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Convergence of the Boundary Parameter for the Three-Dimensional Viscous Primitive Equations of Large-Scale

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Abstract: The main objective of this paper is concerned with the convergence of the boundary parameter for the large-scale, three-dimensional, viscous primitive equations. Such equations are often used for weather prediction and climate change. Under the assumptions of some boundary conditions, we obtain a prior bounds for the solutions of the equations by using the differential inequality technology and method of the energy estimates, and the convergence of the equations on the boundary parameter is proved.

Keywords: viscous primitive equations; convergence; energy estimates

MSC: 49J20; 65N30

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1. Introduction

The primitive system of equations is a classic model for studying climate change and weather prediction. Studying the equations is of great significance, because the ever-changing flow phenomena are all determined by the equations. Firstly, L.F. Richardson [1] introduced the primitive equations of the ocean and atmosphere, the models mainly include hydrodynamic equations with Coriolis force, thermodynamic equations and equations of state, as can be easily deduced in Animasaun et al. [2]. In view of this, researchers have started using various strategies to simplify the model since the equations are too complex.

Lions et al. [3] established the primitive equation model for dry atmosphere by introducing some technical treatments, such as viscosity, and established the primitive equations of the ocean in [4,5], as well as introducing some mathematical theories of coupled atmospheric–oceanic models in [6]. For more information on the development of the primitive equations of the atmosphere and ocean, readers may refer to [7]. the three-dimensional viscous primitive equations of large-scale are used to describe the turbulent behavior of long-term weather prediction and climate changes, researchers are first concerned about whether these equations have inherent logical unity in mathematics, that is, well-posedness. For example, Guo et al. [8] obtained the global existence of smooth solutions in the primitive system of equations for dry atmospheres using refined energy estimates. Guo et al. [9] obtained the problem of global well-posedness for the primitive system of equations for moist atmospheres. More results on the well-posedness of the primitive equations can be found in [10–16]. This field of research is developing rapidly, and some researchers have been attracted to it in recent decades.

These equations are too complex to be handled, both theoretically and computationally. The most common physical simplification is that researchers have taken a static approximation based on hydrostatics due to the observation that the vertical dimensions are often much smaller than the horizontal dimensions. Therefore, in the process of establishing mathematical models for weather forecasting and climate changes, and in the process of model simplification, some errors will inevitably arise. We need to know whether these errors will cause huge changes in the solutions of the equations. This kind of research is called stability research.

Recently, attention has been paid to the research on the stability of the primitive equations. Li [17] proved the continuous dependence of a large-scale, oceanic, 3D primitive equation system under oscillating random forces on the viscosity coefficient. Li [18] proved continuous dependence on the boundary parameters for the 3D viscous primitive equation of large-scale ocean and atmosphere dynamics by using the technique of differential inequality and the method of energy estimation. The convergence of this paper is different from continuous dependence. The main content of this paper is the influence of the coefficient, boundary parameter or known function of the equations approaching zero. Being able to obtain the continuous dependence of the system of equations does not necessarily mean that the convergence of the system of equations can be obtained. For example, we can obtain the continuous dependence of the system of equations on the viscosity coefficient, but its convergence cannot be deduced. There have been many articles on convergence in fluid equations (see [19–32]). At present, there are no relevant results about the convergence of the coefficients and parameters of the primitive equations. In this paper, the convergence results were obtained by using the differential inequality technology and a prior bounds. The method we used will provide a reference for research on other types of primitive equations.

2. Preliminary

In this section, we provide some our preliminary work, which will be used frequently. M denotes a bounded, smooth region on R^2, $\Omega = M \times (0,1)$ denotes a columnar domain in R^3, $\Gamma_0 := \{(x_1,x_2,x_3)|(x_1,x_2) \in M,\ x_3 = 0\}$, $\Gamma_1 := \{(x_1,x_2,x_3)|(x_1,x_2) \in M,\ x_3 = 1\}$, $\Gamma_s := \{(x_1,x_2,x_3)|(x_1,x_2) \in \partial M, 0 \leq\ x_3 \leq 1\}$, and n denotes the outer unit normal vector on Γ_s. $\frac{\partial}{\partial n}$ denotes the outer unit normal derivative on Γ_s, and $\partial_z = \frac{\partial}{\partial z}$ is the partial derivative to z. $\nabla := (\partial_{x_1}, \partial_{x_2})$, $\Delta := \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}$. Commas are used to denote derivation, repeated Greek subscripts denote summation from 1 to 2, and repeated English subscripts denote summation from 1 to 3. e.g., $u_{,\alpha}u_{,\alpha} = \sum_{\alpha=1}^{2}\left(\frac{\partial u}{\partial x_\alpha}\right)^2$, $u_{,i}u_{,i} = \sum_{i=1}^{3}\left(\frac{\partial u}{\partial x_i}\right)^2$.

This paper studies the 3D viscous primitive equations defined on $\Omega \times (0,\infty)$ as follows (see [19]):

$$v_t + (v\cdot\nabla)v + w\partial_{x_3}v + \nabla p + fv^{\perp} - \mu_1\Delta v - \mu^2\partial_{x_3x_3}v = 0, \tag{1}$$

$$\partial_{x_3}p + T = 0, \tag{2}$$

$$\nabla\cdot v + \partial_{x_3}w = 0, \tag{3}$$

$$T_t + v\cdot\nabla T + w\partial_{x_3}T - k_1\Delta T - k^2\partial_{x_3x_3}T = 0. \tag{4}$$

where $v = (v_1, v^2)$ denotes the horizontal velocity field, w denotes the vertical velocity, T denotes temperature, p denotes pressure, μ_1 and μ^2 denote the viscosity coefficients in the horizontal and vertical directions, respectively, k_1 and k_2 denote the thermal diffusivity in the horizontal and vertical directions, respectively, and f is the Coriolis parameter: $v^{\perp} = (-v_2, v_1)$. Equations (1)–(4) satisfy on the boundary of Ω. Because the direction of heat transfer is opposite to the direction of the gradient of the temperature, the gradient of the temperature at the boundary is negative.

