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Notes on Convergence Results for Parabolic Equations with Riemann–Liouville Derivatives

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Abstract: Fractional diffusion equations have applications in various fields and in this paper we consider a fractional diffusion equation with a Riemann–Liouville derivative. The main objective is to investigate the convergence of solutions of the problem when the fractional order tends to 1^-. Under some suitable conditions on the Cauchy data, we prove the convergence results in a reasonable sense.

Keywords: fractional diffusion equation; Riemman–Liouville; regularity; convergence rate

MSC: 35A05; 35A08

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1. Introduction

Fractional differential equations arise in models in many fields such as economics, biology, mechanics, geology, and heat transfer, and usually there are two types of derivatives that are of interest, namely the Caputo and the Riemman–Liouville derivatives. Both of these derivatives are defined by non-local integrals. Note that the Riemann–Liouville derivative of a constant is not zero and if an arbitrary function is constant at the origin, then its fractional derivative has a singularity at the origin, for example, exponential and Mittag–Leffler functions. However, if we study the initial value problem with the Caputo derivative, we often treat the initial values as normal functions, just like the initial value problem with the classical derivative. However, when considering a problem with the Riemman–Liouville derivative its initial value condition usually involves a fractional integral or a fractional derivative (see [1]). In this paper, we examine:

$$\begin{cases} \mathbf{D}^{\alpha}_{0+} u - u_{xx} = G(x,t), & (x,t) \in (0,\pi) \times (0,T), \\ u(0,t) = u(\pi,t) = 0, & 0 < t < T, \\ t^{1-\alpha} u|_{t=0} = f(x), \end{cases} \tag{1}$$

where $\mathbf{D}^{\alpha}_{0+} v$ is called the Riemann–Liouville fractional derivative of v with order α, $0 < \alpha \le 1$ and it is defined by

$$\mathbf{D}^{\alpha}_{0+} v(t) = \frac{d}{dt}\left(\frac{1}{\Gamma(1-\alpha)} \int_0^t (t-r)^{-\alpha} v(r) dr\right), \tag{2}$$

and $\mathbf{D}^{\alpha}_{0+} v(t) =: \frac{d}{dt} v(t)$ if $\alpha = 1$. The diffusion equation arises from many diffusion phenomena that occur in nature (e.g., phase transitions, biochemistry) and equations of time fractional reactions occur in describing "memory" in physics, for example plasma turbulence [2], fractal geometry [3,4], and single-molecular protein dynamics [5] (we also refer the reader to Refs. [6–22]).

In Ref. [23], the authors focused on:

$$\begin{cases} \mathbf{D}^{\alpha}_{0+}u = F(t,u(t)), & (x,t) \in (0,\pi) \times (0,T), \\ t^{1-\alpha}u|_{t=0} = \lambda \int_0^T u(t)dt + h, & h \in \mathbb{R}, \end{cases} \tag{3}$$

where $\lambda \geq 0$ and in [24] the following non-local problem was studied:

$$\begin{cases} \mathbf{D}^{\alpha}_{0+}u + F(t,v(t)) = a, \quad t \in (0,T), \\ \mathbf{D}^{\beta}_{0+}v + F(t,u(t)) = b, \quad t \in (0,T), \\ u(0) = 0, \quad u(T) = \int_0^T \phi(t)u(t)dt, \\ v(0) = 0, \quad v(T) = \int_0^T \phi(t)v(t)dt, \end{cases} \tag{4}$$

where $1 < \alpha, \beta < 2$. In Ref. [25], the authors studied the existence and continuation of solutions, and several global existence results were given to a general fractional differential equation (FDE) with a Riemann–Liouville derivative.

Recently, the authors in Ref. [26] investigated the following nonlinear diffusion equation with the Riemann–Liouville derivative

$$\mathbf{D}^{\alpha}_{0+}u - u_{xx} = F(x,t,u),$$

and obtained the existence and regularity of mild solutions using the Banach fixed point theorem. In Ref. [27], Luc studied the fractional diffusion equation with the Riemann–Liouville derivative in the form

$$\mathbf{D}^{\alpha}_{0+}u - u_{xx} = F(x,t), \tag{5}$$

and under some assumptions on the input data, he obtained the wellposedness of problem (5).

Returning to Problem (1) we see that if $f \in \mathbb{H}^s(\Omega)$ for any $s \geq 0$, Problem (1) has a unique solution. The solution to the problem depends on the fractional order α and we denote the solution as u_α. The main goal of this paper is to see what happens to u_α when $\alpha \to 1^-$, and this question is inherently difficult.

In this paper, we prove for the first time that the solution u_α to our problem converges to the solution of the problem with a classical derivative. Theorem 1 gives a result concerning this convergence in the homogeneous case and Theorem 2 is a convergent result for the linear inhomogeneous case. The main technique of the paper is to use a Lemma in Ref. [28] combined with the evaluation in Hilbert scale spaces.

2. Preliminaries

With $\Omega = (0,\pi)$, note the spectral problem

$$\begin{cases} \Delta e_j(x) = -\lambda_j e_j(x), & x \in \Omega, \\ e_j(x) = 0, & x \in \partial\Omega, \end{cases}$$

with the eigenvalues

$$0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_j \leq \cdots \text{ with } \lambda_j \to \infty \text{ for } j \to \infty$$

and the corresponding eigenfunctions $e_j \in H^1_0(\Omega)$.

Definition 1. *The Mittag–Leffler function, which is defined by*

$$E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(n\alpha+\beta)}, (z \in \mathbb{C}),$$

for $\alpha > 0$ and $\beta \in \mathbb{R}$. When $\beta = 1$, it is abbreviated as $E_\alpha(z) = E_{\alpha,1}(z)$.

We note following lemmas (see for example Ref. [2]).

