U_1) + \mu g(U_2, U_3)A(U_1)].\end{aligned} \tag{25}$$

Contracting (25) over U_1 and U_2 leads to

$$\partial r(U_3) = A(U_3)[2b\lambda + 2d\mu] + B(U_3)[2c\mu + 2d\lambda]. \tag{26}$$

From (7), it follows that

$$\partial r(U_1) = 0. \tag{27}$$

In view of (27), (26) turns to

$$A(U_3)[2b\lambda + 2d\mu] + B(U_3)[2c\mu + 2d\lambda] = 0. \tag{28}$$

Thus, by virtue of (28), (5) takes the form

$$Ric(U_1, U_2) = ag(U_1, U_2) + \left[b + c\left(\frac{(b\lambda + d\mu)}{(c\mu + d\lambda)}\right)^2 - 2d\frac{(b\lambda + d\mu)}{(c\mu + d\lambda)}\right]A(U_1)A(U_2) \tag{29}$$

which is a quasi-Einstein manifold. Thus, we can state the following theorem:

Theorem 2. *Let $MG(QE)_n$ be a mixed generalized quasi-Einstein manifold. If the associated vector fields of $MG(QE)_n$ are concurrent and the associated scalars are constants, then the manifold reduces to a quasi-Einstein manifold.*

4. $MG(QE)_n$ Admitting Einstein's Field Equations

The Einstein's field equations with and without cosmological constants are given by

$$Ric(U_1, U_2) - \frac{r}{2}g(U_1, U_2) + \lambda g(U_1, U_2) = \kappa T(U_1, U_2), \tag{30}$$

and

$$Ric(U_1, U_2) - \frac{r}{2}g(U_1, U_2) = \kappa T(U_1, U_2), \tag{31}$$

respectively; κ is a gravitational constant, λ is a cosmological constant, and T is the energy–momentum tensor.

Using (6) in (31), it follows that

$$\begin{aligned} &\left(a - \frac{r}{2}\right)g(U_1, U_2) + bA(U_1)A(U_2) + cB(U_1)B(U_2) \\ &+ d[A(U_1)B(U_2) + A(U_2)B(U_1)] = \kappa T(U_1, U_2). \end{aligned} \tag{32}$$

Now, taking the covariant derivative of (32) with respect to U_3, we arrive at

$$\begin{aligned} &b[(D_{U_3}A)(U_1)A(U_2) + A(U_1)(D_{U_3}A)(U_2)] \\ &+ c[(D_{U_3}B)(U_1)B(U_2) + B(U_1)(D_{U_3}B)(U_2)] \\ &+ d[(D_{U_3}A)(U_1)B(U_2) + A(U_1)(D_{U_3}B)(U_2) \\ &+ (D_{U_3}B)(U_1)A(U_2) + B(U_1)(D_{U_3}A)(U_2)] = \kappa(D_{U_3}T)(U_1, U_2). \end{aligned} \tag{33}$$

Thus, we have a result.

Theorem 3. *Let $MG(QE)_n$ admit Einstein's field equation without a cosmological constant. If the associated 1-forms A and B are covariantly constant, then the energy–momentum tensor is also covariantly constant.*

5. $MG(QE)_4$ Spacetime Admitting Space-Matter Tensor

In 1969, Petrov [19] introduced and studied the space–matter tensor $\overline{P}$ of type $(0, 4)$ and defined by

$$\overline{P} = \overline{K} + \frac{\kappa}{2}g \wedge T - \nu G, \tag{34}$$

where $\overline{K}$ is the curvature tensor of type $(0, 4)$, T is the energy–momentum tensor of type $(0, 2)$, κ is the gravitational constant, and ν is the energy density. Furthermore, G and $g \wedge T$ are, respectively, defined by

$$G(U_1, U_2, U_3, U_4) = g(U_2, U_3)g(U_1, U_4) - g(U_1, U_3)g(U_2, U_4), \tag{35}$$

and

$$\begin{aligned} (g \wedge T)(U_1, U_2, U_3, U_4) = &\, g(U_2, U_3)T(U_1, U_4) + g(U_1, U_4)T(U_2, U_3) \\ &- g(U_1, U_3)T(U_2, U_4) - g(U_2, U_4)T(U_1, U_3), \end{aligned} \tag{36}$$

for all U_1, U_2, U_3, U_4 on M.

