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The Intrinsic Structure of High-Dimensional Data According to the Uniqueness of Constant Mean Curvature Hypersurfaces

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Abstract: In this paper, we study the intrinsic structures of high-dimensional data sets for analyzing their geometrical properties, where the core message of the high-dimensional data is hiding on some nonlinear manifolds. Using the manifold learning technique with a particular focus on the mean curvature, we develop new methods to investigate the uniqueness of constant mean curvature spacelike hypersurfaces in the Lorentzian warped product manifolds. Furthermore, we extend the uniqueness of stochastically complete hypersurfaces using the weak maximum principle. For the more general cases, we propose some non-existence results and a priori estimates for the constant higher-order mean curvature spacelike hypersurface.

Keywords: intrinsic geometry structure; Laplacian operator; weak maximum principle; uniqueness; spacelike slice; spacelike hypersurface; constant mean curvature

MSC: 53C42; 53C40

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1. Introduction

Manifold learning is a promising tool in dimensionality reduction with many applications in medical imaging, pattern recognition, data mining and machine learning. Manifold learning algorithms utilize the variables underlying the large and complex data set with nonlinear distribution characteristics. In order to capture the geometrical features of the original data, the manifold method assigns the original data points in a hypersurface to a warped-product space using an isometric mapping and then uses the properties of the inner-product space to determine the intrinsic dimension of the original data space. After the nonlinear transformation, linear techniques can be used to reveal the nonlinear structures of the original data set. Moreover, since the curvature of the manifold quantifies the geometrical structure, we propose an approach to learn the properties of the constant mean curvature spacelike hypersurface.

Recent advances in the analysis of high-dimensional data have attracted increasing interest in geometry-based methods for nonlinearity data distribution. The interest in the study of constant mean curvature spacelike hypersurfaces is motivated by their mathematical and physical properties. The classical Aleksandrov theorem [1] asserts that any compact embedded hypersurface in \mathbb{R}^n with the constant mean curvature is a round sphere. In this direction, Calabi [2] proved that in \mathbb{R}^{n+1} ($n \leq 4$) the only complete maximal spacelike hypersurfaces are hyperplanes. Moreover, Cheng and Yau [3] extended this theorem for \mathbb{R}^{n+1} ($n > 4$). Proceeding into this branch, Ros [4] provided the uniqueness theorem for the higher order mean curvature and proved that the sphere is the only embedded compact hypersurface in \mathbb{R}^{n+1} with H_k ($k = 1, 2, \dots, n$) constant. Alias et al. [5,6] also proved the uniqueness of zero scalar curvature hypersurfaces and rotational hypersurfaces in \mathbb{R}^{n+1} . On the other hand, Treibergs [7] showed the existence of lots of complete non-zero constant mean curvature spacelike hypersurfaces. In this paper, we are concerned with the

uniqueness and existence of constant mean curvature spacelike hypersurfaces in a class of Lorentzian warped product manifolds.

The uniqueness of constant mean curvature spacelike hypersurfaces immersed in Lorentzian warped product spaces has been studied intensively. Under suitable restrictions on the value of mean curvatures, the uniqueness of compact hypersurfaces was developed by applying Omori-Yau maximum principle for the Laplacian operator [8,9]. Afterwards, Alias et al. [10,11] extended the above results to the higher order mean curvatures using a generalized version of the maximum principal for a new trace type differential operator L_k . Furthermore, Aledo et al. [12] proved the uniqueness of complete parabolic constant mean curvature spacelike hypersurfaces in a Lorentzian warped product with dimension $n + 1 \leq 5$. There are many special properties for the hypersurface of manifolds M^{n+1} , in particular, when $3 \leq n + 1 \leq 7$. In 2019, Zhou [13] proved that for the prescribed mean curvature in an arbitrary closed manifold M^{n+1} ($3 \leq n + 1 \leq 7$), there exists a nontrivial, smooth, closed, almost embedded, constant mean curvature hypersurface of any given mean curvature. However, all of the uniqueness studies were based on the maximum principle. In this paper, we extend the uniqueness of complete hypersurfaces with dimension $3 \leq n + 1 \leq 7$ using the weak maximum principle of Grigor'yan [14] and Pigola et al. [15,16]. Combining with appropriate geometric assumptions, we obtain some non-existence of constant mean curvature spacelike hypersurfaces as follows:

Let $-I \times_f M^n$ be a generalized Robertson–Walker (GRW) spacetime with $n \geq 2$, where I is bounded and $(\log f)'' \leq 0$. Then, there is no stochastically complete spacelike hypersurface with non-zero constant mean curvature H such that $\langle N, \partial_t \rangle < 1 - n$.

Using the weak maximum principle [16,17] and its equivalent forms, we obtain the following main result of uniqueness for the spacelike hypersurface in spacetimes $-I \times_f M^n$ of dimension $3 \leq n + 1 \leq 7$.

Let $-I \times_f M^n$ be a generalized Robertson–Walker spacetime satisfying the time-like convergence condition (TCC) and the dimension $3 \leq n + 1 \leq 7$, where $I = (a, b)$ with $-\infty \leq a < b < +\infty$. Suppose that $\psi : \Sigma^n \rightarrow -I \times_f M^n$ is a stochastically complete spacelike hypersurface with non-zero constant mean curvature H . Then, the hypersurface is a slice.

This paper is organized as follows. In Section 2, we outline the notations about the hypersurface in GRW spacetimes, the weak maximum principal and its equivalent form, which are our main analytical tools. In Section 3, we obtain the sign relationship between the mean curvature and the derivative of the warping function, then present the application of the curvature estimate. Furthermore, we provide the uniqueness and non-existence results of constant mean curvature spacelike hypersurfaces. In Section 4, combining with the results of curvature estimates and uniqueness, we prove the uniqueness of constant k -mean curvature spacelike hypersurfaces in GRW spacetimes.

2. Materials and Methods

Let M^n be a connected n -dimensional Riemannian manifold, and $f : \mathbb{R} \rightarrow \mathbb{R}^+$ be a positive smooth function. In the warped product differentiable manifold $\overline{M}^{n+1} = -I \times_f M^n$, let π_I and π_M denote the projections onto the fibers \mathbb{I} and M , respectively. A particular class of Lorentzian manifold is the one obtained by furnishing \overline{M} with the metric

$$\langle v, w \rangle_p = -\langle (\pi_{\mathbb{R}})_*v, (\pi_I)_*w \rangle_I + (f \circ \pi_I)^2(p) \langle (\pi_M)_*v, (\pi_M)_*w \rangle_M,$$

for all $p \in \overline{M}^{n+1}$ and all $v, w \in T_p\overline{M}$, where $\langle \cdot, \cdot \rangle_I$ and $\langle \cdot, \cdot \rangle_M$ stand for the metrics of I and M^n , respectively. Such a space is called a Lorentzian warped product space. In what follows, we shall denote it as $\overline{M}^{n+1} = -\mathbb{R} \times_f M^n$. For simplicity of notation, we denote the warped metric as

$$\langle , \rangle = -dt^2 + f^2(t)\langle , \rangle_M.$$

Under this condition, for a fixed $t_0 \in \mathbb{R}$, we say that $M_{t_0}^n = \{t_0\} \times M^n$ is a slice of \overline{M}^{n+1} .

