

Article

On Balanced Host Games: A Sufficient Condition for Non-Emptiness of the Core

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Abstract: Non-transferable utility (NTU) games arise from many economic situations. A classic example is the exchange economy. By pooling and redistributing their initial endowments, coalitions can achieve certain distributions of gains (utilities) that make up the coalition's feasible set. This paper studies a new class of NTU games called host games. A host game is an agent-parametrized family of NTU games, and an NTU game is associated with any agent (called the host in that case). We provide an adequate presumption for the existence of an allocation that is part of the host game's core.

Keywords: NTU games; host games; balanced games

MSC: 91A12; 91A40; 91B06; 91B16



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1. Introduction

Non-transferable utility (NTU) games originate from various economic situations. For instance, by gathering and reallocating its original endowments under an exchange economy, coalitions could achieve certain payoff distributions that form the feasible set for that coalition. More generally, an NTU game describes some feasible set of payoff vectors for every coalition. It is given by its set of players and the sets of outcomes that are feasible for each subset (coalition) of players. That is, an NTU game is an assignment of a set of feasible utility allocations to each coalition of players.

The players face the problem of choosing a payoff that is feasible for the grand coalition. This is a bargaining condition, and its solution may be rationally required to satisfy various criteria. The *core* is the easiest to understand of all the solution concepts for cooperative games, and it is the set of all feasible outcomes without players or coalitions that can improve by acting alone. In other words, no individual or group can reap the benefits of restructuring once an agreement is reached. In a free market, outcomes should be paramount and economic activity should benefit all parties. Unfortunately, for many games, there may be no feasible outcomes that cannot be improved—the pie may not be big enough. Hence, a condition for a game that has a nonempty core is important. Scarf [1] proved that each balanced NTU game has a nonempty core, and Shapley [2] further generalized the concept of balanced games. In Shapley's [2] work, the balanced condition is defined for the coalitional vector system π , which specifies the weight of each participant in each coalition. For the core to be nonempty, the coalitional vectors system π must exist for the game to be π -balanced. Later, Hwang [3] treats the Cartesian product of these NTU games as a multiple-NTU game. Furthermore, Hwang [3] extended the concept of the π -balanced condition for NTU games to multiple-NTU games. Hwang [3] proved that each π -balanced multiple-NTU game has a nonempty core.

In the framework of the NTU games, we would like to utilize the core to analyze the coalition structure and host selection called a "host game". Several game-theoretical results related to the notion of the host have been applied to several topics, such as computer science, economics, decision making, and pathology. Related applications could be found

in Choisy and de Roode [4], Garg and Grosu [5], Ng [6], and so on. Here, we treat these NTU games as host games, which differ from pre-existing results. Moreover, different NTU games will be derived from different hosts. The main notions and relative results are as follows.

- We explore the notion of the π -balanced condition and rely heavily on Shapley’s method [2] to prove the coincident result in the framework of host games. We assume a symmetry condition requiring no difference in overall interests regardless of the host (Equation (1) in Section 4).
- Moreover, participants do not decide who the host is; therefore, all participants will receive a stable allocation, and no participants form a different coalition. Therefore, if such a stable allocation exists, the participants do not have to decide who the host is because they would receive a stable allocation no matter who the host is. In order to cause the participants to not worry about who the host is, we propose a sufficient condition for such an allocation by combining the mild condition of “no difference in overall interests” with the condition of “stable allocation” will.

2. Preliminary

2.1. Definitions and Notations

$N = \{1, 2, \dots, n\}$ denotes the set of players. For any set S , $\mathcal{P}(S)$ denotes the collection of all non-empty subsets of S , and $|S|$ denotes the cardinality of S . Let \mathbb{R}^n denote the n -dimensional real vector space and \mathbb{R}_+^n denote the set of all non-negative vectors in \mathbb{R}^n . For $x \in \mathbb{R}^n$, we make x_S be the restriction of x at S for every $S \subseteq N$. For $E \subseteq \mathbb{R}^n$, $CH(E)$ denotes the convex hull of E and $InS(E)$ denotes the interior set of E .