Lemma 1 (see Ref. [28])**.** *Let $\frac{3}{4} \le \alpha < 1$ and $\alpha \le \beta \le 1$. Then there exists a constant C which is independent of α, β, z such that, for any $z < 0$,*

$$\left|E_{\alpha,\beta}(z) - e^z\right| \le \frac{C}{1+|z|}(1-\alpha). \tag{6}$$

The proof can be found in Ref. [28].

Lemma 2 (see Ref. [29])**.** *Let $0 < \alpha < 1$. Then the function $z \mapsto E_{\alpha,\alpha}(z)$ has no negative root. Moreover, there exists a constant $\overline{C}_\alpha$ such that*

$$0 \le E_{\alpha,\alpha}(-z) \le \frac{\overline{C}_\alpha}{1+z}, \quad z > 0. \tag{7}$$

For a positive number $r \ge 0$, we define the Hilbert scale space

$$H^r(0,\pi) = \left\{ f \in L^2(0,\pi) : \sum_{j=1}^{\infty} j^{2r} \left\langle f, \sqrt{\frac{2}{\pi}} \sin(jx) \right\rangle^2 < +\infty \right\}, \tag{8}$$

with the following norm $\|u\|_{H^r(0,\pi)} = \left(\sum_{j=1}^{\infty} j^{2r} \left\langle f, \sqrt{\frac{2}{\pi}} \sin(jx) \right\rangle^2 \right)^{\frac{1}{2}}$.

Theorem 1. *Let the Cauchy data $f \in \mathbb{H}^{s-2\beta}(\Omega) \cap \mathbb{H}^{s-2\theta}(\Omega) \cap \mathbb{H}^{s+2\varepsilon}(\Omega)$ any $0 < \varepsilon < 1, s \ge 0$, $0 < \beta, \theta < 1$, $\frac{3}{4} \le \alpha < 1$, $\beta, \theta \le \frac{s}{2}$ and $\theta < \alpha$. Let u_α be the mild solution to Problem (1) and u^* be the mild solution to Problem (1) with $\alpha = 1$, i.e.,*

$$\begin{cases} u^*_t - u^*_{xx} = 0, & (x,t) \in (0,\pi) \times (0,T), \\ u^*(0,t) = u^*(\pi,t) = 0, & t \times (0,T), \\ u^*(x,0) = f(x), \end{cases} \tag{9}$$

Then we get

$$\begin{aligned} \left\|u_\alpha - u^*\right\|_{L^1(0,T;\mathbb{H}^s(\Omega))} \le \Big(& \frac{C\Gamma(\alpha)T^{\alpha-\alpha\beta}}{\alpha-\alpha\beta}(1-\alpha)\Big)\|f\|_{\mathbb{H}^{s-2\beta}(\Omega)} \\ & + C(\theta,\mu,T)\Big[(1-\alpha)^\mu + (T^*)^{1-\alpha} - 1\Big]\|f\|_{\mathbb{H}^{s-2\theta}(\Omega)} \\ & + \frac{T^{1-\beta}}{1-\beta}\Big|\Gamma(\alpha) - 1\Big|\|f\|_{\mathbb{H}^{s-2\beta}(\Omega)} \\ & + \left[(1-\alpha)^{\frac{\alpha\varepsilon}{2}} + \Big((T^*)^{1-\alpha} - 1\Big)^{\varepsilon}\right]\|f\|_{\mathbb{H}^{s+2\varepsilon}(\Omega)}, \end{aligned} \tag{10}$$

where $T^* = \max(T, 1)$.

Proof. From the paper Ref. [27], we obtain the representation of the mild solution u_α

$$\int_\Omega u_\alpha(x,t)e_j(x)dx = \Gamma(\alpha)t^{\alpha-1}E_{\alpha,\alpha}\big(-j^2t^\alpha\big)\big(\int_\Omega f(x)e_j(x)dx\big). \tag{11}$$

The mild solution of the classical problem is defined by

$$\int_\Omega u^*(x,t)e_j(x)dx = \exp\big(-j^2t\big)\Big(\int_\Omega f(x)e_j(x)dx\Big). \tag{12}$$

Subtracting both sides of the two equations above we get

$$\begin{aligned}\int_\Omega (u_\alpha(x,t)-u^*(x,t))e_j(x)dx &= \Gamma(\alpha)t^{\alpha-1}\Big(E_{\alpha,\alpha}\Big(-j^2t^\alpha\Big)-\exp\Big(-j^2t\Big)\Big)\Big(\int_\Omega f(x)e_j(x)dx\Big)\\ &+\Gamma(\alpha)t^{\alpha-1}\Big(\exp\Big(-j^2t^\alpha\Big)-\exp\Big(-j^2t\Big)\Big)\Big(\int_\Omega f(x)e_j(x)dx\Big)\\ &+\Gamma(\alpha)\Big(t^{\alpha-1}-1\Big)\exp\Big(-j^2t\Big)\Big(\int_\Omega f(x)e_j(x)dx\Big)\\ &+\Big(\Gamma(\alpha)-1\Big)\exp\Big(-j^2t\Big)\Big(\int_\Omega f(x)e_j(x)dx\Big).\end{aligned} \tag{13}$$

Thus, we get

$$\begin{aligned}u_\alpha(x,t)-u^*(x,t) &= \sum_j \Gamma(\alpha)t^{\alpha-1}\Big(E_{\alpha,\alpha}\Big(-j^2t^\alpha\Big)-\exp\Big(-j^2t\Big)\Big)\Big(\int_\Omega f(x)e_j(x)dx\Big)e_j(x)\\ &\quad+\sum_j \Gamma(\alpha)\Big(t^{\alpha-1}-1\Big)\exp\Big(-j^2t\Big)\Big(\int_\Omega f(x)e_j(x)dx\Big)e_j(x)\\ &\quad+\sum_j \Big(\Gamma(\alpha)-1\Big)\exp\Big(-j^2t\Big)\Big(\int_\Omega f(x)e_j(x)dx\Big)e_j(x)\\ &\quad+\sum_j \Gamma(\alpha)t^{\alpha-1}\Big(\exp\Big(-j^2t^\alpha\Big)-\exp\Big(-j^2t\Big)\Big)\Big(\int_\Omega f(x)e_j(x)dx\Big)e_j(x)\\ &= J_1(x,t)+J_2(x,t)+J_3(x,t)+J_4(x,t).\end{aligned} \tag{14}$$