Let $\psi : \Sigma^n \rightarrow -I \times_f M^n$ be a spacelike hypersurface and A be the second fundamental form of the immersion with respect to the past-pointing Gauss map. In this setting, at each $p \in \Sigma^n$, A is restricted to a self-adjoint linear map $A_p : T_p\Sigma \rightarrow T_p\Sigma$. For $0 \leq k \leq n$, let $S_k(p)$ denote the k th elementary symmetric function on the eigenvalues of A_p . We define n smooth functions $S_k : \Sigma^n \rightarrow \mathbb{R}$ such that

$$\det(tI - A) = \sum_{k=0}^n (-1)^k S_k t^{n-k},$$

where $S_0 = 1$ by definition. If $p \in \Sigma^n$ and $\{e_k\}$ is a basis of $T_p\Sigma$ formed by eigenvectors of A_p , with the corresponding eigenvalues $\{\lambda_k\}$, then

$$S_k(p) = \sigma_k(\lambda_1(p), \lambda_2(p), \dots, \lambda_n(p)) = \sum_{i_1 < \dots < i_k} \lambda_{i_1}(p) \cdots \lambda_{i_k}(p),$$

and the k -mean curvature H_k of the hypersurface is defined by

$$\binom{n}{k} H_k = (-1)^k S_k = (-1)^k \sum_{i_1 < \dots < i_k} \lambda_{i_1} \cdots \lambda_{i_k}. \tag{1}$$

Thus, $H_0 = 1$ and $H_1 = -\frac{1}{n} \text{tr}(A) = H$ is the mean curvature of Σ^n .

In what follows, we work with the so-called Newton transformations $P_k : \mathfrak{X}(\Sigma) \rightarrow \mathfrak{X}(\Sigma)$, which are defined from A by setting $P_0 = I$ (the identity of $\mathfrak{X}(\Sigma)$) and for $1 \leq k \leq n$,

$$k = \binom{n}{k} H_k I + A \circ P_{k-1}. \tag{2}$$

Observe that the Newton transformations P_k are all self-adjoint operators which commute with the shape operator A . Even more, if $\{e_k\}$ is an orthonormal frame on $T_p\Sigma$ which is diagonalizable with A_p and $A_p(e_i) = \lambda_i(p)e_i$, then

$$P_{kp}(e_i) = \mu_{i,k}(p)e_i, \tag{3}$$

where

$$\mu_{i,k} = (-1)^k \sum_{i_1 < \dots < i_k, i_j \neq i} \lambda_{i_1} \cdots \lambda_{i_k}.$$

For each k , $1 \leq k \leq n - 1$, we have

$$\text{tr}(P_k) = c_k H_k, \quad \text{tr}(A \circ P_k) = -c_k H_{k+1},$$

where $c_k = (n - k) \binom{n}{k} = (k + 1) \binom{n}{k + 1}$.

We refer the reader to [18–20] for further details about the classical Newton transformations for hypersurfaces in Riemannian and Lorentzian spaces.

Associated with each Newton transformations P_k , we consider the second-order linear differential operator $L_k : C^\infty(\Sigma) \rightarrow C^\infty(\Sigma)$, given by

$$L_k(f) = \text{tr}(P_k \circ \nabla^2 f) = \text{tr}(P_k \circ \text{hess}(f)).$$

In particular, $L_0 = \Delta$. Here, $\nabla^2 f : \mathfrak{X}(\Sigma) \rightarrow \mathfrak{X}(\Sigma)$ denotes the self-adjoint linear operator which is metrically equivalent to the hessian of f . It is given by

$$\langle \nabla^2 f(X), Y \rangle = \langle \nabla_X(\nabla f), Y \rangle, \quad X, Y \in \mathfrak{X}(\Sigma).$$

Let $\psi : \Sigma^n \rightarrow -I \times_f M^n$ be a Riemannian immersion with Σ oriented by the unit vector field N . If N is in the opposite time orientation as ∂_t , such that $\langle N, \partial_t \rangle < 0$, the normal vector field N is past-pointing Gauss map of the hypersurface. In what follows, we assume hyperbolic angle function $\langle N, \partial_t \rangle$ does not change the sign on Σ^n . Let h denote the (vertical) height function naturally attached to Σ^n , namely, $h = (\pi_I)|_{\Sigma}$.

Let $\bar{\nabla}$ and ∇ denote gradients with respect to the metrics of $-I \times_f M^n$ and Σ^n , respectively. A simple computation shows that the gradient of $\pi_{\mathbb{R}}$ on $-I \times_f M^n$ is given by

$$\bar{\nabla} \pi_I = -\partial_t,$$

so that the gradient of h on Σ^n is

$$\nabla h = (\bar{\nabla} \pi_{\mathbb{R}})^{\top} = -\partial_t - \langle N, \partial_t \rangle N.$$

In particular, we have

$$|\nabla h|^2 = \langle N, \partial_t \rangle^2 - 1, \tag{4}$$

where $|\cdot|$ denotes the norm of a vector field on Σ^n .

The main results of this paper are based on the particular case of the following lemma for the trace operator, which is an operator of the form

$$L_T(u) = \text{tr}(T \circ \text{hess}(u)) = \text{div}(T(\nabla u)) - \langle \text{div} T, \nabla u \rangle.$$

We recall that stochastic completeness is the property for a stochastic process to have an infinite lifetime as follows:

Definition 1. A Riemannian manifold $(M, \langle \cdot, \cdot \rangle)$ is said to be stochastically complete if for some (and therefore, for any) $(x, t) \in M \times (0, +\infty)$,

$$\int_M p(x, y, t) dy = 1,$$

where $p(x, y, t)$ is the heat kernel of the Laplace-Beltrami operator Δ .

Note that the metric $\langle \cdot, \cdot \rangle$ is not assumed to be complete in the above definition. For a more detailed introduction to Definition 1, we refer to the book by Emery [21]. The following Lemma 1 can be found in Grigor'yan [14] and Pigola et al. [15].

Lemma 1. Let $(M, \langle \cdot, \cdot \rangle)$ be a Riemannian manifold. The following statements are equivalent:

- M is stochastically complete ;
- For every $u \in C^2(M)$ with $\sup_M u < +\infty$, there exists a sequence $\{x_n\}$, $n = 1, 2, \dots$, such that, for every n , $u(x_n) \geq \sup_M u - \frac{1}{n}$ and $\Delta u(x_n) \leq \frac{1}{n}$, which is the Weak Maximum Principle (WMP) for the Laplacian operator;
- For every $g \in C^0(M)$ and every $u \in C^2(M)$ with $u^* = \sup_M u < +\infty$ satisfying $\Delta u \geq g(u)$, we have $g(u^*) \leq 0$.

Equivalently, for any $u \in C^2(M)$ with $u_* = \inf_M u > -\infty$, there exists a sequence $\{y_n\}$, which satisfies the corresponding geometric conditions. Furthermore, as the generalization of Lemma 1, the following conclusion can be found in [15,17].

Lemma 2. *The weak maximum principal holds on M for the operator L if and only if one of the following statements holds on:*

- For every $u \in C^2(M)$ with $u^* = \sup_M u < +\infty$ and every $\gamma < u^*$, we have

$$\inf_{\Omega_\gamma} Lu \leq 0,$$

where $\Omega_\gamma = \{x \in M : u(x) > \gamma\}$;

- For every $g \in C^0(\mathbb{R})$ and $u \in C^2(M)$ which solve the differential inequality $Lu \geq b(x)g(u)$ ($b(x) > 0$), we have $g(u^*) \leq 0$;
- For each $g \in C^0(\mathbb{R})$, for each open set $\Omega \subset M$ with $\partial\Omega \neq \emptyset$, for each $v \in C^0(\bar{\Omega}) \cap C^1(\Omega)$ satisfying $Lv \geq g(v)$ on Ω and $\sup_\Omega v < +\infty$, we have that either

$$\sup_\Omega v = \sup_{\partial\Omega} v$$

or

$$g(\sup_\Omega v) \leq 0.$$

Here, $L = \text{trace}(T \circ \text{hess}(u)) = \text{div}(T(\nabla u)) - \langle \text{div}T, \nabla u \rangle$ is a trace operator with T being a positive definite and symmetric endomorphism on the tangent bundle TM of manifold M .

We already know that $L_0 = \Delta$ is always elliptic. According to Lemma 3.10 of [22], if $H_2 > 0$ on Σ^n , the operator L_1 is elliptic, equivalently, P_1 is positive definite. When $k \geq 2$, the lemma below establishes sufficient conditions which guarantee the ellipticity of the operator L_r . The details about the lemma can be found in [23].