Definition 1. A non-transferable utility (NTU) game (see Aumann and Peleg [7], Billera [8], Scarf [1], and Shapley [2]) is a pair (N, V) , where N is a finite set of players and V is a set-valued function, $V : \mathcal{P}(N) \rightarrow \mathcal{P}(\mathbb{R}^n)$, that assigns to every coalition $S \in \mathcal{P}(N)$ a non-empty subset $V(S)$ of feasible payoff vectors for S satisfying the following five conditions:

1. $V(S)$ is closed;
2. $V(S)$ is comprehensive, i.e., $V(S) = V(S) - \mathbb{R}_+^n$;
3. $V(S)$ is cylindrical, i.e., if $x \in V(S)$ and $y \in \mathbb{R}^n$ such that $y_S = x_S$, then $y \in V(S)$;
4. $V(S)$ is bounded, i.e., there exists a positive number $M > 0$ such that if $x \in V(S)$ and $x_S \geq 0$, then $x_i \leq M$ for every $i \in S$;
5. $V(\{i\}) = \{x \in \mathbb{R}^n : x_i \leq 0\}$ for every $i \in N$.

The core of an NTU game (N, V) consists of all payoff vectors that are feasible for the grand coalition N and that cannot be improved upon by any coalition, including N itself. If $x \in V(N)$, then S can improve upon x if there is a $y \in V(S)$ with $y_i > x_i$ for every $i \in S$. Hence, the formal definition of the core of the game (N, V) is

$$C(N, V) = V(N) \setminus \bigcup_{S \in \mathcal{P}(N)} InS(V(S)).$$

To state the Scarf–Billera–Shapley theorem (Scarf [1]; Billera [8]; Shapley [2]), we introduce additional notation. The unit $(n - 1)$ -simplex in \mathbb{R}^n is denoted by I^{n-1} . The union of all its proper faces is denoted by ∂I^{n-1} , and it is called the boundary of I^{n-1} . Hence, $InS(I^{n-1}) = I^{n-1} \setminus \partial I^{n-1}$. For every $S \in \mathcal{P}(N)$, let

$$B^S = CH(\{e^i : i \in S\}) \text{ and}$$

$$b_S = \sum_{i \in S} \frac{1}{|S|} e^i,$$

where $e^i \in \mathbb{R}^n$ with $e^i_i = 1$ and $e^i_j = 0$ for all $j \neq i$; b_S is known as the *barycenter* of B_S . If $\pi : \mathcal{P}(N) \rightarrow I^{n-1}$ and $\beta \subseteq \mathcal{P}(N)$, then the set

$$CH(\pi(\beta)) = CH(\{\pi(S) : S \in \beta\})$$

is denoted by $B(\pi; \beta)$.

For a given map $\pi : \mathcal{P}(N) \rightarrow I^{n-1}$, $\beta \subseteq \mathcal{P}(N)$ is said to be π -balanced if $b_N \in B(\pi; \beta)$. An NTU game (N, V) is π -balanced for some π if for every π -balanced family β ,

$$\bigcap_{S \in \beta} V(S) \subseteq V(N).$$

The Scarf [1], Billera [8], and Shapley [2] theorem is as follows:

Theorem 1. (Scarf–Billera–Shapley)

Every π -balanced NTU game has a non-empty core.

2.2. Applications of NTU Games and Host Games

Situations applying the notion of host games appear in public sports events, such as the Asian Cup, the European Cup, the America’s Cup, the World Cup, and the Olympics. Various host townships/cities will bring other interests from the participating townships/cities. In this paper, we utilize an NTU game to model the problem. Our game considers the coalition structure and host selection called a “host game.” In addition to the situations described above, the government choosing to build public buildings is also an interesting social issue. For example, the government pondered over the construction of youth residences, national residences, libraries, incinerators, and coronavirus disease 2019 anti-epidemic hotels in Taiwan. If people adopt a chosen location, they will not be averse to it. Therefore, the governance sustainability is high. In this description, we examine a simple question. Can people accept a public building no matter where it is built? We can thus eliminate the political risk of choosing a location. How do we ensure this? As observed in many of the situations mentioned above, one should be dedicated to fully satisfying every participating individual or group in the context of joint decision making.

Based on the statement provided by the core in the previous sections and the related mathematical definitions, we can see that the core is a reasonable allocation given to all participants under the consideration of efficient allocation of resources that are unsatisfactory but acceptable to all individuals or groups. This provides motivation and rationality for analyzing the allocation using the core in this paper. However, the resource allocation using the core is mathematically equivalent to the intersection of all corresponding sets formed by individuals or groups. Therefore, there is a possibility of forming an empty set, which in reality means that a consensus allocation cannot be achieved. Therefore, in the following sections of this paper, we would like to discuss the conditions under which the allocation derived using the core is not an empty set.