Step 1. Estimate of J_1. In view of the inequality in Lemma 1, we have for $z > 0$ that

$$\Big|E_{\alpha,\alpha}(-z)-\exp(-z)\Big| \le \frac{C}{1+|z|}(1-\alpha),$$

and we find that

$$\Big|E_{\alpha,\alpha}\Big(-j^2t^\alpha\Big)-\exp\Big(-j^2t^\alpha\Big)\Big| \le \frac{C}{1+j^2t^\alpha}(1-\alpha). \tag{15}$$

From $0 < \beta < 1$, we get

$$\frac{C}{1+j^2t^\alpha} \le \frac{C}{(1+j^2t^\alpha)^\beta} \le Cj^{-2\beta}t^{-\alpha\beta}. \tag{16}$$

Hence, we obtain the following estimate for $0 < \beta < 1$,

$$\big|E_{\alpha,\alpha}\big(-j^2t^\alpha\big)-\exp\big(-j^2t^\alpha\big)\big| \le Cj^{-2\beta}t^{-\alpha\beta}(1-\alpha). \tag{17}$$

This implies that

$$
\begin{aligned}
\|J_1(.,t)\|^2_{\mathbb{H}^s(\Omega)} &= \sum_j j^{2s}|\Gamma(\alpha)|^2 t^{2\alpha-2}\Big(E_{\alpha,\alpha}\Big(-j^2t^\alpha\Big) - \exp\Big(-j^2t\Big)\Big)^2\Big(\int_\Omega f(x)e_j(x)dx\Big)^2 \\
&\le C^2|\Gamma(\alpha)|^2 t^{2\alpha-2-2\alpha\beta}(1-\alpha)^2\sum_j j^{2s-4\beta}\Big(\int_\Omega f(x)e_j(x)dx\Big)^2. \qquad (18)
\end{aligned}
$$

Using Parseval's equality, we have for any $0 < \beta < 1$ that

$$\|J_1(.,t)\|_{\mathbb{H}^s(\Omega)} \le C\Gamma(\alpha)t^{\alpha-1-\alpha\beta}(1-\alpha)\|f\|_{\mathbb{H}^{s-2\beta}(\Omega)}. \qquad (19)$$

Thus

$$\|J_1\|_{L^1(0,T;\mathbb{H}^s(\Omega))} = \int_0^T \Big\|J_1(.,t)\Big\|_{\mathbb{H}^s(\Omega)}dt \le C\Gamma(\alpha)(1-\alpha)\|f\|_{\mathbb{H}^{s-2\beta}(\Omega)}\Big(\int_0^T t^{\alpha-1-\alpha\beta}dt\Big). \qquad (20)$$

Note the integral $\int_0^T t^{\alpha-1-\alpha\beta}dt$ is convergent and

$$\int_0^T t^{\alpha-1-\alpha\beta}dt = \frac{T^{\alpha-\alpha\beta}}{\alpha-\alpha\beta}.$$

Hence, we get that

$$\Big\|J_1\Big\|_{L^1(0,T;\mathbb{H}^s(\Omega))} \le \frac{C\Gamma(\alpha)T^{\alpha-\alpha\beta}}{\alpha-\alpha\beta}(1-\alpha)\Big\|f\Big\|_{\mathbb{H}^{s-2\beta}(\Omega)}. \qquad (21)$$

Step 2. Estimate of J_2. In this step (at the end) since $\theta < \alpha$ we choose $0 < \mu < 1$ so that $\mu + \theta < \alpha$. Note

$$\Big\|J_2(.,t)\Big\|^2_{\mathbb{H}^s(\Omega)} = \sum_j j^{2s}\Big|t^{\alpha-1}-1\Big|^2\exp\Big(-2j^2t\Big)\Big(\int_\Omega f(x)e_j(x)dx\Big)^2. \qquad (22)$$

We now consider the term $\big|t^{\alpha-1}-1\big|$. If $0 < t \le 1$, then we use the inequality $1-e^{-z} \le C_\mu z^\mu,\ 0<\mu<1$ to obtain

$$
\begin{aligned}
\Big|t^{\alpha-1}-1\Big| &= t^{\alpha-1}\Big(1-t^{1-\alpha}\Big) = t^{\alpha-1}(1-\exp(-(1-\alpha)\log(1/t))) \\
&\le t^{\alpha-1}C_\mu(1-\alpha)^\mu\Big(\log(\frac{1}{t})\Big)^\mu. \qquad (23)
\end{aligned}
$$

In view of the inequality $\log(z) \le z$ for any $z \ge 1$, we have that

$$\Big(\log(\frac{1}{t})\Big)^\mu \le t^{-\mu}.$$

Therefore, we obtain for the case $0 < t \le 1$ that

$$\Big|t^{\alpha-1}-1\Big| \le C_\mu(1-\alpha)^\mu t^{\alpha-\mu-1}. \qquad (24)$$

