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Spheres and Tori as Elliptic Linear Weingarten Surfaces

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Abstract: The linear Weingarten condition with ellipticity for the mean curvature and the extrinsic Gaussian curvature on a surface in the three-sphere can define a Riemannian metric which is called the elliptic linear Weingarten metric. We established some local characterizations of the round spheres and the tori immersed in the 3-dimensional unit sphere, along with the Laplace operator, the spherical Gauss map and the Gauss map associated with the elliptic linear Weingarten metric.

Keywords: elliptic linear Weingarten metric; finite-type immersion; spherical Gauss map; isoparametric surface; torus

MSC: 53C40; 53B25

1. Introduction

The complete surfaces of the unit 3-sphere $\mathbb{S}^3$ which has rich topological and geometrical properties together with the Poincaré's conjecture have unique and special geometric properties, such as no complete surfaces being immersed in $\mathbb{S}^3$ with constant extrinsic Gaussian curvature K_N satisfying $K_N < -1$ and $-1 < K_N < 0$ ([1], p. 138). However, there are infinitely many complete and flat surfaces in $\mathbb{S}^3$, such as the tori $\mathbb{S}^1(r_1) \times \mathbb{S}^1(r_2)$ and the product of two plane circles, where $r_1^2 + r_2^2 = 1$. In particular, the Clifford torus $\mathbb{S}^1(1/\sqrt{2}) \times \mathbb{S}^1(1/\sqrt{2})$ is the only minimal and flat surface immersed in $\mathbb{S}^3$ in the 4-dimensional Euclidean space $\mathbb{E}^4$ ([2,3]).

According to Nash's imbedding theorem, a Riemannian manifold is imbedded in a Euclidean space ([4]). Let M be a Riemannian manifold. Due to Nash's idea, we can consider an isometric immersion $x : M \to \mathbb{E}^m$ of M in a Euclidean space $\mathbb{E}^m$. Generalizing Takahashi's eigenvalue problem of an isometric immersion of a submanifold in a Euclidean space, Chen introduced the notion of finite-type immersion ([5,6]). It is said to be of finite-type if the immersion x can be represented as a sum of finitely-many eigenvectors of the Laplace operator Δ of M in the following:

$$x = x_0 + x_1 + \cdots + x_k, \tag{1}$$

where x_0 is a constant vector and $x_1, \ldots, x_k$ are non-constant vectors satisfying $\Delta x_i = \lambda_i x_i$ for some $\lambda_i \in \mathbb{R}$, $i = 1, 2, \ldots, k$. If all of $\lambda_1, \ldots, \lambda_k$ are different, the immersion x is called k-type or the submanifold M is said to be of k-type. Thus, the 1-type immersion is the simplest finite-type one. It is well-known that a submanifold M of the Euclidean space $\mathbb{E}^m$ is of one type if and only if M is a minimal submanifold of $\mathbb{E}^m$ or a minimal submanifold of a hypersphere of $\mathbb{E}^m$ ([6]). Therefore, spherical submanifolds, i.e., submanifolds lying in a sphere, which are of finite-type submanifolds in Euclidean space, are worth studying.

Let $\mathbb{S}^{m-1}$ be a unit hypersphere of $\mathbb{E}^m$ centered at the origin and $x : M \to \mathbb{S}^{m-1}$ be an isometric immersion of a Riemannian manifold M into $\mathbb{S}^{m-1}$. In this case, if the immersion

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x identified with the position vector in the ambient Euclidean space is of finite-type, we call the spherical submanifold M finite type.

The notion of finite-type immersion can be extended to any smooth map $\phi : M \to \mathbb{E}^m$ of M into the Euclidean $\mathbb{E}^m$. A smooth map ϕ is said to be of finite-type if ϕ can be expressed as a sum of finitely many eigenvectors of Δ such as $\phi = \phi_0 + \phi_1 + \ldots + \phi_k$, where ϕ_0 is a constant vector and $\phi_1, \ldots, \phi_k$ are non-constant vectors satisfying $\Delta\phi_i = \lambda_i\phi_i$ for some $\lambda_i \in \mathbb{R}$, $i = 1, 2, \ldots, k$. Among such maps, the Gauss map is one of the most typical and meaningful maps with geometric meaning. The Gauss map of a submanifold M of $\mathbb{E}^m$ is a map of M into a Grassmann manifold $Gr(n, m)$ consisting of all oriented n-planes passing through the origin, which can be defined by $\eta : M \to Gr(n, m) \subset \mathbb{E}^N$ via $\eta(p) = (e_1 \wedge e_2 \wedge \cdots \wedge e_n)(p)$, where $\{e_1, e_2, \cdots, e_m\}$ is an orthonormal frame of $\mathbb{E}^m$ such that $e_1, e_2, \cdots, e_n$ are tangential to M and $e_{n+1}, e_{n+2}, \cdots, e_m$ normal to M ([3,7]).