3. Generalizations of KKM

Shapley [2] used KKMS lemma, which is a generalization of Knaster, Kuratowski, and Mazurkewicz’s [9] (KKM) lemma, to prove that every balanced non-transferable utility game has a non-empty core. Gale [10] used the KKM lemma, which is another generalization of the KKM lemma, to prove the existence of a price equilibrium in a market with indivisible commodities. Shih and Lee [11] combined KKMS with KKM to obtain a common generalization of KKM lemma. Our proof is based upon this common generalization of the KKM lemma.

This common generalization of the KKM lemma of Shih and Lee [11] is as follows:

Lemma 1 (Theorem 5.1, Shih and Lee [11]). Let $M = \{1, 2, \dots, m\}$ and $m \geq n$. For every $i \in M$, let $\mathcal{F}^i = \{F_S^i : S \in \mathcal{P}(N)\}$ be a family of closed subsets of the unit $(n - 1)$ -simplex I^{n-1} such that

$$B^S \subseteq \bigcup_{T \in \mathcal{P}(S)} F_T^i \text{ for every } S \in \mathcal{P}(N).$$

Then, for every $\pi : \mathcal{P}(N) \rightarrow I^{n-1}$ with $\pi(S) \in B^S$ for every $S \in \mathcal{P}(N)$, there exist a π -balanced set $\{S_1, S_2, \dots, S_n\}$ and n distinct elements m_1, m_2, \dots, m_n of M such that

$$\bigcap_{i=1}^n F_{S_i}^{m_i} \neq \emptyset.$$

Remark 1. In Lemma 1,

1. if the conditions are restricted to (i) $m = n$, (ii) $\mathcal{F}^i = \mathcal{F}^j$ for every $i, j \in N$, and (iii) $F_S^i = \emptyset$ for every $i \in N$ and for every $S \in \mathcal{P}(N)$ with $|S| \geq 2$, then Lemma 1 is the KKM lemma.
2. if the conditions are restricted to (i) $m = n$ and (ii) $\mathcal{F}^i = \mathcal{F}^j$ for every $i, j \in N$, then Lemma 1 is the KKMS lemma.
3. if the conditions are restricted to (i) $m = n$ and (ii) $F_S^i = \emptyset$ for every $i \in M$ and for every $S \in \mathcal{P}(N)$ with $|S| \geq 2$, then Lemma 1 is the KKMG lemma.

4. Main Result

4.1. Host Games and the Core

This note studies a new class of NTU games (host games). A host NTU game is a family of NTU games parametrized by a set of agents, and one associates an NTU game with any agent (called the host). We provide a sufficient condition for an allocation that belongs to the supposed core of the host game.

Definition 2. A host game is a pair $(N, (V^k)_{k \in N})$ which satisfies for all $i, j \in N$,

$$V^i(N) \cap \left(\bigcup_{S \in \mathcal{P}(N)} V^i(S) \right) = V^j(N) \cap \left(\bigcup_{S \in \mathcal{P}(N)} V^j(S) \right), \tag{1}$$

where (N, V^k) is an NTU game and k could be treated as a host for every k .

Equation (1) says that no matter who the host is, there is no difference in overall interests. That is, it matters nothing to the participants who the host is.

Definition 3. The core of a host game $(N, (V^k)_{k \in N})$, $C_h(N, (V^k)_{k \in N})$, is defined to be the set of feasible outcomes that cannot be improved upon by any coalition (no matter who the host is), i.e., $x \in C_h(N, (V^k)_{k \in N})$ if and only if

1. $x \in V^k(N)$ for each $k \in N$,
2. there is no $S \subseteq N$ and $y^k \in V^k(S)$ for some k , such that $y_j^k > x_j$, for all $j \in S$.

Remark 2. Let $(N, (V^k)_{k \in N})$ be a host game. It is not difficult to see the relation between $C_h(N, (V^k)_{k \in N})$ and $C(N, V^k)$ as follows:

$$\begin{aligned} C_h(N, (V^k)_{k \in N}) \neq \emptyset &\iff \bigcap_{k \in N} C(N, V^k) \neq \emptyset \\ &\implies C(N, V^k) \neq \emptyset \forall k \in N. \end{aligned}$$

However, " $C(N, V^k) \neq \emptyset \forall k \in N$ " does not imply " $C_h(N, (V^k)_{k \in N}) \neq \emptyset$."

Two examples are offered to interpret Remark 2 as follows.

