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Closed-Loop Solvability of Stochastic Linear-Quadratic Optimal Control Problems with Poisson Jumps

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Abstract: The stochastic linear–quadratic optimal control problem with Poisson jumps is addressed in this paper. The coefficients in the state equation and the weighting matrices in the cost functional are all deterministic but are allowed to be indefinite. The notion of closed-loop strategies is introduced, and the sufficient and necessary conditions for the closed-loop solvability are given. The optimal closed-loop strategy is characterized by a Riccati integral–differential equation and a backward stochastic differential equation with Poisson jumps. A simple example is given to demonstrate the effectiveness of the main result.

Keywords: stochastic linear–quadratic optimal control; Poisson random measure; backward stochastic differential equation with Poisson jumps; Riccati integral–differential equation; closed-loop solvability

MSC: 49N10; 49K45; 60H10; 93E20

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1. Introduction

The stochastic linear–quadratic (SLQ) optimal control problem plays an extremely important role in modern control theory and methodology, because of its elegant structure of solutions and wide applications in engineering, finance, networks, etc. More importantly, SLQ optimal control problems can also reasonably approximate some nonlinear stochastic optimal control problems. In the literature for SLQ optimal control problems, refer to Wonham [1], Bismut [2], Bensoussan [3], Peng [4], Chen et al. [5], Chen and Zhou [6], Chen and Yong [7], Ait Rami et al. [8], Tang [9], Yu [10], Tang [11], Sun et al. [12], Sun and Yong [13], and Sun et al. [14], for journal papers and Davis [15], Anderson and Moore [16], Yong and Zhou [17], and Sun and Yong [18] for monographs.

In the above work, stochastic systems are modelled by Brownian motions. However, in reality, Brownian noises are usually inadequate in a mathematical modeling sense. For example, it is particularly appropriate to use stochastic systems with Poisson jumps or Lévy jumps to describe the large fluctuations in the stock market (Merton [19], Kou [20], Cont and Tankov [21], Oksendal and Sulem [22], Lim [23], Hanson [24]). Moreover, from a mathematical point of view, there exist essential differences between stochastic systems with and without jumps.

SLQ optimal control problems with Poisson jumps (SLQP optimal control problems) are also researched by many authors. Tang and Hou [25] studied an optimal control problem of partially observed linear–quadratic stochastic systems with a Poisson process and obtained an explicit solution of this problem by the partially observed maximum principle. Wu and Wang [26] studied a kind of SLQP optimal control problem, the explicit form of optimal controls is obtained by the solutions to a forward–backward stochastic differential equation with Poisson jumps (FBSDEP) and a generalized Riccati equation system. Hu and Oksendal [27] studied an SLQP optimal control problem with partial information. Meng [28] considered an SLQP optimal control problem with random coefficients. The state feedback representation was obtained for the open-loop optimal control by a matrix-valued backward stochastic Riccati equations with jumps (BSREJ), and the solvability of it in a special case was discussed. The solvability of BSREJ in the general case was studied in Zhang et al. [29]. Note that Li et al. [30] gave the concept of relax compensator, which is used to describe indefinite BSREJ, then they investigated the solvability of BSREJ and gave the optimal control. Moon and Chung [31] studied the indefinite SLQP optimal control problem with random coefficients by a completion of squares approach.

Our interest in this paper is the closed-loop solvability of the SLQP optimal control problem, which, as far as we know, is not researched in the literature. In 2014, the notions of open-loop and closed-loop solvabilities for SLQ optimal control problems were introduced in Sun and Yong [32], where they concentrated on the LQ zero-sum stochastic differential game, in which SLQ optimal control problem is a special case when there is only one player/controller is considered. Sun et al. [12] further gave more detailed necessary and sufficient conditions of the open-loop and closed-loop solvability for SLQ optimal control problems. Sun and Yong [13] studied the open-loop and closed-loop solvability of SLQ optimal control problems in the infinite horizon and showed that open-loop and closed-loop solvabilities are equivalent in the infinite horizon. For more details and complete content, please also refer to their book [18]. Li et al. [33] studied the SLQ optimal control problem of mean-field type and gave the characterization of the closed-loop optimal strategy. Lv [34,35] researched the closed-loop solvabilities of SLQ optimal control problems for systems governed by stochastic evolution equations (SEEs) and SEEs of mean-field type, respectively. Tang et al. [36] studied the open-loop and closed-loop solvability for indefinite SLQP optimal control problem of mean-field type and its application in finance. Sun et al. [14] considered the indefinite SLQ optimal control problem with random coefficients and investigated the closed-loop representation of open-loop optimal controls.

Our work differs from the existing results in the following respects. (1) We consider an SLQP optimal control problem with deterministic coefficients in a general framework (**problem (SLQP)** in Section 2), where the weighting matrices in the cost functional are allowed to be indefinite. Moreover, cross-product terms in the control and state processes are present in the cost functional. Non-homogenous terms also appear in the controlled state equation and cost functional.The model considered in this paper is a nontrivial generalization of those in [12,32]. (2) Characterization of the closed-loop solvability for the SLQP optimal control problem is obtained, via the Riccati integral–differential equation (RIDE). For the SLQ optimal control problem without Poisson jumps, Sun et al. [12] first found two matrix-valued SDE of $\mathbb{X}(\cdot)$, $\mathbb{Y}(\cdot)$, then they applied Itô's formula to find $\mathbb{X}^{-1}(\cdot)$. The solution to the related Riccati equation is defined as $P(\cdot) = \mathbb{Y}(\cdot)\mathbb{X}^{-1}(\cdot)$. However, this method fails in our SLQP optimal control problem, as the Poisson jumps appear in the controlled system and difficulty is encountered. In detail, when we take the inverse of the matrix-valued SDEP as that of the matrix-valued SDE in [12] and apply Itô–Wentzell's formula (Oksendal and Zhang [37]), terms such as $\left[F(e) + G(e)\bar{\Theta} + 1\right]^{-1}$ will appear in $\mathbb{X}^{-1}(\cdot)$. Because we do not have any restrictions on the coefficients in our system and $\bar{\Theta}(\cdot)$ is the closed-loop optimal strategy that we are going to seek, there is no reason to arbitrarily presume that $F(e) + G(e)\bar{\Theta} + 1$ is invertible. From Lv [34,35], we overcome this difficulty by transforming the original problem (SLQP) into a problem of solving the open-loop optimal control of **problem (SLQP)**$^0_{\bar{\Theta}}$ in Section 2. Thus, a Lyapunov integral–differential Equation (25) is given first and then the RIDE (29) is obtained. Note that the technique used in this paper is also different from that in Tang et al. [36], where a matrix minimum principle by Athans [38] was used when dealing with the SLQP problem of mean-field type.

The rest of this paper is organized as follows. Section 2 begins with the preparation work, including giving some basic knowledge and presenting the formulation of the SLQP optimal control problem. In Section 3, characterizations of closed-loop solvability of SLQP optimal control problems are presented and the concrete proofs are given. Section 4 gives an example to demonstrate the effectiveness of the main result. Finally, in Section 5, some concluding remarks are given.

2. Problem Formulation and Preliminaries

First, let us introduce some notations that will be used throughout this paper.

Let $T > 0$ be a constant, and $[0, T]$ is a finite time duration. Let $\mathbb{R}^{n\times m}$ be the collection of all $(n \times m)$ matrices, and $\mathbb{S}^{n\times n}$ be the collection of all $(n \times n)$ symmetric matrices. We let I be the identity matrix with a suitable size. We use $\langle \cdot, \cdot \rangle$ to denote inner products in possibly different Hilbert spaces and $|\cdot|$ to denote the norm induced by these inner products. Let $M^\top$ and $\mathcal{R}(M)$ be the transpose and range of a matrix M, respectively. For $M, N \in \mathbb{S}^{n\times n}$, $M \geqslant N$ (respectively, $M > N$) implies that $M - N$ is a positive semi-definite matrix (respectively, positive definite matrix). Let $M^\dagger$ denote the pseudo-inverse of a matrix $M \in \mathbb{R}^{m\times n}$. If the inverse M^{-1} of $M \in \mathbb{R}^{n\times n}$ exists, then the pseudo-inverse is equal to the inverse. See Penrose [39] for the definition and some basic properties of the pseudo-inverse.

For any Banach space H (for example, $H = \mathbb{R}^n, \mathbb{R}^{n\times m}, \mathbb{S}^{n\times n}$) and $t \in [0, T)$, let $L^p(t, T; H)$ $(1 \leqslant p \leqslant \infty)$ be the space of all L^p-integrable functions valued from H on $[t, T]$, $C([t, T]; H)$ be the space of all continuous functions valued from H on $[t, T]$, and $L^\infty(t, T; H)$ be the space of Lebesgue measurable, essentially bounded functions from $[t, T]$ into H.

Let $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ be a completed filtered probability space, where $\mathbb{F} = \{\mathcal{F}_t\}_{t\geqslant 0}$ is filtration generated by the following two mutually independent stochastic processes and augmented by all the $\mathbb{P}$-null sets in $\mathcal{F}$:

- A standard one-dimensional Brownian motion $W = \{W(t); 0 \leqslant t < \infty\}$.
- A Poisson random measure N defined on $\mathbf{E} \times \mathbf{R}_+$, where $\mathbf{E} \subset \mathbf{R}^l$ is a nonempty open set and its Borel field is $\mathcal{B}(\mathbf{E})$. The compensator of N is $\hat{N}(dedt) = \pi(de)dt$, satisfying $\pi(A) < \infty$, such that $\tilde{N}(A \times [0, t]) = (N - \hat{N})(A \times [0, t])_{t\geqslant 0}$ is a martingale for any $A \in \mathcal{B}(\mathbf{E})$; π is assumed to be a σ-finite measure on $(\mathbf{E}, \mathcal{B}(\mathbf{E}))$ and is called the characteristic measure.

For $t \in [0, T)$, we introduce some notation for spaces of random variables and stochastic processes:

$$L^2_{\mathcal{F}_s}(\Omega; H) = \Big\{ \xi : \Omega \to H \,\big|\, \xi \text{ is } \mathcal{F}_s\text{-measurable random variable, } s.t.,\ \mathbb{E}|\xi|^2 < \infty \Big\},\ s \in [t, T],$$

$$L^2_{\mathbb{F}}(t, T; H) = \Big\{ f : [t, T] \times \Omega \to H \,\big|\, f \text{ is } \mathbb{F}\text{-progressively measurable, } s.t.,\ \mathbb{E}\int_t^T |f(s)|^2 ds < \infty \Big\},$$

$$L^2_{\mathbb{F},p}(t, T; H) = \Big\{ f : [t, T] \times \Omega \to H \,\big|\, f \text{ is } \mathbb{F}\text{-predictable, } s.t.,\ \mathbb{E}\int_t^T |f(s)|^2 ds < \infty \Big\},$$

$$F^\infty_p(t, T; H) = \Big\{ f : [t, T] \times \mathbf{E} \to H \,\big|\, f \text{ satisfies } \sup_{t\leqslant s\leqslant T,\, e\in\mathbf{E}} |f(s, e)| < \infty \Big\},$$

$$F^2_{\mathbb{F},p}(t, T; H) = \Big\{ f : \Omega \times [t, T] \times \mathbf{E} \to H \,\big|\, f \text{ is } \mathbb{F}\text{-predictable,} s.t.,\ \mathbb{E}\int_t^T \int_{\mathbf{E}} |f(\cdot, s, e)|^2 \pi(de) ds < \infty \Big\}.$$

We consider the following controlled linear SDEP on $[t, T]$:

$$\begin{cases} dX(s) = \big[A(s)X(s) + B(s)u(s) + b(s)\big]ds + \big[C(s)X(s) + D(s)u(s) + \sigma(s)\big]dW(s) \\ \qquad\qquad + \displaystyle\int_{\mathbf{E}} \big[F(s, e)X(s-) + G(s, e)u(s) + f(s, e)\big]\tilde{N}(deds), \quad s \in [t, T], \\ X(t) = x, \end{cases} \tag{1}$$

where $t \in [0, T)$ is the initial time and $x \in L^2_{\mathcal{F}_t}(\Omega; \mathbb{R}^n)$ is the given initial state; $A(\cdot)$, $B(\cdot)$, $C(\cdot)$, $D(\cdot)$ are given deterministic matrix-valued functions of proper dimensions, and $F(\cdot, \cdot), G(\cdot, \cdot)$ are independent of ω. The expressions $b(\cdot)$, $\sigma(\cdot)$ are $\mathbb{F}$-progressively measurable processes and $f(\cdot,\cdot)$ is also random; $u(\cdot)$ is the control process. We define the admissible control set:

$$\mathcal{U}[t,T]=\left\{u:[t,T]\times\Omega\to\mathbb{R}^m \;\middle|\; u \text{ is } \mathbb{F}\text{-predictable}\,,\ \mathbb{E}\int_t^T|u(s)|^2ds<\infty\right\}. \tag{2}$$

The control process $u(\cdot)\in\mathcal{U}[t,T]$ is called an admissible control.