Using (35) and (36) in (34), it follows that

$$\begin{aligned} \overline{P}(U_1, U_2, U_3, U_4) = &\, \overline{K}(U_1, U_2, U_3, U_4) + \frac{\kappa}{2}[g(U_2, U_3)T(U_1, U_4) \\ &+ g(U_1, U_4)T(U_2, U_3) - g(U_1, U_3)T(U_2, U_4) \\ &- g(U_2, U_4)T(U_1, U_3)] - \nu[g(U_2, U_3)g(U_1, U_4) \\ &- g(U_1, U_3)g(U_2, U_4)]. \end{aligned} \tag{37}$$

If $\overline{P} = 0$, then (37) gives

$$\begin{aligned} \overline{K}(U_1, U_2, U_3, U_4) = &-\frac{\kappa}{2}[g(U_2, U_3)T(U_1, U_4) + g(U_1, U_4)T(U_2, U_3) \\ &- g(U_1, U_3)T(U_2, U_4) - g(U_2, U_4)T(U_1, U_3)] \\ &+ \nu[g(U_2, U_3)g(U_1, U_4) - g(U_1, U_3)g(U_2, U_4)]. \end{aligned} \tag{38}$$

In view of (5), from (31), it follows that

$$\kappa T(U_1,U_2) = \Big(a - \frac{r}{2}\Big)g(U_1,U_2) + bA(U_1)A(U_2) + cB(U_1)B(U_2) \\ + d[A(U_1)B(U_2) + A(U_2)B(U_1)]. \tag{39}$$

Thus, from (38) and (39), we obtain

$$\begin{aligned}\overline{K}(U_1,U_2,U_3,U_4) = {} & f_1[g(U_2,U_3)g(U_1,U_4) - g(U_1,U_3)g(U_2,U_4)] \\ & + f_2[g(U_1,U_4)A(U_2)A(U_3) - g(U_2,U_4)A(U_1)A(U_3) \\ & + g(U_2,U_3)A(U_1)A(U_4) - g(U_1,U_3)A(U_2)A(U_4)] \\ & + f_3[g(U_1,U_4)B(U_2)B(U_3) - g(U_2,U_4)B(U_1)B(U_3) \\ & + g(U_2,U_3)B(U_1)B(U_4) - g(U_1,U_3)B(U_2)B(U_4)] \\ & + f_4[g(U_1,U_4)\{A(U_2)B(U_3) + B(U_2)A(U_3)\} \\ & - g(U_2,U_4)\{A(U_1)B(U_3) + B(U_1)A(U_3)\} \\ & + g(U_2,U_3)\{A(U_1)B(U_4) + B(U_1)A(U_4)\} \\ & - g(U_1,U_3)\{A(U_2)B(U_4) + B(U_2)A(U_4)\}],\end{aligned} \tag{40}$$

where $f_1 = (\nu - a + \frac{r}{2})$, $f_2 = -\frac{b}{2}$, $f_3 = -\frac{c}{2}$, $f_4 = -\frac{d}{2}$. Thus, we can state the following theorem:

Theorem 4. *For a vanishing space–matter tensor, $MG(QE)_4$ spacetime satisfying Einstein's field equation without a cosmological constant is a $MG(QC)_4$ spacetime.*

Next, we investigate the existence of a sufficient condition under which $MG(QE)_4$ can be a divergence-free space–matter tensor.