Example 1. Define a host game $(N, (V^k)_{k \in N})$ to be $N = \{1, 2, 3\}$ and

$$\begin{aligned} V^1(\{1, 2, 3\}) &= V^2(\{1, 2, 3\}) = V^3(\{1, 2, 3\}) = \{(x_1, x_2, x_3) \in \mathbb{R}^N \mid x_1 + x_2 + x_3 \leq 1\}, \\ V^1(\{1, 2\}) &= \{(x_1, x_2, x_3) \in \mathbb{R}^N \mid x_1 + x_2 \leq 1\}, \\ V^1(\{2, 3\}) &= \{(x_1, x_2, x_3) \in \mathbb{R}^N \mid x_2 + x_3 \leq 0\}, \\ V^1(\{1, 3\}) &= \{(x_1, x_2, x_3) \in \mathbb{R}^N \mid x_1 + x_3 \leq 0\}, \\ V^1(\{i\}) &= \{(x_1, x_2, x_3) \in \mathbb{R}^N \mid x_i \leq 0\} \text{ for all } i \in N, \\ V^2(\{1, 2\}) &= \{(x_1, x_2, x_3) \in \mathbb{R}^N \mid x_1 + x_2 \leq 0\}, \\ V^2(\{2, 3\}) &= \{(x_1, x_2, x_3) \in \mathbb{R}^N \mid x_2 + x_3 \leq 0\}, \\ V^2(\{1, 3\}) &= \{(x_1, x_2, x_3) \in \mathbb{R}^N \mid x_1 + x_3 \leq 1\}, \\ V^2(\{i\}) &= \{(x_1, x_2, x_3) \in \mathbb{R}^N \mid x_i \leq 0\} \text{ for all } i \in N. \\ V^3(\{1, 2\}) &= \{(x_1, x_2, x_3) \in \mathbb{R}^N \mid x_1 + x_2 \leq 0\}, \\ V^3(\{2, 3\}) &= \{(x_1, x_2, x_3) \in \mathbb{R}^N \mid x_2 + x_3 \leq 1\}, \\ V^3(\{1, 3\}) &= \{(x_1, x_2, x_3) \in \mathbb{R}^N \mid x_1 + x_3 \leq 0\}, \\ V^3(\{i\}) &= \{(x_1, x_2, x_3) \in \mathbb{R}^N \mid x_i \leq 0\} \text{ for all } i \in N. \end{aligned}$$

It is easy to have that $(\frac{1}{2}, \frac{1}{2}, 0) \in C(N, V^1)$, $(\frac{1}{2}, 0, \frac{1}{2}) \in C(N, V^2)$, $(0, \frac{1}{2}, \frac{1}{2}) \in C(N, V^3)$, but $C_h(N, (V^k)_{k \in N}) = \emptyset$.

Example 2. Define a host game $(N, (V^k)_{k \in N})$ to be $N = \{1, 2, 3\}$ and

$$\begin{aligned} V^1(\{1, 2, 3\}) &= V^2(\{1, 2, 3\}) = V^3(\{1, 2, 3\}) = \{(x_1, x_2, x_3) \in \mathbb{R}^N \mid x_1 + x_2 + x_3 \leq 2\}, \\ V^1(\{1, 2\}) &= \{(x_1, x_2, x_3) \in \mathbb{R}^N \mid x_1 + x_2 \leq 1\}, \\ V^1(\{2, 3\}) &= \{(x_1, x_2, x_3) \in \mathbb{R}^N \mid x_2 + x_3 \leq 0\}, \\ V^1(\{1, 3\}) &= \{(x_1, x_2, x_3) \in \mathbb{R}^N \mid x_1 + x_3 \leq 0\}, \\ V^1(\{i\}) &= \{(x_1, x_2, x_3) \in \mathbb{R}^N \mid x_i \leq 0\} \text{ for all } i \in N, \\ V^2(\{1, 2\}) &= \{(x_1, x_2, x_3) \in \mathbb{R}^N \mid x_1 + x_2 \leq 0\}, \\ V^2(\{2, 3\}) &= \{(x_1, x_2, x_3) \in \mathbb{R}^N \mid x_2 + x_3 \leq 0\}, \\ V^2(\{1, 3\}) &= \{(x_1, x_2, x_3) \in \mathbb{R}^N \mid x_1 + x_3 \leq 1\}, \\ V^2(\{i\}) &= \{(x_1, x_2, x_3) \in \mathbb{R}^N \mid x_i \leq 0\} \text{ for all } i \in N. \\ V^3(\{1, 2\}) &= \{(x_1, x_2, x_3) \in \mathbb{R}^N \mid x_1 + x_2 \leq 0\}, \\ V^3(\{2, 3\}) &= \{(x_1, x_2, x_3) \in \mathbb{R}^N \mid x_2 + x_3 \leq 1\}, \\ V^3(\{1, 3\}) &= \{(x_1, x_2, x_3) \in \mathbb{R}^N \mid x_1 + x_3 \leq 0\}, \\ V^3(\{i\}) &= \{(x_1, x_2, x_3) \in \mathbb{R}^N \mid x_i \leq 0\} \text{ for all } i \in N. \end{aligned}$$