Then we define the cost functional:

$$\begin{aligned} J(t,x;u(\cdot))=\mathbb{E}\bigg\{\int_t^T\Big[&\big\langle Q(s)X(s),X(s)\big\rangle+2\big\langle S(s)X(s),u(s)\big\rangle+\big\langle R(s)u(s),u(s)\big\rangle\\ &+2\big\langle q(s),X(s)\big\rangle+2\big\langle \rho(s),u(s)\big\rangle\Big]ds+\big\langle HX(T),X(T)\big\rangle+2\big\langle g,X(T)\big\rangle\bigg\}, \end{aligned} \tag{3}$$

where H is a symmetric matrix and $Q(\cdot),S(\cdot)$, and $R(\cdot)$ are deterministic matrix-valued functions of proper dimensions that satisfy $Q(\cdot)^\top=Q(\cdot),\ R(\cdot)^\top=R(\cdot)$. The expression g is an $\mathcal{F}_T$-measurable random variable, $q(\cdot)$ is an $\mathbb{F}$-progressively measurable process, and $\rho(\cdot)$ is an $\mathbb{F}$-predictable process.

In order to make the given (1) and (3) meaningful, we adopt some assumptions for coefficients as follows.

Hypothesis 1. *The coefficients of the state Equation (1) satisfy the following:*

$$\begin{cases} A(\cdot)\in L^\infty(t,T;\mathbb{R}^{n\times n}),\ B(\cdot)\in L^\infty(t,T;\mathbb{R}^{n\times m}),\ b(\cdot)\in L^2_{\mathbb{F}}(t,T;\mathbb{R}^n),\\ C(\cdot)\in L^\infty(t,T;\mathbb{R}^{n\times n}),\ D(\cdot)\in L^\infty(t,T;\mathbb{R}^{n\times m}),\ \sigma(\cdot)\in L^2_{\mathbb{F}}(t,T;\mathbb{R}^n),\\ F(\cdot,\cdot)\in F^\infty_p(t,T;\mathbb{R}^{n\times n}),\ G(\cdot,\cdot)\in F^\infty_p(t,T;\mathbb{R}^{n\times m}),\ f(\cdot,\cdot)\in F^2_{\mathbb{F},p}(t,T;\mathbb{R}^n). \end{cases}$$

Hypothesis 2. *The weighting coefficients of the cost functional (3) satisfy the following:*

$$\begin{cases} Q(\cdot)\in L^\infty(t,T;\mathbb{S}^{n\times n}),\ S(\cdot)\in L^\infty(t,T;\mathbb{R}^{m\times n}),\ R(\cdot)\in L^\infty(t,T;\mathbb{S}^{m\times m}),\\ q(\cdot)\in L^2_{\mathbb{F}}(t,T;\mathbb{R}^n),\ \rho(\cdot)\in L^2_{\mathbb{F},p}(t,T;\mathbb{R}^m),\ g\in L^2_{\mathcal{F}_T}(\Omega;\mathbb{R}^n),\ H\in\mathbb{S}^{n\times n}. \end{cases}$$

For simplicity, we denote the above Hypothesis 1 and Hypothesis 2 as (H1) and (H2), respectively.

Under (H1) and (H2), for any given $x\in L^2_{\mathcal{F}_t}(\Omega;\mathbb{R}^n)$ and $u(\cdot)\in\mathcal{U}[t,T]\equiv L^2_{\mathbb{F},p}(t,T;\mathbb{R}^m)$, state Equation (1) admits a unique adapted solution $X(\cdot)\in L^2_{\mathbb{F}}(t,T;\mathbb{R}^n)$ and the cost functional is well-defined. Therefore, the following problem is meaningful.

Problem 1. ***(SLQP)***. *For given initial pair $(t,x)\in[0,T]\times L^2_{\mathcal{F}_t}(\Omega;\mathbb{R}^n)$, find a $\bar{u}(\cdot)\in\mathcal{U}[t,T]$ such that*

$$J(t,x;\bar{u}(\cdot))=\inf_{u(\cdot)\in\mathcal{U}[t,T]}J(t,x;u(\cdot))\equiv V(t,x). \tag{4}$$

Any $\bar{u}(\cdot)\in\mathcal{U}[0,T]$ satisfying (4) is called an open-loop optimal control of problem (SLQP) for (t,x), the corresponding $\bar{X}(\cdot)\equiv X(\cdot;t,x,\bar{u}(\cdot))$ is called an open-loop optimal state, and $(\bar{X}(\cdot),\bar{u}(\cdot))$ is called an open-loop optimal pair. The map $V(\cdot,\cdot)$ is called the value function of problem (SLQP).

In particular, when $b(\cdot)$, $\sigma(\cdot)$, $f(\cdot,\cdot)$, $q(\cdot)$, $\rho(\cdot)$, and g are all zero, we refer to the above problem as the problem (SLQP)^0.

Definition 1. *For given initial pair $(t,x)\in[0,T]\times L^2_{\mathcal{F}_t}(\Omega;\mathbb{R}^n)$, if there exists a (unique) $\bar{u}(\cdot)\in\mathcal{U}[t,T]$ such that (4) holds, then we say that problem (SLQP) is (uniquely) open-loop solvable for (t,x).*

Next, take $\Theta(\cdot) \in L^2(t,T;\mathbb{R}^{m\times n}) \equiv \mathcal{Q}[t,T]$ and $v(\cdot) \in \mathcal{U}[t,T]$. For given initial pair $(t,x) \in [0,T) \times L^2_{\mathcal{F}_t}(\Omega;\mathbb{R}^n)$, let us consider the following equation (some time variables are usually omitted):

$$\begin{cases} dX = \big[(A+B\Theta)X + Bv + b\big]ds + \big[(C+D\Theta)X + Dv + \sigma\big]dW \\ \qquad\qquad + \displaystyle\int_{\mathbf{E}} \big[(F(e)+G(e)\Theta)X_- + G(e)v + f(e)\big]\tilde{N}(deds), \\ X(t) = x, \end{cases} \tag{5}$$

which admits a unique solution $X(\cdot) \equiv X(\cdot;t,x,\Theta(\cdot),v(\cdot))$, depending on the $\Theta(\cdot)$ and $v(\cdot)$; $(\Theta(\cdot),v(\cdot)) \in \mathcal{Q}[t,T] \times \mathcal{U}[t,T]$ is called a closed-loop strategy and the above Equation (5) is called a closed-loop system of the original state Equation (1) under $(\Theta(\cdot),v(\cdot))$. We point out that $(\Theta(\cdot),v(\cdot))$ is independent of the initial state $x \in L^2_{\mathcal{F}_t}(\Omega;\mathbb{R}^n)$. With the above solution $X(\cdot)$, we define

$$\begin{aligned} J(t,x;\Theta(\cdot)X(\cdot)+v(\cdot)) &= \mathbb{E}\bigg\{\int_t^T \bigg[\bigg\langle \begin{pmatrix} Q & S^\top \\ S & R \end{pmatrix}\begin{pmatrix} X \\ \Theta X + v \end{pmatrix}, \begin{pmatrix} X \\ \Theta X + v \end{pmatrix}\bigg\rangle \\ &\quad + 2\bigg\langle \begin{pmatrix} q \\ \rho \end{pmatrix}, \begin{pmatrix} X \\ \Theta X + v \end{pmatrix}\bigg\rangle\bigg]ds + \big\langle HX(T),X(T)\big\rangle + 2\big\langle g,X(T)\big\rangle\bigg\} \\ &= \mathbb{E}\bigg\{\int_t^T \Big[\big\langle (Q + \Theta^\top S + S^\top\Theta + \Theta^\top R\Theta)X,X\big\rangle + 2\big\langle (S+R\Theta)X,v\big\rangle + \big\langle Rv,v\big\rangle \\ &\quad + 2\big\langle q + \Theta^\top\rho, X\big\rangle + 2\big\langle \rho,v\big\rangle\Big]ds + \big\langle HX(T),X(T)\big\rangle + 2\big\langle g,X(T)\big\rangle\bigg\}. \end{aligned} \tag{6}$$

We now introduce the following definition.

Definition 2. *If a closed-loop strategy pair $(\bar\Theta(\cdot),\bar v(\cdot)) \in \mathcal{Q}[t,T] \times \mathcal{U}[t,T]$ satisfies the following inequality*

$$\begin{aligned} J(t,x;\bar\Theta(\cdot)\bar X(\cdot)+\bar v(\cdot)) \leqslant J(t,x;\Theta(\cdot)X(\cdot)+v(\cdot)), \\ \forall x \in L^2_{\mathcal{F}_t}(\Omega;\mathbb{R}^n),\ \Theta(\cdot) \in \mathcal{Q}[t,T],\ v(\cdot) \in \mathcal{U}[t,T], \end{aligned} \tag{7}$$

where $\bar X(\cdot) \equiv X(\cdot;t,x,\bar\Theta(\cdot),\bar v(\cdot))$ on the left and $X(\cdot) \equiv X(\cdot;t,x,\Theta(\cdot),v(\cdot))$ on the right, then$(\bar\Theta(\cdot),\bar v(\cdot))$ is called a closed-loop optimal strategy of problem (SLQP) on $[t,T]$, and we say that problem (SLQP) is closed-loop solvable on $[t,T]$.

We emphasize that the pair $(\bar\Theta(\cdot),\bar v(\cdot))$ is required to be independent of the initial state $x \in L^2_{\mathcal{F}_t}(\Omega;\mathbb{R}^n)$. We have the following equivalence theorem.

Theorem 1. *Let (H1) and (H2) hold and let $(\bar\Theta(\cdot),\bar v(\cdot)) \in \mathcal{Q}[t,T] \times \mathcal{U}[t,T]$. Then the following statements are equivalent:*

(i) $(\bar\Theta(\cdot),\bar v(\cdot))$ is a closed-loop optimal strategy of problem (SLQP) on $[t,T]$.

(ii) For any given $x \in L^2_{\mathcal{F}_t}(\Omega;\mathbb{R}^n)$ and $v(\cdot) \in \mathcal{U}[t,T]$,

$$J(t,x;\bar\Theta(\cdot)\bar X(\cdot)+\bar v(\cdot)) \leqslant J(t,x;\bar\Theta(\cdot)X(\cdot)+v(\cdot)),$$

where $X(\cdot) \equiv X(\cdot;t,x,\bar\Theta(\cdot),v(\cdot))$ on the right.

(iii) For any given $x \in L^2_{\mathcal{F}_t}(\Omega;\mathbb{R}^n)$ and $u(\cdot) \in \mathcal{U}[t,T]$,

$$J(t,x;\bar\Theta(\cdot)\bar X(\cdot)+\bar v(\cdot)) \leqslant J(t,x;u(\cdot)). \tag{8}$$

Proof. (i) ⇒ (ii). From the definition of closed-loop optimal strategy, it can be proved. (ii) ⇒ (iii). For given $x \in L^2_{\mathcal{F}_t}(\Omega;\mathbb{R}^n)$ and $u(\cdot) \in \mathcal{U}[t,T]$, $X(\cdot)$ is the adapted solution to the following SDEP:

$$\begin{cases} dX(s) = \big[A(s)X(s) + B(s)u(s) + b(s)\big]ds + \big[C(s)X(s) + D(s)u(s) + \sigma(s)\big]dW(s) \\ \qquad\qquad + \int_{\mathbf{E}} \big[F(s,e)X(s-) + G(s,e)u(s) + f(s,e)\big]\tilde{N}(deds), \quad s \in [t,T], \\ X(t) = x. \end{cases}$$

Taking $v(\cdot) = u(\cdot) - \bar{\Theta}(\cdot)X(\cdot)$, it is easy to see that $v(\cdot) \in \mathcal{U}[t,T]$. Thus

$$u(\cdot) = \bar{\Theta}(\cdot)X(\cdot) + v(\cdot).$$

From the existence and uniqueness of the adapt solution to SDEP, we obtain that

$$X(\cdot) = X(\cdot;t,x,u(\cdot)) = X(\cdot;t,x,\bar{\Theta}(\cdot),v(\cdot)) \equiv \tilde{X}(\cdot).$$

Therefore, by (ii), one has

$$J(t,x;\bar{\Theta}(\cdot)\bar{X}(\cdot) + \bar{v}(\cdot)) \leqslant J(t,x;\bar{\Theta}(\cdot)\tilde{X}(\cdot) + v(\cdot)) = J(t,x;u(\cdot)),$$

proving (iii). (iii) ⇒ (i). For a given $x \in L^2_{\mathcal{F}_t}(\Omega;\mathbb{R}^n)$, $u(\cdot) \in \mathcal{U}[t,T]$ and $(\Theta(\cdot),v(\cdot)) \in \mathcal{Q}[t,T] \times \mathcal{U}[t,T]$, let $X(\cdot) \equiv X(\cdot;t,x,\Theta(\cdot),v(\cdot))$. Taking

$$u(\cdot) = \Theta(\cdot)X(\cdot) + v(\cdot),$$

from the existence and uniqueness of the adapt solution of SDEP, we know that

$$X(\cdot;t,x,\Theta(\cdot),v(\cdot)) = X(\cdot;t,x,u(\cdot)).$$

Therefore, by (iii), we have

$$J(t,x;\bar{\Theta}(\cdot)\bar{X}(\cdot) + \bar{v}(\cdot)) \leqslant J(t,x;u(\cdot)) = J(t,x;\Theta(\cdot)X(\cdot) + v(\cdot)).$$

□

For the closed-loop optimal strategy $(\bar{\Theta}(\cdot),\bar{v}(\cdot))$ of problem (SLQP) on $[t,T]$ and the corresponding closed-loop optimal state $\bar{X}(\cdot) = \bar{X}(\cdot;t,x,\bar{\Theta}(\cdot),\bar{v}(\cdot))$, we can define the outcome

$$\bar{u}(\cdot) = \bar{\Theta}(\cdot)\bar{X}(\cdot) + \bar{v}(\cdot) \in \mathcal{U}[t,T].$$

From the third part of Theorem 1, we see that for any given initial state $x \in L^2_{\mathcal{F}_t}(\Omega;\mathbb{R}^n)$ and $u(\cdot) \in \mathcal{U}[t,T]$, $\bar{u}(\cdot)$ is an open-loop optimal control of problem (SLQP) for x. Therefore, if problem (SLQP) is closed-loop solvable on $[t,T]$, it must also be open-loop solvable, and the outcome of the closed-loop optimal strategy is the open-loop optimal control for any $x \in L^2_{\mathcal{F}_t}(\Omega;\mathbb{R}^n)$.