$$\partial_{x_3}v = 0, w = 0, \partial_{x_3}T = -\lambda T,\ on\quad \Gamma_1 \times (0,\infty), \tag{5}$$

$$\partial_{x_3}v = 0,\ w = 0,\ \partial_{x_3}T = 0,\ \ on\ \ \Gamma_0 \times (0,\infty), \tag{6}$$

$$v\cdot n = 0,\ \frac{\partial v}{\partial n}\times n = 0,\ w = 0,\ \frac{\partial T}{\partial n} = 0,\ \ on\ \ \Gamma_s \times (0,\infty). \tag{7}$$

where λ is a constant greater than zero. In addition, the system of Equations (1)–(4) has the following initial conditions:

$$v(x,0) = v_0(x),\ T(x,0) = T_0(x),\ on\ \ \Omega \times \{0\}. \tag{8}$$

where $v_0(x), T_0(x)$ are nonnegative, continuous functions. Using Equation (3), we have

$$w(x_1,x_2,x_3,t) = w(x_1,x_2,0,t) + \int_0^{x_3} \partial_\xi w(x_1,x_2,\xi,t)d\xi = -\int_0^{x_3} \nabla \cdot v(x_1,x_2,\xi,t)d\xi. \tag{9}$$

because, again, $w\big|_{\Gamma_0} = w\big|_{\Gamma_1} = 0$,

$$\int_0^1 \nabla \cdot v(x_1,x_2,\xi,t)d\xi = 0. \tag{10}$$

Similarly, integrating Equation (2) from 0 to x_3, we have

$$p(x_1,x_2,x_3,t) = p_0(x_1,x_2,0,t) - \int_0^{x_3} T(x_1,x_2,\xi,t)d\xi = 0, \tag{11}$$

where $p_0(x_1,x_2,0,t)$ represents the approximation of the pressure on the earth's surface. Substituting Equations (9) and (11) into Equations (1)–(4), we have

$$\begin{aligned} v_t + (v\cdot\nabla)v - (\textstyle\int_0^{x_3} \nabla\cdot v(x_1,x_2,\xi,t)d\xi)\partial_{x_3}v + fv^{\perp} + \nabla p_0 \\ -\textstyle\int_0^{x_3} \nabla T(x_1,x_2,\xi,t)d\xi - \mu_1\Delta v - \mu^2\partial_{x_3x_3}v = 0, \end{aligned} \tag{12}$$

$$T_t + v\cdot\nabla T - (\int_0^{x_3} \nabla\cdot v(x_1,x_2,\xi,t)d\xi)\partial_{x_3}T - k_1\Delta T - k^2\partial_{x_3x_3}T = 0. \tag{13}$$

To prove the convergence of problems (1)–(8) to the boundary parameter λ, we suppose that (v^*, T^*, p_0^*) is another solution of the system of Equations (12) and (13) with boundary conditions (5)–(7) and initial conditions (8), when λ = 0. Let

$$\widetilde{v} = v - v^*, \widetilde{T} = T - T^*, \pi = p_0 - p_0^*,$$

then, $(\widetilde{v}, \widetilde{T}, \pi)$ satisfies the system of equations:

$$\begin{aligned} \widetilde{v}_t + (\widetilde{v}\cdot\nabla)v - (\textstyle\int_0^{x_3}\nabla\cdot\widetilde{v}(x_1,x_2,\xi,t)d\xi)\partial_{x_3}v + (v^*\cdot\nabla)\widetilde{v} - (\textstyle\int_0^{x_3}\nabla\cdot v^*(x_1,x_2,\xi,t)d\xi)\partial_{x_3}\widetilde{v} \\ +f\widetilde{v}^{\perp} + \nabla\pi - \textstyle\int_0^{x_3}\nabla\widetilde{T}(x_1,x_2,\xi,t)d\xi - \mu_1\Delta\widetilde{v} - \mu_2\partial_{x_3x_3}\widetilde{v} = 0, \end{aligned} \tag{14}$$

$$\begin{aligned} \widetilde{T}_t + \widetilde{v}\cdot\nabla T - (\textstyle\int_0^{x_3}\nabla\cdot\widetilde{v}(x_1,x_2,\xi,t)d\xi)\partial_{x_3}T + v^*\cdot\nabla\widetilde{T} \\ -(\textstyle\int_0^{x_3}\nabla\cdot v^*(x_1,x_2,\xi,t)d\xi)\partial_{x_3}\widetilde{T} - k_1\Delta\widetilde{T} - k_2\partial_{x_3x_3}\widetilde{T} = 0, \end{aligned} \tag{15}$$

with the following initial boundary value conditions:

$$\partial_{x_3}\widetilde{v} = 0, \partial_{x_3}\widetilde{T} = -\lambda\widetilde{T} - \lambda T^*, \qquad on\ \ \Gamma_0 \times (0,\infty), \tag{16}$$

$$\partial_{x_3}\widetilde{v} = 0, \partial_{x_3}\widetilde{T} = 0, \qquad on\ \ \Gamma_1 \times (0,\infty), \tag{17}$$

$$\widetilde{v}\cdot n = 0, \frac{\partial\widetilde{v}}{\partial n} \times n = 0, \frac{\partial\widetilde{T}}{\partial n} = 0,\ \ on \qquad \Gamma_s \times (0,\infty), \tag{18}$$

$$\widetilde{v}(x,0) = \widetilde{T}(x,0) = 0, \qquad on\ \ \Omega \times \{0\}. \tag{19}$$

Next, we provide some our preliminary work, which will be used in this paper frequently.

Lemma 1 ([33–35]). *If $\omega(x_3) \in C^1(0,1)$, and $\omega(1) = \omega(0) = 0$, then*

$$\int_0^1 \omega^2 dx_3 \le \frac{1}{\pi^2}\int_0^1 (\frac{d\omega}{dx_3})^2 dx_3.$$

Lemma 2 ([36]). *Assuming $\omega \in H^1(\Omega)$, then*

$$\|\omega\|_4^4 \le C\|\omega\|_2\|\nabla\omega\|_2^3,$$

where C is a constant greater than zero.

3. Main Results

We are now in the position to formulate the main results of this article.

Theorem 1. (the convergence of the equations on the boundary parameter). *Assume that $(\widetilde{v}, \widetilde{T}, \pi)$ is the solution of Equations (14)–(19). When $\lambda \to 0$,*

$$(v, T, p_0) \to (v^*, T^*, p_0^*),$$

the following inequality holding:

$$\int_0^t \int_\Omega \left|\widetilde{v}\right|^2 dx\, d\eta + \int_0^t \int_\Omega \widetilde{T}^2 dx \le \lambda \int_0^t b_2(s) e^{\int_s^t b_1(\eta) d\eta} ds,$$

where $b_1(t)$ and $b_2(t)$ are non-negative functions that depend on t. Their definitions will be given below.