It is easy to have that $(\frac{2}{3}, \frac{2}{3}, \frac{2}{3}) \in C(N, V^1)$, $(\frac{2}{3}, \frac{2}{3}, \frac{2}{3}) \in C(N, V^2)$, $(\frac{2}{3}, \frac{2}{3}, \frac{2}{3}) \in C(N, V^3)$, and $(\frac{2}{3}, \frac{2}{3}, \frac{2}{3}) \in C_h(N, (V^k)_{k \in N})$.

From Remark 2, we can observe that even though each NTU game (N, V^k) is π -balanced for each k , it cannot guarantee that a host game $(N, (V^k)_{k \in N})$ has a non-empty core. Thus, we extend the idea of the π -balanced criterion for NTU games to host games in the following manner.

Definition 4. A host game $(N, (V^k)_{k \in N})$ is π -balanced for some π if for each π -balanced family $\beta = \{S^1, S^2, \dots, S^n\}$,

$$\bigcap_{k=1}^n V^k(S^k) \subseteq \bigcap_{k=1}^n V^k(N).$$

We already knew that in Shapley’s [2] proof, each π -balanced NTU game has a non-empty core based on the KKMS lemma (see Remark 1). Here, based on Lemma 1, we will demonstrate that each π -balanced host game has a non-empty core. Shapley’s [2] approach

is prominently applied in this proof. Moreover, we present the evidence to be thorough. Our main theorem is as follows:

Theorem 2. *Every π -balanced host game has a non-empty core.*

Proof. Let $(N, (V^k)_{k \in N})$ be a π -balanced host game. By boundedness condition 4, there exists a positive number $M > 0$ such that if $x \in V^k(S)$ with $x_S \geq 0$, then $x_i \leq M$ for every $i \in S$, for every $S \in \mathcal{P}(N)$, and for every $k = 1, 2, \dots, n$. For every $i = 1, 2, \dots, n$, take $t^i = -nMe^i$ for $i = 1, 2, \dots, n$, i.e.,

$$t_j^i = 0, \text{ if } j \neq i, \text{ and } t_i^i = -nM.$$

For every $S \in \mathcal{P}(N)$, let $B^S = CH(\{t^i : i \in S\})$. For every $k = 1, 2, \dots, n$, define a real-value function $r^k : \mathbb{R}^n \rightarrow \mathbb{R}$ by

$$r^k(a) = \sup \left\{ \lambda : a + \lambda e^N \in V^k(N) \cap \left(\bigcup_{S \in \mathcal{P}(N)} V^k(S) \right) \right\},$$

where e^N is the vector of all 1 s. Since Equation (1), for all $i, j \in N$,

$$V^i(N) \cap \left(\bigcup_{S \in \mathcal{P}(N)} V^i(S) \right) = V^j(N) \cap \left(\bigcup_{S \in \mathcal{P}(N)} V^j(S) \right),$$

we can assume that $r = r^k$ for all $k = 1, 2, \dots, n$.

Since every $V^k(S)$ satisfies all five conditions in Definition 1, the supremum in the definition of r is actually a maximum, and defines a continuous function. Let $f : B^N \rightarrow \mathbb{R}^n$ by

$$f(a) = a + r(a)e^N.$$

For every $k = 1, 2, \dots, n$, define

$$F_S^k = (f)^{-1}(V^k(S)) = \{a \in B^N : f(a) \in V^k(S)\}.$$

Since r is continuous, this implies that f is continuous. Thus, for every k and for every S , F_S^k is closed. For every k , let $\mathcal{F}^k = \{F_S^k : S \in \mathcal{P}(N)\}$. Next, we will show that for every k , \mathcal{F}^k satisfies the conditions of KKMS. In other words, for every k and for every $S \in \mathcal{P}(N)$,

$$B^S \subseteq \bigcup_{T \in \mathcal{P}(S)} F_T^k.$$

First, we show that if $a \in F_T^k \cap B^S$, then $T \subseteq S$. If $S = N$, then we are done. Suppose $S \neq N$. As $a \in B^S$, we have $\sum_{i \in S} a_i = -nM$. This implies that for at least one, $j \in S$, such that we have $a_j \leq -nM|S| < -M$. Considering just $T = \{j\}$ in the definition of r , we obtain $r(a) > M$. Hence, $[a + r(a)e^N] \in V^k(T)$ and $[a + r(a)e^N] \notin \text{InS}(V^k(P))$ for every $P \in \mathcal{P}(N)$. In particular, $[a + r(a)e^N] \notin \text{InS}(V^k(\{i\}))$ for every $i \in T$. Hence, by boundedness, for every $i \in T$,