The following result is concerned with open-loop solvability of problem (SLQP) for given initial state.

Proposition 1. *Let (H1) and (H2) hold. For given initial pair $(t,x) \in [0,T] \times L^2_{\mathcal{F}_t}(\Omega;\mathbb{R}^n)$, a control $\bar{u}(\cdot)$ is an open-loop optimal control of problem (SLQP) if and only if the following hold:*

(i) The stationarity condition holds:

$$B^\top \bar{Y} + D^\top \bar{Z} + \int_{\mathbf{E}} G(e)^\top \bar{K}(e)\pi(de) + S\bar{X} + R\bar{u} + \rho = 0, \quad \textit{a.e., } \mathbb{P}\textit{-a.s.}, \tag{9}$$

where $(\bar{X}(\cdot), \bar{Y}(\cdot), \bar{Z}(\cdot), \bar{K}(\cdot,\cdot)) \in L^2_{\mathbb{F}}(t,T;\mathbb{R}^n) \times L^2_{\mathbb{F}}(t,T;\mathbb{R}^n) \times L^2_{\mathbb{F},p}(t,T;\mathbb{R}^n) \times F^2_{\mathbb{F},p}(t,T;\mathbb{R}^n)$ *is the adapted solution to the following FBSDEP:*

$$\begin{cases} d\bar{X} = \big[A\bar{X} + B\bar{u} + b\big]ds + \big[C\bar{X} + D\bar{u} + \sigma\big]dW \\ \qquad + \int_{\mathbf{E}} \big[F(e)\bar{X}_- + G(e)\bar{u} + f(e)\big]\tilde{N}(deds), \\ d\bar{Y} = -\big[A^\top \bar{Y} + C^\top \bar{Z} + \int_{\mathbf{E}} F^\top(e)\bar{K}(e)\pi(de) + Q\bar{X} + S^\top \bar{u} + q\big]ds \\ \qquad + \bar{Z}dW + \int_{\mathbf{E}} \bar{K}(e)\tilde{N}(deds), \\ \bar{X}(t) = x, \quad \bar{Y}(T) = H\bar{X}(T) + g. \end{cases} \tag{10}$$

(ii) The convexity condition holds: For any $u(\cdot) \in \mathcal{U}[t,T]$,

$$\mathbb{E}\bigg\{\big\langle Hx_0(T), x_0(T)\big\rangle + \int_t^T \Big[\big\langle Qx_0, x_0\big\rangle + 2\big\langle Sx_0, u\big\rangle + \big\langle Ru, u\big\rangle\Big]ds\bigg\} \geqslant 0, \tag{11}$$

where $x_0(\cdot) \in L^2_{\mathbb{F}}(t,T;\mathbb{R}^n)$ *is the adapted solution to the following SDEP:*

$$\begin{cases} dx_0(s) = \big[A(s)x_0(s) + B(s)u(s)\big]ds + \big[C(s)x_0(s) + D(s)u(s)\big]dW(s) \\ \qquad + \int_{\mathbf{E}} \big[F(s,e)x_0(s-) + G(s,e)u(s)\big]\tilde{N}(deds), \quad s \in [t,T], \\ x_0(t) = 0. \end{cases} \tag{12}$$

Proof. For any $u(\cdot) \in \mathcal{U}[t,T]$ and $\epsilon \in \mathbb{R}$, let $u^\epsilon(\cdot) = \bar{u}(\cdot) + \epsilon u(\cdot)$, thus $X^\epsilon(\cdot) \equiv X(\cdot;t,x,\bar{u}(\cdot) + \epsilon u(\cdot))$ is the corresponding state that satisfies

$$\begin{cases} dX^\epsilon = \big[AX^\epsilon + B(\bar{u} + \epsilon u) + b\big]ds + \big[CX^\epsilon + D(\bar{u} + \epsilon u) + \sigma\big]dW \\ \qquad + \int_{\mathbf{E}} \big[F(e)X^\epsilon_- + G(e)(\bar{u} + \epsilon u) + f(e)\big]\tilde{N}(deds), \\ X^\epsilon(t) = x. \end{cases}$$

Then $\frac{X^\epsilon(\cdot) - X(\cdot)}{\epsilon}$ satisfies the following SDEP:

$$\begin{cases} d\Big(\dfrac{X^\epsilon - X}{\epsilon}\Big) = \Big[A\Big(\dfrac{X^\epsilon - X}{\epsilon}\Big) + Bu\Big]ds + \Big[C\Big(\dfrac{X^\epsilon - X}{\epsilon}\Big) + Du\Big]dW \\ \qquad + \int_{\mathbf{E}} \Big[F(e)\Big(\dfrac{X^\epsilon_- - X_-}{\epsilon}\Big) + G(e)u\Big]\tilde{N}(deds), \\ \dfrac{X^\epsilon - X}{\epsilon}(t) = 0. \end{cases}$$

From the existence and uniqueness of the solution to SDEP, we know that $x_0 \equiv \frac{X^\epsilon - X}{\epsilon}$. Then

$$\begin{aligned} &J(t,x;\bar{u}(\cdot) + \epsilon u(\cdot)) - J(t,x;\bar{u}(\cdot)) \\ &= 2\epsilon\mathbb{E}\bigg\{\big\langle H\bar{X}(T) + g, x_0(T)\big\rangle + \int_t^T \Big[\big\langle Qx_0, \bar{X}\big\rangle + \big\langle S\bar{X}, u\big\rangle \\ &\qquad + \big\langle Sx_0, \bar{u}\big\rangle + \big\langle R\bar{u}, u\big\rangle + \big\langle q, x_0\big\rangle + \big\langle \rho, u\big\rangle\Big]ds\bigg\} \\ &\quad + \epsilon^2\mathbb{E}\bigg\{\big\langle Hx_0(T), x_0(T)\big\rangle + \int_t^T \Big[\big\langle Qx_0, x_0\big\rangle + 2\big\langle Sx_0, u\big\rangle + \big\langle Ru, u\big\rangle\Big]ds\bigg\}. \end{aligned}$$

Applying Itô's formula to $\langle \bar{Y}(\cdot), x_0(\cdot)\rangle$, we get

$$\begin{aligned}
&J(t,x;\bar{u}(\cdot)+\epsilon u(\cdot)) - J(t,x;\bar{u}(\cdot))\\
&= 2\epsilon\mathbb{E}\int_t^T \Big\langle B^\top \bar{Y} + D^\top \bar{Z} + \int_{\mathbf{E}} G(e)^\top \bar{K}(e)\pi(de) + S\bar{X} + R\bar{u} + \rho, u\Big\rangle\, ds\\
&\quad + \epsilon^2\mathbb{E}\Big\{\big\langle Hx_0(T), x_0(T)\big\rangle + \int_t^T \Big[\big\langle Qx_0, x_0\big\rangle + 2\big\langle Sx_0, u\big\rangle + \big\langle Ru, u\big\rangle\Big] ds\Big\}.
\end{aligned}$$

Therefore, $(\bar{X}(\cdot), \bar{u}(\cdot))$ is an open-loop optimal pair of problem (SLQP) if and only if (9) and (11) hold. □

On the other hand, from the second part of Theorem 1, we can see that $(\bar{\Theta}(\cdot), \bar{v}(\cdot))$ being a closed-loop optimal strategy of problem (SLQP) is equivalent to $\bar{v}(\cdot)$ being an open-loop optimal control of the SLQP optimal control problems (5) and (6) with $\Theta(\cdot) = \bar{\Theta}(\cdot)$, which we denote by **problem (SLQP)**$_{\bar{\Theta}}$. In particular, when $b(\cdot)$, $\sigma(\cdot)$, $f(\cdot,\cdot)$, $q(\cdot)$, $\rho(\cdot)$ and g are all zero, we refer to it as problem (SLQP)$^0_{\bar{\Theta}}$. Similar to Proposition 1, we can give the following result.

Proposition 2. *Let (H1) and (H2) hold. For any given $x \in L^2_{\mathcal{F}_t}(\Omega;\mathbb{R}^n)$, $\bar{v}(\cdot)$ is an open-loop optimal control of problem (SLQP)$_{\bar{\Theta}}$ if and only if the following stationarity condition holds:*

$$B^\top \bar{Y} + D^\top \bar{Z} + \int_{\mathbf{E}} G(e)^\top \bar{K}(e)\pi(de) + (S + R\bar{\Theta})\bar{X} + R\bar{v} + \rho = 0, \quad \text{a.e., } \mathbb{P}\text{-a.s.}, \tag{13}$$

where $(\bar{X}(\cdot), \bar{Y}(\cdot), \bar{Z}(\cdot), \bar{K}(\cdot,\cdot)) \in L^2_{\mathbb{F}}(t,T;\mathbb{R}^n) \times L^2_{\mathbb{F}}(t,T;\mathbb{R}^n) \times L^2_{\mathbb{F},p}(t,T;\mathbb{R}^n) \times F^2_{\mathbb{F},p}(t,T;\mathbb{R}^n)$ is the adapted solution to the following FBSDEP:

$$\begin{cases}
d\bar{X} = \big[(A + B\bar{\Theta})\bar{X} + B\bar{v} + b\big]ds + \big[(C + D\bar{\Theta})\bar{X} + D\bar{v} + \sigma\big]dW\\
\qquad + \displaystyle\int_{\mathbf{E}} \big[(F(e) + G(e)\bar{\Theta})\bar{X}_- + G(e)\bar{v} + f(e)\big]\tilde{N}(deds),\\
d\bar{Y} = -\Big[(A + B\bar{\Theta})^\top \bar{Y} + (C + D\bar{\Theta})^\top \bar{Z} + \displaystyle\int_{\mathbf{E}} (F(e) + G(e)\bar{\Theta})^\top \bar{K}(e)\pi(de)\\
\qquad\quad + (Q + S^\top\bar{\Theta} + \bar{\Theta}^\top S + \bar{\Theta}^\top R\bar{\Theta})\bar{X} + (S + R\bar{\Theta})^\top \bar{v} + q + \bar{\Theta}^\top \rho\Big] ds\\
\qquad + \bar{Z}dW + \displaystyle\int_{\mathbf{E}} \bar{K}(e)\tilde{N}(deds),\\
\bar{X}(t) = x, \quad \bar{Y}(T) = H\bar{X}(T) + g,
\end{cases} \tag{14}$$

and the following convexity condition holds: For any $v(\cdot) \in \mathcal{U}[t,T]$,

$$\begin{aligned}
\mathbb{E}\Big\{\int_t^T \Big[&\big\langle (Q + \bar{\Theta}^\top S + S^\top \bar{\Theta} + \bar{\Theta}^\top R\bar{\Theta})X, X\big\rangle + 2\big\langle (S + R\bar{\Theta})X, v\big\rangle\\
&+ \big\langle Rv, v\big\rangle\Big] ds + \big\langle HX(T), X(T)\big\rangle\Big\} \geq 0,
\end{aligned}$$

where $X(\cdot) \in L^2_{\mathbb{F}}(t,T;\mathbb{R}^n)$ is the adapted solution to the following SDEP:

$$\begin{cases}
dX = \big[(A + B\bar{\Theta})X + Bv\big]ds + \big[(C + D\bar{\Theta})X + Dv\big]dW\\
\qquad + \displaystyle\int_{\mathbf{E}} \big[(F(e) + G(e)\bar{\Theta})X_- + G(e)v\big]\tilde{N}(deds),\\
X(t) = 0.
\end{cases}$$

3. Main Results

In this section, we will study the necessary and sufficient conditions for problem (SLQP) to be closed-loop solvable.

Making use of (13), we may rewrite the BSDEP in (14) and obtain

$$\begin{cases} d\bar{X} = \big[(A+B\bar{\Theta})\bar{X} + B\bar{v} + b\big]ds + \big[(C+D\bar{\Theta})\bar{X} + D\bar{v} + \sigma\big]dW \\ \qquad\quad + \int_{\mathbf{E}} \big[(F(e)+G(e)\bar{\Theta})\bar{X}_- + G(e)\bar{v} + f(e)\big]\tilde{N}(deds), \\ d\bar{Y} = -\Big[A^\top\bar{Y} + C^\top\bar{Z} + \int_{\mathbf{E}} F(e)^\top\bar{K}(e)\pi(de) + (Q+S^\top\bar{\Theta})\bar{X} + S^\top\bar{v} + q\Big]ds \\ \qquad\quad + \bar{Z}dW + \int_{\mathbf{E}} \bar{K}(e)\tilde{N}(deds), \quad s\in[t,T], \\ \bar{X}(t) = x, \quad \bar{Y}(T) = H\bar{X}(T) + g. \end{cases} \tag{15}$$

We first have the following result.