For the case $t > 1$, we see that

$$\big|t^{\alpha-1}-1\big| = t^{\alpha-1}\Big(t^{1-\alpha}-1\Big) \le t^{\alpha-1}\Big((T^*)^{1-\alpha}-1\Big), \qquad (25)$$

where $T^* = \max(T, 1)$. From (24) and (25), we have that

$$\left|t^{\alpha-1} - 1\right| \le t^{\alpha-1}\Big[C_\mu (1-\alpha)^\mu t^{-\mu} + (T^*)^{1-\alpha} - 1\Big], \tag{26}$$

for any $0 < \mu < 1$. In view of the inequality $e^{-z} \le C_\theta z^{-\theta}$, $0 < \theta < 1$, one has the following inequality

$$\exp\Big(-2j^2 t\Big) \le C_\theta^2 t^{-2\theta} j^{-4\theta}. \tag{27}$$

Combining (22), (26), and (27), we deduce that

$$\begin{aligned}\Big\|J_2(.,t)\Big\|^2_{\mathbb{H}^s(\Omega)} &\le C(\theta,\mu) t^{2\alpha-2-2\theta}\Big[C_\mu (1-\alpha)^\mu t^{-\mu} + (T^*)^{1-\alpha} - 1\Big]^2 \\ &\qquad \sum_j j^{2s-4\theta}\Big(\int_\Omega f(x)e_j(x)dx\Big)^2 \\ &= C(\theta,\mu) t^{2\alpha-2-2\theta}\Big[C_\mu (1-\alpha)^\mu t^{-\mu} + (T^*)^{1-\alpha} - 1\Big]^2 \|f\|^2_{\mathbb{H}^{s-2\theta}(\Omega)}.\end{aligned} \tag{28}$$

Thus

$$\Big\|J_2(.,t)\Big\|_{\mathbb{H}^s(\Omega)} \le C(\theta,\mu) t^{\alpha-1-\theta}\Big[(1-\alpha)^\mu t^{-\mu} + (T^*)^{1-\alpha} - 1\Big] \|f\|_{\mathbb{H}^{s-2\theta}(\Omega)}, \tag{29}$$

where $C(\theta,\mu)$ represents a constant that depends on μ, θ. Therefore, we have

$$\begin{aligned}\|J_2\|_{L^1(0,T;\mathbb{H}^s(\Omega))} &= \int_0^T \Big\|J_2(.,t)\Big\|_{\mathbb{H}^s(\Omega)} dt \\ &\le C(\theta,\mu)(1-\alpha)^\mu \Big\|f\Big\|_{\mathbb{H}^{s-2\theta}(\Omega)}\Big(\int_0^T t^{\alpha-1-\theta-\mu}dt\Big) \\ &+ C(\theta,\mu)\Big((T^*)^{1-\alpha} - 1\Big)\Big\|f\Big\|_{\mathbb{H}^{s-2\theta}(\Omega)}\Big(\int_0^T t^{\alpha-1-\theta}dt\Big).\end{aligned} \tag{30}$$

Note (with μ chosen so that $\theta + \mu < \alpha$)

$$\int_0^T t^{\alpha-1-\theta-\mu} dt = \frac{T^{\alpha-\theta-\mu}}{\alpha-\theta-\mu},$$

and

$$\int_0^T t^{\alpha-1-\theta} dt = \frac{T^{\alpha-\theta}}{\alpha-\theta}.$$

The above imply that

$$\|J_2\|_{L^1(0,T;\mathbb{H}^s(\Omega))} \le C(\theta,\mu,T)\Big[(1-\alpha)^\mu + (T^*)^{1-\alpha} - 1\Big]\Big\|f\Big\|_{\mathbb{H}^{s-2\theta}(\Omega)}. \tag{31}$$

Step 3. Estimate of J_3. Using Parseval's equality, we have

$$\|J_3(.,t)\|^2_{\mathbb{H}^s(\Omega)} = \Big(\Gamma(\alpha) - 1\Big)^2 \sum_j j^{2s} \exp\Big(-2j^2 t\Big)\Big(\int_\Omega f(x)e_j(x)dx\Big)^2. \tag{32}$$

In view of the inequality $e^{-z} \le C_\beta z^{-\beta}$, we have

$$\exp\Big(-2j^2 t\Big) \le C_\beta^2 j^{-4\beta} t^{-2\beta}.$$

This inequality together with (32) gives

$$\|J_3(.,t)\|_{\mathbb{H}^s(\Omega)} \leq \Big|\Gamma(\alpha)-1\Big| t^{-\beta}\|f\|_{\mathbb{H}^{s-2\beta}(\Omega)}. \tag{33}$$

Thus

$$\|J_3\|_{L^1(0,T;\mathbb{H}^s(\Omega))} = \int_0^T \|J_3(.,t)\|_{\mathbb{H}^s(\Omega)}dt \leq \Big|\Gamma(\alpha)-1\Big|\|f\|_{\mathbb{H}^{s-2\beta}(\Omega)}\Big(\int_0^T t^{-\beta}dt\Big). \tag{34}$$

Since $0<\beta<1$, then the proper integral $\int_0^T t^{-\beta}dt$ is convergent and

$$\int_0^T t^{-\beta}dt = \frac{T^{1-\beta}}{1-\beta}. \tag{35}$$

Combining (34) and (35), we obtain

$$\|J_3\|_{L^1(0,T;\mathbb{H}^s(\Omega))} \leq \frac{T^{1-\beta}}{1-\beta}|\Gamma(\alpha)-1|\ \|f\|_{\mathbb{H}^{s-2\beta}(\Omega)}. \tag{36}$$

Step 4. Estimate of J_4. We will choose $\mu = \frac{\alpha}{2}$ below. In view of Parseval's equality, we get

$$\|J_4(.,t)\|^2_{\mathbb{H}^s(\Omega)} = \big(\Gamma(\alpha)\big)^2 t^{2\alpha-2}\sum_j j^{2s}\Big(\exp\Big(-j^2t^\alpha\Big)-\exp\Big(-j^2t\Big)\Big)^2\Big(\int_\Omega f(x)e_j(x)dx\Big)^2. \tag{37}$$