It is also interesting to consider the case of the Gauss map η satisfying some differential equations, such as $\Delta\eta = f\eta$ or $\Delta\eta = f(\eta + C)$, for some non-zero smooth function f and a constant vector C. For example, the helicoid and the right cone in $\mathbb{R}^3$ have the Gauss map η, which satisfy, respectively, $\Delta\eta = f\eta$ and $\Delta\eta = f(\eta + C)$ for some non-vanishing function f and a non-zero constant vector C. Inspired by this, in [8], the notion of a pointwise 1-type Gauss map was introduced. The Gauss map η of a submanifold M in the Euclidean space $\mathbb{E}^m$ is said to be of *pointwise 1-type* if it satisfies $\Delta\eta = f(\eta + C)$ for some non-zero smooth function f and a constant vector C. In particular, it is said to be of pointwise 1-type of the first kind if the constant vector C is zero. If C is non-zero, it is said to be of pointwise 1-type of the second kind.

For a spherical surface M lying in a unit hypersphere $\mathbb{S}^3$, the position vector x of each point p of $\mathbb{S}^3$ and an orthonormal basis $\{e_1, e_2\}$ of the tangent space T_pM determine an oriented 3-plane in $\mathbb{E}^4$. Thus, we can have a map $\eta^S : M \to G(3, 4) \subset \mathbb{S}^3 \subset \mathbb{E}^4$ via $\eta^S(p) = x \wedge e_1 \wedge e_2$. We call η^S the spherical Gauss map of M in $\mathbb{S}^3$. We now define the pointwise 1-type spherical Gauss map of the spherical submanifold ([9,10]). We also call the spherical Gauss map η^S pointwise 1-type if it satisfies $\Delta\eta^S = f(\eta^S + C)$ for some non-zero smooth function f and a constant vector C. If $C = 0$, it is called pointwise 1-type of the first kind, and pointwise 1-type of the second kind otherwise.

In the present paper, a sphere $\mathbb{S}^2(r)$ $(0 < r \le 1)$ and the tori in $\mathbb{S}^3$ are characterized locally with the notion of ELW metric and its Laplace operator.

We assume that a surface of the sphere $\mathbb{S}^3$ is complete and connected unless stated otherwise, and a compact surface means that it is closed without boundary.

2. Preliminaries

Let $\mathbb{E}^4$ be the 4-dimensional Euclidean space with the canonical metric $\langle\cdot,\cdot\rangle$ and $\mathbb{S}^3$ be the unit hypersphere centered at the origin in $\mathbb{E}^4$. Let M be a surface in $\mathbb{S}^3$. We denote the Levi–Civita connection by $\widetilde{\nabla}$ of $\mathbb{E}^4$ and the induced connection ∇ of M of M in $\mathbb{S}^3$.

The Gauss and Weingarten formulas of M in $\mathbb{E}^4$ are, respectively, given by

$$\tilde{\nabla}_X Y = \nabla_X Y + \langle SX, Y\rangle N - \langle X, Y\rangle x, \tag{2}$$

$$\tilde{\nabla}_X N = -SX, \quad \tilde{\nabla}_X x = X \tag{3}$$

for vector fields X, Y and Z tangent to M, where N is the unit normal vector field associated with the orientation of M in $\mathbb{S}^3$, and $S : TM \to TM$ is the shape operator, where TM is the tangent bundle of M. Let R be the curvature tensor of M. The Gauss equation is then given by

$$\langle R(X, Y)Z, W\rangle = \langle h(X, W), h(Y, Z)\rangle - \langle h(Y, W), h(X, Z)\rangle \tag{4}$$