Lemma 3. *Let $\psi : \Sigma^n \rightarrow \bar{M}^{n+1}$ be a spacelike hypersurface immersed into a semi-Riemannian manifold \bar{M}^{n+1} . If Σ^n has an elliptic point with respect to an appropriate choice of the Gauss map N and $H_{r+1} > 0$ on Σ^n for some $2 \leq r \leq n - 1$, then P_k is positive definite and H_k is positive for all $1 \leq k \leq r$.*

Recall that by an elliptic point in the spacelike hypersurface we mean a point $p_0 \in \Sigma$ where all principal curvature $\lambda_i(p_0)$ have the same sign.

3. Uniqueness of Spacelike Hypersurfaces in Generalized Robertson Spacetimes

We consider a hypersurface Σ^n with a Gauss map satisfying $\langle N, \partial_t \rangle \leq -1$. Under this setting, we define the hyperbolic angle function Θ between Σ^n and ∂_t which is a smooth function and satisfies

$$-\infty < \Theta = \langle N, \partial_t \rangle \leq -1 < 0,$$

thus

$$|\nabla h|^2 = \Theta^2 - 1.$$

The following are some useful expressions for the second-order linear differential operator L_k , which can be found in [10].

Lemma 4. *Let $\psi : \Sigma^n \rightarrow -I \times_f M^n$ be a complete spacelike hypersurface, $g(t) = \int_{t_0}^t f(s) ds$ be defined on I , $\langle N, \partial_t \rangle$ be the angle function for some fixed $t_0 \in I$. We have that*

$$\begin{cases} L_k h = -\frac{f'}{f}(h)(c_k H_k + \langle P_k \nabla h, \nabla h \rangle) - c_k \langle N, \partial_t \rangle H_{k+1}, \\ L_k(g(h)) = -c_k f(h) \left(\frac{f'}{f}(h) H_k + \langle N, \partial_t \rangle H_{k+1} \right), \end{cases} \tag{5}$$

and

$$\begin{aligned}
 L_{k-1}(f(h)\langle N, \partial_t \rangle) &= \binom{n}{k} f(h)\langle \nabla h, \nabla H_k \rangle + f'(h)c_{k-1}H_k \\
 &+ \frac{\langle N, \partial_t \rangle}{f(h)} \sum_{i=1}^n \mu_{k-1,i} K_M(E_i^*, N^*) \|E_i^* \wedge N^*\|^2 \\
 &+ \frac{\langle N, \partial_t \rangle}{f(h)} (f'^2 - f''f)(c_{k-1}H_{k-1}|\nabla h|^2 - \langle P_{k-1}\nabla h, \nabla h \rangle) \\
 &+ f(h)\langle N, \partial_t \rangle \binom{n}{k} (nH_1H_k - (n-k)H_{k+1}).
 \end{aligned}$$

In particular, when $k = 1$, we have

$$\begin{aligned}
 \Delta(f(h)\langle N, \partial_t \rangle) &= f(h)\langle \nabla h, \nabla H \rangle + f'(h)nH + nf(h)\langle N, \partial_t \rangle (nH^2 - (n-1)H_2) \\
 &+ f(h)\langle N, \partial_t \rangle \left(\text{Ric}_M(N^*, N^*) - (n-1)(f'^2 - f''f)(h)\langle N^*, N^* \rangle_M \right),
 \end{aligned}$$

where N^* and E_i^* denote the projections onto P^n of N and E_i , respectively, and $\{E_i\}_1^n$ is a local orthonormal frame such that the operators A and P_{k-1} are commute, that is $P_{k-1}E_i = \mu_{k-1,i} E_i$.

Observe that

$$L_k(f(h)) = f''(h)\langle P_k\nabla h, \nabla h \rangle + f'(h)L_k h,$$

which implies that

$$L_k(f(h)) = \frac{f''f - f'^2}{f}(h)\langle P_k\nabla h, \nabla h \rangle - c_k f'(h)H_k \left(\frac{f'}{f}(h) + \langle N, \partial_t \rangle \frac{H_{k+1}}{H_k} \right). \tag{6}$$

Combining with the formula above, we have

$$L_k(f(h)\langle N, \partial_t \rangle) = \langle N, \partial_t \rangle L_k f(h) + f(h)L_k \langle N, \partial_t \rangle + 2\langle P_k\nabla f(h), \nabla \langle N, \partial_t \rangle \rangle.$$

Thus,

$$\begin{aligned}
 L_{k-1}\langle N, \partial_t \rangle &= \binom{n}{k} \langle \nabla h, \nabla H_k \rangle + \langle N, \partial_t \rangle \binom{n}{k} (nH_1H_k - (n-k)H_{k+1}) \\
 &+ \frac{f'(h)}{f(h)} c_{k-1}H_k + \frac{\langle N, \partial_t \rangle}{f^2(h)} \sum_{i=1}^n \mu_{k-1,i} K_M(E_i^*, N^*) \|E_i^* \wedge N^*\|^2 \\
 &+ \frac{\langle N, \partial_t \rangle}{f^2(h)} (f'^2 - f''f) (c_{k-1}H_{k-1}|\nabla h|^2 - \langle P_{k-1}\nabla h, \nabla h \rangle) \\
 &+ \langle N, \partial_t \rangle \frac{f'^2 - f''f}{f^2} \langle P_{k-1}\nabla h, \nabla h \rangle \\
 &+ \langle N, \partial_t \rangle \left(\frac{f'(h)}{f(h)} \right)^2 c_{k-1}H_{k-1} + \frac{f'(h)}{f(h)} \langle N, \partial_t \rangle^2 c_{k-1}H_k \\
 &+ 2\langle N, \partial_t \rangle \left(\frac{f'(h)}{f(h)} \right)^2 \langle P_{k-1}\nabla h, \nabla h \rangle - 2\frac{f'(h)}{f(h)} \langle P_{k-1}\nabla h, A\nabla h \rangle.
 \end{aligned}$$

On the other hand, since

$$\text{hess}(X) = -\frac{f'(h)}{f(h)} (X + \langle X, \nabla h \rangle \nabla h) + \langle N, \partial_t \rangle AX,$$

the square algebraic trace-norm of the hessian tensor of h is

$$|\text{hess}(h)|^2 = \left(\frac{f'(h)}{f(h)}\right)^2(n - 1 + \Theta^4) + \Theta^2\|A\|^2 + 2nH\frac{f'(h)}{f(h)}\Theta - 2\frac{f'(h)}{f(h)}\Theta\langle A\nabla h, \nabla h \rangle. \tag{7}$$

Moreover,

$$\begin{aligned} \langle \text{hess}(h), P_{k-1} \circ \text{hess}(h) \rangle &= \left(\frac{f'(h)}{f(h)}\right)^2 c_{k-1} H_{k-1} + 2\Theta \frac{f'(h)}{f(h)} c_{k-1} H_k \\ &\quad + \Theta^2 \binom{n}{k} \left(nH_1 H_k - (n - k)H_{k+1} \right) \\ &\quad + \left(\frac{f'(h)}{f(h)}\right)^2 \langle P_{k-1} \nabla h, \nabla h \rangle + \left(\frac{f'(h)}{f(h)}\right)^2 \Theta^2 \langle P_{k-1} \nabla h, \nabla h \rangle \\ &\quad - 2\Theta \langle A\nabla h, P_{k-1} \nabla h \rangle. \end{aligned}$$

It is easy to see that $\|\text{hess}(h)\|^2 \geq 0$ and $\langle \text{hess}(h), P_{k-1} \circ \text{hess}(h) \rangle \geq 0$ when P_{k-1} is positive. Subsequently, we obtain the following result that will be used in our computations.