Using the inequality $\left|e^{-a}-e^{-b}\right| \leq C_\varepsilon|a-b|^\varepsilon$, see Refs. [30,31] page 12–13, for any $a,b>0$, $0<\varepsilon<1$, we get

$$\Big|\exp\big(-j^2t^\alpha\big)-\exp\big(-j^2t\big)\Big| \leq C_\varepsilon j^{2\varepsilon}|t^\alpha - t|^\varepsilon. \tag{38}$$

From (26), we obtain the following inequality

$$\begin{aligned} |t^\alpha - t| = t\big|t^{\alpha-1}-1\big| &\leq t^\alpha\Big[C_\mu(1-\alpha)^\mu t^{-\mu} + (T^*)^{1-\alpha}-1\Big] \\ &\leq C_\mu(1-\alpha)^\mu t^{\alpha-\mu} + \Big((T^*)^{1-\alpha}-1\Big)t^\alpha, \end{aligned} \tag{39}$$

for any $0<\mu<1$. Thus, we obtain that

$$|t^\alpha - t|^{2\varepsilon} \leq C(\mu,\varepsilon)\Big((1-\alpha)^{2\varepsilon\mu}t^{2\varepsilon\alpha-2\varepsilon\mu} + \big((T^*)^{1-\alpha}-1\big)^{2\varepsilon}t^{2\varepsilon\alpha}\Big). \tag{40}$$

From some previous observations, we have

$$\begin{aligned} & t^{2\alpha-2}\sum_j j^{2s}\Big(\exp\Big(-j^2t^\alpha\Big)-\exp\Big(-j^2t\Big)\Big)^2\Big(\int_\Omega f(x)e_j(x)dx\Big)^2 \\ & \leq C(\mu,\varepsilon)(1-\alpha)^{2\varepsilon\mu}t^{2\varepsilon\alpha-2\varepsilon\mu+2\alpha-2}\sum_j j^{2s+4\varepsilon}\Big(\int_\Omega f(x)e_j(x)dx\Big)^2 \\ & + C(\mu,\varepsilon)\Big((T^*)^{1-\alpha}-1\Big)^{2\varepsilon}t^{2\varepsilon\alpha+2\alpha-2}\sum_j j^{2s+4\varepsilon}\Big(\int_\Omega f(x)e_j(x)dx\Big)^2. \end{aligned} \tag{41}$$

We now choose $\mu = \dfrac{\alpha}{2}$. Using the inequality $\sqrt{a+b} \le \sqrt{a} + \sqrt{b}$ for any $a, b \ge 0$, we have

$$\begin{aligned}\Big\|J_4(.,t)\Big\|_{\mathbb{H}^s(\Omega)} &\le \Gamma(\alpha)C(\alpha,\varepsilon)(1-\alpha)^{\frac{\alpha\varepsilon}{2}} t^{\frac{\varepsilon\alpha}{2}+\alpha-1}\|f\|_{\mathbb{H}^{s+2\varepsilon}(\Omega)} \\ &+ \Gamma(\alpha)C(\alpha,\varepsilon)\Big((T^*)^{1-\alpha}-1\Big)^{\varepsilon} t^{\varepsilon\alpha+\alpha-1}\|f\|_{\mathbb{H}^{s+2\varepsilon}(\Omega)}. \end{aligned} \tag{42}$$

The latter estimate implies that

$$\begin{aligned}\|J_4\|_{L^1(0,T;\mathbb{H}^s(\Omega))} &= \int_0^T \Big\|J_4(.,t)\Big\|_{\mathbb{H}^s(\Omega)} dt \\ &\le \Gamma(\alpha)C(\alpha,\varepsilon)(1-\alpha)^{\frac{\alpha\varepsilon}{2}}\|f\|_{\mathbb{H}^{s+2\varepsilon}(\Omega)}\Big(\int_0^T t^{\frac{\varepsilon\alpha}{2}+\alpha-1}dt\Big) \\ &+ \Gamma(\alpha)C(\alpha,\varepsilon)\Big((T^*)^{1-\alpha}-1\Big)^{\varepsilon}\|f\|_{\mathbb{H}^{s+2\varepsilon}(\Omega)}\Big(\int_0^T t^{\varepsilon\alpha+\alpha-1}dt\Big). \end{aligned} \tag{43}$$

We note that the two integrals $\int_0^T t^{\frac{\varepsilon\alpha}{2}+\alpha-1}dt$ and $\int_0^T t^{\varepsilon\alpha+\alpha-1}dt$ are convergent. Thus, it follows from (43) that

$$\|J_4\|_{L^1(0,T;\mathbb{H}^s(\Omega))} \lesssim \Big[(1-\alpha)^{\frac{\alpha\varepsilon}{2}} + \Big((T^*)^{1-\alpha}-1\Big)^{\varepsilon}\Big]\|f\|_{\mathbb{H}^{s+2\varepsilon}(\Omega)}, \tag{44}$$

where the hidden constant depends on α, ε. □

Remark 1. *Note that the right hand side of* (10) *goes to* 0 *when* $\alpha \to 1^-$.

3. Inhomogeneous Case

In this section, we consider the following diffusion equation with the inhomogeneous term

$$\begin{cases} \mathbf{D}^{\alpha}_{0^+}u - u_{xx} = G(x,t), & (x,t) \in (0,\pi)\times(0,T), \\ u = 0, & (x,t) \in \partial\Omega \times (0,T), \\ t^{1-\alpha}u|_{t=0} = f(x). \end{cases} \tag{45}$$

In order to reduce the calculation duplication process, we take the function $f = 0$. The goal of this section is to prove the convergence of the mild solution to problem (45) when $\alpha \to 1^-$.