$$a_i + r(a) \leq M.$$

Since $r(a) > M$, this implies that $a_i < 0$ for all $i \in T$. However, $a \in B^S$ implies $a_i = 0$ for all $i \notin S$. It forces that $T \subseteq S$. Combining above this with the fact that every $a \in B^N$ belongs to at least one set F_T^k , we have that $B^S \subseteq \bigcup_{T \in \mathcal{P}(S)} F_T^k$ for each $S \in \mathcal{P}(N)$.

Let $m = n$ in Lemma 1. Lemma 1 asserts for any π the existence of a point $a \in B^N$ and a π -balanced family $\beta = \{S^1, S^2, \dots, S^n\}$ such that $a \in F_{S^k}^k$ for all $k = 1, 2, \dots, n$. Let $\hat{x} = a + r(a)e^N$. This implies that for all $k = 1, 2, \dots, n$, $\hat{x} \in V^k(S^k)$. That is, $\hat{x} \in \bigcap_{k=1}^n V^k(S^k)$.

Since $(N, (V^k)_{k \in N})$ is π -balanced, this implies that $\hat{x} \in \bigcap_{k=1}^n V^k(N)$. Next, we will show that $\hat{x} \in C_h(N, (V^k)_{k \in N})$.

Assume there exist k, S , and $y^k \in V^k(S)$ such that $y_i^k > \hat{x}_i$ for all $i \in S$. This is in contradiction with the definition of $r(a)$. Hence, the core is non-empty. \square

Remark 3. In Example 1, it is easy to see that $C_h(N, (V^k)_{k \in N}) = \emptyset$ implies that $(N, (V^k)_{k \in N})$ is not π -balanced. In Example 2, it is clear to see that $C_h(N, (V^k)_{k \in N}) \neq \emptyset$ if $(N, (V^k)_{k \in N})$ is π -balanced.

4.2. Differences among Host Games and Multiple-NTU Games

Hwang [3] created a multiple-NTU game from the Cartesian product of these NTU games. Thus, we treat these NTU games differently from multiple-NTU games by treating them as hosts. Through the non-emptiness of the cores in various NTU games, we will compare the variations between the non-emptiness of the core in a multiple-NTU game and the non-emptiness of the core in a host game. A multiple-NTU game is defined as follows:

Definition 5. A multiple-NTU game is a pair (N, v) with $v(S) = \prod_{i=1}^n V^k(S)$ for each $S \subseteq N$, where (N, V^k) is an NTU game for each k .

Note that $v : \mathcal{P}(N) \rightarrow \prod_{k=1}^n \mathcal{P}(\mathbb{R}^n)$. For convenience, we will use the lower case letter v and the capital letter V to denote the characteristic functions of multiple-NTU games and NTU games, respectively.

The core of a multiple-NTU game (N, v) is defined as follows:

Definition 6. The core of a multiple-NTU game (N, v) with $v(S) = \prod_{k=1}^n V^k(S)$, $C_p(N, v)$, is defined to be the set of feasible outcomes that cannot be improved upon by any coalition, i.e., $x \in C_p(N, v)$ if and only if

1. $x = \prod_{k=1}^n x^k \in v(N) = \prod_{k=1}^n V^k(N)$,
2. there is no $S \subseteq N$, $S \neq \emptyset$, and $y^k \in V^k(S)$ for some k , such that $y_j^k > x_j^k$, for all $j \in S$.

Remark 4. Let (N, v) with $v(S) = \prod_{k=1}^n V^k(S)$ be a multiple-NTU game. It is not difficult to see the relation between $C_p(N, v)$ and $C(N, V^k)$, as follows:

$$C_p(N, v) = \prod_{k=1}^n C(N, V^k), \text{ and}$$

$$C_p(N, v) \neq \emptyset \iff C(N, V^k) \neq \emptyset \forall k \in N.$$

Two examples are offered to interpret Remark 4.

