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Technology for Obtaining the Approximate Value of Moving Singular Points for a Class of Nonlinear Differential Equations in a Complex Domain

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Abstract: In previous studies, the authors formulated precise criteria for finding moving singular points of one class of nonlinear differential equations with a second degree polynomial right-hand side for a real domain. In this paper, the authors generalize these exact criteria to a complex one by using phase spaces. The proposed technology for obtaining an approximate value of moving singular points is necessary for developing PC programs. This technology has been tested in a manual version based on a numerical experiment.

Keywords: nonlinear differential equations; mobile singular point; exact criteria of existence; necessary and sufficient conditions

MSC: 34G20; 35A05

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1. Introduction

As is known, differential equations are the basis of mathematical models in many areas, including building structures [1–3], as well as the study of wave processes in elastic beams [4,5]. If in the case of linear equations there are no problems, then for nonlinear differential equations the problem is related to finding moving singular points. Let us note that, for nonlinear differential equations, there is a possibility for an asymptotic approach [6,7] in single cases of solvability in quadratures [8–11]. Let us present the development of the authors' approach in the method of analytical approximate solution for a certain class of nonlinear equations [12–19].

On the basis of the authors' results in [19], this paper comes up with necessary and sufficient criteria for the existence of moving singular points for a complex domain. It should be noted that these criteria are divided into point and interval. If the first category is necessary to confirm the fact of the existence of a moving singular point, then the second allows to obtain these singular points with a given accuracy.

2. Research Results

In [18] for a nonlinear differential form:

$$y'''(z) = \sum_{n=0}^{2} a_n(z) y^n(z),$$

reduced to normal form with the help of some substition, we consider the Cauchy problem

$$y'''(z) = y^2(z) + r(z), \quad (1)$$

$$\begin{cases} y(z_0) = y_0, \\ y'(z_1) = y_1, \\ y''(z_2) = y_2. \end{cases} \quad (2)$$

By using the Singular Point Regularization Technique with the change of variable

$$y(z) = \frac{1}{w(z)}, \quad (3)$$

we procede to the inverse problem for the function $w(z)$:

$$w'''w^2 = -6w'^3 + 6ww'w'' - w^2 - r(z)w^4, \quad (4)$$

$$\begin{cases} w(z_0) = w_0, \\ w'(z_0) = w_1, \\ w''(z_0) = w_2. \end{cases} \quad (5)$$

Function $w(z)$ is represented in the form $w(z) = u(x,y) + i \cdot v(x,y)$, where functions $u(x,y)$ and $v(x,y)$ are characterized accordingly by the phase spaces $\Phi_1(x,y,u(x,y))$ and $\Phi_2(x,y,v(x,y))$. Let us use terminology [17] to determine *the correct* and *incorrect* lines, necessary to facilitate the presentation of the following provisions.

Theorem 1. *A necessary and sufficient condition for the existence of a moving singular point z^* of multiplicity 3, for a solution $y(z)$ to the Cauchy problem (1) and (2), is the fulfillment of the conditions:*

1. *$Re(w(z))$ and $Im(w(z))$ $w(z)$ function $w(z)$, are functions of some domain G, which is a neighborhood of a regular point $z^*(x^*, y^*)$ for the function $w(z)$, phase spaces Φ_1 and Φ_2 are continuous with respect to their arguments.*
2. *Functions $Re(w(z))$ and $Im(w(z))$ change signs, passing through $z^*(x^*, y^*)$, moving along the correct line l in the direction of the axes Ox and Oy, while $l : \{z^* \in l \subset G, l \in (\Phi_1 \cup \Phi_2)\}$.*

Proof of Necessity. Knowing that z^* is a moving singular point for $y(z)$ of the Cauchy Problem (1) and (2), let us prove that in this case $Re(w(z))$ and $Im(w(z))$ satisfy Theorem 1.

From [18], if we replace the real variable x with the complex variable z, it follows that the function has the structure:

$$y(z) = (z^* - z)^{-3} \sum_{n=0}^{\infty} C_n (z^* - z)^n. \quad (6)$$

In this case, based on [18], we have

$$r(z) = \sum_{n=0}^{\infty} A_n (z^* - z)^n, C_0 = -60, C_1 = 0, C_2 = 0, C_3 = 0, C_4 = 0, C_5 = 0, C_6 = \frac{A_0}{126}.$$

Taking into account the main part of the function $y(z)$, $y(z) = O\left(-60/(z^* - z)^3\right)$ and the regularization of the moving singular point (3), we can write down for the function $w(z)$ in the domain G: $w(z) = o\left(-(z^* - z)^3/60\right)$.