Theorem 2. *Let* $G \in L^\infty(0,T;\mathbb{H}^{s+2\varepsilon}(\Omega)) \cap L^\infty(0,T;\mathbb{H}^{s-2\beta}(\Omega)) \cap L^\infty(0,T;\mathbb{H}^{s-\theta}(\Omega))$, *for any* $\varepsilon > 0$, $s \ge 0$, $0 < \beta, \theta < 1$, $\frac{3}{4} \le \alpha < 1$, $\beta, \theta \le \dfrac{s}{2}$, $\beta < \dfrac{1}{2}$ *and* $\theta < \alpha$. *Let* u_α *be the mild solution of Problem* (45) *and let* u^* *be the mild solution to the classical problem, i.e.,*

$$\begin{cases} u^*_t - u^*_{xx} = G(x,t), & (x,t) \in (0,\pi)\times(0,T), \\ u^* = 0, & (x,t) \in \partial\Omega \times (0,T), \\ u^*(x,0) = f(x), & x \in \Omega. \end{cases} \tag{46}$$

Then we have the following estimate

$$\begin{aligned}\|u_\alpha(.,t) - u^*(.,t)\|_{\mathbb{H}^s(\Omega)} &\le (1-\alpha)\|G\|_{L^\infty(0,T;\mathbb{H}^{s-2\beta}(\Omega))} \\ &+ \Big((1-\alpha)^{\frac{\alpha\varepsilon}{2}} + \Big((T^*)^{1-\alpha}-1\Big)^{\varepsilon}\Big)\ \|G\|_{L^\infty(0,T;\mathbb{H}^{s+2\varepsilon}(\Omega))} \\ &+ \sqrt{C_\alpha T^{\frac{\alpha}{2}}(1-\alpha)^{\frac{\alpha}{2}} + \frac{T^\alpha}{\alpha}((T^*)^{1-\alpha}-1)}\ \|G\|_{L^\infty(0,T;\mathbb{H}^{s-\theta}(\Omega))}. \end{aligned} \tag{47}$$

Proof. The mild solution u_α of Problem (45) is defined by (see Ref. [26])

$$\int_\Omega u_\alpha(x,t)e_j(x)dx = \int_0^t (t-r)^{\alpha-1}E_{\alpha,\alpha}\Big(-j^2(t-r)^\alpha\Big)\Big(\int_\Omega G(x,r)e_j(x)dx\Big)dr. \tag{48}$$

The mild solution to the classical problem is given by

$$\int_\Omega u^*(x,t)e_j(x)dx = \int_0^t \exp(-j^2(t-r))\Big(\int_\Omega G(x,r)e_j(x)dx\Big)dr. \tag{49}$$

From two above equalities, we get that

$$\begin{aligned}
&u_\alpha(x,t) - u^*(x,t)\\
&= \sum_{j=1}^\infty \left[\int_0^t (t-r)^{\alpha-1}\Big[E_{\alpha,\alpha}\Big(-j^2(t-r)^\alpha\Big) - \exp(-j^2(t-r))\Big]\Big(\int_\Omega G(x,r)e_j(x)dx\Big)dr\right]e_j(x)\\
&\quad + \sum_{j=1}^\infty \left[\int_0^t (t-r)^{\alpha-1}\Big[\exp(-j^2(t-r)^\alpha) - \exp(-j^2(t-r))\Big]\Big(\int_\Omega G(x,r)e_j(x)dx\Big)dr\right]e_j(x)\\
&\quad + \sum_{j=1}^\infty \left[\int_0^t \Big[(t-r)^{\alpha-1} - 1\Big]\exp(-j^2(t-r))\Big(\int_\Omega G(x,r)e_j(x)dx\Big)dr\right]e_j(x)\\
&= \mathbb{H}_1(x,t) + \mathbb{H}_2(x,t) + \mathbb{H}_3(x,t).
\end{aligned} \tag{50}$$

Step 1. Estimate of $\mathbb{H}_1$. In order to estimate $\mathbb{H}_1$, we use the bound (17) and obtain

$$\Big|E_{\alpha,\alpha}\big(-j^2(t-r)^\alpha\big) - \exp(-j^2(t-r)^\alpha)\Big| \le Cj^{-2\beta}(t-r)^{-\alpha\beta}(1-\alpha). \tag{51}$$

The term $\mathbb{H}_1$ is bounded by

$$\begin{aligned}
\|\mathbb{H}_1\|^2_{\mathbb{H}^s(\Omega)} &= \sum_{j=1}^\infty j^{2s}\left[\int_0^t (t-r)^{\alpha-1}\Big[E_{\alpha,\alpha}\Big(-j^2(t-r)^\alpha\Big) - \exp(-j^2(t-r))\Big]\Big(\int_\Omega G(x,r)e_j(x)dx\Big)dr\right]^2\\
&\le \sum_{j=1}^\infty j^{2s}\Big(\int_0^t (t-r)^{\alpha-1}dr\Big)\\
&\left[\int_0^t (t-r)^{\alpha-1}\Big[E_{\alpha,\alpha}\Big(-j^2(t-r)^\alpha\Big) - \exp(-j^2(t-r))\Big]\right]^2\Big(\int_\Omega G(x,r)e_j(x)dx\Big)^2 dr,
\end{aligned} \tag{52}$$

where we have used the Hölder inequality. In view of (51), we obtain

$$\begin{aligned}
\|\mathbb{H}_1\|^2_{\mathbb{H}^s(\Omega)} &\lesssim (1-\alpha)^2\sum_{j=1}^\infty j^{2s-4\beta}\left(\int_0^t (t-r)^{\alpha-1-2\alpha\beta}\Big(\int_\Omega G(x,r)e_j(x)dx\Big)^2 dr\right)\\
&= (1-\alpha)^2\int_0^t (t-r)^{\alpha-1-2\alpha\beta}\Big\|G(.,r)\Big\|^2_{\mathbb{H}^{s-2\beta}(\Omega)}dr\\
&\le (1-\alpha)^2\|G\|^2_{L^\infty(0,T;\mathbb{H}^{s-2\beta}(\Omega))}\Big(\int_0^t (t-r)^{\alpha-1-2\alpha\beta}dr\Big).
\end{aligned} \tag{53}$$

Note

$$\int_0^t (t-r)^{\alpha-1-2\alpha\beta}dr = \frac{t^{\alpha-2\alpha\beta}}{\alpha-2\alpha\beta},$$

and we recall that $\beta < \frac{1}{2}$. Thus, we can deduce that

$$\|\mathbb{H}_1\|_{\mathbb{H}^s(\Omega)} \lesssim (1-\alpha)\|G\|_{L^\infty(0,T;\mathbb{H}^{s-2\beta}(\Omega))}. \tag{54}$$

Step 2. Estimate of $\mathbb{H}_2$.