for tangent vector fields X, Y, Z and W on M. We also obtained the Codazzi equation

$$(\bar{\nabla}h)(X, Y, Z) - (\bar{\nabla}h)(Y, X, Z) = 0 \tag{5}$$

for all vector fields X, Y, Z tangent to M. Here, $(\bar{\nabla}h)(X, Y, Z)$ is defined by $(\bar{\nabla}h)(X, Y, Z) = (\bar{\nabla}_X h)(Y, Z) = \nabla^{\perp}_X h(Y, Z) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z)$. The Codazzi Equation (5) can be written as follows:

$$(\nabla_X S)Y = (\nabla_Y S)X$$

for all tangent vector fields X and Y of M.

Let H and K_N be the mean curvature and the extrinsic Gaussian curvature of M in $\mathbb{S}^3$ defined by $H = \frac{1}{2}\mathrm{tr}S$ and $K_N = \det S$ of M, respectively. M is said to be flat if its Gaussian curvature $K = 1 + K_N$ in $\mathbb{E}^4$ vanishes, and M is said to be minimal (in $\mathbb{S}^3$) if the mean curvature H vanishes. In particular, the Clifford torus $\mathbb{S}^1(1/\sqrt{2}) \times \mathbb{S}^1(1/\sqrt{2})$ is minimal in $\mathbb{S}^3$ and flat in $\mathbb{E}^4$, which is of 1-type in the usual sense in $\mathbb{E}^4$ ([2,6]).

A surface M in $\mathbb{S}^3$ is called Weingarten if some relationship between its two principal curvatures $\kappa_1, \ \kappa_2$ is satisfied, namely, if there is a smooth function (the Weingarten function) of two variables satisfying $W(\kappa_1, \kappa_2) = 0$. It implies $Z(K_N, H) = 0$ for some function Z. Especially, a surface in $\mathbb{S}^3$ is called linear Weingarten if its mean curvature H and the extrinsic Gaussian curvature K_N satisfy

$$2aH + bK_N = c \tag{6}$$

for some constants a, b and c, which are not all zero at the same time. Particularly, $a^2 + bc > 0$ gives the ellipticity for the differential equations of the coordinate functions of a parametrization $x = x(s, t)$ relative to the principal curvatures, and it enables the symmetric tensor $\sigma = aI + bII$ to define a Riemannian metric on the surface, where I is the induced metric on M, and II is the second fundamental form. Briefly speaking, choose an orthonormal basis $\{e_1, e_2\}$ at a point $p \in M$ diagonalizing the shape operator S; i.e.,

$$Se_i = \kappa_i e_i,$$

where $i = 1, 2$. Then,

$$\begin{aligned} &\sigma(e_1, e_1)\sigma(e_2, e_2) - \sigma(e_1, e_2)^2 \\ &= (a + b\kappa_1)(a + b\kappa_2) \\ &= a^2 + b(2aH + bK_N) \\ &= a^2 + bc > 0. \end{aligned} \tag{7}$$

If necessary, the unit normal vector can be chosen as $-\eta$ for σ to be positive definite. We call the surface (M, σ) with the Riemannian metric σ an elliptic linear Weingarten surface abbreviated by an ELW surface and σ an elliptic linear Weingarten metric or simply an ELW metric ([11–13]).

3. The Gauss Map of ELW Surface of $\mathbb{S}^3$ in $\mathbb{E}^4$

Let $x : M \to \mathbb{S}^3$ be an isometric immersion induced from $\mathbb{E}^4$ in a natural manner, and we assume that $\{u, v\}$ is a local coordinate system of M. We may regard x as the position vector of the point of M in $\mathbb{E}^4$.

We use the components of the first fundamental form I by

$$E_1 = \langle x_u, x_u \rangle, F_1 = \langle x_u, x_v \rangle, G_1 = \langle x_v, x_v \rangle$$

and those of the second fundamental forms by

$$E_2 = \langle x_{uu}, N \rangle = \langle Sx_u, x_u \rangle, F_2 = \langle x_{uv}, N \rangle = \langle Sxu, x_v \rangle, G_2 = \langle x_{vv}, N \rangle = \langle Sx_v, x_v \rangle,$$

from which,

$$I = E_1 du^2 + 2F_1 dudv + G_1 dv^2, \quad II = E_2 du^2 + 2F_2 dudv + G_2 dv^2.$$

As was explained in (7), the first and second fundamental forms I and II define a Riemannian metric

$$\sigma = aI + bII \tag{8}$$

on M.