Remark 1. *This theorem shows that the solutions of Equations (1)–(8) converge with the boundary parameter. This shows that small changes in the boundary parameter will not have a significant impact on the solution.*

Proof. Step 1, mark

$$E_1(t) = \int_0^t \int_\Omega \left|\widetilde{v}\right|^2 dx\, d\eta.$$

So, taking the derivative of $E_1(t)$ and using Equation (14), we have

$$\begin{aligned} E_1'(t) = 2\textstyle\int_0^t \int_\Omega \widetilde{v}\widetilde{v}_\eta dx d\eta = -2\int_0^t \int_\Omega \widetilde{v}[(\widetilde{v}\cdot\nabla)v - (\int_0^{x_3} \nabla\cdot\widetilde{v}(x_1, x_2, \xi, t)d\xi)\partial_{x_3} v] dx d\eta \\ -2\textstyle\int_0^t \int_\Omega (\int_0^{x_3} \nabla\widetilde{T}(x_1, x_2, \xi, t)d\xi)\widetilde{v} dx d\eta \\ -2\mu_1 \textstyle\int_0^t \int_\Omega |\nabla\widetilde{v}|_2 dx d\eta - 2\mu_2 \int_0^t \int_\Omega |\partial_{x_3}\widetilde{v}|_2 dx d\eta, \end{aligned} \tag{20}$$

where we have used the following two Equations (21) and (22):

$$\int_0^t \int_\Omega \widetilde{v}[(v^*\cdot\nabla)\widetilde{v} - (\int_0^{x_3} \nabla\cdot v^*(x_1, x_2, \xi, t)d\xi)\partial_{x_3}\widetilde{v}] dx d\eta = 0, \tag{21}$$

$$\int_0^t \int_\Omega f\widetilde{v}^\perp \cdot \widetilde{v} dx d\eta = \int_0^t \int_\Omega \nabla\pi \cdot \widetilde{v} dx d\eta = 0. \tag{22}$$

Using the divergence theorem and boundary conditions, Equations (21) and (22) can be proved. Using the Hölder inequality, Lemma 1, Lemma 2, and arithmetic–geometric mean inequality, we can obtain

$$\begin{aligned} &-2\textstyle\int_0^t \int_\Omega \widetilde{v}[(\widetilde{v}\cdot\nabla)v - (\int_0^{x_3} \nabla\cdot\widetilde{v}(x_1, x_2, \xi, t)d\xi)\partial_{x_3} v] dx d\eta \\ &\le 2(\textstyle\int_0^t \int_\Omega |\widetilde{v}|^4 dx d\eta)^{\frac{1}{2}} (\int_0^t \int_\Omega |\nabla v|^2 dx d\eta)^{\frac{1}{2}} \\ &+2(\textstyle\int_0^t \int_\Omega |\widetilde{v}|^4 dx d\eta)^{\frac{1}{4}} (\int_0^t \int_\Omega |\partial_{x_3} v|^4 dx d\eta)^{\frac{1}{4}} (\int_0^t \int_\Omega (\int_0^{x_3} \nabla\cdot\widetilde{v}(x_1, x_2, \xi, t)d\xi)^2 dx d\eta)^{\frac{1}{2}} \\ &\le 2\sqrt{C}(\textstyle\int_0^t \int_\Omega |\widetilde{v}|^2 dx d\eta)^{\frac{1}{4}} (\int_0^t \int_\Omega |\nabla\widetilde{v}|^2 dx d\eta)^{\frac{3}{4}} (\int_0^t \int_\Omega |\nabla v|^2 dx d\eta)^{\frac{1}{2}} \\ &+\tfrac{2}{\pi}\sqrt[4]{C}(\textstyle\int_0^t \int_\Omega |\partial_{x_3} v|^4 dx d\eta)^{\frac{1}{4}} (\int_0^t \int_\Omega |\widetilde{v}|^2 dx d\eta)^{\frac{1}{8}} (\int_0^t \int_\Omega |\nabla\widetilde{v}|^2 dx d\eta)^{\frac{7}{8}} \\ &\le \left\{\tfrac{1}{2}\varepsilon_1^{-3} C^2 (\textstyle\int_0^t \int_\Omega |\nabla v|_2 dx d\eta)^2 + \frac{2^6}{\pi^8}\varepsilon_2^{-7} C^2 (\int_0^t \int_\Omega |\partial_{x_3} v|_4 dx d\eta)^2\right\} E_1(t) \\ &+[\tfrac{3}{2}\varepsilon_1 + \tfrac{7}{4}\varepsilon_2] \textstyle\int_0^t \int_\Omega |\nabla\widetilde{v}|_2 dx d\eta, \end{aligned} \tag{23}$$

where ε_1 and ε_2 are arbitrary constants greater than zero. Using the divergence theorem, Hölder inequality, and Lemma 1, we have

$$\begin{aligned} & -2\int_0^t \int_\Omega \widetilde{v}\int_0^{x_3} \nabla\widetilde{T}(x_1,x_2,\xi,t)d\xi dxd\eta \\ & \leq 2(\int_0^t \int_\Omega \widetilde{v}^2 dxd\eta)^{\frac{1}{2}}(\int_0^t \int_\Omega (\int_0^{x_3} \nabla\widetilde{T}(x_1,x_2,\xi,t)d\xi)^2 dxd\eta)^{\frac{1}{2}} \\ & \leq \tfrac{2}{\pi}(\int_0^t \int_\Omega \widetilde{v}^2 dxd\eta)^{\frac{1}{2}}(\int_0^t \int_\Omega \left|\nabla\widetilde{T}\right|_2 dx\,d\eta)^{\frac{1}{2}} \\ & \leq \tfrac{1}{\pi^2\varepsilon_3}E_1(t) + \varepsilon_3\int_0^t \int_\Omega \left|\nabla\widetilde{T}\right|^2 dxd\eta, \end{aligned} \tag{24}$$

where ε_3 is an arbitrary constant greater than zero. Substituting Equations (23) and (24) into Equation (20) and choosing

$$\varepsilon_1 = \frac{1}{3}\mu_1, \text{ we}\varepsilon_2 = \frac{2}{7}\mu_1$$

we have

$$\begin{aligned} E_1'(t) \leq & \left\{ \tfrac{1}{2}\varepsilon_1^{-3}C^2(\int_0^t \int_\Omega |\nabla v|^2 dx\,d\eta)^2 + \tfrac{2^6}{\pi^8}\varepsilon_2^{-7}C^2(\int_0^t \int_\Omega |\partial_{x_3}v|^4 dx\,d\eta)^2 + \tfrac{1}{\pi^2\varepsilon_3} \right\} E_1(t) \\ & -\mu_1\int_0^t \int_\Omega |\nabla\widetilde{v}|^2 dx\,d\eta - 2\mu_2\int_0^t \int_\Omega |\partial_{x_3}\widetilde{v}|^2 dx\,d\eta + \varepsilon_3\int_0^t \int_\Omega \left|\nabla\widetilde{T}\right|_2 dx\,d\eta. \end{aligned} \tag{25}$$