Theorem 2. *Let (H1) and (H2) hold. If $(\bar{\Theta}(\cdot),\bar{v}(\cdot)) \in \mathcal{Q}[t,T]\times\mathcal{U}[t,T]$ is an optimal closed-loop strategy of problem (SLQP) on $[t,T]$, then $(\bar{\Theta}(\cdot),0)$ is an optimal closed-loop strategy of problem (SLQP)^0 on $[t,T]$.*

Proof. By Proposition 2, we can see that $(\bar{\Theta}(\cdot),\bar{v}(\cdot))$ is an optimal closed-loop strategy of problem (SLQP) on $[t,T]$ if and only if for any $x\in L^2_{\mathcal{F}_t}(\Omega;\mathbb{R}^n)$, the stationarity condition holds:

$$B^\top\bar{Y} + D^\top\bar{Z} + \int_{\mathbf{E}} G(e)^\top\bar{K}(e)\pi(de) + (S+R\bar{\Theta})\bar{X} + R\bar{v} + \rho = 0, \quad a.e.,\ \mathbb{P}\text{-}a.s., \tag{16}$$

where $(\bar{X}(\cdot),\bar{Y}(\cdot),\bar{Z}(\cdot),\bar{K}(\cdot,\cdot))$ is the adapted solution to the following FBSDEP:

$$\begin{cases} d\bar{X} = \big[(A+B\bar{\Theta})\bar{X} + B\bar{v} + b\big]ds + \big[(C+D\bar{\Theta})\bar{X} + D\bar{v} + \sigma\big]dW \\ \qquad\quad + \int_{\mathbf{E}} \big[(F(e)+G(e)\bar{\Theta})\bar{X}_- + G(e)\bar{v} + f(e)\big]\tilde{N}(deds), \\ d\bar{Y} = -\Big[(A+B\bar{\Theta})^\top\bar{Y} + (C+D\bar{\Theta})^\top\bar{Z} + \int_{\mathbf{E}}(F(e)+G(e)\bar{\Theta})^\top\bar{K}(e)\pi(de) \\ \qquad\qquad + (Q+S^\top\bar{\Theta}+\bar{\Theta}^\top S+\bar{\Theta}^\top R\bar{\Theta})\bar{X} + (S+R\bar{\Theta})^\top\bar{v} + q + \bar{\Theta}^\top\rho\Big]ds \\ \qquad\quad + \bar{Z}dW + \int_{\mathbf{E}} \bar{K}(e)\tilde{N}(deds), \\ \bar{X}(t) = x, \quad \bar{Y}(T) = H\bar{X}(T) + g, \end{cases} \tag{17}$$

and the following convexity condition holds: For any $v(\cdot)\in\mathcal{U}[t,T]$,

$$\mathbb{E}\bigg\{\int_t^T \Big[\big\langle(Q+\bar{\Theta}^\top S+S^\top\bar{\Theta}+\bar{\Theta}^\top R\bar{\Theta})X_0, X_0\big\rangle + 2\big\langle(S+R\bar{\Theta})X_0, v\big\rangle \\ + \langle Rv, v\rangle\Big]ds + \big\langle HX_0(T), X_0(T)\big\rangle\bigg\} \geq 0,$$

where $X_0(\cdot)$ is the adapted solution to the following SDEP:

$$\begin{cases} dX_0 = \big[(A+B\bar{\Theta})X_0 + Bv\big]ds + \big[(C+D\bar{\Theta})X_0 + Dv\big]dW \\ \qquad\qquad + \int_{\mathbf{E}} \big[(F(e)+G(e)\bar{\Theta})X_{0-} + G(e)v\big]\tilde{N}(deds), \quad s\in[t,T], \\ X_0(t) = 0. \end{cases}$$

Since the FBSDEP (17) admits a solution for each $x \in L^2_{\mathcal{F}_t}(\Omega;\mathbb{R}^n)$ and $(\bar{\Theta}(\cdot), \bar{v}(\cdot))$ is independent of x, by subtracting solutions corresponding x and 0, the latter from the former, we see that, for any $x \in L^2_{\mathcal{F}_t}(\Omega;\mathbb{R}^n)$, the following FBSDEP:

$$\begin{cases} dX = \big[(A+B\bar{\Theta})X\big]ds + \big[(C+D\bar{\Theta})X\big]dW + \int_{\mathbf{E}} \big[(F(e)+G(e)\bar{\Theta})X_-\big]\tilde{N}(deds), \\ dY = -\Big[(A+B\bar{\Theta})^\top Y + (C+D\bar{\Theta})^\top Z + \int_{\mathbf{E}} (F(e)+G(e)\bar{\Theta})^\top K(e)\pi(de) \\ \qquad + (Q + S^\top\bar{\Theta} + \bar{\Theta}^\top S + \bar{\Theta}^\top R\bar{\Theta})X\Big]ds + ZdW + \int_{\mathbf{E}} K(e)\tilde{N}(deds), \\ X(t) = x, \quad Y(T) = HX(T), \end{cases} \tag{18}$$

and in this time $(X(\cdot), Y(\cdot), Z(\cdot), K(\cdot,\cdot))$ satisfies

$$B^\top Y + D^\top Z + \int_{\mathbf{E}} G(e)^\top K(e)\pi(de) + (S+R\bar{\Theta})X = 0, \quad a.e., \ \mathbb{P}\text{-}a.s..$$

It follows, again from Theorem 1 and Proposition 2, that $(\bar{\Theta}(\cdot), 0)$ is an optimal closed-loop strategy of problem (SLQP)^0 on $[t,T]$. □

To summarize the relationship between problem (SLQP), problem (SLQP)^0, problem (SLQP)$_{\bar{\Theta}}$, and problem (SLQP)$^0_{\bar{\Theta}}$, we plot the following diagram in Figure 1:

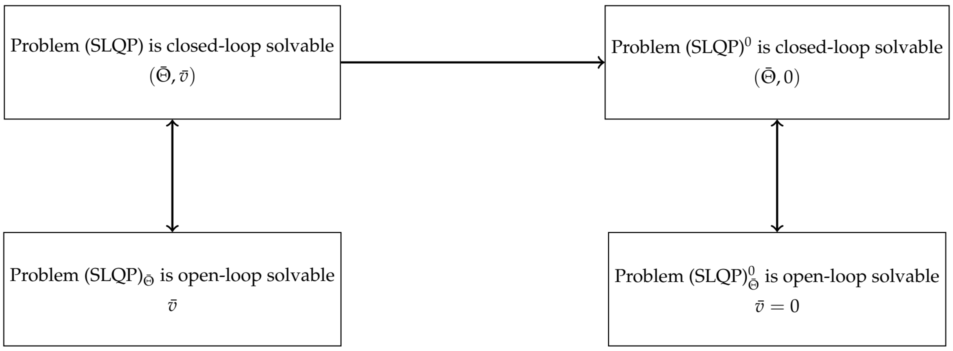

Figure 1. The relationship among problems.

It is clear that when we want to study the necessary conditions for the closed-loop solvability of problem (SLQP), we can transform the original problem into the open-loop solvability of problem (SLQP)$^0_{\bar{\Theta}}$ where the open-loop optimal control is $\bar{v}(\cdot) \equiv 0$. Thus from Proposition 1, we can know that the optimal system of problem (SLQP)$^0_{\bar{\Theta}}$ is the following FBSDEP:

$$\begin{cases} d\bar{X} = \big[(A+B\bar{\Theta})\bar{X}\big]ds + \big[(C+D\bar{\Theta})\bar{X}\big]dW + \int_{\mathbf{E}} \big[(F(e)+G(e)\bar{\Theta})\bar{X}_-\big]\tilde{N}(deds), \\ d\bar{Y} = -\Big[A^\top\bar{Y} + C^\top\bar{Z} + \int_{\mathbf{E}} F(e)^\top\bar{K}(e)\pi(de) + (Q+S^\top\bar{\Theta})\bar{X}\Big]ds \\ \qquad + \bar{Z}dW + \int_{\mathbf{E}} \bar{K}(e)\tilde{N}(deds), \\ \bar{X}(t) = x, \quad \bar{Y}(T) = H\bar{X}(T), \end{cases} \tag{19}$$

and in this time $(\bar{X}(\cdot), \bar{Y}(\cdot), \bar{Z}(\cdot), \bar{K}(\cdot,\cdot))$ satisfies

$$B^\top\bar{Y} + D^\top\bar{Z} + \int_{\mathbf{E}} G(e)^\top\bar{K}(e)\pi(de) + (S+R\bar{\Theta})\bar{X} = 0, \quad a.e., \ \mathbb{P}\text{-}a.s.. \tag{20}$$

In the light of $\bar{Y}(T) = H\bar{X}(T)$, we assume that

$$\bar{Y}(\cdot) = P(\cdot)\bar{X}(\cdot), \tag{21}$$

where $P(\cdot) : [t, T] \to \mathbb{S}^{n\times n}$ is a matrix-valued differential function satisfying $P(T) = H$. Applying Itô–Wentzell's formula to (21), we have

$$\begin{aligned} d\bar{Y} &= \dot{P}\bar{X}ds + Pd\bar{X} \\ &= \big[\dot{P}\bar{X} + P(A + B\bar{\Theta})\bar{X}\big]ds + P(C + D\bar{\Theta})\bar{X}dW + \int_{\mathbf{E}} P(F(e) + G(e)\bar{\Theta})\bar{X}_{-}\tilde{N}(deds). \end{aligned} \tag{22}$$

Comparing the diffusion coefficient of the above equation and of the second equation in (19), we can see

$$\begin{cases} \bar{Z} = P(C + D\bar{\Theta})\bar{X}, \\ \bar{K}(e) = P(F(e) + G(e)\bar{\Theta})\bar{X}. \end{cases} \tag{23}$$

Plugging (21) and (23) into (20), we obtain

$$\left[B^\top P + D^\top P(C + D\bar{\Theta}) + \int_{\mathbf{E}} G(e)^\top P(F(e) + G(e)\bar{\Theta})\pi(de) + (S + R\bar{\Theta})\right]\bar{X} = 0. \tag{24}$$

Now comparing the drift coefficient of (22) and of the second equation in (19), noting that (21), (23) and (24), we have

$$\begin{aligned} 0 = &\bigg[\dot{P} + P(A + B\bar{\Theta}) + A^\top P + C^\top P(C + D\bar{\Theta}) \\ &+ \int_{\mathbf{E}} F(e)^\top P(F(e) + G(e)\bar{\Theta})\pi(de) + Q + S^\top\bar{\Theta}\bigg]\bar{X} \\ = &\bigg[\dot{P} + P(A + B\bar{\Theta}) + A^\top P + C^\top P(C + D\bar{\Theta}) + Q + S^\top\bar{\Theta} \\ &+ \int_{\mathbf{E}} F(e)^\top P(F(e) + G(e)\bar{\Theta})\pi(de) + \bar{\Theta}^\top\Big(B^\top P + D^\top P(C + D\bar{\Theta}) + S \\ &+ R\bar{\Theta} + \int_{\mathbf{E}} G(e)^\top P(F(e) + G(e)\bar{\Theta})\pi(de)\Big)\bigg]\bar{X} \\ = &\bigg[\dot{P} + P(A + B\bar{\Theta}) + (A + B\bar{\Theta})^\top P + (C + D\bar{\Theta})^\top P(C + D\bar{\Theta}) + Q + S^\top\bar{\Theta} \\ &+ \bar{\Theta}^\top S + \bar{\Theta}^\top R\bar{\Theta} + \int_{\mathbf{E}} (F(e) + G(e)\bar{\Theta})^\top P(F(e) + G(e)\bar{\Theta})\pi(de)\bigg]\bar{X}. \end{aligned}$$

Thus, we let $P(\cdot)$ satisfy the following Lyapunov integral–differential equation:

$$\begin{cases} 0 = \dot{P} + P(A + B\bar{\Theta}) + (A + B\bar{\Theta})^\top P + (C + D\bar{\Theta})^\top P(C + D\bar{\Theta}) + Q + S^\top\bar{\Theta} + \bar{\Theta}^\top S \\ \qquad + \bar{\Theta}^\top R\bar{\Theta} + \int_{\mathbf{E}} (F(e) + G(e)\bar{\Theta})^\top P(F(e) + G(e)\bar{\Theta})\pi(de), \\ P(T) = H. \end{cases} \tag{25}$$

Proposition 3. *Let $P(\cdot)$ be the solution to (25). Then for any $s \in [t, T]$, we have*

$$\begin{cases} R + D^\top PD + \int_{\mathbf{E}} G(e)^\top PG(e)\pi(de) \geqslant 0, \\ 0 = \left[R + D^\top PD + \int_{\mathbf{E}} G(e)^\top PG(e)\pi(de)\right]\bar{\Theta} \\ \qquad + B^\top P + D^\top PC + S + \int_{\mathbf{E}} G(e)^\top PF(e)\pi(de). \end{cases} \tag{26}$$