Lemma 5. *Let $\psi : \Sigma^n \rightarrow -I \times_f M^n$ be a complete spacelike hypersurface with constant k -mean curvature H_k , $\Theta = \langle N, \partial_t \rangle$ be the angle function and h be the height function, then we have*

$$\begin{aligned} \Delta\Theta &= n\langle \nabla h, \nabla H \rangle + \Theta \left(\text{Ric}_M(N^*, N^*) - (n - 1)(ff'' - f'^2)|\nabla h|^2 \right) - \Theta(\log f)''|\nabla h|^2 \\ &\quad + \frac{\|\text{hess}(h)\|^2}{\Theta} + \frac{f'(h)}{\Theta} |\nabla h|^2 \left(\frac{f'}{f}(h)\Theta^2 + nH\Theta + (n - 1)\frac{f'}{f}(h) \right) \end{aligned}$$

and

$$\begin{aligned} L_{k-1}\Theta &= \binom{n}{k} \langle \nabla h, \nabla H_k \rangle + \frac{\Theta}{f^2(h)} \sum_{i=1}^n \mu_{k-1,i} K_M(E_i^*, N^*) \|E_i^* \wedge N^*\|^2 \\ &\quad + \Theta \frac{f'^2 - f''f}{f^2} \left(c_{k-1} H_{k-1} |\nabla h|^2 - \langle P_{k-1} \nabla h, \nabla h \rangle \right) \\ &\quad + \Theta \frac{f'^2 - f''f}{f^2} \langle P_{k-1} \nabla h, \nabla h \rangle + \frac{\langle \text{hess}(h), P_{k-1} \circ \text{hess}(h) \rangle}{\Theta} \\ &\quad + \Theta \frac{f'(h)}{f(h)} \left(\frac{f'}{f}(h) c_{k-1} H_{k-1} + c_{k-1} H_k \Theta + \frac{f'}{f}(h) \langle P_{k-1} \nabla h, \nabla h \rangle \right) \\ &\quad - \frac{f'(h)}{\Theta} \left(\frac{f'}{f}(h) c_{k-1} H_{k-1} + c_{k-1} H_k \Theta + \frac{f'}{f}(h) \langle P_{k-1} \nabla h, \nabla h \rangle \right), \end{aligned}$$

where $|\nabla h|^2 = f^2(h)\langle N^*, N^* \rangle_M$.

Using Lemma 1, we prove the following sign relationship between mean curvature H and the derivative of warping function f . Furthermore, we obtain the result about mean curvature estimates.

Lemma 6. *Let $\psi : \Sigma^n \rightarrow -I \times_f M^n$ ($n \geq 2$) be a stochastically complete spacelike hypersurface with non-zero constant mean curvature H and $(\log f)'' \leq 0$, where $I = (a, b)$ with $-\infty \leq a < b \leq +\infty$. If $H > 0$ and $h^* < b$, then $f' > 0$; similarly, if $H < 0$ and $h_* > a$, then $f' < 0$.*

Proof. First, we prove the case for $H > 0$ and $f' > 0$. Since the weak maximum principle holds on the hypersurface, $\langle N, \partial_t \rangle \leq -1$ and h is bounded from above, then using Lemma 1, there exists a sequence $\{p_j\}$ such that

$$h(p_j) > h^* - \frac{1}{j}, \quad \Delta h(p_j) \leq \frac{1}{j},$$

i.e.,

$$\Delta h(p_j) = -\frac{f'}{f}(h(p_j))(n + |\nabla h(p_j)|^2) - nH\langle N, \partial_t \rangle(p_j) \leq \frac{1}{j}.$$

The inequality $(\log f)'' \leq 0$ implies $\frac{f'}{f}(h) \geq \frac{f'}{f}(h^*)$. This further leads to

$$-\frac{f'}{f}(h(p_j)) \leq \frac{n}{n + |\nabla h(p_j)|^2} \langle N, \partial_t \rangle H + \frac{\frac{1}{j}}{n + |\nabla h(p_j)|^2} < \frac{n}{n + |\nabla h(p_j)|^2} \langle N, \partial_t \rangle H + \frac{1}{j},$$

where $\langle N, \partial_t \rangle \leq -1$, H is a positive constant and $0 \leq |\nabla h|^2 = \langle N, \partial_t \rangle^2 - 1$. Therefore, there exists a positive integer N , such that

$$\frac{f'}{f}(h(p_j)) > 0, \quad \forall j > N.$$

Moreover, since f' is non-vanishing, we have $f' > 0$.

Following a similar argument as before, we can show that if the constant mean curvature H is strictly negative, then $f' < 0$. \square

From the proof of Lemma 6, one can ensure that $\langle N, \partial_t \rangle H f' < 0$ either f' or H is non-vanishing.

Theorem 1. Let $\psi : \Sigma^n \rightarrow -I \times_f M^n$ ($n \geq 2$) be a stochastically complete spacelike hypersurface with non-zero constant mean curvature H and $(\log f)'' \leq 0$, where $I = (a, b)$ with $-\infty \leq a < b \leq +\infty$. If $H > 0$ and $h^* < b$, then $0 < H \leq \frac{f'}{f}(h^*)$; similarly, if $H < 0$ and $h_* > a$, then $\frac{f'}{f}(h_*) \leq H < 0$.

Proof of Theorem 1. If h is constant, it is easy to show that Σ is a slice $\{h^*\} \times M^n$ with constant H . Now, we assume that h is non-constant. Since H is non-vanishing, we just consider the case $H > 0$. We can complete the proof in a similar way when $H < 0$. Combining with Lemma 6, we can assume that $\frac{f'}{f}(h^*) < H$, then we can prove it by contradiction. From the hypothesis, we consider $\gamma < h^*$ such that $\partial\Omega_\gamma \neq \emptyset$, where $\Omega_\gamma = \{x \in \Sigma : h(x) > \gamma\}$, and there exists some $\varepsilon > 0$, such that

$$H - \frac{f'}{f}(h) \geq \varepsilon,$$

and

$$\begin{aligned} \Delta g(h) &= -nf(h)\left(\frac{f'}{f}(h) + \Theta H\right) \\ &\geq -nf(h)\left(\frac{f'}{f}(h) - H\right) \\ &\geq -nf(h)(-\varepsilon) \\ &\geq f(\gamma)\varepsilon > 0. \end{aligned}$$

Since the hypersurface is stochastically complete, from Lemma 1, the weak maximum principle holds on Σ^n for the Laplace–Beltrami operator Δ . Applying it, we obtain

$$\sigma = f(\gamma)\varepsilon \leq 0,$$

which is a contradiction. Thus, we complete the proof. \square

In order to extend the result above to the higher-order mean curvature, we introduce two second-order elliptic differential operators, which will be used in our computation as follows. The more detailed version can be found in [11].

Lemma 7. *Let $\psi : \Sigma^n \rightarrow -I \times_f M^n$ be a complete spacelike hypersurface with an elliptic point and non-vanishing k -mean curvature H_k ($2 \leq k \leq n$). Now, we define the operators \mathcal{P}_{k-1} as follows*

$$\mathcal{P}_{k-1} = \sum_{i=0}^{k-1} \frac{c_{k-1}}{c_i} \left(\frac{f'}{f}(h)\right)^{k-1-i} (-\Theta)^i P_i,$$

where P_i is defined in Equation (2). Then

$$\mathcal{L}_{k-1} = \sum_{i=0}^{k-1} \frac{c_{k-1}}{c_i} \left(\frac{f'}{f}(h)\right)^{k-1-i} (-\Theta)^i L_i = \text{Tr}(\mathcal{P}_{k-1} \circ \text{hess}),$$

and

$$\mathcal{L}_{k-1}(g(h)) = -c_{k-1}f(h) \left(\left(\frac{f'}{f}(h)\right)^k - (-\Theta)^k H_k \right).$$

Remark 1. *Combing with the definition of the elliptic point and Lemma 3, we obtain that if N is the right orientation such that the k -mean curvature H_k is positive, then the operator P_k is always positive definite. Thus, using Lemma 3, we have L_k is elliptic. Using the assumption that $f' > 0$ and $\langle N, \partial_t \rangle < 0$, it is easy to see that \mathcal{P}_{k-1} is positive definite, equivalently, the operator \mathcal{L}_{k-1} is elliptic.*

Next, we generalize the result of Theorem 1 to the case of constant higher-order mean curvatures. Alías et al. proved the case of $f' > 0$ in [17] (Theorem 4.4). We extend the conclusion to $f' \neq 0$, which is presented below. In order to apply Lemma 3, we assume the existence of the elliptic point. Now, we deduce the following result, which is about the higher-order mean curvature estimate for spacelike hypersurfaces.