Step 2, mark

$$E_2(t) = \int_0^t \int_\Omega \widetilde{T}^2 dxd\eta. \tag{26}$$

Using Equation (15), we have

$$\begin{aligned} & E_2'(t) = 2\int_0^t \int_\Omega \widetilde{T}\widetilde{T}_\eta dxd\eta = -2\int_0^t \int_\Omega \widetilde{T}[(\widetilde{v}\cdot\nabla T - (\int_0^{x_3} \nabla\cdot\widetilde{v}(x_1,x_2,\xi,t)d\xi)\partial_{x_3}T]dxd\eta \\ & -2k_1\int_0^t \int_\Omega \left|\nabla\widetilde{T}\right|_2 dxd\eta - 2k_2\int_0^t \int_\Omega \left|\partial_{x_3}\widetilde{T}\right|_2 dxd\eta - 2k_2\lambda\int_0^t \int_{\Gamma_1} \widetilde{T}^2 dAd\eta - 2k_2\lambda\int_0^t \int_{\Gamma_1} \widetilde{T}T^* dAd\eta. \end{aligned} \tag{27}$$

Similar to the derivation of (23), we can obtain

$$\begin{aligned} & -\int_0^t \int_\Omega \widetilde{T}[\widetilde{v}\cdot\nabla T - (\int_0^{x_3} \nabla\cdot\widetilde{v}(x_1,x_2,\xi,t)d\xi)\partial_{x_3}T]dxd\eta \\ & \leq (\int_0^t \int_\Omega |\nabla T|_2 dx\,d\eta)^{\frac{1}{2}}(\int_0^t \int_\Omega \widetilde{T}^4 dxd\eta)^{\frac{1}{4}}(\int_0^t \int_\Omega |\widetilde{v}|^4 dxd\eta)^{\frac{1}{4}} \\ & +\tfrac{1}{\pi}(\int_0^t \int_\Omega |\nabla\widetilde{v}|_2 dx\,d\eta)^{\frac{1}{2}}(\int_0^t \int_\Omega \widetilde{T}^4 dxd\eta)^{\frac{1}{4}}(\int_0^t \int_\Omega |\partial_{x_3}T|^4 dxd\eta)^{\frac{1}{4}} \\ & \leq \sqrt{C}(\int_0^t \int_\Omega |\nabla T|^2 dx\,d\eta)^{\frac{1}{2}}[E_2(t)]^{\frac{1}{8}}(\int_0^t \int_\Omega \left|\nabla\widetilde{T}\right|_2 dx\,d\eta)^{\frac{3}{8}}[E_1(t)]^{\frac{1}{8}}(\int_0^t \int_\Omega |\nabla\widetilde{v}|^2 dx\,d\eta)^{\frac{3}{8}} \\ & +\tfrac{1}{\pi}\sqrt[4]{C}(\int_0^t \int_\Omega |\partial_{x_3}T|_4 dx\,d\eta)^{\frac{1}{4}}(\int_0^t \int_\Omega |\nabla\widetilde{v}|^2 dx\,d\eta)^{\frac{1}{2}}[E_2(t)]^{\frac{1}{8}}(\int_0^t \int_\Omega \left|\nabla\widetilde{T}\right|_2 dx\,d\eta)^{\frac{3}{8}} \\ & \leq [\tfrac{1}{8}\varepsilon_4{}^{-3}C^2(\int_0^t \int_\Omega |\nabla T|_2 dx\,d\eta)^2 + \tfrac{1}{8}C(\int_0^t \int_\Omega |\partial_{x_3}T|_4 dx\,d\eta)^2\varepsilon_6{}^{-4}\varepsilon_7{}^{-3}]E_2(t) \\ & +\tfrac{3}{8}[\varepsilon_4+\varepsilon_7]\int_0^t \int_\Omega \left|\nabla\widetilde{T}\right|_2 dx\,d\eta \\ & +\tfrac{1}{8}\varepsilon_5{}^{-3}C^2(\int_0^t \int_\Omega |\nabla T|^2 dxd\eta)^2 E_1(t) + [\tfrac{3}{8}\varepsilon_5 + \tfrac{1}{2}\varepsilon_6]\int_0^t \int_\Omega |\nabla\widetilde{v}|^2 dxd\eta. \end{aligned} \tag{28}$$

As well as

$$\int_0^t \int_{\Gamma_1} \widetilde{T}T^* dAd\eta \leq \int_0^t \int_{\Gamma_1} \widetilde{T}^2 dAd\eta + \frac{1}{4}\int_0^t \int_{\Gamma_1} (T^*)^2 dAd\eta, \tag{29}$$

where $\varepsilon_4, \varepsilon_5, \varepsilon_6$, and ε_7 are arbitrary constants greater than zero. Combining with (27), (28), and (29) and choosing

$$\varepsilon_4 = \varepsilon_7 = \frac{2}{3}k_1. \tag{30}$$

we have

$$\begin{aligned} E_2'(t) \le & [\tfrac{1}{8}\varepsilon_4{}^{-3}C^2(\int_0^t\int_\Omega|\nabla T|^2dxd\eta)^2+\tfrac{1}{8}C(\int_0^t\int_\Omega|\partial_{x_3}T|^4dxd\eta)^2\varepsilon_6{}^{-4}\varepsilon_7{}^{-3}]E_2(t) \\ & +\tfrac{1}{8}\varepsilon_5{}^{-3}C^2(\int_0^t\int_\Omega|\nabla T|^2dxd\eta)^2E_1(t)+[\tfrac{3}{8}\varepsilon_5+\tfrac{1}{2}\varepsilon_6]\int_0^t\int_\Omega|\nabla\widetilde{v}|^2dxd\eta \\ & -k_1\int_0^t\int_\Omega\left|\nabla\widetilde{T}\right|^2dxd\eta-2k_2\int_0^t\int_\Omega\left|\partial_{x_3}\widetilde{T}\right|^2dxd\eta+\tfrac{1}{2}k_2\lambda\int_0^t\int_{\Gamma_1}(T^*)^2dAd\eta. \end{aligned} \tag{31}$$