Proof. Let us consider problem (SLQP)$^0_{\bar\Theta}$. For any $v(\cdot) \in \mathcal{U}[t,T]$, the state equation and the cost functional are:

$$\begin{cases} dX = \big[(A+B\bar\Theta)X + Bv\big]ds + \big[(C+D\bar\Theta)X + Dv\big]dW \\ \qquad\qquad + \displaystyle\int_{\mathbf{E}} \big[(F(e)+G(e)\bar\Theta)X_- + G(e)v\big]\tilde N(deds), \\ X(t) = x, \end{cases} \tag{27}$$

and

$$\begin{aligned} \tilde J(t,x;v(\cdot)) = \mathbb{E}\bigg\{ \int_t^T \Big[\big\langle (Q + \bar\Theta^\top S + S\bar\Theta + \bar\Theta^\top R\bar\Theta)X, X\big\rangle + 2\big\langle (S+R\bar\Theta)X, v\big\rangle \\ + \big\langle Rv, v\big\rangle\Big]ds + \big\langle HX(T), X(T)\big\rangle\bigg\}. \end{aligned} \tag{28}$$

Applying Itô–Wentzell's formula to $\langle PX(\cdot), X(\cdot)\rangle$, we get

$$\begin{aligned} d\langle PX, X\rangle &= \Big\langle -P(A+B\bar\Theta)X - (A+B\bar\Theta)^\top PX - (C+D\bar\Theta)^\top P(C+D\bar\Theta)X \\ &\quad - \int_{\mathbf{E}} \big[(F(e)+G(e)\bar\Theta)^\top P(F(e)+G(e)\bar\Theta)X\big]\pi(de) \\ &\quad - (Q + \bar\Theta^\top S + S^\top\bar\Theta + \bar\Theta^\top R\bar\Theta)X, X\Big\rangle ds + 2\big\langle P\big[(A+B\bar\Theta)X + Bv\big], X\big\rangle\, ds \\ &\quad + \big\langle P\big[(C+D\bar\Theta)X + Dv\big], (C+D\bar\Theta)X + Dv\big\rangle\, ds \\ &\quad + \int_{\mathbf{E}} \big\langle P\big[(F(e)+G(e)\bar\Theta)X + G(e)v\big], (F(e)+G(e)\bar\Theta)X + G(e)v\big\rangle \pi(de)ds \\ &\quad + [\cdots]dW + \int_{\mathbf{E}}[\cdots]\tilde N(deds) \\ &= \bigg\{ -\big\langle\big[Q + \bar\Theta^\top S + S^\top\bar\Theta + \bar\Theta^\top R\bar\Theta\big]X, X\big\rangle + 2\big\langle\big[B^\top P + D^\top P(C+D\bar\Theta)\big]X, v\big\rangle \\ &\quad + \int_{\mathbf{E}} \big\langle\big[2G(e)^\top P\big(F(e)+G(e)\bar\Theta\big)X + G(e)^\top PG(e)v\big], v\big\rangle\pi(de) + \big\langle D^\top PDv, v\big\rangle\bigg\}ds \\ &\quad + [\cdots]dW + \int_{\mathbf{E}}[\cdots]\tilde N(deds). \end{aligned}$$

Thus, we have

$$\begin{aligned} &\mathbb{E}\big\langle HX(T), X(T)\big\rangle - \mathbb{E}\big\langle P(t)x, x\big\rangle \\ &= \mathbb{E}\int_t^T \bigg\{ -\Big\langle\big[Q + \bar\Theta^\top S + S^\top\bar\Theta + \bar\Theta^\top R\bar\Theta\big]X, X\Big\rangle + \Big\langle\big[D^\top PD + \int_{\mathbf{E}} G(e)^\top PG(e)\pi(de)\big]v, v\Big\rangle \\ &\qquad + 2\Big\langle\big[B^\top P + D^\top P(C+D\bar\Theta) + \int_{\mathbf{E}} G(e)^\top P\big(F(e)+G(e)\bar\Theta\big)\pi(de)\big]X, v\Big\rangle\bigg\}ds. \end{aligned}$$

Putting the above equation into the cost functional, we have

$$\begin{aligned} \tilde J(t,x;v(\cdot)) = \mathbb{E}\big\langle P(t)x, x\big\rangle + \mathbb{E}\int_t^T \bigg\{ \Big\langle\big[R + D^\top PD + \int_{\mathbf{E}} G(e)^\top PG(e)\pi(de)\big]v, v\Big\rangle \\ + 2\Big\langle\big[B^\top P + D^\top PC + S + \int_{\mathbf{E}} G(e)^\top PF(e)\pi(de) \\ + \big(R + D^\top PD + \int_{\mathbf{E}} G(e)^\top PG(e)\pi(de)\big)\bar\Theta\big]X, v\Big\rangle\bigg\}ds. \end{aligned}$$

From the previous analysis, we know that $\bar{v}(\cdot) = 0$ is the open-loop optimal control for problem $(\text{SLQP})^0_{\bar{\Theta}}$, i.e.,

$$\tilde{J}(t,x;v(\cdot)) \geqslant \tilde{J}(t,x;0) = \mathbb{E}\big\langle P(t)x, x\big\rangle,$$

then

$$\begin{aligned}0 \leqslant \tilde{J}(t,x;v(\cdot)) - \mathbb{E}\big\langle P(t)x,x\big\rangle = \mathbb{E}\int_t^T \bigg\{ \Big\langle \big[R + D^\top PD + \int_{\mathbf{E}} G(e)^\top PG(e)\pi(de)\big]v, v\Big\rangle \\ + 2\Big\langle \big[B^\top P + D^\top PC + S + \int_{\mathbf{E}} G(e)^\top PF(e)\pi(de) \\ + \big(R + D^\top PD + \int_{\mathbf{E}} G(e)^\top PG(e)\pi(de)\big)\bar{\Theta}\big]X, v\Big\rangle \bigg\} ds.\end{aligned}$$

For simplicity, we give the following notations:

$$\begin{cases} \mathcal{R} := R + D^\top PD + \int_{\mathbf{E}} G(e)^\top PG(e)\pi(de), \\ \mathcal{L} := B^\top P + D^\top PC + S + \int_{\mathbf{E}} G(e)^\top PF(e)\pi(de), \end{cases}$$

thus,

$$\mathbb{E}\int_t^T \Big[\big\langle \mathcal{R}v, v\big\rangle + 2\big\langle (\mathcal{L} + \mathcal{R}\bar{\Theta})X, v\big\rangle\Big] ds \geqslant 0,\ \forall v(\cdot) \in \mathcal{U}[t,T].$$

Choose the initial pair $(t,x) = (0,0)$ and $v(\cdot) = v_0 I_{[r,r+h]}(\cdot)$, $0 \leqslant r < r+h \leqslant T$, $v_0 \in \mathbb{R}^m$. In this time, $v(\cdot)$ is a deterministic function, thus

$$\begin{aligned}&\mathbb{E}\int_0^T \Big[\big\langle \mathcal{R}v, v\big\rangle + 2\big\langle (\mathcal{L} + \mathcal{R}\bar{\Theta})X, v\big\rangle\Big] ds = \int_0^T \Big[\big\langle \mathcal{R}v, v\big\rangle + 2\big\langle (\mathcal{L} + \mathcal{R}\bar{\Theta})\mathbb{E}X, v\big\rangle\Big] ds \\ &= \int_r^{r+h} \Big[\big\langle \mathcal{R}v_0, v_0\big\rangle + 2\big\langle (\mathcal{L} + \mathcal{R}\bar{\Theta})\mathbb{E}X, v_0\big\rangle\Big] ds,\end{aligned}$$

where $\mathbb{E}X(\cdot)$ is solution to the following ordinary differential equation (ODE):

$$\begin{cases} d\mathbb{E}X(s) = \Big\{ \big[A(s) + B(s)\bar{\Theta}(s)\big]\mathbb{E}X(s) + B(s)v_0 I_{[r,r+h]}(s) \Big\} ds, \quad s \in [0,T], \\ \mathbb{E}X(0) = 0. \end{cases}$$

Then

$$\mathbb{E}X(t) = \int_0^t B(s)v_0 I_{[r,r+h]}(s) e^{-\int_0^s \left(A(r) + B(r)\bar{\Theta}(r)\right)dr} ds \cdot e^{\int_0^t \left(A(r)+B(r)\bar{\Theta}(r)\right)dr},\ t \in [0,T].$$

It is easy to see that

$$\mathbb{E}X(t) \to 0, \text{ as } h \to 0,\ t \in [0,T],$$

thus

$$\lim_{h\to 0} \frac{1}{h}\int_r^{r+h} \Big[\big\langle \mathcal{R}(s)v_0, v_0\big\rangle + 2\big\langle (\mathcal{L}(s) + \mathcal{R}(s)\bar{\Theta}(s))\mathbb{E}X(s), v_0\big\rangle\Big] ds \geqslant 0.$$

We know

$$\big\langle \mathcal{R}(r)v_0, v_0\big\rangle \geqslant 0,\ r \in [0,T],\ \forall v_0 \in \mathbb{R}^m \quad \Rightarrow \quad \mathcal{R}(r) \geqslant 0,\ r \in [0,T].$$

Thus, the first inequality in (26) is obtained, and now let us prove the second equation. We take the initial pair $(t,x) \in [0,T) \times \mathbb{R}^n$ and $v(\cdot) = \frac{v_0}{n} I_{[r,r+h]}(\cdot)$, $v_0 \in \mathbb{R}^m$, $t \leqslant r < r+h \leqslant T$. In this time,

$$\mathbb{E}\int_t^T \Big[\langle \mathcal{R}v, v\rangle + 2\big\langle (\mathcal{L}+\mathcal{R}\bar{\Theta})X, v\big\rangle\Big]ds = \mathbb{E}\int_r^{r+h}\Big[\langle \mathcal{R}\frac{v_0}{n}, \frac{v_0}{n}\rangle + 2\big\langle (\mathcal{L}+\mathcal{R}\bar{\Theta})X, \frac{v_0}{n}\big\rangle\Big]ds$$
$$= \int_r^{r+h}\Big[\frac{1}{n^2}\langle \mathcal{R}v_0, v_0\rangle + \frac{2}{n}\big\langle (\mathcal{L}+\mathcal{R}\bar{\Theta})\mathbb{E}X, v_0\big\rangle\Big]ds,$$

where $\mathbb{E}X(\cdot)$ is solution to the following ODE:

$$\begin{cases} d\mathbb{E}X(s) = \Big\{[A(s)+B(s)\bar{\Theta}(s)]\mathbb{E}X(s) + \dfrac{B(s)v_0}{n} I_{[r,r+h]}(s)\Big\}ds, \quad s \in [t,T], \\ \mathbb{E}X(t) = \mathbb{E}x. \end{cases}$$

As we know, this is an inhomogeneous linear ODE, and we get for $s \in [t,T]$,

$$\mathbb{E}X(s) = \Big[\mathbb{E}x + \int_t^s B(y)\frac{v_0}{n} I_{[r,r+h]}(y) e^{-\int_t^y \left(A(z)+B(z)\bar{\Theta}(z)\right)dz} dy\Big] \cdot e^{\int_t^s \left(A(z)+B(z)\bar{\Theta}(z)\right)dz}.$$

It is easy to see

$$\mathbb{E}X(s) \to \mathbb{E}x \cdot e^{\int_t^s \left(A(z)+B(z)\bar{\Theta}(z)\right)dz} \triangleq \mathbb{E}x \cdot \Psi(s), \text{ as } n \to \infty.$$

Thus

$$\lim_{n\to\infty}\int_r^{r+h} n \cdot \Big[\frac{1}{n^2}\langle \mathcal{R}v_0, v_0\rangle + \frac{2}{n}\big\langle (\mathcal{L}(s)+\mathcal{R}(s)\bar{\Theta}(s))\mathbb{E}X(s), v_0\big\rangle\Big]ds$$
$$= \lim_{n\to\infty}\int_r^{r+h} \cdot \Big[\frac{1}{n}\langle \mathcal{R}v_0, v_0\rangle + 2\big\langle (\mathcal{L}(s)+\mathcal{R}(s)\bar{\Theta}(s))\mathbb{E}X(s), v_0\big\rangle\Big]ds$$
$$= 2\int_r^{r+h} \big\langle (\mathcal{L}(s)+\mathcal{R}(s)\bar{\Theta}(s))\Psi(s)\mathbb{E}x, v_0\rangle ds \geqslant 0.$$

Since $v_0 \in \mathbb{R}^m$ is arbitrary,

$$\int_r^{r+h} \big\langle (\mathcal{L}(s)+\mathcal{R}(s)\bar{\Theta}(s))\Psi(s)\mathbb{E}x, v_0\rangle ds = 0.$$

Dividing both sides by h and let $h \to 0$, we have

$$\big\langle [\mathcal{L}(r)+\mathcal{R}(r)\bar{\Theta}(r)]\Psi(r)\mathbb{E}x, v_0\big\rangle = 0, \ \forall r \in [t,T].$$

Since $v_0 \in \mathbb{R}^m$ is arbitrary,

$$\mathcal{L}(r)+\mathcal{R}(r)\bar{\Theta}(r) = 0, \ \forall r \in [t,T].$$

This is the second equation in (26). □

The following theorem is the main result in this paper, which characterizes the closed-loop solvability of problem (SLQP).