Theorem 2. *Let $\psi : \Sigma^n \rightarrow -I \times_f M^n$ be a complete spacelike hypersurface with non-vanishing constant k -mean curvature H_k ($2 \leq k \leq n$) and $\sup_{\Sigma} |H| < +\infty$, where $I = (a, b)$ with $-\infty \leq a < b \leq +\infty$. Suppose that the weak maximum principle is valid on Σ^n for the operator L_k , and there exists an elliptic point on Σ^n and $(\log f)'' \leq 0$. Then, if $H_k > 0$ and $h^* < b$, we have $0 < H_k^{\frac{1}{k}} \leq \frac{f'}{f}(h)$; similarly, if $H_k < 0$ and $h_* > a$, then $\frac{f'}{f}(h) \leq H_k^{\frac{1}{k}} < 0$.*

Proof of Theorem 2. Since H_k ($k \geq 2$) is non-vanishing, there exists an elliptic point on Σ . Using Lemma 6 and Cauchy–Schwarz inequality, we have

$$H \geq H_2^{\frac{1}{2}} \geq \dots \geq H_k^{\frac{1}{k}} > 0.$$

Similarly, if $H_k < 0$, then n must be odd, so we have $H < 0$ and $(-1)^{k+1}H_k < 0$.

The operator P_k is positive by Lemma 3. Combining with the Remark 1, we have that the operator \mathcal{L}_{k-1} is elliptic. Furthermore, since $\sup_{\Sigma} |H| < +\infty$ and the weak maximum principle is valid on Σ^n for the operator L_k , then the weak maximum principle is also valid on Σ^n for the operator \mathcal{L}_{k-1} .

If h is constant, then it is easy to see Σ is a slice $h^* \times M^n$ such that $\frac{f'^k}{f^k}(h) = H_k$. Now, we consider the case of non-constant h and non-vanishing f' . We will prove the case of $0 < H_k^{\frac{1}{k}} \leq \frac{f'}{f}(h)$.

We assume that $\frac{f'}{f}(h^*) < H_k^{\frac{1}{k}}$, then there exists $\gamma < h^*$ and $\varepsilon > 0$ such that $\partial\Omega_\gamma \neq \emptyset$, where $\Omega_\gamma = \{x \in \Sigma : h(x) > \gamma\}$ and

$$H_k - \frac{f'^k}{f^k}(h) \geq \varepsilon.$$

Denoting $g(h) = \int_{t_0}^t f(s) ds$, then $g(h)$ is an increasing function and $g(h)^* = g(h^*) < +\infty$. Since $(\log f)'' \leq 0$ and $\langle N, \partial_t \rangle \leq -1$, we obtain

$$\frac{f'^k}{f^k}(h) - (-\Theta)^k H_k \leq \frac{f'^k}{f^k}(\gamma) - H_k \leq -\varepsilon$$

on Ω_γ . Since $f' > 0$, then

$$\begin{aligned} \mathcal{L}_{k-1}(g(h)) &= -c_{k-1}f(h)\left(\left(\frac{f'}{f}(h)\right)^k - (-\Theta)^k H_k\right) \\ &\geq c_{k-1}f(h)\varepsilon \\ &\geq c_{k-1}f(\gamma)\varepsilon > 0 \end{aligned}$$

on Ω_γ . Then, applying Lemma 2 to Ω_γ for the elliptic operator \mathcal{L}_k , with $v \equiv c_{k-1}f(\gamma)\varepsilon$ a positive constant, we obtain

$$c_k f(\gamma)\varepsilon \leq 0.$$

Therefore, it is in contradiction with the hypothesis, then we obtain the conclusion.

If $H < 0$, i.e., $(-1)^{k+1}H_k < 0$, we denote the operator $(-1)^{k-1}\mathcal{L}_{k-1}$ is elliptic, where the operator $\text{Tr}((-1)^{k-1}\mathcal{P}_{k-1})$ is positive definite. The proof follows a similar method as $H > 0$. \square

Remark 2. For the case of $k = 1$, there is no need for the existence of an elliptic point. If $H_2 > 0$, then $H^2 \geq H_2 > 0$. We have $H > 0$ by choosing the appropriate Gauss map. Since $n^2H^2 = \sum k_i^2 + n(n-1)H_2 > k_i^2$, then $\mu_{i,1} = nH_1 + K_i > 0$ for any i , so P_1 is positive definite, which guarantees the ellipticity of L_1 .

If the hyperbolic angle function $\Theta < 0$, recall that the constant mean curvature spacelike hypersurface in \bar{M} is the spacelike slice $\tau \times M^n$ of the Lorentzian warped product $\bar{M} = -I \times_f M^n$, if and only if it satisfies $H = \frac{f'(\tau)}{f(\tau)}$, which is extended to the higher order mean curvature $H_k = \left(\frac{f'(\tau)}{f(\tau)}\right)^k$.

Theorem 3. Let $\psi : \Sigma^2 \rightarrow -I \times_f M^2$ be a stochastically complete spacelike surface with non-zero constant mean curvature H , and be contained in a slab and on which $(\log f)'' \leq 0$. Then, the surface Σ^2 must be a slice.

Proof. Since the mean curvature is non-zero constant, from Lemma 6, we have $f' \neq 0$. Using Lemma 4, we have

$$\begin{aligned} \Delta h &= -\left(\frac{f'}{f}(h)\Theta^2 + n\Theta H + (n-1)\frac{f'}{f}(h)\right) \\ &= -\frac{1}{\frac{f'}{f}(h)}\left(\left(\frac{f'}{f}(h)\Theta\right)^2 + n\Theta H\frac{f'}{f}(h) + (n-1)\frac{f'^2}{f^2}(h)\right) \\ &= -\frac{1}{\frac{f'}{f}(h)}\left(\left(\frac{f'}{f}(h)\Theta + \frac{nH}{2}\right)^2 + (n-1)\left(\frac{f'^2}{f^2}(h) - \frac{n^2H^2}{4(n-1)}\right)\right). \end{aligned}$$

If $n = 2$, combining with Theorem 1, we have

$$\Delta h = -\frac{1}{\frac{f'}{f}(h)}\left(\left(\frac{f'}{f}(h)\Theta + H\right)^2 + \left(\frac{f'^2}{f^2}(h) - H^2\right)\right) \leq 0.$$

Since the surface is contained in a slab, we have $h_* > -\infty$, then applying Lemma 1, we obtain that

$$g(h_*) = -\frac{1}{\frac{f'}{f}(h_*)}\left(\left(\frac{f'}{f}(h_*)\Theta + H\right)^2 + \left(\frac{f'^2}{f^2}(h_*) - H^2\right)\right) \geq 0.$$

From Theorem 1, we have $\frac{f'}{f}(h_*) \geq \frac{f'}{f}(h) \geq H > 0$. Clearly, the only solution of this inequality is $\frac{f'}{f}(h_*) = H$ and $\Theta = -1$, which implies that the surface is a slice. \square

Under the assumption of Theorem 3, we have that the constant mean curvature spacelike slice $\{\gamma\} \times M^n$ is given by $H = \frac{f'}{f}(\gamma)$, therefore, we obtain some non-existence results as follows.

Corollary 1. *Let $\psi : \Sigma^n \rightarrow -I \times_f M^n$ ($n \geq 3$) be a spacelike hypersurface with non-zero constant mean curvature H and $(\log f)'' \leq 0$, where $I = (a, b)$ with $-\infty \leq a < b < +\infty$. Then, there is no stochastically complete spacelike hypersurface with non-zero constant mean curvature H such that*

$$H^2 < \frac{4(n-1)}{n^2} \frac{f'^2}{f^2}(h).$$

Considering the warped product manifold $-\mathbb{R} \times_{e^t} M^2$, the warping function $f = e^t$ satisfies $(\log f)'' \leq 0$. In particular, using the proof of Theorem 1, we obtain that $\frac{f'}{f} = 1$ can replace the assumption of the constant mean curvature. As an application of Theorem 3, we obtain the following corollary.