Example 3. Define a multiple-NTU game (N, v) to be $N = \{1, 2, 3\}$ with $v(S) = \prod_{i=1}^3 V^k(S)$ for each $S \subseteq N$ and

$$\begin{aligned} V^1(\{1, 2, 3\}) &= V^2(\{1, 2, 3\}) = V^3(\{1, 2, 3\}) = \{(x_1, x_2, x_3) \in \mathbb{R}^N \mid x_1 + x_2 + x_3 \leq 1\}, \\ V^1(\{1, 2\}) &= \{(x_1, x_2, x_3) \in \mathbb{R}^N \mid x_1 + x_2 \leq 1\}, \\ V^1(\{2, 3\}) &= \{(x_1, x_2, x_3) \in \mathbb{R}^N \mid x_2 + x_3 \leq 1\}, \\ V^1(\{1, 3\}) &= \{(x_1, x_2, x_3) \in \mathbb{R}^N \mid x_1 + x_3 \leq 1\}, \end{aligned}$$

$$\begin{aligned}
 V^1(\{i\}) &= \{(x_1, x_2, x_3) \in \mathbb{R}^N \mid x_i \leq 0\} \text{ for all } i \in N, \\
 V^2(\{1, 2\}) &= \{(x_1, x_2, x_3) \in \mathbb{R}^N \mid x_1 + x_2 \leq 0\}, \\
 V^2(\{2, 3\}) &= \{(x_1, x_2, x_3) \in \mathbb{R}^N \mid x_2 + x_3 \leq 0\}, \\
 V^2(\{1, 3\}) &= \{(x_1, x_2, x_3) \in \mathbb{R}^N \mid x_1 + x_3 \leq 1\}, \\
 V^2(\{i\}) &= \{(x_1, x_2, x_3) \in \mathbb{R}^N \mid x_i \leq 0\} \text{ for all } i \in N. \\
 V^3(\{1, 2\}) &= \{(x_1, x_2, x_3) \in \mathbb{R}^N \mid x_1 + x_2 \leq 0\}, \\
 V^3(\{2, 3\}) &= \{(x_1, x_2, x_3) \in \mathbb{R}^N \mid x_2 + x_3 \leq 1\}, \\
 V^3(\{1, 3\}) &= \{(x_1, x_2, x_3) \in \mathbb{R}^N \mid x_1 + x_3 \leq 0\}, \\
 V^3(\{i\}) &= \{(x_1, x_2, x_3) \in \mathbb{R}^N \mid x_i \leq 0\} \text{ for all } i \in N.
 \end{aligned}$$

It is easy to have that $C(N, V^1) = \emptyset$, $(\frac{1}{2}, 0, \frac{1}{2}) \in C(N, V^2)$, $(0, \frac{1}{2}, \frac{1}{2}) \in C(N, V^3)$, and $C_p(N, v) = \emptyset$.

Example 4. Define a multiple-NTU game (N, v) to be $N = \{1, 2, 3\}$ with $v(S) = \prod_{i=1}^3 V^k(S)$ for

each $S \subseteq N$ and

$$\begin{aligned}
 V^1(\{1, 2, 3\}) &= V^2(\{1, 2, 3\}) = V^3(\{1, 2, 3\}) = \{(x_1, x_2, x_3) \in \mathbb{R}^N \mid x_1 + x_2 + x_3 \leq 2\}, \\
 V^1(\{1, 2\}) &= \{(x_1, x_2, x_3) \in \mathbb{R}^N \mid x_1 + x_2 \leq 1\}, \\
 V^1(\{2, 3\}) &= \{(x_1, x_2, x_3) \in \mathbb{R}^N \mid x_2 + x_3 \leq 0\}, \\
 V^1(\{1, 3\}) &= \{(x_1, x_2, x_3) \in \mathbb{R}^N \mid x_1 + x_3 \leq 0\}, \\
 V^1(\{i\}) &= \{(x_1, x_2, x_3) \in \mathbb{R}^N \mid x_i \leq 0\} \text{ for all } i \in N, \\
 V^2(\{1, 2\}) &= \{(x_1, x_2, x_3) \in \mathbb{R}^N \mid x_1 + x_2 \leq 0\}, \\
 V^2(\{2, 3\}) &= \{(x_1, x_2, x_3) \in \mathbb{R}^N \mid x_2 + x_3 \leq 0\}, \\
 V^2(\{1, 3\}) &= \{(x_1, x_2, x_3) \in \mathbb{R}^N \mid x_1 + x_3 \leq 1\}, \\
 V^2(\{i\}) &= \{(x_1, x_2, x_3) \in \mathbb{R}^N \mid x_i \leq 0\} \text{ for all } i \in N. \\
 V^3(\{1, 2\}) &= \{(x_1, x_2, x_3) \in \mathbb{R}^N \mid x_1 + x_2 \leq 0\}, \\
 V^3(\{2, 3\}) &= \{(x_1, x_2, x_3) \in \mathbb{R}^N \mid x_2 + x_3 \leq 1\}, \\
 V^3(\{1, 3\}) &= \{(x_1, x_2, x_3) \in \mathbb{R}^N \mid x_1 + x_3 \leq 0\}, \\
 V^3(\{i\}) &= \{(x_1, x_2, x_3) \in \mathbb{R}^N \mid x_i \leq 0\} \text{ for all } i \in N.
 \end{aligned}$$