From (31) and (37), we obtain

$$\begin{aligned}(div\overline{P})(U_1,U_2,U_3) = {} & (divK)(U_1,U_2,U_3) + \frac{1}{2}[(D_{U_1}Ric)(U_2,U_3) \\ & - (D_{U_2}Ric)(U_1,U_3)] - g(U_2,U_3)[\frac{1}{4}\partial r(U_1) + \partial\nu(U_1)] \\ & + g(U_1,U_3)[\frac{1}{4}\partial r(U_2) + \partial\nu(U_2)].\end{aligned} \tag{41}$$

By using $(divK)(U_1,U_2,U_3) = (D_{U_1}Ric)(U_2,U_3) - (D_{U_2}Ric)(U_1,U_3)$ in (41), we obtain

$$\begin{aligned}(div\overline{P})(U_1,U_2,U_3) = {} & \frac{3}{2}[(D_{U_1}Ric)(U_2,U_3) - (D_{U_2}Ric)(U_1,U_3)] \\ & - g(U_2,U_3)[\frac{1}{4}\partial r(U_1) + \partial\nu(U_1)] \\ & + g(U_1,U_3)[\frac{1}{4}\partial r(U_2) + \partial\nu(U_2)].\end{aligned} \tag{42}$$

Let $(div\overline{P})(U_1,U_2,U_3) = 0$; then, contracting (42) over U_2 and U_3, we obtain $\partial\nu(U_1) = 0$, where (27) is used. Hence, we can state the following theorem:

Theorem 5. *For a divergence-free space–matter tensor, the energy density in $MG(QE)_4$ spacetime satisfying Einstein's field equation without a cosmological constant is constant.*

Now, by using (5) in (42), we obtain

$$
\begin{aligned}
(div\overline{P})(U_1,U_2,U_3) = &\frac{3}{2}[\partial a(U_1)g(U_2,U_3) - \partial a(U_2)g(U_1,U_3)] \\
&+ \frac{3}{2}[\partial b(U_1)A(U_2)A(U_3) - \partial b(U_2)A(U_1)A(U_3)] \\
&+ \frac{3b}{2}[(D_{U_1}A)(U_2)A(U_3) + A(U_2)(D_{U_1}A)(U_3) \\
&- (D_{U_2}A)(U_1)A(U_3) - (D_{U_2}A)(U_3)A(U_1)] \\
&+ \frac{3}{2}[\partial c(U_1)B(U_2)B(U_3) - \partial c(U_2)B(U_1)B(U_3)] \\
&+ \frac{3c}{2}[(D_{U_1}B)(U_2)B(U_3) + B(U_2)(D_{U_1}B)(U_3) \\
&- (D_{U_2}B)(U_1)B(U_3) - (D_{U_2}B)(U_3)B(U_1)] \\
&+ \frac{3}{2}[\partial d(U_1)\{A(U_2)B(U_3) + B(U_2)A(U_3)\} \\
&- \partial d(U_2)\{A(U_1)B(U_3) + B(U_1)A(U_3)\}] \\
&+ \frac{3d}{2}[(D_{U_1}A)(U_2)B(U_3) + A(U_2)(D_{U_1}B)(U_3) \\
&+ (D_{U_1}A)(U_3)B(U_2) + A(U_3)(D_{U_1}B)(U_2) \\
&- (D_{U_2}A)(U_1)B(U_3) - A(U_1)(D_{U_2}B)(U_3) \\
&- (D_{U_2}A)(U_3)B(U_1) - A(U_3)(D_{U_2}B)(U_1)] \\
&- g(U_2,U_3)[\frac{1}{4}\partial r(U_1) + \partial \nu(U_1)] \\
&+ g(U_1,U_3)[\frac{1}{4}\partial r(U_2) + \partial \nu(U_2)].
\end{aligned} \tag{43}
$$

By assuming that ν, a, b, c, and d are constants and the generator ρ is a parallel vector field, i.e., $D_{U_1}\rho = 0$, we obtain

$$\partial r(U_1) = 0, \quad \partial \nu(U_1) = 0, \quad (D_{U_1}A)(U_2) = 0. \tag{44}$$

In view of (44), we derive

$$a + b = 0, \quad c = 0, \quad d = 0. \tag{45}$$

Using (44) and (45), (43) reduces to

$$(div\overline{P})(U_1,U_2,U_3) = 0.$$

Thus, we can state the following theorem:

Theorem 6. *In $MG(QE)_4$ spacetimes admitting parallel vector field ρ satisfying Einstein's field equation without a cosmological constant, if the energy density and associated scalars constant are constants, then the divergence of the space–matter tensor vanishes.*