Step 3, mark

$$E(t)=E_1(t)+E_2(t).$$

Combining (25) and (30), and choosing

$$\varepsilon_3=k_1,\varepsilon_5=\frac{4}{3}\mu_1,\varepsilon_6=\mu_1$$

we have

$$E'(t)\le b_1(t)E(t)+b_2(t)\lambda, \tag{32}$$

where

$$\begin{aligned} b_1(t)= & \tfrac{1}{2}\varepsilon_1{}^{-3}C^2(\int_0^t\int_\Omega|\nabla v|^2dxd\eta)^2+\tfrac{2^6}{\pi^8}\varepsilon_2{}^{-7}C^2(\int_0^t\int_\Omega|\partial_{x_3}v|^4dxd\eta)^2+\tfrac{1}{\pi^2\varepsilon_3} \\ & +\tfrac{1}{8}(\varepsilon_4{}^{-3}+\varepsilon_5{}^{-3})C^2(\int_0^t\int_\Omega|\nabla T|^2dxd\eta)^2+\tfrac{1}{8}C(\int_0^t\int_\Omega|\partial_{x_3}T|^4dxd\eta)^2\varepsilon_6{}^{-4}\varepsilon_7{}^{-3} \\ b_2(t)= & k_2\int_0^t\int_{\Gamma_1}(T^*)^2dAd\eta. \end{aligned}$$

Using the Gronwall inequality again, we can obtain

$$E(t)\le\lambda\int_0^t b_2(s)e^{\int_s^t b_1(\eta)d\eta}ds, \tag{33}$$

combining with the definition of E(t), Theorem 1 is proved. □

Note: In order to make (32) meaningful according to the definitions of $b_1(t)$ and $b_2(t)$, we must estimate the upper bound of $\int_0^t\int_\Omega|\nabla v|^2dxd\eta$, $\int_0^t\int_\Omega|\partial_{x_3}v|^4dxd\eta$, $\int_0^t\int_\Omega|\nabla T|^2dxd\eta$, $\int_0^t\int_\Omega|\partial_{x_3}T|^4dxd\eta$, $\int_0^t\int_{\Gamma_1}(T^*)^2dAd\eta$. We will give the derivation process in the next section.

4. A Prior Estimates

We first derive a prior estimates on the L^2 norm for ∇v and ∇T.

4.1. L^2 Estimates for ∇v and ∇T

We take the inner product of Equation (12) with v in $\Omega\times(0,t)$, and using the integral by parts and the initial boundary value conditions (5)–(8), we obtain

$$\begin{aligned} \int_\Omega v^2dx= & \int_\Omega v_0{}^2dx-2\int_0^t\int_\Omega[(v\cdot\nabla)v-(\int_0^{x_3}\nabla\cdot v(x_1,x_2,\xi,t)d\xi)\partial_{x_3}v]vdxd\eta \\ & -2\int_0^t\int_\Omega fv^\perp\cdot vdxd\eta+2\int_0^t\int_\Omega(\int_0^{x_3}\nabla T(x_1,x_2,\xi,t)d\xi)\cdot vdxd\eta \\ & -2\int_0^t\int_\Omega\nabla p_0\cdot vdxd\eta+2\int_0^t\int_\Omega\mu_1\nabla v\cdot vdxd\eta+2\int_0^t\int_\Omega\mu_2\partial_{x_3x_3}v\cdot vdxd\eta \\ \le & \int_\Omega v_0{}^2dx-2\mu_1\int_0^t\int_\Omega|\nabla v|^2dxd\eta-2\mu_2\int_0^t\int_\Omega|\partial_{x_3}v|^2dxd\eta \\ & +\tfrac{1}{2k_1}\int_0^t\int_\Omega v^2dxd\eta+2k_1\int_0^t\int_\Omega|\nabla T|^2dxd\eta. \end{aligned} \tag{34}$$

Similarly, we take the inner product of Equation (13) with T in $\Omega\times(0,t)$, and using integral by-parts and initial boundary value conditions (5)–(8), we obtain

$$\int_\Omega T^2dx=\int_\Omega T_0{}^2dx-2k_1\int_0^t\int_\Omega|\nabla T|^2dxd\eta-2k_2\int_0^t\int_\Omega|\partial_{x_3}T|^2dxd\eta-2k_2\lambda\int_0^t\int_{\Gamma_1}T^2dAd\eta. \tag{35}$$

It is easy to determine from Equation (34) that

$$\int_\Omega T^2dx, 2k_1\int_0^t\int_\Omega|\nabla T|^2dxd\eta, 2k_2\int_0^t\int_\Omega|\partial_{x_3}T|^2dxd\eta, 2k_2\lambda\int_0^t\int_{\Gamma_1}T^2dAd\eta\le\int_\Omega {T_0}^2dx := m_1. \tag{36}$$

Combining Equation (35), from Equation (33), we obtain

$$\int_\Omega v^2dx + 2\mu_1\int_0^t\int_\Omega|\nabla v|^2dxd\eta + 2\mu_2\int_0^t\int_\Omega|\partial_{x_3}v|^2dxd\eta\le\frac{1}{2k_1}\int_0^t\int_\Omega v^2dxd\eta + \int_\Omega v_0^2dx + m_1.$$

Finally, by Gronwall inequality, we can obtain

$$\int_\Omega v^2dx + 2\mu_1\int_0^t\int_\Omega|\nabla v|^2dxd\eta + 2\mu_2\int_0^t\int_\Omega|\partial_{x_3}v|^2dxd\eta \le m_2(t), \tag{37}$$

where $m_2(t) = [\int_\Omega {v_0}^2dx + m_1]e^{\frac{1}{2k_1}t}$.

4.2. L^4 Estimates for $\partial_{x_3}v$ and $\partial_{x_3}T$

Lemma 3, which follows, will be used frequently in this section.

