mathematics

Article

Generalized Contractions and Fixed Point Results in Spaces with Altering Metrics

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Abstract: In this paper, we have provided some fixed point results for self-mappings fulfilling generalized contractive conditions on altered metric spaces. In addition, some applications of the main results to continuous data dependence of the fixed points of operators defined on these spaces were shown.

Keywords: generalized contractions; fixed point theorems; spaces with altering metrics; data dependence

MSC: 47H10; 47H09; 54E50

Citation: Branga, A.N.; Olaru, I.M. Generalized Contractions and Fixed Point Results in Spaces with Altering Metrics. *Mathematics* **2022**, *10*, 4083. https://doi.org/10.3390/math10214083

Academic Editor: Timilehin Opeyemi Alakoya

Received: 25 September 2022
Accepted: 31 October 2022
Published: 2 November 2022

1. Introduction

The fixed point theorems for an operator $T : X \to X$ related to altering distances between points in complete metric space were originally achieved by Delbosco [1], Skof [2], M.S. Khan, M. Swaleh and S. Sessa [3] by using some suitable distance control function $\mu : \mathbb{R}_+ \to \mathbb{R}_+$, where $\mathbb{R}_+$ is the real interval $[0, \infty)$, and contractive conditions of type

$$\mu(d(T(x), T(y))) \leq a \cdot \mu(d(x, y)) + b \cdot \mu(d(x, T(x))) + c \cdot \mu(d(y, T(y))), 0 \leq a + b + c < 1,$$

or more general

$$\mu(d(T(x), T(y))) \leq$$
$$a(d(x, y)) \cdot \mu((d(x, y)) + b(d(x, y)) \cdot \{\mu(d(x, T(x))) + \mu(d(y, T(y)))\}+$$
$$c(d(x, y)) \cdot \min\{\mu(x, T(y)), \mu(y, T(x))\},$$

for all $x, y \in X$, $x \neq y$ and $a, b, c : {\mathbb{R}_+}^* \to [0, 1)$ being decreasing functions in order that $a(t) + 2 \cdot b(t) + c(t) < 1$ for every $t > 0$. Also in [4], the authors considered a contractive condition of type

$$\mu(d(T(x), T(y))) \leq \alpha(d(x, y)) \cdot \mu(d(x, y)), \forall x, y \in X,$$

where $\alpha : \mathbb{R}_+ \to [0, 1)$ is the order that $\limsup_{s \to t} \alpha(s) < 1$. Further Akkouchi et al. [5], Pant et al. [6–8] and Sastry et al. [9] have obtained common fixed point results by altering the distance between the points of a metric space. Moreover, the fixed point results by altering distance between the points was extended to the setup of generalized metric spaces (fuzzy metrics spaces Masmali et al. [10], orthogonal complete metric Gungor [11], partially ordered metric spaces Gupta et al. [12]) or to cyclic operators, see Khaleel et al. [13]. Recently, Branga and Olaru [14] extended the above results by altering the distance between two points and considering a contractive condition of type

$$\mu(d(T(x), T(y))) \leq \eta(\mu(d(x, y))), \tag{1}$$

for all $x, y \in X$, $x \neq y$ and $\eta : [0, \infty) \to [0, \infty)$ is a monotone increasing, right continuous and satisfies $\eta(t) < t$ for each $t > 0$. A survey work on some fixed point theorems by altering distances between points on a metric space can be found on Jha et al. [15]. Some recent applications of fixed point theory may be found on Rezapour et al. [16], Zareen et al. [17] and Turab et al. [18]. Next, our aim is to extend the results from [14] by considering a contractive condition of type (1), η being a right upper semi-continuous function.

2. Preliminaries

Next, we recall the definitions of the upper semi-continuous and right upper semi-continuous functions.