Further, in this situation we have:

$$sign(u(x,y)) = sign((x^* - x)), \quad (7)$$

$$sign(v(x,y)) = sign((y^* - y)). \quad (8)$$

Analysis of the right-hand part $w(z)$: the sign of the function $u(x,y)$ is determined by the sign of x, and the sign of the function $v(x,y)$ is determined by the sign of y. Thus, we establish the fulfillment of Conditions 1 and 2 of Theorem 1. □

Proof of Sufficiency. Conditions 1 and 2 of **Theorem 1** are satisfied. Taking into account the substitution (3) and formula (6), the function $w(z)$ must have the structure as follows: $w(z) = (z^* - z)^3 \sum_0^\infty D_n (z^* - z)^n$.

Then, from (3), we find $y(z) : y(z) = (z^* - z)^{-3} \sum_0^\infty C_n (z^* - z)^n$. It follows from the last equality that z^* is a moving singular point of the function $y(z)$ with multiplicity 3. □

Theorem 2. *Let z^* be a moving singular point, with multiplicity 3, of the solution $y(z)$ to the Cauchy problem (1) and (2), then the necessary and sufficient condition is:*

1. *The functions $Re(w(z))$ and $Im(w(z))$ are continuous with respect to their arguments in some domain G, the neighborhood of a regular point $z^*(x^*, y^*)$ for the function $w(z)$, phase spaces Φ_1 and Φ_2.*
2. *$Re(w(z))$ and $Im(w(z))$ change signs, passing through $z^*(x^*, y^*)$, moving sequentially along the wrong line l_1, l_2 in the direction of the axes Ox and Oy, provided that*

$$l_1, l_2 : \{z^* \in l_1 \subset G, z^* \in l_2 \subset G, l_1 \in \Phi_1, l_1 \in \Phi_2, l_2 \in \Phi_1, l_2 \in \Phi_2\}.$$

Proof of Necessity. Knowing that z^* is a moving singular point with multiplicity 3 of the function $y(z)$, we will prove that the functions $Re(w(z))$ and $Im(w(z))$ also satisfy **Theorem 2.**

Based on previously obtained results [18], if we replace the real variable x with the complex variable z, the function $y(z)$ can be represented in the form (6), where $C_0 = -60$, $C_1 = 0, C_2 = 0, C_3 = 0, C_4 = 0, C_5 = 0, C_6 = \frac{A_0}{126}$.

Then, the main part of the function is $y(z) = O\left(-60/(z^* - z)^3\right)$. Taking into account the substitution (3), for the inverse function of the Cauchy Problem (4) and (5) we have: $w(z) = o\left(-(z^* - z)^3/60\right)$.

Let us take a straight line $l_1 : y = const$ as an irregular line. Moving along the specified line, taking into account expression (7), the function of the real part $u(x, const) \in C(G)$ will certainly change sign when passing through $z^*(x, const)$. Therefore, by using the bisection procedure for the integration step over the coordinate $''x''$, we obtain it with a given accuracy. Acting similarly, we set $x = x^*$, and, having $l_2 : x = x^*$ as an irregular line, we obtain the second coordinate of the moving singular point y^* in the analysis of the function $v(x^*, y)$. □

Proof of Sufficiency. Let the imaginary and real parts of the function $w(z)$ in some neighborhood of the region G of phase spaces Φ_1 and Φ_2 satisfy conditions 1, 2 of **Theorem 2.** Let us prove that z^* is a moving singular point of the solution $y(z)$ to the Cauchy Problem (1) and (2).

Based on Conditions 1 and 2 of Theorem 2, the connection between the original Problem (1) and (2) and the inverse Problem (4) and (5), as well as the results of [18], if we replace the real variable x with the complex variable z, it follows that z^* is a regular point for solving Problem (4) and (5). In this case, the function can be represented as:

$$w(z) = (z^* - z)^3 \sum_0^\infty C_n (z^* - z)^n.$$

Taking into account the regularization procedure $y(z) = \frac{1}{w(z)}$, we have

$$y(z) = (z^* - z)^{-3} \sum_0^\infty C_n (z^* - z)^n.$$

Here, it follows that z^* is a moving singular point of multiplicity 3 of the solution to the Cauchy Problem (1) and (2). □

Theorem 3. *(point criterion for the existence of movable singular points). A necessary and sufficient condition for the fact that z^* is a movable singular point of multiplicity 3 belonging to the function $y(z)$, a solution to the Cauchy Problem (1) and (2), is the fulfillment of the conditions*

$$z(0) = z^*, z'(0) = 0, z''(0) = 0, z'''(0) = \frac{1}{10}, \tag{9}$$

where $z(w)$ is the inverse function for the inverse Cauchy Problem (4)–(5) solution.