6. $MG(QE)_4$ Spacetime Admitting General Relativistic Viscous Fluid

Ellis [20] defined the energy–momentum tensor for a perfect fluid distribution with heat conduction as

$$
\begin{aligned}
T(U_1,U_2) = &\omega g(U_1,U_2) + (\nu+\omega)A(U_1)A(U_2) + B(U_1)B(U_2) \\
&+ A(U_1)B(U_2) + A(U_2)B(U_1),
\end{aligned} \tag{46}
$$

where $g(U_1,\rho) = A(U_1)$, $g(U_1,\sigma) = B(U_1)$, $A(\rho) = -1$, $B(\sigma) > 0$, $g(\rho,\sigma) = 0$, and ν, ω are called the isotropic pressure and the energy density, respectively. σ is the heat conduction vector field perpendicular to the velocity vector field ρ. Assuming a mixed generalized quasi-Einstein spacetime satisfying Einstein's field equation without a cosmological constant whose matter content is viscous fluid, then, from (31) and (46), the Ricci tensor takes the form

$$\begin{aligned} Ric(U_1,U_2) = (\kappa\omega + \frac{r}{2})g(U_1,U_2) + \kappa(\nu+\omega)A(U_1)A(U_2) \\ + \kappa B(U_1)B(U_2) + \kappa[A(U_1)B(U_2) + A(U_2)B(U_1)]. \end{aligned} \tag{47}$$

By comparing (5) and (47), we obtain

$$a = \kappa\omega + \frac{r}{2}, \quad b = \kappa(\nu+\omega), \quad c = \kappa, \quad d = \kappa. \tag{48}$$

Taking a frame field to contract (48) over U_1 and U_2, we obtai

$$r = \kappa(\nu - 3\omega). \tag{49}$$

In view of (49), (47) turns to

$$\begin{aligned} Ric(U_1,U_2) = \frac{\kappa(\nu-\omega)}{2}g(U_1,U_2) + \kappa(\nu+\omega)A(U_1)A(U_2) \\ + \kappa B(U_1)B(U_2) + \kappa[A(U_1)B(U_2) + A(U_2)B(U_1)]. \end{aligned} \tag{50}$$

Now, let R be the Ricci operator given by $g(R(U_1),U_2) = Ric(U_1,U_2)$ and $Ric(R(U_1),U_2) = Ric^2(U_1,U_2)$. Then, we have $A(R(U_1)) = g(R(U_1),\rho) = Ric(U_1,\rho)$ and $B(R(U_1)) = g(R(U_1),\sigma) = Ric(U_1,\sigma)$. Thus, we obtain

$$\begin{aligned} Ric(R(U_1),U_2) = \frac{\kappa(\nu-\omega)}{2}Ric(U_1,U_2) + \kappa(\nu+\omega)Ric(U_1,\rho)A(U_2) \\ + \kappa Ric(U_1,\sigma)B(U_2) + \kappa[Ric(U_1,\rho)B(U_2) \\ + A(U_2)Ric(U_1,\sigma)]. \end{aligned} \tag{51}$$

Now, contracting (51) over U_1 and U_2, we obtain

$$\begin{aligned} Ric(U_1,U_1) = ||R||^2 = \frac{\kappa(\nu-\omega)r}{2} + \kappa(\nu+\omega)Ric(\rho,\rho) \\ + \kappa Ric(\sigma,\sigma) + \kappa[Ric(\rho,\sigma) + Ric(\sigma,\rho)]. \end{aligned} \tag{52}$$

For a mixed generalized quasi-Einstein spacetime, from (5), it follows that

$$Ric(U_1,\rho) = (a-b)A(U_1) - dB(U_1), \quad Ric(U_1,\sigma) = (a+c)B(U_1) + dA(U_1). \tag{53}$$

In view of (48), (49), and (53), we find that

$$Ric(\rho,\rho) = \frac{\kappa(\nu+3\omega)}{2}, \;\; Ric(\sigma,\rho) = Ric(\rho,\sigma) = -\kappa, \;\; Ric(\sigma,\sigma) = \frac{\kappa(\nu-\omega+2)}{2}. \tag{54}$$

By making use of (54), from (52), it follows that

$$||R||^2 = \kappa^2(\nu^3\omega^2 + \nu + \omega - 3). \tag{55}$$

Thus, we can state the following theorem:

Theorem 7. *If $MG(QE)_4$ spacetime admitting viscous fluid satisfies Einstein's field equation without a cosmological constant, then the square of the length of Ricci operator is $\kappa^2(\nu^3\omega^2 + \nu + \omega - 3)$.*

7. Example of $MG(QE)_4$ Spacetime

In this section, we constructed a non-trivial concrete example to prove the existence of a $MG(QE)_4$ spacetime.