Definition 1 ([19])**.** *Let us consider A a subset of $\mathbb{R}$, $a \in A$ a point and $f : A \to \mathbb{R}$ a function. The following can be affirmed:*

(1) f is upper semicontinuous at a if for every $\varepsilon > 0$ there is $\delta(\varepsilon) > 0$ in order that

$$f(x) < f(a) + \varepsilon \text{ for all } x \in (a - \delta(\varepsilon), a + \delta(\varepsilon)) \cap A;$$

(2) f is upper semicontinuous if it is upper semicontinuous at every point $a \in A$;

(3) f is right upper semicontinuous at a if for each $\varepsilon > 0$ there is $\delta(\varepsilon) > 0$ in order that

$$f(x) < f(a) + \varepsilon \text{ for all } x \in (a, a + \delta(\varepsilon)) \cap A;$$

(4) f is right upper semicontinuous if it is right upper semicontinuous at every point $a \in A$.

Remark 1. *Let us consider A a subset of $\mathbb{R}$, $a \in A$ a point and $f : A \to \mathbb{R}$ a function. The following can be remarked:*

(1) if f is right-continuous at a, then f is right upper semi-continuous at a;

(2) if f is right upper semi-continuous at a and f is monotonically increasing, then f is right-continuous at a;

(3) if f is upper semi-continuous at a, then f is right upper semi-continuous at a.

The following results will be used in order to proof Lemma 2:

Theorem 1 ([19])**.** *Let A be a subset of $\mathbb{R}$, $a \in A'$, (the set of accumulation points of A) and $f : A \to \mathbb{R}$ a function. Then:*

(1) f is upper semi-continuous at a if and only if

$$\limsup_{x \to a} f(x) \leq f(a);$$

(2) f is right upper semi-continuous at a if and only if

$$\limsup_{x \searrow a} f(x) \leq f(a).$$

Theorem 2 ([19])**.** *Let us consider A a subset of $\mathbb{R}$, $a \in A$ a point and $f : A \to \mathbb{R}$ a function. Then:*

(1) f is upper semi-continuous at a if and only if, for each sequence $(a_n)_{n \in \mathbb{N}} \subseteq A$ satisfying $a_n \to a$ as $n \to \infty$, we have

$$\limsup_{n \to \infty} f(a_n) \leq f(a);$$

(2) f is right upper semicontinuous at a if and only if, for every sequence $(a_n)_{n \in \mathbb{N}} \subseteq A$ satisfying $a_n \to a$ as $n \to \infty$, $a_n \geq a$ for all $n \in \mathbb{N}$, we have

$$\limsup_{n \to \infty} f(a_n) \leq f(a).$$

Theorem 3 ([19]). *If A is a subset of $\mathbb{R}$ and $f : A \to \mathbb{R}$ a function, then f is upper semi-continuous if and only if the superlevel set $U_y(f) := \{x \in A \mid f(x) \geq y\}$ is closed in A for every $y \in \mathbb{R}$.*

Theorem 4 ([19]). *Let $(a_n)_{n\in\mathbb{N}} \subseteq \mathbb{R}$ be a sequence and $a \in \mathbb{R}$. Then,*

$$\limsup_{n\to\infty} a_n \leq a,$$

if and only if there is a number $n_0 \in \mathbb{N}$ in order that

$$a_n \leq a \text{ for all } n \geq n_0.$$

Boyd and Wong [20] extend the contraction principle (the Picard–Banach theorem) in complete metric spaces.

Theorem 5 ([20]). *Let $\eta : \mathbb{R}_+ \to \mathbb{R}_+$ be a function fulfilling the statements: η is right upper semicontinuous and $\eta(t) < t$ for all $t > 0$. If (X, d) is a complete metric space and $T : X \to X$ is an operator in order that*

$$d(T(x), T(y)) \leq \eta(d(x, y)), \ \forall x, y \in X,$$

then T has a unique fixed point $x^* \in X$ and the sequence $T^m(x_0) \to x^*$ as $m \to \infty$, for any arbitrary point $x_0 \in X$.