Proof of Necesity. Knowing that z^* is a moving singular point of the function $y(z)$, which is the solution to the Cauchy Problem (1) and (2), we will prove that the function satisfies the conditions:

$$z(0) = z^*, z'(0) = 0, z''(0) = 0, z'''(0) = \frac{1}{10}.$$

Based on substitution (3), we represent the function as a regular series:

$$w(z) = D_0(z^* - z)^3 + D_1(z^* - z)^4 + D_2(z^* - z)^5 + \dots. \tag{10}$$

Taking into account that $w(z^*) = 0$, based on the Lagrange theorem on the inversion of series [20], we obtain:

$$z^* - z = \sum_{3}^{\infty} B_n \cdot w^n,$$

where $B_3 = \frac{1}{C_0}$.

When $w = 0$ we obtain $z(0) = z^*$. Differentiating with respect to w, we obtain:

$$z' = -3B_3 \cdot w^2 - 4B_4 w^3 - 5B_5 w^4 + \dots.$$

Finding the value $z'(0) = 0$. Differentiating the previous expression with respect to w, we get: $z'' = -6B_3 w - 12B_4 w^2 - 20B_5 w^3 + \dots$.

We obtainen what was sought $z(0) = z^*, z'(0) = 0, z''(0) = 0, z'''(0) = \frac{1}{10}$. □

Proof of Sufficiency. Taking into account the conditions of the theorem, the function $z(w)$, which is the inverse function to the solution of the inverse Cauchy Problem (4) and (5), satisfies the conditions: $z(0) = z^*, z'(0) = 0, z''(0) = 0, z'''(0) = \frac{1}{10}$. Let us prove that the function $y(z)$ has a moving singular point of multiplicity 3.

Based on the conditions of the theorem, we conclude that the function is represented by a regular series:

$$z(w) = \tilde{B}_0 + \tilde{B}_1 w + \tilde{B}_2 w^2 + \dots \tag{11}$$

and in accordance with the condition of the theorem $\tilde{B}_0 = z^*$. Differentiating (11), we obtain:

$$z' = \tilde{B}_1 + 2\tilde{B}_2 w + 3\tilde{B}_3 w^2 + \dots. \tag{12}$$

From (12) it follows that $B_1 = 0$. Further, differentiating (12), we have:

$$z'' = 2\tilde{B}_2 + 6\tilde{B}_3 w + 12\tilde{B}_4 w^2 + \dots, \tag{13}$$

wherein it follows that $B_2 = 0$. Differentiating (13): we get:

$$z''' = 6\tilde{B}_3 + 24\tilde{B}_4 w + \dots \tag{14}$$

Taking into account the initial conditions, we obtain $\tilde{B}_3 = \frac{1}{60}$.

Thus, for $z(w)$ we obtain the decomposition:

$$z(w) = z^* + \frac{1}{60} w^3 + \dots.$$

or

$$z^* - z = -\frac{1}{60}w^3 - B_4 w^4 - \ldots. \quad (15)$$

Based on the series inversion theorem [20], it follows from (15):

$$w(z) = D_0(z^* - z)^3 + D_1(z^* - z)^4 + D_2(z^* - z)^5 + \ldots,$$

where $D_0 = C_0 = -60$.

By virtue of the singular point regularization technology (3), we obtain the following representation for the function $y(z)$:

$y(z) = \frac{1}{w(z)} = (z^* - z)^{-3} \sum_0^\infty C_n (z^* - z)^n$, wherey it follows that $B_2 = 0$.

The latter proves that the solution of the Cauchy Problem (1) and (2) has a moving singular point of multiplicity 3. □

Next, we turn to the application of the above theorems and theorems from [19] for compiling an algorithm that allows us to find a moving singular point of the equation under consideration with any predetermined accuracy. We illustrate the operation of the algorithm with the help of a block diagram and a description of each step.