We will choose $\mu = \dfrac{\alpha}{2}$ below. Using Hölder's inequality, the term $\mathbb{H}_2$ is bounded by

$$\|\mathbb{H}_2\|^2_{\mathbb{H}^s(\Omega)} = \sum_{j=1}^{\infty} j^{2s} \left[\int_0^t (t-r)^{\alpha-1} \Big[\exp\big(-j^2(t-r)^{\alpha}\big) - \exp(-j^2(t-r)) \Big] \left(\int_\Omega G(x,r) e_j(x) dx \right) dr \right]^2$$
$$\leq \sum_{j=1}^{\infty} j^{2s} \left(\int_0^t (t-r)^{\alpha-1} dr \right)$$
$$\left[\int_0^t (t-r)^{\alpha-1} \Big[\exp\Big(-j^2(t-r)^{\alpha}\Big) - \exp(-j^2(t-r)) \Big]^2 \left(\int_\Omega G(x,r) e_j(x) dx \right)^2 dr. \quad (55)$$

Using (38) and (40) and the inequality $(a+b)^{\varepsilon} \leq C_{\varepsilon}(a^{\varepsilon} + b^{\varepsilon})$ for any $a, b \geq 0$, $\epsilon > 0$, we find for any $0 \leq r \leq t$ that

$$\Big| \exp\Big(-j^2(t-r)^{\alpha}\Big) - \exp\Big(-j^2(t-r)\Big) \Big| \leq C_{\varepsilon} j^{2\varepsilon} |(t-r)^{\alpha} - (t-r)|^{\varepsilon}$$
$$\lesssim j^{2\varepsilon} \Big[(1-\alpha)^{\mu\varepsilon} (t-r)^{(\alpha-\mu)\varepsilon} + \big((T^*)^{1-\alpha} - 1\big)^{\varepsilon} (t-r)^{\alpha\varepsilon} \Big]. \quad (56)$$

Now choose $\mu = \dfrac{\alpha}{2}$. Thus, we see

$$\Big| \exp\Big(-j^2(t-r)^{\alpha}\Big) - \exp\Big(-j^2(t-r)\Big) \Big|^2$$
$$\lesssim j^{4\varepsilon} (1-\alpha)^{2\mu\varepsilon} (t-r)^{\alpha\varepsilon} + j^{4\varepsilon} \big((T^*)^{1-\alpha} - 1\big)^{2\varepsilon} (t-r)^{2\alpha\varepsilon}. \quad (57)$$

This implies that

The right hand side of (55)

$$\lesssim (1-\alpha)^{\alpha\varepsilon} \sum_{j=1}^{\infty} j^{2s+4\varepsilon} \int_0^t (t-r)^{\alpha-1+\alpha\varepsilon} \left(\int_\Omega G(x,r) e_j(x) dx \right)^2 dr$$
$$+ \Big((T^*)^{1-\alpha} - 1 \Big)^{2\varepsilon} \sum_{j=1}^{\infty} j^{2s+4\varepsilon} \int_0^t (t-r)^{\alpha-1+2\alpha\varepsilon} \left(\int_\Omega G(x,r) e_j(x) dx \right)^2 dr. \quad (58)$$

Hence, we obtain that

$$\|\mathbb{H}_2\|^2_{\mathbb{H}^s(\Omega)} \lesssim (1-\alpha)^{\alpha\varepsilon} \int_0^t (t-r)^{\alpha-1+\alpha\varepsilon} \|G(.,r)\|^2_{\mathbb{H}^{s+2\varepsilon}(\Omega)} dr$$
$$+ \Big((T^*)^{1-\alpha} - 1 \Big)^{2\varepsilon} \int_0^t (t-r)^{\alpha-1+2\alpha\varepsilon} \|G(.,r)\|^2_{\mathbb{H}^{s+2\varepsilon}(\Omega)} dr. \quad (59)$$

Note the convergence of the two proper integrals $\int_0^t (t-r)^{\alpha-1+\alpha\varepsilon} dr$ and $\int_0^t (t-r)^{\alpha-1+2\alpha\varepsilon} dr$, and we have the following estimate

$$\|\mathbb{H}_2\|^2_{\mathbb{H}^s(\Omega)} \lesssim \left((1-\alpha)^{\alpha\varepsilon} + \Big((T^*)^{1-\alpha} - 1 \Big)^{2\varepsilon} \right) \|G\|^2_{L^{\infty}(0,T;\mathbb{H}^{s+2\varepsilon}(\Omega))}. \quad (60)$$

Thus, we deduce that

$$\|\mathbb{H}_2\|_{\mathbb{H}^s(\Omega)} \lesssim \left((1-\alpha)^{\frac{\alpha\varepsilon}{2}} + \Big((T^*)^{1-\alpha} - 1 \Big)^{\varepsilon} \right) \|G\|_{L^{\infty}(0,T;\mathbb{H}^{s+2\varepsilon}(\Omega))}. \quad (61)$$

Step 3. Estimate of $\mathbb{H}_3$.