We now assume that $M = (M, \sigma)$ is an ELW surface of $\mathbb{S}^3$ with the ELW metric σ. Let (u, v) be the isothermal coordinates for the metric σ. Then, we have

$$\sigma = (aE_1 + bE_2)du^2 + 2(aF_1 + bF_2)dudv + (aG_1 + bG_2)dv^2 = \lambda(du^2 + dv^2) \tag{9}$$

for some positive valued function λ. From the first and second fundamental forms I and II, we have the shape operator S of the form

$$S = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix} \tag{10}$$

where

$$S_{11} = \frac{1}{E_1G_1 - F_1^2}(G_1E_2 - F_1F_2), \quad S_{12} = \frac{1}{E_1G_1 - F_1^2}(G_1F_2 - F_1G_2), \tag{11}$$

$$S_{21} = \frac{1}{E_1G_1 - F_1^2}(-E_2F_1 + E_1F_2), S_{22} = \frac{1}{E_1G_1 - F_1^2}(E_1G_2 - F_1F_2). \tag{12}$$

Then, Equation (9) gives the Laplacian Δ^σ with respect to the Riemannian metric σ by

$$\begin{aligned} \Delta^\sigma &= -\frac{1}{\sqrt{\det \sigma}}(\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2}) \\ &= -\frac{1}{\lambda}(\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2}). \end{aligned} \tag{13}$$

If we compute λ^2 by using (9), we have

$$\lambda^2 = (aE_1 + bE_2)(aG_1 + bG_2) - (aF_1 + bF_2)^2,$$

from which

$$\lambda^2 = \{a^2 + b(2aH + bK_N)\}(E_1G_1 - F_1^2).$$

Since $2aH + bK_N = c$, we get

$$\lambda^2 = (a^2 + bc)(E_1G_1 - F_1^2).$$

Without loss of generality, we may assume that $a^2 + bc = 1$. Then, we get

$$\lambda = \sqrt{E_1G_1 - F_1^2}. \tag{14}$$

We now define the Gauss map $\eta : M \to \Lambda^2\mathbb{E}^4 \cong \mathbb{E}^6$ of M and the spherical Gauss map $\eta^S : M \to \Lambda^3\mathbb{E}^4$ by

$$\eta = \frac{x_u \wedge x_v}{||x_u \wedge x_v||} = \frac{x_u \wedge x_v}{\lambda} \quad \text{and} \quad \eta^S = \frac{x \wedge x_u \wedge x_v}{||x \wedge x_u \wedge x_v||} = \frac{x \wedge x_u \wedge x_v}{\lambda}. \tag{15}$$

4. Great Spheres as ELW Surfaces of $\mathbb{S}^3(1)$

Let $x : M \to \mathbb{S}^3$ be a ELW surface immersed in $\mathbb{S}^3$ with the isothermal coordinates (u, v) associated with the ELW metric σ.

Since the vector space $\Lambda^3\mathbb{E}^4 = \{X \wedge Y \wedge Z | X, Y, Z \in \mathbb{E}^4\}$ is naturally identified with $\mathbb{E}^4$, we can define an inner product $(X_1 \wedge X_2 \wedge X_3, Y)$ with $X_1 \wedge X_2 \wedge X_3 \in \Lambda^3\mathbb{E}^4$ and $Y \in \mathbb{E}^4$ as

$$(X_1 \wedge X_2 \wedge X_3, Y) = \det \begin{pmatrix} Y \\ X_1 \\ X_2 \\ X_3 \end{pmatrix}, \tag{16}$$

where the determinant is taken by the 4×4-matrix made up of the components of the vectors X_1, X_2, X_3 and Y in $\mathbb{E}^4$. Thus, the spherical Gauss map η^S of M can be viewed as a unit normal vector field N in $\mathbb{S}^3$. Without loss of generality, we may assume that η^S is a unit vector field normal to M in $\mathbb{S}^3$. Let us consider an example of a surface with 1-type spherical Gauss map in $\mathbb{S}^3$.