3. Description of the Algorithm for Finding an Approximate Value of a Moving Singular Point with a Given Accuracy

1. We find the radius of convergence of the solution to the Cauchy Problem (1) and (2) R_l , by admitting $z_{1,l} = z_{i+j}$. Then we move on to point 2.
2. We choose a step for the analytical approximate solution $y(z)$, taking into account the condition that the step should not exceed the radius of convergence for the analytical approximate solution. Let us move on to point 3.
3. Calculation of function values $y(z)$ at the corresponding points: $y(z_i + H), z_{i+1} = z_i + H, y'(z_{i+1}), y''(z_{i+1})$. Next, we go to point 4.
4. Verification of the condition for falling into the neighborhood of a moving singular point $y(z_i) > 0, y'(z_i) > 0, y''(z_i) > 0$. If these conditions are met, go to step 7; otherwise, go to step 5.
5. When the system of inequalities $|z_{i+1}| + |H| \leq |z_{1,l}| + R_l$, $|z_{i+1}| + |H| \leq b$ is fulfilled, go to point 3; otherwise go to the next point.
6. If the conditions $|z_{i+1}| + |H| > |z_{1,l}| + R_l$ and $|z_{i+1}| + |H| \leq b$ are met, we return to point 1. In the case $|z_{i+1}| + |H| > b$, to point 11.
7. We consider the value of the solution of the inverse Cauchy Problem (4) and (5) and the fulfillment of the corresponding conditions $w(z_{i+j}) \cdot w(z_{i+j+1}) < 0$. If the condition is met, go to step 8; otherwise, go to step 10.
8. We consider the value of the solution of the inverse Cauchy Problem (4) and (5) at the corresponding points $w(z_{i+1}), w'(z_{i+1}), w''(z_{i+1}), w'''(z_{i+1})$. Let us move on to point 9.
9. If $|w(z_{i+1})| < \varepsilon_1, |w'(z_{i+1})| < \varepsilon_2, |w''(z_{i+1})| < \varepsilon_3, |w'''(z_{i+1}) + 360| < \varepsilon_4$, go to point 11; otherwise, go to point 10.
10. Let us reduce the step $H = H/2, z_{i+j+1} = z_{i+j} + H, j = j + 1$ and go to step 7.
11. End of the algorithm.

The block diagram of the moving singular point search algorithm is shown in Figure 1.

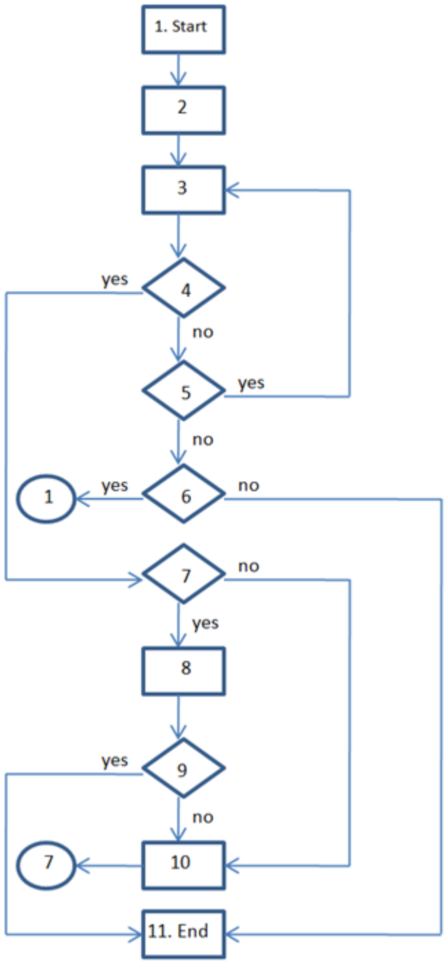

Figure 1. Block diagram of the algorithm of searching for a moving singular point.

4. Discussion

On the basis of theoretical provisions, an algorithm for finding a moving singular point with a given accuracy has been compiled. For practical application, Theorem 2 is the best option, since Theorem 1 is related to the solution of an additional problem of optimizing the deviation when moving along the correct line. Theorem 3 is applied to confirm the obtained value of the moving singular point. This algorithm was tested in a manual version in Matlab for a specific task $r(z) = 0; y(0) = \frac{1}{4}; y'(0) = i; y''(0) = 1$, an approximate value $z^* = 2.652717$ of the moving singular point was obtained with an accuracy of $\varepsilon = 10^{-6}$.

5. Conclusions

This paper presents a technology for obtaining moving singular points based on formulated and proven theorems related to the necessary and sufficient condition for the existence of moving singular points in a complex domain. The results obtained are the theoretical basis of the program for finding a moving singular point with a given accuracy. The results of the study were tested for a specific Cauchy Problem (1) and (2) in a manual version in Matlab, taking into account the initial data discussed in Section 3.

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