We assume a Lorentzian manifold (M^4, g) endowed with the Lorentzian metric g given by

$$ds^2 = g_{ij}du^i du^j = (1+2p)[(du^1)^2 + (du^2)^2 + (du^3)^2 - (du^4)^2], \tag{56}$$

where u^1, u^2, u^3, u^4 are standard coordinates of M^4, $i, j = 1,2,3,4$, and $p = e^{u^1}k^{-2}$, and k is a non-zero constant. Here, the signature of g is $(+,+,+,-)$, which is Lorentzian. Then, the only non-vanishing components of the Christoffel symbols and the curvature tensors are

$$\left\{\begin{matrix} 1 \\ 11 \end{matrix}\right\} = \left\{\begin{matrix} 1 \\ 44 \end{matrix}\right\} = \left\{\begin{matrix} 2 \\ 12 \end{matrix}\right\} = \left\{\begin{matrix} 3 \\ 13 \end{matrix}\right\} = \left\{\begin{matrix} 4 \\ 14 \end{matrix}\right\} = \frac{p}{1+2p}, \ \left\{\begin{matrix} 1 \\ 22 \end{matrix}\right\} = \left\{\begin{matrix} 1 \\ 33 \end{matrix}\right\} = \frac{-p}{1+2p}. \tag{57}$$

$$\overline{K}_{1212} = \overline{K}_{1313} = \frac{-p}{1+2p}, \ K_{1414} = \frac{p}{1+2p},$$

$$\overline{K}_{3232} = \frac{-p^2}{1+2p}, \ \overline{K}_{4242} = \overline{K}_{4343} = \frac{p^2}{1+2p}$$

and the components are obtained by the symmetry properties.

The non-vanishing components of the Ricci tensors are

$$R_{11} = \frac{3p}{(1+2p)^2}, \ R_{22} = R_{33} = \frac{p}{(1+2p)^2}, \ R_{44} = \frac{-p}{(1+2p)^2},$$

Thus, the scalar curvature r is $\frac{6q(1+q)}{(1+2q)^3}$.

Let us consider the associated scalars a, b, c, and d defined by

$$a = \frac{p}{(1+2p)^3}, \ b = \frac{1}{(1+2p)}, \ c = \frac{-1}{(1+2p)^3}, \ d = \frac{-p}{(1+2p)^2}$$

and the 1-forms are defined by

$$A_1 = B_1 = \sqrt{1+2p}, \quad A_i = B_i = 0 \quad \forall \quad i = 2,3,4,$$

where the generators are unit vector fields; then, from (5), we have

$$R_{11} = ag_{11} + bA_1A_1 + cB_1B_1 + d(A_1B_1 + A_1B_1), \tag{58}$$

$$R_{22} = ag_{22} + bA_2A_2 + cB_2B_2 + d(A_2B_2 + A_2B_2), \tag{59}$$

$$R_{33} = ag_{33} + bA_3A_3 + cB_3B_3 + d(A_3B_3 + A_3B_3), \tag{60}$$

$$R_{44} = ag_{44} + bA_4A_4 + cB_4B_4 + d(A_4B_4 + A_4B_4). \tag{61}$$

$$\begin{aligned} Now, R.H.S. \, of \ (58) \quad &= \quad ag_{11} + bA_1A_1 + cB_1B_1 + d(A_1B_1 + A_1B_1) \\ &= \frac{3p}{(1+2p)^2} \\ &= R_{11} \\ &= L.H.S. \, of \ (58). \end{aligned}$$

Similarly, it can easily be show that (59), (60), and (61) are also true. Hence, $(\mathbb{R}^4, g)$ is a $MG(QE)_4$.

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