Corollary 2. *Let $\varphi : \Sigma^2 \rightarrow -\mathbb{R} \times_{e^t} M^2$ be a stochastically complete spacelike surface which is contained in a bounded slab. Assume that the non-vanishing mean curvature H is bounded, then the surface $\varphi(\Sigma^2)$ is a slice of $-\mathbb{R} \times_{e^t} M^2$, and $H = 1$.*

Theorem 4. *Let $-I \times_f M^n$ be a GRW spacetime with $n \geq 2$, where I is bounded and $(\log f)'' \leq 0$. Then, there is no stochastically complete spacelike hypersurface with non-zero constant mean curvature H such that $\Theta < 1 - n$.*

Proof of Theorem 4. Assume that there exists a constant mean curvature hypersurface with $\Theta < 1 - n$, now we can prove it by contradiction. From the assumption that there is an $\varepsilon > 0$ such that $\Theta - (1 - n) \leq -\varepsilon < 0$, combining with the proof of Theorem 3, we have

$$\begin{aligned} \Delta h &= -\left(\frac{f'}{f}(h)\Theta^2 + n\Theta H + (n - 1)\frac{f'}{f}(h)\right) \\ &\leq -H\left(\Theta^2 + n\Theta + (n - 1)\right) \\ &= -H(\Theta + 1)(\Theta + n - 1) \\ &\leq -H(\Theta + 1)(-\varepsilon) \\ &\leq -H(-n + 2 - \varepsilon)(-\varepsilon) < 0. \end{aligned}$$

It is easy to obtain $\Delta h \leq -H(\Theta + 1)(\Theta + n - 1) < 0$ when $n \geq 2$. Applying Lemma 1, we obtain

$$v = g(h_*) = -H\varepsilon(n - 2 + \varepsilon) \geq 0,$$

which is a contradiction, so we complete the proof. \square

If we denote $\{e_k\}$ as an orthogonal frame on $T_p\Sigma$ which is diagonalizable with A_p , such that $A_p(e_i) = \lambda_i(p)e_i$, then $P_k(e_i) = \mu_{i,k}e_i$ and $\nabla h(e_i) = \sigma_i e_i$. In order to extend the uniqueness to higher-order mean curvatures, we need to estimate the range of $\langle P_{k-1} \nabla h, \nabla h \rangle$. Using Chebyshev inequality, we have that

$$\min \langle P_{k-1} \nabla h, \nabla h \rangle \leq \frac{c_{k-1}H_{k-1}}{n} |\nabla h|^2 \leq \max \langle P_{k-1} \nabla h, \nabla h \rangle,$$

where the inequality holds if and only if $\mu_{k-1,i} = \mu_{k-1,j}$ or $\sigma_i = \sigma_s$ with $1 \leq i, j, s, t \leq n$ and $i \neq j, s \neq t$. Thus, we can assume that there exists $\theta \geq 1$ satisfying that

$$\langle P_{k-1} \nabla h, \nabla h \rangle \geq \frac{c_{k-1}H_{k-1}}{\theta n} |\nabla h|^2. \tag{8}$$

Now, we extend the result of Theorem 4 to higher order mean curvatures. For simplicity, we just focus on the case of $f' > 0$, the case of $f' < 0$ can be obtained in a similar way.

Theorem 5. Let $-I \times_f M^n (n \geq 2)$ be a GRW spacetime satisfying $(\log f)'' \leq 0$ and I be bounded. Then, there is no stochastically complete spacelike hypersurface with non-zero constant k -mean curvature H_k ($2 \leq k \leq n$) and $\sup_\Sigma |H| < \infty$, such that $\Theta < 1 - \theta n$; especially, if $\theta = 1$, then $\Theta < 1 - n$.

Proof of Theorem 5. From the hypothesis, the weak maximum principle for L_k holds on the hypersurface. From Theorem 2, we obtain

$$\begin{aligned} L_{k-1}h &= -\frac{f'}{f}(h)\left(c_{k-1}H_{k-1} + \langle P_{k-1} \nabla h, \nabla h \rangle\right) - c_{k-1}\langle N, \partial_t \rangle H_k \\ &\leq -\frac{f'}{f}(h)c_{k-1}H_{k-1} - \frac{f'}{f}(h)\frac{c_{k-1}H_{k-1}}{\theta n} |\nabla h|^2 - \Theta c_{k-1}H_{k-1}H_k^{\frac{1}{k}} \\ &= -\frac{1}{\theta n}c_{k-1}H_{k-1}\left(\frac{f'}{f}(h)\Theta^2 + \theta n\Theta H_k^{\frac{1}{k}} + (\theta n - 1)\frac{f'}{f}(h)\right) \\ &\leq -\frac{1}{\theta n}c_{k-1}H_{k-1}\left(\frac{f'}{f}(h)\Theta^2 + \theta n\Theta\frac{f'}{f}(h) + (\theta n - 1)\frac{f'}{f}(h)\right) \\ &= -\frac{1}{\theta n}\frac{f'}{f}(h)(\Theta + \theta n - 1)(\Theta + 1). \end{aligned}$$

Now, using Lemma 2, we can finish the proof in a similar way as the proof in Theorem 4. \square

4. The Application of Prior Estimation of the Constant k-th Order Mean Curvature Spacelike Hypersurface

In this section, we consider the generalized Robertson–Walker spacetime $-I \times_f M^n$ satisfying the following (see [24]) null convergence condition (NCC):

$$\text{Ric}_M \geq (n - 1) \sup_I (f''f - f'^2) \langle \cdot, \cdot \rangle_M, \tag{9}$$

where Ric_M is the Ricci tensor of the fiber M^n ; furthermore, recall that the spacetime $-I \times_f M^n$ obeys the timelike convergence condition (TCC) when its Ricci curvature is non-negative on timelike directions, i.e.,

$$f'' \leq 0, \text{Ric}_M \geq (n - 1) \sup_I (f''f - f'^2) \langle \cdot, \cdot \rangle_M. \tag{10}$$

Theorem 6. *Let $-I \times_f M^n$ be a generalized Robertson–Walker spacetime satisfying the TCC and the dimension $3 \leq n + 1 \leq 7$, where $I = (a, b)$ with $-\infty \leq a < b < +\infty$. Suppose that $\psi : \Sigma^n \rightarrow -I \times_f M^n$ is a stochastically complete spacelike hypersurface with non-zero constant mean curvature H . Then, the hypersurface is a slice.*

Proof of Theorem 6. Consider the formula for $\Delta\Theta$ of Lemma 5, we have

$$\begin{aligned} \Delta\Theta &= \Theta(\text{Ric}_M(N^*, N^*) - (n - 1)(\log f)''|\nabla h|^2) - \Theta(\log f)''|\nabla h|^2 \\ &\quad + \frac{\|\text{hess}(h)\|^2}{\Theta} + \frac{f'(h)}{\Theta} |\nabla h|^2 \left(\frac{f'}{f}(h)\Theta^2 + nH\Theta + (n - 1)\frac{f'}{f}(h) \right), \end{aligned}$$

where N^* denotes the projection of N onto the fiber M^n . Observe that

$$\|\nabla h\|^2 = \|N^*\|^2 = f^2(h)\langle N^*, N^* \rangle_M.$$

Therefore, by the timelike convergence condition (10), we have

$$\text{Ric}_M(N^*, N^*) - (n - 1)(\log f)''|\nabla h|^2 \geq 0,$$

and

$$-\Theta(\log f)''|\nabla h|^2 = -\Theta \frac{f''f}{f^2} |\nabla h|^2 + \Theta \frac{f'^2}{f^2} |\nabla h|^2 \leq \Theta \frac{f'^2}{f^2} |\nabla h|^2.$$