Lemma 1. *Let M be an ELW surface of $\mathbb{S}^3$. The spherical Gauss map η^S satisfies*

$$\Delta^\sigma \eta^S = \Phi_1 x_u + \Phi_2 x_v - (1/\lambda) tr(S^2 I)\eta^S + (1/\lambda)(E_2 + G_2)x, \tag{17}$$

for some functions Φ_1 and Φ_2 defined on M.

Proof. For the ELW metric σ of M, the Laplacian $\Delta^\sigma \eta^S$ with respect to the isothermal coordinates (u, v) is given by

$$\begin{aligned} \Delta^\sigma \eta^S &= -\frac{1}{\lambda}\left(\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2}\right)\eta^S \\ &= -\frac{1}{\lambda}(\tilde{\nabla}_{x_u}\tilde{\nabla}_{x_u} + \tilde{\nabla}_{x_v}\tilde{\nabla}_{x_v})\eta^S. \end{aligned} \tag{18}$$

Note that η^S can be regarded as the unit normal vector field of M in $\mathbb{S}^3$. If we apply the Gauss and Weingarten formulas (2) and (3) to (18), we get

$$\begin{aligned} \Delta^\sigma \eta^S &= \Phi_1 x_u + \Phi_2 x_v - (1/\lambda)\mathrm{tr}(S^2 I)\eta^S + (1/\lambda)\mathrm{tr}(SI)x \\ &= \Phi_1 x_u + \Phi_2 x_v - (1/\lambda)\mathrm{tr}(S^2 I)\eta^S + (1/\lambda)(E_2 + G_2)x \end{aligned}$$

for some functions Φ_1 and Φ_2 defined on M. □

From Lemma 17, we immediately have

Proposition 1. *Let M be a ELW surface of $\mathbb{S}^3$ with the spherical Gauss map η^S. Then, the spherical Gauss map η^S is harmonic with respect to the ELW metric σ, i.e., $\Delta^\sigma \eta^S = 0$, if and only if M is part of a great sphere $\mathbb{S}^2$.*

Proof. If the spherical Gauss map η^S is harmonic with respect to the ELW metric σ, we see that M is totally geodesic from (17). Thus, M is part of a great sphere.

Conversely, suppose that M is part of great sphere $\mathbb{S}^2$. Then, the shape operator S vanishes, and so does the second fundamental form II. For the isothermal coordinates (u, v) for the ELW metric σ, (8) gives $\sigma = I$ with $a = 1$, $E_1 = G_1$ and $F_1 = 0$. A direct computation yields $\Delta^\sigma \eta^S = 0$. □

Theorem 1. *Let M be a ELW surface of $\mathbb{S}^3$ with the spherical Gauss map η^S of pointwise 1-type of the first kind with respect to the ELW metric σ. Then, we have*
(1) If $b \neq 0$, the mean curvature H of M is constant.
(2) M is minimal if and only if the induced metric I coincides with the ELW metric σ, i.e., $b = 0$ and $a = 1$.

Proof. Since the spherical Gauss map η^S is of pointwise 1-type of the first kind with respect to the ELW metric σ, i.e.,

$$\Delta^\sigma \eta^S = f\eta^S, \tag{19}$$

for some non-zero smooth function f. Since the vector fields x_u, x_v, $N = \eta^S$ and x are linearly independent, Lemma 1 implies that $\Phi_1 = \Phi_2 = 0$ and $E_2 + G_2 = 0$. Since $aI + bII = \sigma$, we get

$$aE_1 + bE_2 = aG_1 - bE_2 = \lambda.$$

It follows $2\lambda = a(E_1 + G_1)$ and $2b/aE_2 = G_1 - E_1$. Since $aF_1 + bF_2 = 0$, the mean curvature H is given by

$$H = (1/2\lambda^2)(E_2G_1 - 2F_1F_2 + G_2E_1) = (a/4b\lambda^2)\{(E_1 - G_1)^2 + 4F_1^2\}.$$

(1) $\Rightarrow$ (2) If M is minimal, $E_1 = G_1$ and $F_1 = 0$. If $b \neq 0$, $E_2 = G_2 = 0$ and $F_2 = 0$ and II vanish. Therefore, $\Delta^\sigma \eta^S = 0$, a contradiction. Thus, $b = 0$ and the induced metric I is the same as the ELW metric σ.