The following result will represent a generalization of the above Boyd's result and it will be used in order to prove Lemma 3 and Theorem 8.

Definition 2 ([21]). *A function $\eta : \mathbb{R}_+^k \to \mathbb{R}_+$, $k \geq 1$ is a comparison function if:*

(i) *η is increasing with respect to each variable, i.e., the mapping $t_i \to \eta(t_1, \cdots, t_i, \cdots, t_k)$ is increasing for every $i \in \{1, \ldots, k\}$;*
(ii) *the iterates sequence $\mu^n(t) \to 0$ as $n \to \infty$, for every $t > 0$, where $\mu : \mathbb{R}_+ \to \mathbb{R}_+$ is defined by $\mu(t) := \eta(t, t, \cdots, t)$.*

Theorem 6 ([21]). *Let us consider (X, d) a complete metric space, $\eta : \mathbb{R}_+^5 \to \mathbb{R}_+$ a comparison function and $T : X \to X$ be an operator in order that*

$$d(T(x), T(y)) \leq \eta(d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)), \ \forall x, y \in X,$$

T has a unique fixed point $x^* \in X$ and the sequence $T^m(x_0) \to x^*$ as $m \to \infty$, for any arbitrary point $x_0 \in X$.

3. Results

Definition 3 ([3]). *A function $\gamma : \mathbb{R}_+ \to \mathbb{R}_+$ belongs to the class Γ, if:*

(i) *γ is continuous;*
(ii) *γ is monotonically increasing;*
(iii) *$\gamma(t) = 0$ if and only if $t = 0$.*

Let us consider (X, d) a metric space. When the metric d is changed by a function $\gamma \in \Gamma$, it can be seen that, in the majority of cases, the application $\gamma \circ d$ does not keep the metric properties.

Example 1. *Let us consider $d : \mathbb{R} \times \mathbb{R} \to \mathbb{R}_+$, $d(x, y) = |x - y|$ and $\gamma : \mathbb{R}_+ \to \mathbb{R}_+$, $\gamma(t) = t^4$. The following can be affirmed:*

(1) *$\gamma \in \Gamma$;*
(2) *$\gamma \circ d$ is not a metric on X.*

Proof.

(1) It is obvious that γ verifies the conditions from Definition 3.
(2) By taking $x = 2$, $y = 3$ and $z = 2.1$, we observe that the triangle inequality is not verified for $\gamma \circ d$, and consequently it is not a metric on $\mathbb{R}$.

□

Lemma 1. *Let $\eta : \mathbb{R}_+ \to \mathbb{R}_+$ be a function, under the following hypothesis:*

(1) is right upper semicontinuous;
(2) $\eta(t) < t$ for all $t > 0$.

Then:

$$\liminf_{s \searrow t}(s - \eta(s)) > 0 \text{ for every } t > 0.$$

Proof. By using the hypothesis (2), it follows that $\liminf_{s \searrow t}(s - \eta(s)) \geq 0$ for every $t > 0$. Suppose that there exists $t_0 > 0$ such that $\liminf_{s \searrow t_0}(s - \eta(s)) = 0$. Taking into consideration the properties of the limit inferior and limit superior of a function, the fact that η is right upper semi-continuous, applying Theorem 1 (2) and the hypothesis (2), we obtain

$$t_0 = \liminf_{s \searrow t_0} \eta(s) \leq \limsup_{s \searrow t_0} \eta(s) \leq \eta(t_0) < t_0,$$

which is a contradiction. Consequently, $\liminf_{s \searrow t}(s - \eta(s)) > 0$ for every $t > 0$. □

Lemma 2. *Let be $\gamma \in \Gamma$, $\mu : \mathbb{R}_+ \to \mathbb{R}_+$ defined by:*

$$\mu(t) = \sup\{s \in \mathbb{R}_+ \mid \gamma(s) \leq \eta(\gamma(t))\}, \tag{2}$$

and $\eta : \mathbb{R}_+ \to \mathbb{R}_+$ a function, under the following hypothesis:

(1) $\eta(0) = 0$;
(2) η is right upper semicontinuous;
(3) $\eta(t) < t$ for all $t > 0$.