Furthermore, using the inequality $|\text{hess}(h)|^2 \geq 0$, and the result of Theorem 1, we have for every $3 \leq n + 1 \leq 7$,

$$\begin{aligned} \Delta\Theta &\leq \frac{|\nabla h|^2}{\Theta} \left(\frac{f'^2}{f^2}(h)\Theta^2 + nH\Theta \frac{f'}{f}(h) + (n - 1)\frac{f'^2}{f^2}(h) \right) + \frac{|\nabla h|^2}{\Theta} \frac{f'^2}{f^2}(h)\Theta^2 \\ &= \frac{|\nabla h|^2}{\Theta} \left(2\frac{f'^2}{f^2}(h)\Theta^2 + nH\Theta \frac{f'}{f}(h) + (n - 1)\frac{f'^2}{f^2}(h) \right) \\ &= \frac{\Theta^2 - 1}{\Theta} \left(2\left(\frac{f'}{f}(h)\Theta + \frac{nH}{4}\right)^2 + (n - 1)\left(\frac{f'^2}{f^2}(h) - \frac{n^2}{8(n - 1)}H^2\right) \right) \leq 0. \end{aligned}$$

Furthermore, since the hypersurface is stochastically complete, if we denote

$$g(\Theta) = \frac{\Theta^2 - 1}{\Theta} \left(2\left(\frac{f'}{f}(h)\Theta + \frac{nH}{4}\right)^2 + (n - 1)\left(\frac{f'^2}{f^2}(h) - \frac{n^2}{8(n - 1)}H^2\right) \right),$$

then using the weak maximum principle of Lemma 1, we have

$$g(\Theta_*) = \frac{\Theta_*^2 - 1}{\Theta_*} \left(2 \left(\frac{f'}{f}(h) \Theta_* + \frac{nH}{4} \right)^2 + (n-1) \left(\frac{f'^2}{f^2}(h) - \frac{n^2}{8(n-1)} H^2 \right) \right) \geq 0,$$

where the inequality holds if and only if $\Theta_*^2 = 1$. Thus, combining with $\Theta \leq -1$, we have $\Theta = -1$, i.e., $|\nabla h|^2 = \Theta_*^2 - 1 = 0$. It means that when $2 \leq n \leq 6$ the hypersurfaces are slices. \square

Motivated by the previous results, we extend the result of Theorem 6 to constant higher order mean curvatures.

Theorem 7. Let $-I \times_f M^n$ be a generalized Robertson–Walker spacetime with dimension $3 \leq n+1 \leq 7$, where $I = (a, b)$ and $-\infty \leq a < b < +\infty$. Suppose that $\psi : \Sigma^n \rightarrow -I \times_f M^n$ is a complete spacelike hypersurface with non-zero constant k -mean curvature H_k ($2 \leq k \leq n$) and there exists an elliptic point on Σ . Suppose that the Ricci curvature of M^n satisfies $K_M \geq \sup_I (ff'' - f'^2)$ and $(f)'' \leq 0$, the WMP holds on the operator L_k and $\sup_\Sigma H < +\infty$. Furthermore, $\langle P_{k-1} \nabla h, \nabla h \rangle \geq \frac{c_{k-1} H_{k-1}}{n} |\nabla h|^2$, i.e., $\theta = 1$. Then, the hypersurface is a slice.

Proof of Theorem 7. Now, we consider the hypersurface with the constant k -th order mean curvature. We already know that

$$\begin{aligned} L_{k-1} \langle N, \partial_t \rangle &= \frac{\Theta}{f^2(h)} \sum_{i=1}^n \mu_{k-1,i} K_M(E_i^*, N^*) \|E_i^* \wedge N^*\|^2 \\ &\quad + \Theta \frac{f'^2 - f''f}{f^2} (c_{k-1} H_{k-1} |\nabla h|^2 - \langle P_{k-1} \nabla h, \nabla h \rangle) \\ &\quad + \Theta \frac{(f'^2 - f''f)}{f^2} \langle P_{k-1} \nabla h, \nabla h \rangle + \frac{\langle \text{hess}(h), P_{k-1} \cdot \text{hess}(h) \rangle}{\Theta} \\ &\quad + \Theta \frac{f'(h)}{f(h)} \left(\frac{f'}{f}(h) c_{k-1} H_{k-1} + c_{k-1} H_k \Theta + \frac{f'}{f}(h) \langle P_{k-1} \nabla h, \nabla h \rangle \right) \\ &\quad - \frac{f'(h)}{\Theta} \left(\frac{f'}{f}(h) c_{k-1} H_{k-1} + c_{k-1} H_k \Theta + \frac{f'}{f}(h) \langle P_{k-1} \nabla h, \nabla h \rangle \right). \end{aligned}$$

Using the decomposition

$$N = N^* + \langle N, \partial_t \rangle \partial_t, E_i = E_i^* + \langle E_i, \partial_t \rangle \partial_t, \partial_t = \nabla h + \langle N, \partial_t \rangle N,$$

we have

$$|E_i^* \wedge N^*|^2 = |\nabla h|^2 - \langle \nabla h, E_i \rangle^2 \leq |\nabla h|^2 = \langle N, \partial_t \rangle^2 - 1.$$

If we denote $\eta := \sup_I \{f'^2 - f''f\}$, we have

$$\begin{aligned} &\sum_{i=1}^n \mu_{k-1,i} K_M(E_i^*, N^*) \|E_i^* \wedge N^*\|^2 \geq \eta \sum_{i=1}^n \mu_{k-1,i} \|E_i^* \wedge N^*\|^2 \\ &= \eta (c_{k-1} H_{k-1} |\nabla h|^2 - \langle P_{k-1} \nabla h, \nabla h \rangle). \end{aligned}$$

Thus,

$$\begin{aligned} &\frac{1}{f^2(h)} \sum_{i=1}^n \mu_{k-1,i} K_M(E_i^*, N^*) \|E_i^* \wedge N^*\|^2 - \frac{f''f - f'^2}{f^2(h)} (c_{k-1} H_{k-1} |\nabla h|^2 - \langle P_{k-1} \nabla h, \nabla h \rangle) \\ &\geq \frac{\eta - (f''f - f'^2)}{f^2(h)} (c_{k-1} H_{k-1} |\nabla h|^2 - \langle P_{k-1} \nabla h, \nabla h \rangle) \geq 0. \end{aligned}$$

Therefore, we have

$$\begin{aligned}
 L_{k-1}\langle N, \partial_t \rangle &\leq \frac{|\nabla h|^2}{\Theta} \frac{f'}{f}(h) \left(\frac{f'}{f}(h) c_{k-1} H_{k-1} + c_{k-1} H_k \Theta + \frac{f'}{f}(h) \langle P_{k-1} \nabla h, \nabla h \rangle \right) \\
 &\quad + \Theta \frac{f'^2}{f^2} \langle P_{k-1} \nabla h, \nabla h \rangle \\
 &\leq \frac{c_{k-1} H_{k-1} |\nabla h|^2}{n \Theta} \left(2 \frac{f'^2}{f^2}(h) \Theta^2 + n \Theta H_k^{\frac{1}{k}} + (n-1) \frac{f'^2}{f^2}(h) \right) \\
 &= \frac{c_{k-1} H_{k-1} |\nabla h|^2}{n \Theta} \left(2 \left(\frac{f'}{f}(h) \Theta + \frac{n H_k^{\frac{1}{k}}}{4} \right)^2 + (n-1) \left(\frac{f'^2}{f^2}(h) - \frac{n^2}{8(n-1)} H_k^{\frac{2}{k}} \right) \right).
 \end{aligned}$$

When $3 \leq n+1 \leq 7$, we obtain $\frac{n^2}{8(n-1)} < 1$, thus $L_{k-1}\Theta \leq 0$. Combining with Theorem 2, we can finish the proof by Lemma 2 in a similar way as in the proof of Theorem 6. \square

As an application of Theorems 6 and 7, we obtain the following corollary.