(2) $\Rightarrow$ (1) is obvious. □

Example 1. *The Clifford torus $\mathbb{S}^1(1/\sqrt{2}) \times \mathbb{S}^1(1/\sqrt{2})$ is a minimal and flat compact surface of $\mathbb{S}^3$ which is of 1-type. A parametrization x of the Clifford torus is given by*

$$x(s,t) = 1/\sqrt{2}(\cos s, \sin s, \cos t, \sin t).$$

Choose

$$\eta^S = 1/\sqrt{2}(\cos s, \sin s, -\cos t, -\sin t).$$

We easily see that

$$I = \begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix}, \quad II = \begin{pmatrix} -1/2 & 0 \\ 0 & 1/2 \end{pmatrix}, \quad \sigma = I = \begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix}$$

It is straightforward to show that

$$\Delta^\sigma \eta^S = 2\eta^S.$$

Therefore, the spherical Gauss map η^S is of 1-type with respect to the Laplacian associated with the induced metric and the ELW metric.

5. Characterization of the Flat Tori with ELW Metric

In [10], Chen at al. studied and classified the spherical submanifolds with a 1-type spherical Gauss map which was associated with the eigenvalues of the Laplacian defined by the induced metrics and that of the ambient manifold. In this section, we will characterize the flat tori in $\mathbb{S}^3$ associated with the ELW metric and its associated Laplacian.

Let M be a ELW surface of $\mathbb{S}^3$ with the metric σ defined by (8). As we discussed in the previous section, the vector space $\Lambda^3\mathbb{E}^4$ can be identified with $\mathbb{E}^4$.

Theorem 2. *Let M be a ELW surface of $\mathbb{S}^3$. Then, M is part of a torus $\mathbb{S}^1(r_1) \times \mathbb{S}^1(r_2)$ with $r_1^2 + r_2^2 = 1$ if and only if the spherical Gauss map η^S is of 1-type with respect to the ELW metric σ.*

Proof. Suppose that the spherical Gauss map η^S is of 1-type with respect to σ; i.e., η^S is expressed as $\eta^S = \eta_0^S + \eta_1^S$, where η_0^S is a constant vector and η_1^S is a non-constant vector satisfying $\Delta^\sigma \eta_1^S = k\eta_1^S$ for some $k \in \mathbb{R}$ $(\neq 0)$. It follows that $\Delta^\sigma \eta^S = k\eta^S + \mathbb{C}$ for some constant vector $\mathbb{C}$. Together with (17), we get

$$\langle \mathbb{C}, x\rangle = g \quad \text{and} \quad \langle \mathbb{C}, \eta^S\rangle = k, \tag{20}$$

where $g = 1/\lambda(E_2 + G_2)$. Taking the covariant derivative to the second equation of (20) with respect to a tangent vector field X, we get $\langle \mathbb{C}, SX \rangle = 0$. Thus,

$$\kappa_1 \langle \mathbb{C}, X_1 \rangle = \kappa_2 \langle \mathbb{C}, X_2 \rangle = 0, \tag{21}$$

where k_1 and κ_2 are the principal curvatures of S corresponding to the principal directions X_1 and X_2, respectively.
Case (1). $\kappa_1 = \kappa_2 = 0$; i.e., M is totally geodesic. Thus, M is part of a great sphere $\mathbb{S}^2$. Taking into account (17), part of great sphere does not have 1-type spherical Gauss map.
Case (2). $\kappa_1 \neq 0$ or $\kappa_2 \neq 0$. Suppose that the open subset $O = \{p \in M | (\kappa_2(p) \neq 0\}$ is not empty. From (21), one can obtain $X_2 g = 0$ on O. Taking the covariant differentiation to $\langle \mathbb{C}, X_2 \rangle = 0$ with respect to X_1, we obtain

$$\omega_2^1(X_1) \langle \mathbb{C}, X_1 \rangle = 0, \tag{22}$$

where ω_2^1 is the connection form defined by $\nabla_{X_1} X_2 = \omega_2^1(X_1) X_1$. Suppose that there exists a point $q \in O$ such that $\omega_2^1(X_1)(q) \neq 0$. Then, there exists an open subset $U \subset O$ such that $\omega_2^1(X_1) \neq 0$ everywhere on U. Therefore, $\langle \mathbb{C}, X_1 \rangle = 0$ everywhere on U. Thus, $g = 1/\lambda(E_2 + G_2)$ is constant on U and