It is easy to have that $(\frac{2}{3}, \frac{2}{3}, \frac{2}{3}) \in C(N, V^1)$, $(\frac{2}{3}, \frac{2}{3}, \frac{2}{3}) \in C(N, V^2)$, $(\frac{2}{3}, \frac{2}{3}, \frac{2}{3}) \in C(N, V^3)$, and $(\frac{2}{3}, \frac{2}{3}, \frac{2}{3}, \frac{2}{3}, \frac{2}{3}, \frac{2}{3}, \frac{2}{3}, \frac{2}{3}) \in C_p(N, v)$.

5. Conclusions

- Some notions and relative techniques proposed by Shapley [2] and Hwang [3] have been applied throughout this paper. In order to present the significance of this paper, one should compare our results with the results due to Shapley [2] and Hwang [3].
 - Shapley [2] presented the idea of the π -balanced condition for NTU games in the literature. Shapley [2] then established that each π -balanced NTU game has a non-empty core by using the KKMS lemma.
 - Hwang [3] treated the Cartesian product of these NTU games as a multiple-NTU game. The π -balanced criteria for NTU games was also expanded by Hwang [3] to include multiple-NTU games. Hwang [3] then demonstrated via Lemma 1 that every π -balanced multiple-NTU game has a non-empty core.
 - Differently from Shapley [2] and Hwang [3], these NTU games were treated as hosts in this note, as opposed to numerous NTU games. From Remark 2, we observe that even though each NTU game (N, V^k) is π -balanced for each k , it cannot guarantee that a host game $(N, (V^k)_{k \in N})$ has a non-empty core. Therefore, it is crucial to get a sufficient result that the host game has a non-empty core. This note was meant to do this.
 - We expanded the idea of the π -balanced requirement for NTU games to host games in Section 4. Subsequently, using Lemma 1, we demonstrated that each hosted game with π -balance has a non-empty core. Shapley's [2] approach is substantially utilized in Hwang's [3] proof, although Lemma 1 is used instead of

the KKMS lemma in Shapley's [2] proof. Our proof is nearly identical to that of Hwang [3] in Theorem 2 under Equation (1).

In conclusion, we want to stress that, although the proving methodology is not novel, it is crucial to produce a sufficient demonstration that the host game has a non-empty core.

2. There are several types of conditions which guarantee non-emptiness of the core for NTU games. For example,
 - Scarf [1] proved that every balanced NTU game has a non-empty core.
 - Billera [8] further generalized the notion of a balanced game. In Billera's work, the balancedness condition is defined with respect to a system π of coalitional vectors specifying the weight of each player within every coalition. The existence of the system of the coalitional vectors π such that the game is π -balanced suffices for the core to be non-empty. Moreover, Billera [8] proposed a necessary and sufficient condition of a nonempty core for all NTU games whose payoff sets are assumed to be convex.
 - Vilkov [12] extended Shapley's definition of convex games [13] to the context of NTU games and proved that such games have non-empty cores. Greenberg [14] generalized Vilkov's result [12] that convex games have non-empty cores, and moreover, that the class of convex games is not contained in the class of balanced games.
 - An extension of the π -balancedness condition is reached by allowing the system of coalitional vectors π to depend on the utility distributions. Predtetchinski and JeanJacques Herings [15] introduced the balancedness of the game with respect to a correspondence Π that assigns to each distribution of utilities a set of coalitional vectors π : they allowed the weight of a player within a coalition to depend on the utility distribution that is proposed. They showed that the core of an NTU game is non-empty if and only if it is balanced with respect to some correspondence Π . An alternative necessary and sufficient balancedness condition for non-emptiness of the core was defined by Keiding and Thorlund-Petersen [16]. Predtetchinski and Jean-Jacques Herings [15] pointed out the difference between the two works: the advantage of the Π -balancedness condition in Predtetchinski and Jean-Jacques Herings' paper [15] over the condition in Keiding and Thorlund-Petersen's paper [16] is that it applies directly to the game of interest and avoids the construction of any auxiliary games or sequences of approximating games.

At present, the extension of the Π -balancedness condition to host games is an open problem. We plan to propose some more extensions in subsequent researches.

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