Theorem 3. *Let (H1) and (H2) hold. Problem (SLQP) admits an optimal closed-loop strategy $(\bar{\Theta}(\cdot), \bar{v}(\cdot)) \in \mathcal{Q}[t,T] \times \mathcal{U}[t,T]$ if and only if the following RIDE:*

$$
\begin{cases}
0 = \dot{P} + PA + A^\top P + C^\top PC + \int_{\mathbf{E}} F(e)^\top PF(e)\pi(de) + Q \\
\qquad - \Big(B^\top P + D^\top PC + \int_{\mathbf{E}} G(e)^\top PF(e)\pi(de) + S\Big)^\top \Big(R + D^\top PD \\
\qquad + \int_{\mathbf{E}} G(e)^\top PG(e)\pi(de)\Big)^\dagger \Big(B^\top P + D^\top PC + \int_{\mathbf{E}} G(e)^\top PF(e)\pi(de) + S\Big), \\
\mathcal{R}\Big(B^\top P + D^\top PC + \int_{\mathbf{E}} G(e)^\top PF(e)\pi(de) + S\Big) \\
\quad \subseteq \mathcal{R}\Big(R + D^\top PD + \int_{\mathbf{E}} G(e)^\top PG(e)\pi(de)\Big), \\
R + D^\top PD + \int_{\mathbf{E}} G(e)^\top PG(e)\pi(de) \geqslant 0, \quad P(T) = H,
\end{cases}
\tag{29}
$$

admits a solution $P(\cdot) \in C([t,T]; \mathbb{S}^{n\times n})$ such that

$$
\begin{aligned}
&\Big(R + D^\top PD + \int_{\mathbf{E}} G(e)^\top PG(e)\pi(de)\Big)^\dagger \Big(B^\top P + D^\top PC + \int_{\mathbf{E}} G(e)^\top PF(e)\pi(de) + S\Big) \\
&\quad \in L^2(t,T;\mathbb{R}^{m\times n}),
\end{aligned}
\tag{30}
$$

and the following BSDEP:

$$
\begin{cases}
\quad d\eta = -\bigg\{ \Big[A - B\Big(R + D^\top PD + \int_{\mathbf{E}} G(e)^\top PG(e)\pi(de)\Big)^\dagger \\
\qquad\qquad \times \Big(B^\top P + D^\top PC + \int_{\mathbf{E}} G(e)^\top PF(e)\pi(de) + S\Big)\Big]^\top \eta \\
\qquad + \Big[C - D\Big(R + D^\top PD + \int_{\mathbf{E}} G(e)^\top PG(e)\pi(de)\Big)^\dagger \\
\qquad\qquad \times \Big(B^\top P + D^\top PC + \int_{\mathbf{E}} G(e)^\top PF(e)\pi(de) + S\Big)\Big]^\top (\zeta + P\sigma) \\
\qquad + \int_{\mathbf{E}} \Big[F(e) - G(e)\Big(R + D^\top PD + \int_{\mathbf{E}} G(e)^\top PG(e)\pi(de)\Big)^\dagger \\
\qquad\qquad \times \Big(B^\top P + D^\top PC + \int_{\mathbf{E}} G(e)^\top PF(e)\pi(de) + S\Big)\Big]^\top (\psi(e) + Pf(e))\pi(de) \\
\qquad + Pb + q - \Big(B^\top P + D^\top PC + \int_{\mathbf{E}} G(e)^\top PF(e)\pi(de) + S\Big)^\top \\
\qquad\quad \times \Big(R + D^\top PD + \int_{\mathbf{E}} G(e)^\top PG(e)\pi(de)\Big)^\dagger \rho \bigg\} ds + \zeta dW + \int_{\mathbf{E}} \psi(e)\tilde{N}(deds), \\
\quad B^\top \eta + D^\top(\zeta + P\sigma) + \int_{\mathbf{E}} G(e)^\top (\psi(e) + Pf(e))\pi(de) + \rho \\
\qquad \in \mathcal{R}\big(R + D^\top PD + \int_{\mathbf{E}} G(e)^\top PG(e)\pi(de)\big), \ \textit{a.e.}, \ \mathbb{P}\textit{-a.s.}, \\
\eta(T) = g,
\end{cases}
\tag{31}
$$

admits a unique solution $(\eta(\cdot), \zeta(\cdot), \psi(\cdot))$, *which satisfies*

$$\begin{aligned}&\left(R + D^\top PD + \int_{\mathbf{E}} G(e)^\top PG(e)\pi(de)\right)^\dagger \\ &\quad\times \left(B^\top \eta + D^\top(\zeta + P\sigma) + \int_{\mathbf{E}} G(e)^\top \big(\psi(e) + Pf(e)\big)\pi(de) + \rho\right) \in L^2_{\mathbb{F},p}(t,T;\mathbb{R}^m).\end{aligned} \tag{32}$$

In this case, the closed-loop optimal strategy $(\bar{\Theta}(\cdot), \bar{v}(\cdot))$ *of problem (SLQP) admits the following representation:*

$$\begin{cases}\bar{\Theta} = -\left(R + D^\top PD + \int_{\mathbf{E}} G(e)^\top PG(e)\pi(de)\right)^\dagger \\ \qquad\times \left(B^\top P + D^\top PC + \int_{\mathbf{E}} G(e)^\top PF(e)\pi(de) + S\right) \\ \qquad + \left[I - \left(R + D^\top PD + \int_{\mathbf{E}} G(e)^\top PG(e)\pi(de)\right)^\dagger \right.\\ \qquad\left.\times \left(R + D^\top PD + \int_{\mathbf{E}} G(e)^\top PG(e)\pi(de)\right)\right]\theta, \\ \bar{v} = -\left(R + D^\top PD + \int_{\mathbf{E}} G(e)^\top PG(e)\pi(de)\right)^\dagger \\ \qquad\times \left[B^\top \eta + D^\top(\zeta + P\sigma) + \int_{\mathbf{E}} G(e)^\top \big(\psi(e) + Pf(e)\big)\pi(de) + \rho\right] \\ \qquad + \left[I - \left(R + D^\top PD + \int_{\mathbf{E}} G(e)^\top PG(e)\pi(de)\right)^\dagger \right.\\ \qquad\left.\times \left(R + D^\top PD + \int_{\mathbf{E}} G(e)^\top PG(e)\pi(de)\right)\right]v,\end{cases} \tag{33}$$

for some $\theta(\cdot) \in \mathcal{Q}[t,T]$, $v(\cdot) \in \mathcal{U}[t,T]$. *Further, the value function* $V(\cdot,\cdot)$ *is given by*

$$\begin{aligned}V(t,x) &\equiv \inf_{u(\cdot)\in\mathcal{U}[t,T]} J(t,x;u(\cdot)) = \mathbb{E}\Bigg\{\langle P(t)x, x\rangle + 2\langle \eta(t), x\rangle \\ &+ \int_t^T \Bigg[2\langle \eta, b\rangle + 2\langle \zeta, \sigma\rangle + 2\int_{\mathbf{E}} \langle \psi(e), f(e)\rangle \pi(de) + \langle P\sigma, \sigma\rangle \\ &\quad + \int_{\mathbf{E}} \langle Pf(e), f(e)\rangle \pi(de) - \Bigg|\left[\left(R + D^\top PD + \int_{\mathbf{E}} G(e)^\top PG(e)\pi(de)\right)^\dagger\right]^{\frac{1}{2}} \\ &\quad \times \left[B^\top \eta + D^\top(\zeta + P\sigma) + \int_{\mathbf{E}} G(e)^\top \big(\psi(e) + Pf(e)\big)\pi(de) + \rho\right]\Bigg|^2\Bigg] ds\Bigg\}.\end{aligned} \tag{34}$$

Proof. We first prove the necessity. Let $(\bar{\Theta}(\cdot), \bar{v}(\cdot)) \in \mathcal{Q}[t,T] \times \mathcal{U}[t,T]$ be a closed-loop optimal strategy of problem (SLQP) over $[t,T]$. The second equation of (26) implies

$$\mathcal{R}\left(B^\top P + D^\top PC + \int_{\mathbf{E}} G(e)^\top PF(e)\pi(de) + S\right) \subseteq \mathcal{R}\left(R + D^\top PD + \int_{\mathbf{E}} G(e)^\top PG(e)\pi(de)\right), \text{ a.e..}$$

Denoting $\mathcal{R} := R + D^\top PD + \int_{\mathbf{E}} G(e)^\top PG(e)\pi(de)$, since

$$\mathcal{R}^\dagger\left(B^\top P + D^\top PC + \int_{\mathbf{E}} G(e)^\top PF(e)\pi(de) + S\right) = -\mathcal{R}^\dagger\mathcal{R}\bar{\Theta},$$

and $\mathcal{R}^\dagger\mathcal{R}$ is an orthogonal projection, we see that (30) holds and

$$\bar{\Theta} = -\mathcal{R}^\dagger\Big(B^\top P + D^\top PC + \int_{\mathbf{E}} G(e)^\top PF(e)\pi(de) + S\Big) + (I - \mathcal{R}^\dagger\mathcal{R})\theta,$$

for some $\theta(\cdot) \in \mathcal{Q}[t,T]$. Consequently,

$$\begin{aligned}
&\Big(PB + C^\top PD + \int_{\mathbf{E}} F(e)^\top PG(e)\pi(de) + S^\top\Big)\bar{\Theta} \\
&= -\bar{\Theta}^\top\mathcal{R}\bar{\Theta} = \bar{\Theta}^\top\mathcal{R}\mathcal{R}^\dagger\Big(B^\top P + D^\top PC + \int_{\mathbf{E}} G(e)^\top PF(e)\pi(de) + S\Big) \\
&= -\Big(B^\top P + D^\top PC + \int_{\mathbf{E}} G(e)^\top PF(e)\pi(de) + S\Big)^\top\mathcal{R}^\dagger \\
&\quad\times\Big(B^\top P + D^\top PC + \int_{\mathbf{E}} G(e)^\top PF(e)\pi(de) + S\Big).
\end{aligned}$$

Plugging the above into the Lyapunov Equation (25), we obtain the RIDE in (29). To determine $\bar{v}(\cdot)$, we define

$$\begin{cases}
\eta = \bar{Y} - P\bar{X}, \\
\zeta = \bar{Z} - P(C + D\bar{\Theta})\bar{X} - PD\bar{v} - P\sigma, \\
\psi(e) = \bar{K}(e) - P\big(F(e) + G(e)\bar{\Theta}\big)\bar{X} - PG(e)\bar{v} - Pf(e),
\end{cases}$$

where $(\bar{X}(\cdot), \bar{Y}(\cdot), \bar{Z}(\cdot), \bar{K}(\cdot,\cdot))$ is the adapted solution to the FBSDEP (17). Then

$$\begin{aligned}
d\eta &= d\bar{Y} - \dot{P}\bar{X}ds - Pd\bar{X} \\
&= \Big\{ -A^\top\bar{Y} - C^\top\bar{Z} - \int_{\mathbf{E}} F(e)^\top\bar{K}(e)\pi(de) - (Q + S^\top\bar{\Theta})\bar{X} - S^\top\bar{v} - q + A^\top P\bar{X} \\
&\qquad + PA\bar{X} + C^\top PC\bar{X} + \Big(PB + C^\top PD + \int_{\mathbf{E}} F(e)^\top PG(e)\pi(de) + S^\top\Big)\bar{\Theta}\bar{X} \\
&\qquad + Q\bar{X} + \int_{\mathbf{E}} F(e)^\top PF(e)\bar{X}\pi(de) - P(A + B\bar{\Theta})\bar{X} - PB\bar{v} - Pb\Big\}ds \\
&\quad + \big[\bar{Z} - P(C + D\bar{\Theta})\bar{X} - PD\bar{v} - P\sigma\big]dW \\
&\quad + \int_{\mathbf{E}} \big[\bar{K}(e) - P\big(F(e) + G(e)\bar{\Theta}\big)\bar{X}_- - PG(e)\bar{v} - Pf(e)\big]\tilde{N}(deds) \\
&= -\Big\{A^\top(P\bar{X} + \eta) + C^\top\big[\zeta + P(C + D\bar{\Theta})\bar{X} + PD\bar{v} + P\sigma\big] + \int_{\mathbf{E}} F(e)^\top\big[\psi(e) \\
&\qquad + P\big(F(e) + G(e)\bar{\Theta}\big)\bar{X} + PG(e)\bar{v} + Pf(e)\big]\pi(de) + (Q + S^\top\bar{\Theta})\bar{X} + S^\top\bar{v} + q \\
&\qquad - A^\top P\bar{X} - PA\bar{X} - C^\top PC\bar{X} - \Big(PB + C^\top PD + \int_{\mathbf{E}} F(e)^\top PG(e)\pi(de) + S^\top\Big)\bar{\Theta}\bar{X} \\
&\qquad - Q\bar{X} - \int_{\mathbf{E}} F(e)^\top PF(e)\bar{X}\pi(de) + P(A + B\bar{\Theta})\bar{X} + PB\bar{v} + Pb\Big\}ds \\
&\qquad + \zeta dW + \int_{\mathbf{E}} \psi(e)\tilde{N}(deds) \\
&= -\Big\{A^\top\eta + C^\top\zeta + \int_{\mathbf{E}} F(e)^\top\psi(e)\pi(de) + \Big(PB + C^\top PD + \int_{\mathbf{E}} F(e)^\top PG(e)\pi(de) \\
&\qquad + S^\top\Big)\bar{v} + Pb + C^\top P\sigma + \int_{\mathbf{E}} F(e)^\top Pf(e)\pi(de) + q\Big\}ds + \zeta dW + \int_{\mathbf{E}} \psi(e)\tilde{N}(deds).
\end{aligned}$$