$$\langle \mathbb{C}, X \rangle = 0$$

for every tangent vector field X on U. Thus, $\mathbb{C}$ is normal to M on U. If we take the covariant derivative to

$$\langle \mathbb{C}, X_i \rangle = 0$$

with respect to X_i for each $i \in \{1, 2\}$, we see that κ_1 and κ_2 are constant on U. Therefore, the open subset U lies in an isoparametric surface in $\mathbb{S}^3$. Since M is connected, M is part of a torus $S^1(r_1) \times S^1(r_2)$ with $r_1^2 + r_2^2 = 1$ or an ordinary sphere $\mathbb{S}^2(r)$ with $0 < r \leq 1$).

We will show that a sphere $\mathbb{S}^2(r)$ does not have 1-type spherical Gauss map with respect to the ELW metric σ. Suppose that M is an ordinary sphere $\mathbb{S}^2(r)$ of radius r ($0 < r \leq 1$). Choose the isothermal coordinate (u, v) on M. Let the first fundamental form I, second fundamental form II and ELW metric σ be

$$I = \begin{pmatrix} E_1 & F_1 \\ F_1 & G_1 \end{pmatrix}, \quad II = \begin{pmatrix} E_2 & F_2 \\ F_2 & G_2 \end{pmatrix}, \quad \sigma = aI + bII. \tag{23}$$

Since M is totally umbilic in $\mathbb{S}^3$, $SX = -1/rX$ for every tangent vector field X on M. Therefore, $E_2 = -E_1/r$, $F_2 = -F_1/r$, $G_2 = -G_1/r$ and $\lambda = (a - b/r)E_1 = (a - b/r)G_1$ and $(a - b/r)F_1 = 0$. Thus, $F_1 = 0$. Using $\lambda^2 = E_1 G_1 - F_1^2$, we get $(a - b/r)^2 = \lambda^2$ =a constant. We may assume that $a - b/r = 1$. It follows that $\lambda = E_1 = G_1$, which is constant. Therefore, M is flat in $\mathbb{E}^4$, a contradiction. Thus, the spherical Gauss map of $\mathbb{S}^2(r)$ is not of 1-type with respect to the metric σ.

Conversely, suppose that M is part of a product of two circles $S^1(r_1) \times S^1(r_2)$ with $r_1^2 + r_2^2 = 1$, which is flat in $\mathbb{E}^4$. Take the parametrization of M as follows:

$$x(s, t) = (r_1 \cos s, r_1 \sin s, r_2 \cos t, r_2 \sin t). \tag{24}$$

Then, we can choose the spherical Gauss map

$$\eta^S = (r_2 \cos s, r_2 \sin s, -r_1 \cos t, -r_1 \sin t).$$

From the last two equations, we can get

$$E_1 = \langle x_s, x_s \rangle = r_1^2, F_1 = 0, G_1 = \langle x_t, x_t \rangle = r_2^2, E_2 = -r_1 r_2, F_2 = 0, G_2 = r_1 r_2. \tag{25}$$

Then, the ELW metric σ is given by

$$\sigma = a\begin{pmatrix} r_1^2 & 0 \\ 0 & r_2^2 \end{pmatrix} + b\begin{pmatrix} -r_1 r_2 & 0 \\ 0 & r_1 r_2 \end{pmatrix}.$$

Since the shape operator S is given by $S = I^{-1}II$, S is determined as

$$S = \begin{pmatrix} -r_2/r_1 & 0 \\ 0 & r_1/r_2 \end{pmatrix},$$

from which, $\mathrm{tr}(SI) = 0$ and $\mathrm{tr}(S^2 I) = r_1^2 + r_2^2 = 1$. From the fact that all components of I and II are constant, the function λ induced by the components of σ is constant. Since the trace is invariant under the change of basis and M is flat in $\mathbb{E}^4$, Lemma 1 gives $\Delta^\sigma \eta^S = -(1/\lambda)\eta^S$. Thus, η^S is of 1-type with respect to the ELW metric σ. This completes the proof. $\square$

6. Discussion

The topic could be developed further in the higher dimensional cases.

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