We will choose $\mu = \frac{\alpha}{2}$ below and recall $\theta < \alpha$. By Parseval's equality, we obtain that

$$\begin{aligned}\|\mathbb{H}_3\|^2_{\mathbb{H}^s(\Omega)} &= \sum_{j=1}^{\infty} j^{2s}\left[\int_0^t \left[(t-r)^{\alpha-1}-1\right]\exp(-j^2(t-r))\left(\int_\Omega G(x,r)e_j(x)dx\right)dr\right]^2 \\ &\le \left(\int_0^t \left|(t-r)^{\alpha-1}-1\right|dr\right) \\ &\quad \sum_{j=1}^{\infty} j^{2s}\left[\int_0^t \left|(t-r)^{\alpha-1}-1\right|\exp(-2j^2(t-r))\left(\int_\Omega G(x,r)e_j(x)dx\right)^2 dr\right].\end{aligned}$$

In view of (26) (and choosing $\mu = \dfrac{\alpha}{2}$) we have for $0 \le r \le t$ that

$$\left|(t-r)^{\alpha-1}-1\right| \le C_\alpha (t-r)^{\frac{\alpha}{2}-1}(1-\alpha)^{\frac{\alpha}{2}} + \left((T^*)^{1-\alpha}-1\right)(t-r)^{\alpha-1}, \tag{62}$$

where C_α indicates a constant which depends on α. The inequality (62) implies that

$$\begin{aligned}\int_0^t \left|(t-r)^{\alpha-1}-1\right|dr \le C_\alpha(1-\alpha)^{\frac{\alpha}{2}}\left(\int_0^t (t-r)^{\frac{\alpha}{2}-1}dr\right) \\ + \left((T^*)^{1-\alpha}-1\right)\left(\int_0^t (t-r)^{\alpha-1}dr\right).\end{aligned} \tag{63}$$

Note

$$\int_0^t (t-r)^{\frac{\alpha}{2}-1}dr = \frac{2}{\alpha}t^{\frac{\alpha}{2}}, \quad \int_0^t (t-r)^{\alpha-1}dr = \frac{1}{\alpha}t^{\alpha}.$$

Thus, it follows from (63) that

$$\int_0^t \left|(t-r)^{\alpha-1}-1\right|dr \le C_\alpha T^{\frac{\alpha}{2}}(1-\alpha)^{\frac{\alpha}{2}} + \frac{T^\alpha}{\alpha}\left((T^*)^{1-\alpha}-1\right). \tag{64}$$

By applying the inequality $e^{-j^2(t-r)} \le C_\theta j^{-\theta}(t-r)^{-\theta}$ for any $\theta > 0$, we get the following estimate:

$$\begin{aligned}&\sum_{j=1}^{\infty} j^{2s}\left[\int_0^t \left|(t-r)^{\alpha-1}-1\right|\exp(-2j^2(t-r))\left(\int_\Omega G(x,r)e_j(x)dx\right)^2 dr\right. \\ &\le \int_0^t \left|(t-r)^{\alpha-1}-1\right|(t-r)^{-\theta}\left(\sum_{j=1}^{\infty} j^{2s-2\theta}\left(\int_\Omega G(x,r)e_j(x)dx\right)^2\right)dr \\ &\le \|G\|^2_{L^\infty(0,T;\mathbb{H}^{s-\theta}(\Omega))}\left(\int_0^t (t-r)^{\alpha-1-\theta}dr + \int_0^t (t-r)^{-\theta}dr\right),\end{aligned} \tag{65}$$

where we have used the triangle inequality. Then (recall $\theta < \alpha$) from the convergence of the two proper integrals above, we deduce

$$\|\mathbb{H}_3\|^2_{\mathbb{H}^s(\Omega)} \le \left[C_\alpha T^{\frac{\alpha}{2}}(1-\alpha)^{\frac{\alpha}{2}} + \frac{T^\alpha}{\alpha}\left((T^*)^{1-\alpha}-1\right)\right]\|G\|^2_{L^\infty(0,T;\mathbb{H}^{s-\theta}(\Omega))}. \tag{66}$$

Combining (54), (61), and (66), we get

$$\begin{aligned}\left\|u_\alpha(.,t)-u^*(.,t)\right\|_{\mathbb{H}^s(\Omega)} &\le \left\|\mathbb{H}_1\right\|_{\mathbb{H}^s(\Omega)}+\left\|\mathbb{H}_2\right\|_{\mathbb{H}^s(\Omega)}+\left\|\mathbb{H}_3\right\|_{\mathbb{H}^s(\Omega)}\\ &\le (1-\alpha)\left\|G\right\|_{L^\infty(0,T;\mathbb{H}^{s-2\beta}(\Omega))}\\ &\quad+\left((1-\alpha)^{\frac{\alpha\varepsilon}{2}}+\left((T^*)^{1-\alpha}-1\right)^{\varepsilon}\right)\left\|G\right\|_{L^\infty(0,T;\mathbb{H}^{s+2\varepsilon}(\Omega))}\\ &\quad+\sqrt{C_\alpha T^{\frac{\alpha}{2}}(1-\alpha)^{\frac{\alpha}{2}}+\frac{T^\alpha}{\alpha}((T^*)^{1-\alpha}-1)}\left\|G\right\|_{L^\infty(0,T;\mathbb{H}^{s-\theta}(\Omega))}.\end{aligned} \tag{67}$$

□

Remark 2. *Note the right hand side of* (47) *tends to zero when* $\alpha \to 1^-$.

4. Conclusions

In this work, the main objective was to investigate the convergence of solutions to the problem when the fractional order tends to 1^- with the Riemann–Liouville derivative. In the future, we hope to investigate the convergence towards 1^- with the Caputo derivative, the Atangana Baleanu Caputo derivative, and some other non-integer order derivatives.

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