According to the stationarity condition (16), we have

$$
\begin{aligned}
0 &= B^\top \bar{Y} + D^\top \bar{Z} + \int_{\mathbf{E}} G(e)^\top \bar{K}(e)\pi(de) + (S + R\bar{\Theta})\bar{X} + R\bar{v} + \rho \\
&= B^\top (P\bar{X} + \eta) + D^\top \big[\zeta + P(C + D\bar{\Theta})\bar{X} + PD\bar{v} + P\sigma\big] + (S + R\bar{\Theta})\bar{X} + R\bar{v} + \rho \\
&\quad + \int_{\mathbf{E}} G(e)^\top \big[\psi(e) + P\big(F(e) + G(e)\bar{\Theta}\big)\bar{X} + PG(e)\bar{v} + Pf(e)\big]\pi(de) \\
&= B^\top \eta + D^\top \zeta + \int_{\mathbf{E}} G(e)^\top \psi(e)\pi(de) + D^\top P\sigma + \rho + \int_{\mathbf{E}} G(e)^\top Pf(e)\pi(de) \\
&\quad + \left(R + D^\top PD + \int_{\mathbf{E}} G(e)^\top PG(e)\pi(de)\right)\bar{v} + \bigg[B^\top P + D^\top PC \\
&\quad + \int_{\mathbf{E}} G(e)^\top PF(e)\pi(de) + S + \left(R + D^\top PD + \int_{\mathbf{E}} G(e)^\top PG(e)\pi(de)\right)\bar{\Theta}\bigg]\bar{X} \\
&= B^\top \eta + D^\top \zeta + \int_{\mathbf{E}} G(e)^\top \psi(e)\pi(de) + D^\top P\sigma + \rho + \int_{\mathbf{E}} G(e)^\top Pf(e)\pi(de) \\
&\quad + \left(R + D^\top PD + \int_{\mathbf{E}} G(e)^\top PG(e)\pi(de)\right)\bar{v}.
\end{aligned}
\tag{35}
$$

Hence,

$$
\begin{aligned}
&B^\top \eta + D^\top \zeta + \int_{\mathbf{E}} G(e)^\top \psi(e)\pi(de) + D^\top P\sigma + \rho + \int_{\mathbf{E}} G(e)^\top Pf(e)\pi(de) \\
&\qquad \in \mathcal{R}\left(R + D^\top PD + \int_{\mathbf{E}} G(e)^\top PG(e)\pi(de)\right), \quad \textit{a.e., } \mathbb{P}\textit{-a.s.}.
\end{aligned}
\tag{36}
$$

Since

$$
\mathcal{R}^\dagger \left[B^\top \eta + D^\top \zeta + \int_{\mathbf{E}} G(e)^\top \psi(e)\pi(de) + D^\top P\sigma + \rho + \int_{\mathbf{E}} G(e)^\top Pf(e)\pi(de)\right] = -\mathcal{R}^\dagger \mathcal{R}\bar{v}
$$

and $\mathcal{R}^\dagger \mathcal{R}$ is an orthogonal projection, we see that (32) holds and

$$
\bar{v} = -\mathcal{R}^\dagger \left[B^\top \eta + D^\top (\zeta + P\sigma) + \int_{\mathbf{E}} G(e)^\top \big(\psi(e) + Pf(e)\big)\pi(de) + \rho\right] + (I - \mathcal{R}^\dagger \mathcal{R})v,
$$

for some $v(\cdot) \in L^2_{\mathbb{F},p}(t, T; \mathbb{R}^m)$. Consequently,

$$
\begin{aligned}
&\left[PB + C^\top PD + \int_{\mathbf{E}} F(e)^\top PG(e)\pi(de) + S^\top\right]\bar{v} \\
&= -\left[PB + C^\top PD + \int_{\mathbf{E}} F(e)^\top PG(e)\pi(de) + S^\top\right] \\
&\qquad \times \mathcal{R}^\dagger \left[B^\top \eta + D^\top (\zeta + P\sigma) + \int_{\mathbf{E}} G(e)^\top \big(\psi(e) + Pf(e)\big)\pi(de) + \rho\right] \\
&\quad + \left[PB + C^\top PD + \int_{\mathbf{E}} F(e)^\top PG(e)\pi(de) + S^\top\right](I - \mathcal{R}^\dagger \mathcal{R})v \\
&= -\left[PB + C^\top PD + \int_{\mathbf{E}} F(e)^\top PG(e)\pi(de) + S^\top\right]\mathcal{R}^\dagger \bigg[B^\top \eta + D^\top (\zeta + P\sigma) \\
&\qquad + \int_{\mathbf{E}} G(e)^\top \big(\psi(e) + Pf(e)\big)\pi(de) + \rho\bigg] - \bar{\Theta}^\top \mathcal{R}(I - \mathcal{R}^\dagger \mathcal{R})v \\
&= -[PB + C^\top PD + \int_{\mathbf{E}} F(e)^\top PG(e)\pi(de) + S^\top] \\
&\qquad \times \mathcal{R}^\dagger \left[B^\top \eta + D^\top (\zeta + P\sigma) + \int_{\mathbf{E}} G(e)^\top \big(\psi(e) + Pf(e)\big)\pi(de) + \rho\right].
\end{aligned}
$$

Therefore, $(\eta(\cdot), \zeta(\cdot), \psi(\cdot,\cdot))$ is the unique solution to the following BSDEP:

$$\begin{cases} d\eta = -\Big\{A^\top\eta + C^\top\zeta + \int_{\mathbf{E}} F(e)^\top\psi(e)\pi(de) \\ \qquad + \Big[PB + C^\top PD + \int_{\mathbf{E}} F(e)^\top PG(e)\pi(de) + S^\top\Big]\bar{v} \\ \qquad + Pb + C^\top P\sigma + \int_{\mathbf{E}} F(e)^\top Pf(e)\pi(de) + q\Big\}ds + \zeta dW + \int_{\mathbf{E}} \psi(e)\tilde{N}(deds) \\ \quad = -\Big\{A^\top\eta + C^\top\zeta + \int_{\mathbf{E}} F(e)^\top\psi(e)\pi(de) + Pb + C^\top P\sigma + \int_{\mathbf{E}} F(e)^\top Pf(e)\pi(de) + q \\ \qquad - \Big[PB + C^\top PD + \int_{\mathbf{E}} F(e)^\top PG(e)\pi(de) + S^\top\Big]\mathcal{R}^\dagger\Big[B^\top\eta + D^\top(\zeta + P\sigma) \\ \qquad\quad + \int_{\mathbf{E}} G(e)^\top\big(\psi(e) + Pf(e)\big)\pi(de) + \rho\Big]\Big\}ds + \zeta dW + \int_{\mathbf{E}} \psi(e)\tilde{N}(deds), \\ \eta(T) = g. \end{cases}$$

To prove the sufficiency, we take any $u(\cdot) \in \mathcal{U}[t,T]$, and let $X(\cdot) \equiv X(\cdot\,; t, x, u(\cdot))$ be the corresponding state process. Applying Itô–Wentzell's formula to $\langle PX, X\rangle$ and $\langle \eta, X\rangle$, we have

$$\begin{aligned} d\langle PX, X\rangle &= \Big\langle -P(A + B\bar{\Theta})X - (A + B\bar{\Theta})^\top PX - (C + D\bar{\Theta})^\top P(C + D\bar{\Theta})X \\ &\quad - \int_{\mathbf{E}} \big[(F(e) + G(e)\bar{\Theta})^\top P(F(e) + G(e)\bar{\Theta})X\big]\pi(de) - (Q + \bar{\Theta}^\top S + S^\top\bar{\Theta} + \bar{\Theta}^\top R\bar{\Theta})X, X\Big\rangle\, ds \\ &\quad + 2\langle PAX + PBu + Pb, X\rangle ds + \langle P(CX + Du + \sigma), CX + Du + \sigma\rangle\, ds \\ &\quad + \int_{\mathbf{E}} \Big\langle P\big[F(e)X + G(e)u + f(e)\big], F(e)X + G(e)u + f(e)\Big\rangle\pi(de)ds \\ &\quad + [\cdots]dW + \int_{\mathbf{E}} [\cdots]\tilde{N}(deds) \\ &= \Big\{ -\Big\langle\Big[(PB + S^\top)\bar{\Theta} + \bar{\Theta}^\top(B^\top P + S) + (C + D\bar{\Theta})^\top P(C + D\bar{\Theta}) \\ &\qquad + \int_{\mathbf{E}} \big[(F(e) + G(e)\bar{\Theta})^\top P(F(e) + G(e)\bar{\Theta})\big]\pi(de) + Q + \bar{\Theta}^\top R\bar{\Theta}\Big]X, X\Big\rangle \\ &\qquad + 2\langle P(Bu + b), X\rangle + \langle P(CX + Du + \sigma), CX + Du + \sigma\rangle \\ &\qquad + \int_{\mathbf{E}} \Big\langle P\big[F(e)X + G(e)u + f(e)\big], F(e)X + G(e)u + f(e)\Big\rangle\pi(de)\Big\}ds \\ &\quad + [\cdots]dW + \int_{\mathbf{E}} [\cdots]\tilde{N}(deds), \end{aligned}$$

and

$$
\begin{aligned}
d\langle \eta, X\rangle = &\bigg\{\bigg\langle -A^\top \eta - C^\top \zeta - \int_{\mathbf{E}} F(e)^\top \psi(e)\pi(de) \\
&- \bigg[PB + C^\top PD + \int_{\mathbf{E}} F(e)^\top PG(e)\pi(de) + S^\top\bigg]\bar{v} - Pb - C^\top P\sigma \\
&- \int_{\mathbf{E}} F(e)^\top Pf(e)\pi(de) - q, X\bigg\rangle + \langle \eta, AX + Bu + b\rangle + \langle \zeta, CX + Du + \sigma\rangle \\
&+ \int_{\mathbf{E}} \langle \psi(e), F(e)X + G(e)u + f(e)\rangle \pi(de)\bigg\}ds + [\cdots]dW + \int_{\mathbf{E}} [\cdots]\tilde{N}(deds) \\
= &\bigg\{-\bigg\langle \bigg[PB + C^\top PD + \int_{\mathbf{E}} F(e)^\top PG(e)\pi(de) + S^\top\bigg]\bar{v} + Pb + C^\top P\sigma \\
&+ \int_{\mathbf{E}} F(e)^\top Pf(e)\pi(de) + q, X\bigg\rangle + \langle \eta, Bu + b\rangle + \langle \zeta, Du + \sigma\rangle \\
&+ \int_{\mathbf{E}} \langle \psi(e), G(e)u + f(e)\rangle \pi(de)\bigg\}ds + [\cdots]dW + \int_{\mathbf{E}} [\cdots]\tilde{N}(deds).
\end{aligned}
$$

Thus,

$$
\begin{aligned}
J(t, x; u(\cdot)) = &\mathbb{E}\bigg\{\big\langle HX(T), X(T)\big\rangle + 2\big\langle g, X(T)\big\rangle \\
&+ \int_t^T \Big[\big\langle QX, X\big\rangle + 2\big\langle SX, u\big\rangle + \big\langle Ru, u\big\rangle + 2\big\langle q, X\big\rangle + 2\big\langle \rho, u\big\rangle\Big]ds\bigg\} \\
= &\mathbb{E}\bigg\{\big\langle P(t)x, x\big\rangle + 2\big\langle \eta(t), x\big\rangle + \int_t^T \Big[\big\langle QX, X\big\rangle + 2\big\langle SX, u\big\rangle + \big\langle Ru, u\big\rangle + 2\big\langle q, X\big\rangle \\
&+ 2\big\langle \rho, u\big\rangle - 2\big\langle B^\top PX, \bar{\Theta}X\big\rangle - \big\langle C^\top PCX, x\big\rangle - 2\big\langle D^\top PCX, \bar{\Theta}X\big\rangle - \big\langle D^\top PD\bar{\Theta}X, \bar{\Theta}X\big\rangle \\
&- \big\langle QX, X\big\rangle - 2\big\langle SX, \bar{\Theta}X\big\rangle - \big\langle R\bar{\Theta}X, \bar{\Theta}X\big\rangle + 2\big\langle B^\top PX, u\big\rangle + 2\big\langle Pb, X\big\rangle \\
&+ \big\langle C^\top PCX, X\big\rangle + \big\langle D^\top PCX, u\big\rangle + 2\big\langle PCX, \sigma\big\rangle + \big\langle D^\top PXC, u\big\rangle + \big\langle D^\top PDu, u\big\rangle \\
&+ 2\big\langle PDu, \sigma\big\rangle + \big\langle P\sigma, \sigma\big\rangle - 2\bigg\langle \bigg[PB + C^\top PD + \int_{\mathbf{E}} F(e)^\top PG(e)\pi(de) + S^\top\bigg]\bar{v}, X\bigg\rangle \\
&- 2\bigg\langle C^\top P\sigma + Pb + \int_{\mathbf{E}} F(e)^\top Pf(e)\pi(de) + q, X\bigg\rangle + 2\big\langle B^\top \eta, u\big\rangle + 2\big\langle \eta, b\big\rangle + 2\big\langle D^\top \zeta, u\big\rangle \\
&+ 2\big\langle \zeta, \sigma\big\rangle - \int_{\mathbf{E}} \langle F(e)^\top PF(e)X, X\rangle \pi(de) - 2\int_{\mathbf{E}} \langle G(e)^\top PF(e)X, \bar{\Theta}X\rangle \pi(de) \\
&- \int_{\mathbf{E}} \langle G(e)^\top PG(e)\bar{\Theta}X, \bar{\Theta}X\rangle \pi(de) + \int_{\mathbf{E}} \langle F(e)^\top PF(e)X, X\rangle \pi(de) \\
&+ 2\int_{\mathbf{E}} \langle G(e)^\top PF(e)X, u\rangle \pi(de) + 2\int_{\mathbf{E}} \langle F(e)^\top Pf(e), X\rangle \pi(de) \\
&+ \int_{\mathbf{E}} \langle G(e)^\top PG(e)u, u\rangle \pi(de) + 2\int_{\mathbf{E}} \langle G(e)^\top Pf(e), u\rangle \pi(de) + \int_{\mathbf{E}} \langle Pf(e), f(e)\rangle \pi(de) \\
&+ 2\int_{\mathbf{E}} \langle G(e)^\top \psi(e), u\rangle \pi(de) + 2\int_{\mathbf{E}} \langle \psi(e), f(e)\rangle \pi(de)\Big]ds\bigg\}
\end{aligned}
$$