Corollary 3. *Let $-I \times_f M^n$ be a generalized Robertson–Walker spacetime with dimension $n+1 \geq 8$ satisfying the TCC. Suppose that $\psi : \Sigma^n \rightarrow -I \times_f M^n$ is a complete spacelike hypersurface with $I = (a, b)$ and $-\infty \leq a < b < +\infty$. Assume that one of the following holds:*

- (i) *the spacelike hypersurface is stochastically complete with non-zero constant mean curvature H ;*
- (ii) *the spacelike hypersurface has constant k -mean curvature H_k ($k \leq n$), where the WMP holds on Σ for the operator L_k and $\sup_{\Sigma} |H| < +\infty$, there exists an elliptic point in Σ and $\langle P_{k-1} \nabla h, \nabla h \rangle \geq \frac{c_{k-1} H_{k-1}}{n} |\nabla h|^2$.*

Then, there exists no spacelike hypersurface such that

$$H_k^{\frac{2}{k}} < \frac{8(n-1)}{n^2} \frac{f'^2}{f^2}(h) \quad (2 \leq k \leq n).$$

Theorem 8. *Let $\psi : \Sigma^n \rightarrow I \times_f M^n$ be a complete spacelike hypersurface, where $I = (a, b)$ with $-\infty \leq a < b < +\infty$. Assume that one of the following holds:*

- (i) *the stochastically complete spacelike hypersurface has non-zero constant mean curvature H , $(\log f)'' \leq 0$ and $\text{Ric}_M(N^*, N^*) \geq (n-1) \sup_I (f''f - \frac{n}{2(n-1)} f'^2)$;*
- (ii) *the complete spacelike hypersurface has non-zero constant k -mean curvature H_k ($k \leq n$) and there exists an elliptic point on Σ , the Ricci curvature of M^n satisfies $\text{Ric}_M \geq 0$, $\langle P_{k-1} \nabla h, \nabla h \rangle \geq \frac{c_{k-1} H_{k-1}}{n} |\nabla h|^2$ and $(f)'' \leq 0$, and the WMP holds on the operator L_k and $\sup_{\Sigma} H < +\infty$.*

Then, the hypersurface is a slice.

Proof of Theorem 8. (i) Since the hypersurface is stochastically complete, we have the weak maximum principle holds on the Laplace–Beltrami operator. Consider the inequality $\text{Ric}_M(N^*, N^*) \geq (n-1) \sup_I (f''f - \frac{n}{2(n-1)} f'^2)$ holds on the spacetimes, we have

$$\begin{aligned}
 \Delta\Theta &= \Theta\left(\text{Ric}_M(N^*, N^*) - (n-1)(\log f)''|\nabla h|^2\right) - \Theta(\log f)''|\nabla h|^2 \\
 &\quad + \frac{\|\text{hess}(h)\|^2}{\Theta} + \frac{f'(h)}{\Theta}|\nabla h|^2\left(\frac{f'}{f}(h)\Theta^2 + nH\Theta + (n-1)\frac{f'}{f}(h)\right) \\
 &\leq \frac{|\nabla h|^2}{\Theta}\left(\frac{f'^2}{f^2}(h)\Theta^2 + nH\Theta\frac{f'}{f}(h) + (n-1)\frac{f'^2}{f^2}(h)\right) + \frac{n-2}{2}\frac{|\nabla h|^2}{\Theta}\frac{f'^2}{f^2}(h)\Theta^2 \\
 &= \frac{|\nabla h|^2}{\Theta}\left(\frac{n}{2}\frac{f'^2}{f^2}(h)\Theta^2 + nH\Theta\frac{f'}{f}(h) + (n-1)\frac{f'^2}{f^2}(h)\right) \\
 &= \frac{|\nabla h|^2}{\Theta}\left(\frac{n}{2}\left(\frac{f'}{f}(h)\Theta + H\right)^2 + (n-1)\left(\frac{f'^2}{f^2}(h) - \frac{nH^2}{2(n-1)}\right)\right) \leq 0.
 \end{aligned}$$

Thus, if we denote

$$g(\Theta) = \frac{|\nabla h|^2}{\Theta}\left(\frac{n}{2}\left(\frac{f'}{f}(h)\Theta + H\right)^2 + (n-1)\left(\frac{f'^2}{f^2}(h) - \frac{nH^2}{2(n-1)}\right)\right),$$

then using Theorem 2 and Lemma 1, we obtain that $\forall n \geq 2$,

$$g(\Theta_*) = \frac{\Theta_*^2 - 1}{\Theta_*}\left(\frac{n}{2}\left(\frac{f'}{f}(h)\Theta_* + H\right)^2 + (n-1)\left(\frac{f'^2}{f^2}(h) - \frac{nH^2}{2(n-1)}\right)\right) \geq 0.$$

Therefore, we have $\Theta = -1$, which implies the hypersurface is a slice.

(ii) From the hypothesis $\text{Ric}_M \geq 0$ and $(f)'' \leq 0$, combining with the proof of Theorem 7, we have

$$\begin{aligned}
 L_{k-1}\Theta &= \frac{\Theta}{f^2(h)} \sum_{i=1}^n \mu_{k-1,i} K_M(E_i^*, N^*) \|E_i^* \wedge N^*\|^2 \\
 &\quad + \Theta \frac{f'^2 - f''f}{f^2} (c_{k-1}H_{k-1}|\nabla h|^2 - \langle P_{k-1}\nabla h, \nabla h \rangle) \\
 &\quad + \Theta \frac{(f'^2 - f''f)}{f^2} \langle P_{k-1}\nabla h, \nabla h \rangle + \frac{\langle \text{hess}(h), P_{k-1} \cdot \text{hess}(h) \rangle}{\Theta} \\
 &\quad + \Theta \frac{f'(h)}{f(h)} \left(\frac{f'}{f}(h)c_{k-1}H_{k-1} + c_{k-1}H_k\Theta + \frac{f'}{f}(h)\langle P_{k-1}\nabla h, \nabla h \rangle\right) \\
 &\quad - \frac{f'}{\Theta} \left(\frac{f'}{f}(h)c_{k-1}H_{k-1} + c_{k-1}H_k\Theta + \frac{f'}{f}(h)\langle P_{k-1}\nabla h, \nabla h \rangle\right) \\
 &\leq \frac{|\nabla h|^2}{\Theta} \frac{f'}{f}(h) \left(\frac{f'}{f}(h)c_{k-1}H_{k-1} + c_{k-1}H_k\Theta + \frac{f'}{f}(h)\langle P_{k-1}\nabla h, \nabla h \rangle\right) \\
 &\quad + \Theta \frac{f'^2 - f''f}{f^2} c_{k-1}H_{k-1}|\nabla h|^2 \\
 &\leq \frac{c_{k-1}H_{k-1}|\nabla h|^2}{n\Theta} \left((n+1)\frac{f'^2}{f^2}(h)\Theta^2 + n\Theta H_k^{\frac{1}{k}} + (n-1)\frac{f'^2}{f^2}(h)\right) \\
 &= \frac{c_{k-1}H_{k-1}|\nabla h|^2}{n\Theta} \left(n\left(\frac{f'}{f}(h)\Theta + \frac{1}{2}H_k^{\frac{1}{k}}\right)^2 + n\left(\frac{f'^2}{f^2}(h) - \frac{1}{4}H_k^{\frac{2}{k}}\right) + \frac{f'^2}{f^2}(h)|\nabla h|^2\right).
 \end{aligned}$$

Since $\frac{f'^2}{f^2}(h) \geq H_k^{\frac{2}{k}}$, we have $L_{k-1}\Theta \leq 0$. Using the same way as case (i), we can complete the proof. \square

5. Conclusions

Manifold learning for high-dimensional data is becoming increasingly important in many areas. In this paper, we investigated the geometrical structure of the high dimensional manifold by considering the constant mean curvature hypersurface. In particular, we illustrated the uniqueness of constant mean curvature spacelike hypersurfaces. For higher-order mean curvature spacelike hypersurfaces, we demonstrated the prior estimates and non-existence results. For future work, we would like to extend the framework to many real data applications, including biological sequence analysis and gene-disease association studies.

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