Lemma 3. (see [37]). *Assuming that Ω is a bounded convex region, then*

$$\left(\int_0^t\int_\Omega u^4dxd\eta\right)^{\frac{1}{2}} \le k[(1+\frac{1}{4}\delta)\int_0^t\int_\Omega u^2dxd\eta + \frac{3}{4}\delta^{-\frac{1}{3}}\int_0^t\int_\Omega|\nabla u|^2dxd\eta],$$

where k is a constant greater than zero, and δ is an arbitrary constant greater than zero.

By using Lemma 3 ($\delta = 1$), we can obtain

$$\left(\int_0^t\int_\Omega|\partial_{x_3}v|^4dxd\eta\right)^{\frac{1}{2}} \le k[\frac{5}{4}\int_0^t\int_\Omega|\partial_{x_3}v|^2dxd\eta + \frac{3}{4}\int_0^t\int_\Omega|\nabla\partial_{x_3}v|^2dxd\eta], \tag{38}$$

$$\left(\int_0^t\int_\Omega|\partial_{x_3}T|^4dxd\eta\right)^{\frac{1}{2}} \le k[\frac{5}{4}\int_0^t\int_\Omega|\partial_{x_3}T|^2dxd\eta + \frac{3}{4}\int_0^t\int_\Omega|\nabla\partial_{x_3}T|^2dxd\eta], \tag{39}$$

It can be seen from Equations (37) and (38) that we must first derive L^2 estimates for them in order to obtain L^4 estimates for $\partial_{x_3}v$ and $\partial_{x_3}T$. Firstly, taking the derivative of Equation (12), we have

$$\begin{aligned}\partial_{x_3}v_t + (\partial_{x_3}v\cdot\nabla)v - \nabla\cdot v\partial_{x_3}v + (v\cdot\nabla)\partial_{x_3}v - (\textstyle\int_0^{x_3}\nabla\cdot v(x_1,x_2,\xi,t)d\xi)\partial_{x_3x_3}v\\ +f\partial_{x_3}v^\perp - \nabla T - \mu_1\Delta\partial_{x_3}v - \mu_2\partial_{x_3x_3x_3}v = 0.\end{aligned} \tag{40}$$

Taking the inner product of Equation (39) and $\partial_{x_3}v$ on $\Omega\times(0,t)$, and discarding zero items, we have

$$\begin{aligned}\textstyle\int_\Omega|\partial_{x_3}v|^2dx = \int_\Omega|\partial_{x_3}v_0|^2dx - 2\int_0^t\int_\Omega[(\partial_{x_3}v\cdot\nabla)v - \nabla\cdot v\partial_{x_3}v]\partial_{x_3}vdxd\eta\\ \textstyle+2\int_0^t\int_\Omega\nabla T\cdot\partial_{x_3}vdxd\eta - 2\mu_1\int_0^t\int_\Omega|\nabla\partial_{x_3}v|^2dxd\eta - 2\mu_2\int_0^t\int_\Omega|\partial_{x_3x_3}v|^2dxd\eta.\end{aligned} \tag{41}$$

Using the Hölder inequality, Lemma 3, and Equation (36), we can obtain

$$\begin{aligned}\textstyle-2\int_0^t\int_\Omega[(\partial_{x_3}v\cdot\nabla)v - \nabla\cdot v\partial_{x_3}v]\partial_{x_3}vdxd\eta \le 4(\int_0^t\int_\Omega|\nabla v|^2dxd\eta)^{\frac{1}{2}}(\int_0^t\int_\Omega|\partial_{x_3}v|^4dxd\eta)^{\frac{1}{2}}\\ \textstyle\le 4k\sqrt{\frac{m_2(t)}{2\mu_1}}\left[(1+\frac{1}{4}\delta_1)\int_0^t\int_\Omega|\partial_{x_3}v|^2dxd\eta + \frac{3}{4}{\delta_1}^{-\frac{1}{3}}\int_0^t\int_\Omega|\nabla\partial_{x_3}v|^2dxd\eta\right]\\ \textstyle\le 4k\sqrt{\frac{m_2(t)}{2\mu_1}}\left[(1+\frac{1}{4}\delta_1)\frac{m_2(t)}{2\mu_2} + \frac{3}{4}{\delta_1}^{-\frac{1}{3}}\int_0^t\int_\Omega|\nabla\partial_{x_3}v|^2dxd\eta\right],\end{aligned} \tag{42}$$

where δ_1 is an arbitrary constant greater than zero. Similarly, we can obtain

$$2\int_0^t\int_\Omega \nabla T\cdot\partial_{x_3}v dxd\eta \le 2(\int_0^t\int_\Omega |\nabla T|^2dxd\eta)^{\frac{1}{2}}(\int_0^t\int_\Omega |\partial_{x_3}v|^2dxd\eta)^{\frac{1}{2}} \le \sqrt{\frac{m_1m_2(t)}{k_1\mu_2}}. \quad (43)$$

Inserting (41) and (42) into (40), and choosing the appropriate δ_1 so that

$$3k\sqrt{\frac{m_2(t)}{2\mu_1}}\delta_1{}^{-\frac{1}{3}} = \mu_1.$$

we have

$$\begin{aligned} &\int_\Omega |\partial_{x_3}v|^2dx + \mu_1\int_0^t\int_\Omega |\nabla\partial_{x_3}v|^2dxd\eta + 2\mu_2\int_0^t\int_\Omega |\partial_{x_3x_3}v|^2dxd\eta \\ &\le \int_\Omega |\partial_{x_3}v_0|^2dx + 4C\sqrt{\tfrac{m_2(t)}{2\mu_1}}(1+\tfrac{1}{4}\delta_1)\tfrac{m_2(t)}{2\mu_2} + \sqrt{\tfrac{m_1m_2(t)}{k_1\mu_2}} := m_3(t). \end{aligned} \quad (44)$$

Similarly, from Equation (13), we have

$$\begin{aligned} &\int_\Omega |\partial_{x_3}T|^2dx = \int_\Omega |\partial_{x_3}T_0|^2dx - 2\int_0^t\int_\Omega [(\partial_{x_3}v\cdot\nabla T - \nabla\cdot v\partial_{x_3}T]\partial_{x_3}Tdxd\eta \\ &-2k_1\int_0^t\int_\Omega|\nabla\partial_{x_3}T|^2dxd\eta - 2k_2\int_0^t\int_\Omega|\partial_{x_3x_3}T|^2dxd\eta + 2k_2\int_0^t\int_{\Gamma_1}\partial_{x_3x_3}T\partial_{x_3}TdAd\eta \end{aligned} \quad (45)$$