$$
\begin{aligned}
&= \mathbb{E}\bigg\{ \big\langle P(t)x, x \big\rangle + 2\big\langle \eta(t), x \big\rangle + \int_t^T \Big[\big\langle \mathcal{R}u, u \big\rangle - \big\langle \mathcal{R}\bar{\Theta}X, \bar{\Theta}X \big\rangle \\
&\quad + 2\Big\langle B^\top P + D^\top PC + \int_{\mathbf{E}} G(e)^\top PF(e)\pi(de) + S, u - \bar{\Theta}X - \bar{v} \Big\rangle \\
&\quad + 2\Big\langle B^\top \eta + D^\top \zeta + \int_{\mathbf{E}} G(e)^\top \psi(e)\pi(de) + D^\top P\sigma + \rho \\
&\quad + \int_{\mathbf{E}} G(e)^\top Pf(e)\pi(de), u \Big\rangle + \big\langle P\sigma, \sigma \big\rangle + \int_{\mathbf{E}} \big\langle Pf(e), f(e) \big\rangle \pi(de) \\
&\quad + 2\big\langle \eta, b \big\rangle + 2\big\langle \zeta, \sigma \big\rangle + 2\int_{\mathbf{E}} \big\langle \psi(e), f(e) \big\rangle \pi(de) \Big] ds \bigg\} \\
&= \mathbb{E}\bigg\{ \big\langle P(t)x, x \big\rangle + 2\big\langle \eta(t), x \big\rangle + \int_t^T \big\langle \mathcal{R}\big(u - \bar{\Theta}X - \bar{v}\big), u - \bar{\Theta}X - \bar{v} \big\rangle \, ds \\
&\quad + \int_t^T \Big[\big\langle P\sigma, \sigma \big\rangle + \int_{\mathbf{E}} \big\langle Pf(e), f(e) \big\rangle \pi(de) + 2\big\langle \eta, b \big\rangle + 2\big\langle \zeta, \sigma \big\rangle + 2\int_{\mathbf{E}} \big\langle \psi(e), f(e) \big\rangle \pi(de) \\
&\qquad - \Big\langle \mathcal{R}^\dagger \Big[B^\top \eta + D^\top \zeta + \int_{\mathbf{E}} G(e)^\top \psi(e)\pi(de) + D^\top P\sigma + \rho + \int_{\mathbf{E}} G(e)^\top Pf(e)\pi(de) \Big], \\
&\qquad\quad B^\top \eta + D^\top \zeta + \int_{\mathbf{E}} G(e)^\top \psi(e)\pi(de) + D^\top P\sigma + \rho + \int_{\mathbf{E}} G(e)^\top Pf(e)\pi(de) \Big\rangle \Big] ds \bigg\} \\
&= J(t, x; \bar{\Theta}(\cdot)\bar{X}(\cdot) + \bar{v}(\cdot)) + \int_t^T \big\langle \mathcal{R}\big(u - \bar{\Theta}X - \bar{v}\big), u - \bar{\Theta}X - \bar{v} \big\rangle ds.
\end{aligned}
$$

Hence,

$$J(t, x; \bar{\Theta}(\cdot)\bar{X}(\cdot) + \bar{v}(\cdot)) \leqslant J(t, x; u(\cdot)), \quad \forall u(\cdot) \in \mathcal{U}[t, T],$$

if and only if

$$R + D^\top PD + \int_{\mathbf{E}} G(e)^\top PG(e)\pi(de) \geqslant 0, \quad a.e..$$

□

4. Example

In this section, we will give a simple example. Consider the following controlled linear SDE with Poisson jumps:

$$
\begin{cases}
dX(s) = \big[A(s)X(s) + B(s)u(s)\big]ds + \big[C(s)X(s) + D(s)u(s)\big]dW(s) \\
\qquad\qquad + \int_{\mathbf{E}} G(s, e)u(s)\tilde{N}(deds), \quad s \in [t, T], \\
X(t) = x,
\end{cases}
\tag{37}
$$

and the cost functional is defined to be

$$J(t, x; u(\cdot)) = \mathbb{E}\bigg\{ \int_t^T \Big[\big\langle QX, X \big\rangle + 2\big\langle SX, u \big\rangle + \big\langle Ru, u \big\rangle \Big] ds + \big\langle HX(T), X(T) \big\rangle \bigg\}. \tag{38}$$

Then, for any closed-loop strategy pair $(\Theta(\cdot), v(\cdot)) \in \mathcal{Q}[t,T] \times \mathcal{U}[t,T]$, we get the closed-loop system:

$$\begin{cases} dX = \big[(A+B\Theta)X + Bv\big]ds + \big[(C+D\Theta)X + Dv\big]dW \\ \qquad\qquad + \int_{\mathbf{E}} \big[G(e)\Theta X_- + G(e)v\big]\tilde{N}(deds), \quad s \in [t,T], \\ X(t) = x, \end{cases} \tag{39}$$

and the closed-loop cost functional

$$\begin{aligned} J\big(t,x;\Theta(\cdot)X(\cdot)+v(\cdot)\big) = \mathbb{E}\Big\{ \big\langle HX(T), X(T)\big\rangle \\ + \int_t^T \Big[\big\langle (Q + \Theta^\top S + S^\top\Theta + \Theta^\top R\Theta)X, X\big\rangle + 2\big\langle (S+R\Theta)X, v\big\rangle + \big\langle Rv, v\big\rangle\Big]ds\Big\}. \end{aligned} \tag{40}$$

In this case, $(0,0,0)$ is the unique adaptive solution to BSDEP (31). Then, by Theorem 3, we obtain the closed-loop optimal strategy $(\bar{\Theta}(\cdot), \bar{v}(\cdot))$ admits the following representation:

$$\begin{cases} \bar{\Theta} = -\Big(R + D^\top PD + \int_{\mathbf{E}} G(e)^\top PG(e)\pi(de)\Big)^\dagger (B^\top P + D^\top PC + S) \\ \qquad + \Big[I - \Big(R + D^\top PD + \int_{\mathbf{E}} G(e)^\top PG(e)\pi(de)\Big)^\dagger \\ \qquad\quad \times \Big(R + D^\top PD + \int_{\mathbf{E}} G(e)^\top PG(e)\pi(de)\Big)\Big]\theta, \\ \bar{v} = 0, \end{cases} \tag{41}$$

for some $\theta(\cdot) \in \mathcal{Q}[t,T]$, where $P(\cdot) \in C([t,T];\mathbb{S}^{n\times n})$ is the solution to the following RIDE:

$$\begin{cases} 0 = \dot{P} + PA + A^\top P + C^\top PC + Q \\ \quad - \big(B^\top P + D^\top PC + S\big)^\top \Big(R + D^\top PD + \int_{\mathbf{E}} G(e)^\top PG(e)\pi(de)\Big)^\dagger \big(B^\top P + D^\top PC + S\big), \\ \mathcal{R}\big(B^\top P + D^\top PC + S\big) \subseteq \mathcal{R}\Big(R + D^\top PD + \int_{\mathbf{E}} G(e)^\top PG(e)\pi(de)\Big), \\ R + D^\top PD + \int_{\mathbf{E}} G(e)^\top PG(e)\pi(de) \geqslant 0, \quad P(T) = H, \end{cases} \tag{42}$$

such that

$$\Big(R + D^\top PD + \int_{\mathbf{E}} G(e)^\top PG(e)\pi(de)\Big)^\dagger \big(B^\top P + D^\top PC + S\big) \in L^2(t,T;\mathbb{R}^{m\times n}). \tag{43}$$

Further, the value function $V(\cdot,\cdot)$ is given by

$$V(t,x) \equiv \inf_{u(\cdot)\in\mathcal{U}[t,T]} J(t,x;u(\cdot)) = \mathbb{E}\big\langle P(t)x, x\big\rangle. \tag{44}$$

In this time, the closed-loop optimal control of this problem is $\bar{u}(\cdot) = \bar{\Theta}(\cdot)\bar{X}(\cdot)$, where $\bar{X}(\cdot)$ admits the following repesentation:

$$\begin{aligned} \bar{X}(s) = x\exp\Big\{ \int_t^s \Big[(A + B\bar{\Theta}) - \frac{(C + D\bar{\Theta})^2}{2} - \int_{\mathbf{E}} G(e)\bar{\Theta}\pi(de)\Big]dr \\ + \int_t^s (C + D\bar{\Theta})dW\Big\} \cdot \prod_{t<r\leqslant s}\Big[1 + \int_{\mathbf{E}} G(e)^\top\bar{\Theta}N(de,r)\Big], \quad s \in [t,T]. \end{aligned}$$

We further make the following assumptions:

Hypothesis 3. *The coefficients of the drift and diffusion terms of (37) are time-invariant constants, and the coefficient of the jump diffusion term $G(\cdot)$ depends only on the variable e, that is, $G(\cdot)$ is independent of t.*

Hypothesis 4. *The Poisson process N has jumps of unit size, i.e., $\mathbf{E} = \{1\}$.*

Hypothesis 5. *The weighting coefficients of the cost functional (38) satisfy the following:*

$$R \gg 0, \qquad Q - SR^{-1}S^\top \geq 0, \qquad H \geq 0,$$

Under (H4), the compensated Poisson process $\tilde{N}([0,t]) = (N - \hat{N})([0,t])_{t\geqslant 0}$ is a martingale, where $\hat{N}(dedt) = \pi(de)dt = \pi dt$ and $\pi > 0$ is the intensity of the Poisson process N. Then

$$\int_{\mathbf{E}} G(s,e)^\top P(s)G(s,e)\pi(de)ds \equiv \pi G(s)^\top P(s)G(s)ds, \quad s \in [t,T].$$

Under (H4)–(H5), the equation of RIDE (42) can be reduced to the following:

$$\begin{cases} 0 = \dot{P} + PA + A^\top P + C^\top PC + Q \\ \qquad - \left(B^\top P + D^\top PC + S\right)^\top \left(R + D^\top PD + \pi G^\top PG\right)^{-1} \left(B^\top P + D^\top PC + S\right), \\ P(T) = H, \end{cases}$$

where A, B, C, D, G, Q, R, S, and H are all constants.

When we assume that the coefficients are $A = 2$, $B = 3$, $C = 1$, $D = 3$, $G = 3$, $Q = 4$, $R = 2$, $S = 1$, $H = 3$, and $\pi = 1$, respectively, the solution $P(\cdot)$ of the above Riccati equation can be shown in the following Figure 2.

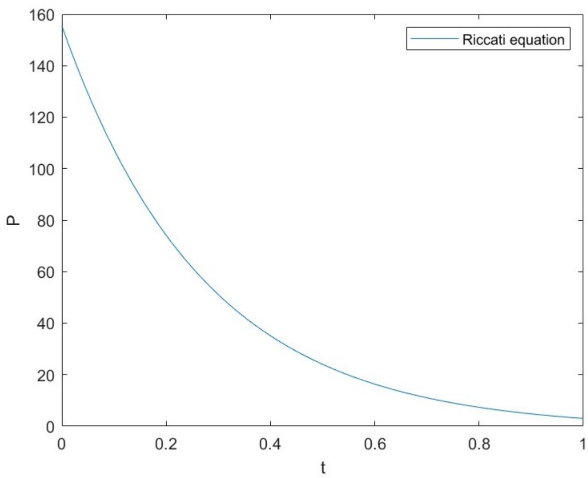

Figure 2. Riccati equation.

5. Concluding Remarks

In this paper, we have investigated the closed-loop solvability of problem (SLQP): the stochastic linear–quadratic optimal control problem with Poisson jumps. We transform the initial problem into a new and simple problem and obtain a Riccati integral–differential equation first. Then we get the characterization of its closed-loop solvability, i.e., the solvabilities of the RIDE and a BSDEP (Theorem 3). We also give a simple example to prove the effectiveness of the main result.

Motivated by [13,14,32,40,41], characterization of the closed-loop solvability for SLQP optimal control problem in infinite horizon, with random coefficients, and of the closed-loop saddle points, Nash equilibria for SLQ zero-sum, nonzero-sum differential games, respectively, are our future research topics. The closed-loop solvability for Stackelberg stochastic LQ differential game is recently studied by Li and Shi [42,43], and we are also interested in that for Stackelberg stochastic LQ differential game of mean-field type. We will consider them in the near future.

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