Applying the Hölder inequality, Lemma 3 ($\delta = 1$), Equations (35), (36) and (43), we can obtain

$$\begin{aligned} &-2\int_0^t\int_\Omega [\partial_{x_3}v\cdot\nabla T - \nabla\cdot v\partial_{x_3}T]\partial_{x_3}Tdxd\eta \\ &\le 2(\int_0^t\int_\Omega |\nabla T|^2dxd\eta)^{\frac{1}{2}}(\int_0^t\int_\Omega |\partial_{x_3}v|^4dxd\eta)^{\frac{1}{4}}(\int_0^t\int_\Omega |\partial_{x_3}T|^4dxd\eta)^{\frac{1}{4}} \\ &+2(\int_0^t\int_\Omega |\nabla v|^2dxd\eta)^{\frac{1}{2}}(\int_0^t\int_\Omega |\partial_{x_3}T|^4dxd\eta)^{\frac{1}{2}} \\ &\le k\sqrt{\tfrac{m_1}{2k_1}}\left[\tfrac{5}{4}\int_0^t\int_\Omega |\partial_{x_3}v|^2dxd\eta + \tfrac{3}{4}\int_0^t\int_\Omega |\nabla\partial_{x_3}v|^2dxd\eta\right] \\ &+k\sqrt{\tfrac{m_1}{2k_1}}\left[(1+\tfrac{1}{4}\delta_2)\int_0^t\int_\Omega |\partial_{x_3}T|^2dxd\eta + \tfrac{3}{4}\delta_2^{-\frac{1}{3}}\int_0^t\int_\Omega |\nabla\partial_{x_3}T|^2dxd\eta\right] \\ &+2k\sqrt{\tfrac{m_2(t)}{2\mu_1}}\left[(1+\tfrac{1}{4}\delta_3)\int_0^t\int_\Omega |\partial_{x_3}T|^2dxd\eta + \tfrac{3}{4}\delta_3^{-\frac{1}{3}}\int_0^t\int_\Omega |\nabla\partial_{x_3}T|^2dxd\eta\right] \\ &\le k\sqrt{\tfrac{m_1}{2k_1}}\left[(\tfrac{5}{4}\tfrac{m_2(t)}{2\mu_2} + \tfrac{3}{4}\tfrac{m_3(t)}{2\mu_1}\right] + k\sqrt{\tfrac{m_1}{2k_1}}\left[(1+\tfrac{1}{4}\delta_2)\tfrac{m_1}{2k_2} + \tfrac{3}{4}\delta_2^{-\frac{1}{3}}\int_0^t\int_\Omega |\nabla\partial_{x_3}T|^2dxd\eta\right] \\ &+2k\sqrt{\tfrac{m_2(t)}{2\mu_1}}\left[(1+\tfrac{1}{4}\delta_3)\tfrac{m_1}{2k_2} + \tfrac{3}{4}\delta_3^{-\frac{1}{3}}\int_0^t\int_\Omega |\nabla\partial_{x_3}T|^2dxd\eta\right], \end{aligned} \quad (46)$$

where δ_2 and δ_3 are arbitrary constants greater than zero. Now, we can deal with the last term of Equation (44). Taking $x_3 = 0$ in Equation (13) and using boundary conditions, we have

$$\begin{aligned} 2k_2\int_0^t\int_{\Gamma_1}\partial_{x_3x_3}T\partial_{x_3}TdAd\eta &= -2\lambda k_2\int_0^t\int_{\Gamma_1}\partial_{x_3x_3}TTdAd\eta \\ &= -2\lambda\int_0^t\int_{\Gamma_1}[T_\eta + v\cdot\nabla T - k_1\Delta T]TdAd\eta \\ &= -2\lambda\int_{\Gamma_1}T^2dA + 2\lambda\int_{\Gamma_1}T_0{}^2dA + \lambda\int_0^t\int_{\Gamma_1}\nabla\cdot vT^2dAd\eta - 2\lambda\int_0^t\int_{\Gamma_1}|\nabla T|^2dAd\eta. \end{aligned} \quad (47)$$

However,

$$\lambda\int_0^t\int_{\Gamma_1}\nabla\cdot vT^2dAd\eta = \lambda\int_0^t\int_{\Gamma_1}T^2[\int_{x_3}^1\nabla\cdot\partial_\xi v(x_1,x_2,\xi)d\xi - \nabla\cdot v]dAd\eta. \quad (48)$$

Integrating Equation (47) from 0 to 1, and by using the Hölder inequality, the arithmetic–geometric mean inequality, and Equation (35), we have

$$\begin{aligned}\lambda\int_0^t\int_{\Gamma_1}\nabla\cdot vT^2dAd\eta &\le \lambda[\int_0^t\int_\Omega|\nabla v|^2dxd\eta+\int_0^t\int_\Omega|\nabla\partial_{x_3}v|^2dxd\eta](\int_0^t\int_{\Gamma_1}T^4dAd\eta)^{\frac{1}{2}}\\ &\le \lambda k[\tfrac{m_1}{2\mu_1}+\tfrac{m_3(t)}{\mu_1}][(1+\tfrac{1}{4}\delta_4)\int_0^t\int_{\Gamma_1}T^2dAd\eta+\tfrac{3}{4}\delta_4^{-\frac{1}{3}}\int_0^t\int_{\Gamma_1}|\nabla T|^2dAd\eta]\\ &\le k[\tfrac{m_1}{2\mu_1}+\tfrac{m_3(t)}{\mu_1}](1+\tfrac{1}{4}\delta_4)\tfrac{m_1}{2k_2}+\tfrac{3}{4}\delta_4^{-\frac{1}{3}}k[\tfrac{m_1}{2\mu_1}+\tfrac{m_3(t)}{\mu_1}]\lambda\int_0^t\int_{\Gamma_1}|\nabla T|^2dAd\eta,\end{aligned}\tag{49}$$

choosing the appropriate δ_4 in (48), so that

$$\frac{3}{4}\delta_4^{-\frac{1}{3}}k[\frac{m_1}{2\mu_1}+\frac{m_3(t)}{\mu_1}]=2,\tag{50}$$

inserting (49) into (46), we can obtain

$$2k_2\int_0^t\int_{\Gamma_1}\partial_{x_3x_3}T\partial_{x_3}TdAd\eta\le 2\lambda\int_{\Gamma_1}T_0^2dA+k[\frac{m_1}{2\mu_1}+\frac{m_3(t)}{\mu_1}](1+\frac{1}{4}\delta_4)\frac{m_1}{2k_2}\tag{51}$$

Choosing appropriate δ_2 and δ_3 in (50), so that

$$k\sqrt{\frac{m_1}{2k_1}}\frac{3}{4}\delta_2^{-\frac{1}{3}}=2k\sqrt{\frac{m_2(t)}{2\mu_1}}\frac{3}{4}\delta_3^{-\frac{1}{3}}=\frac{1}{2}k_1,\tag{52}$$

then, inserting Equations (50) and (51) into Equation (44), we can obtain

$$\int_\Omega|\partial_{x_3}T|^2dx+k_1\int_0^t\int_\Omega|\nabla\partial_{x_3}T|^2dxd\eta+2k_2\int_0^t\int_\Omega|\partial_{x_3x_3}T|^2dxd\eta\le m_4(t),\tag{53}$$

where

$$\begin{aligned}m_4(t)=2\lambda\int_{\Gamma_1}T_0^2dA+\int_\Omega|\partial_{x_3}T_0|^2dx+k\sqrt{\tfrac{m_1}{2k_1}}[\tfrac{5}{4}\tfrac{m_2(t)}{2\mu_2}+\tfrac{3}{4}\tfrac{m_3(t)}{2\mu_1}]+k\sqrt{\tfrac{m_1}{2k_1}}[1+\tfrac{1}{4}\delta_2]\tfrac{m_1}{2k_2}\\ +2k\sqrt{\tfrac{m_2(t)}{2\mu_1}}(1+\tfrac{1}{4}\delta_3)\tfrac{m_1}{2k_2}+k[\tfrac{m_1}{2\mu_1}+\tfrac{m_3(t)}{\mu_1}](1+\tfrac{1}{4}\delta_4)\tfrac{m_1}{2k_2}.\end{aligned}$$

Inserting (44) and (45) into (37) and (38) and combining (35) and (36), we have

$$(\int_0^t\int_\Omega|\partial_{x_3}v|^4dxd\eta)^{\frac{1}{2}}\le k[\frac{5}{4}\frac{m_2(t)}{2\mu_2}+\frac{3}{4}\frac{m_3(t)}{\mu_1}],(\int_0^t\int_\Omega|\partial_{x_3}T|^4dxd\eta)^{\frac{1}{2}}\le k[\frac{5}{4}\frac{m_1}{2k_2}+\frac{3}{4}\frac{m_4(t)}{k_1}].$$

4.3. Estimate of $\int_0^t\int_{\Gamma_1}(T*)^2dAd\eta$

From the second section, when λ = 0, under the same initial condition (8) as (v, T, p_0), (v^*, T^*, p_0^*) is another solution of Equations (12) and (13) with boundary conditions (5)–(7); so, taking $x_3=0$ and using the boundary conditions, Equation (13) can be written as

$$T_t^*+v\cdot\nabla T^*-k_1\Delta T^*-k_2\partial_{x_3x_3}T^*=0,$$

Then,

$$\begin{aligned}\int_{\Gamma_1}(T^*)^2dA+2k_1\int_0^t\int_{\Gamma_1}|\nabla T^*|^2dAd\eta+2k_2\int_0^t\int_{\Gamma_1}|\partial_{x_3}T^*|^2dAd\eta\\ =\int_{\Gamma_1}(T_0)^2dA+\int_0^t\int_{\Gamma_1}\nabla\cdot v(T^*)^2dAd\eta.\end{aligned}\tag{54}$$

Similar to the derivation of (47), we can obtain

$$\begin{aligned}\int_0^t\int_{\Gamma_1}\nabla\cdot v(T^*)^2dAdx&\le[\int_0^t\int_\Omega|\nabla v|^2dxd\eta+\int_0^t\int_\Omega|\nabla\partial_{x_3}v|^2dxd\eta](\int_0^t\int_{\Gamma_1}(T^*)^4dAd\eta)^{\frac{1}{2}}\\ &\le k[\tfrac{m_1}{2\mu_1}+\tfrac{m_3(t)}{\mu_1}][(1+\tfrac{1}{4}\delta_5)\int_0^t\int_{\Gamma_1}(T^*)^2dAd\eta+\tfrac{3}{4}\delta_5^{-\frac{1}{3}}\int_0^t\int_{\Gamma_1}|\nabla T^*|^2dAd\eta].\end{aligned}\tag{55}$$

Choosing δ_5 in (54), so that $\frac{3}{4}k[\frac{m_1}{2\mu_1}+\frac{m_3(t)}{\mu_1}]\delta_5^{-\frac{1}{3}}=2k_1$, Equation (53) can be written as

$$\int_{\Gamma_1}(T^*)^2dA \le \int_{\Gamma_1}(T_0)^2dA + k[\frac{m_1}{2\mu_1}+\frac{m_3(t)}{\mu_1}](1+\frac{1}{4}\delta_5)\int_0^t\int_{\Gamma_1}(T^*)^2dAd\eta. \quad (56)$$

Integrating (55) from 0 to t, we have

$$\int_0^t\int_{\Gamma_1}(T^*)^2dAd\eta \le \int_{\Gamma_1}(T_0)^2dA\cdot\int_0^t \exp\left\{\int_s^t C[\frac{m_1}{2\mu_1}+\frac{m_3(\eta)}{\mu_1}](1+\frac{1}{4}\delta_5)d\eta\right\}ds.$$

5. Conclusions

In this paper, firstly, we obtained estimates on the L^2 norm of ∇v and ∇T of the solutions of the 3D viscous primitive equations by using the differential inequality technology and method of the energy estimates. Next, we obtained estimates on the L^4 norm of $\partial_{x_3}v$ and $\partial_{x_3}T$ of them by using the same method. Under the assumption of some boundary conditions and initial conditions, we also obtained the estimate of $\int_0^t\int_{\Gamma_1}(T^*)^2dAd\eta$. Finally, the convergence of the equations on the boundary parameter was proved. At present, this kind of research has not appeared in the literature, and it can also extend to primitive equations with random forces, ocean primitive equations, atmosphere primitive equations, coupled oceanic and atmosphere primitive equations, and even dry atmosphere primitive equations. We hope that the research in this paper can bring some inspiration to the next research project, which is also a key direction of our forthcoming research.

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