Then:

(i) μ is well defined;
(ii) $\mu(0) = 0$;
(iii) $\mu(t) \leq t$ for all $t \in \mathbb{R}_+$;
(iv) $\gamma(\mu(t)) \leq \eta(\gamma(t))$ for all $t \in \mathbb{R}_+$;
(v) $\mu(t) < t$ for all $t > 0$;
(vi) $\eta \circ \gamma$ is right upper semi-continuous;
(vii) μ is right upper semi-continuous.

Proof. (i) Let us consider $t \in \mathbb{R}_+$ an arbitrary chosen number. We construct the set

$$A_t := \{s \in \mathbb{R}_+ \mid \gamma(s) \leq \eta(\gamma(t))\}. \tag{3}$$

As $\gamma(0) = 0$ (in accordance with Definition 3 (iii)) and $\eta(\gamma(t)) \geq 0$ ($\eta, \gamma : \mathbb{R}_+ \to \mathbb{R}_+$), we obtain $\gamma(0) \leq \eta(\gamma(t))$, therefore $0 \in A_t$, so A_t is a non-empty set. The next cases can be differentiated:

1. $t = 0$:
   As $\gamma(0) = 0$ (in accordance with Definition 3 (iii)) and $\eta(0) = 0$ (by the hypothesis (1)) we obtain $\eta(\gamma(0)) = 0$, therefore $A_0 = \{s \in \mathbb{R}_+ \mid \gamma(s) \leq 0\}$. Taking into account Definition 3 (iii), it is obtained that $A_0 = \{0\}$. It results in $\mu(0) = \sup A_0 = \sup\{0\} = 0$.
2. $t > 0$:
   Select $s \in A_t$ is an arbitrary chosen element. One has $s \in \mathbb{R}_+$ and $\gamma(s) \leq \eta(\gamma(t))$. On

the opposite side, as $t > 0$, considering Definition 3 (iii), we obtain $\gamma(t) > 0$. Applying hypothesis (3), we obtain $\eta(\gamma(t)) < \gamma(t)$. It results that $\gamma(s) < \gamma(t)$. Taking into account that γ is monotonically increasing (using Definition 3 (ii)), it is found that $s < t$. Hence, $s \in [0, t)$. Considering that we have arbitrary selected $s \in A_t$, it follows that $A_t \subseteq [0, t)$. As a result, the set A_t is bounded from above by t. We conclude that, there is $\sup A_t \leq t$. Therefore, $\mu(t) := \sup A_t \leq t$ is well defined and we get $\mu(t) \leq t$.

(ii), (iii) follows from (i).

(iv) Let us consider $t \in \mathbb{R}_+$ an arbitrary selected element. In accordance with (i), the set A_t is bounded from above by t and $\mu(t) := \sup A_t$. It results that, there is a sequence $(s_n)_{n\in\mathbb{N}} \subseteq A_t$ in order that $s_n \to \mu(t)$ as $n \to \infty$ and $s_n \leq \mu(t)$ for all $n \in \mathbb{N}$. Considering that $s_n \in A_t$ for all $n \in \mathbb{N}$, it is concluded that

$$\gamma(s_n) \leq \eta(\gamma(t)) \text{ for all } n \in \mathbb{N}.$$

On the opposite side, as γ is continuous (using Definition 3 (i)), we obtain $\gamma(s_n) \to \gamma(\mu(t))$ as $n \to \infty$. Hence, from the previous inequality, we conclude that $\gamma(\mu(t)) \leq \eta(\gamma(t))$.

Specifically, $\mu(t) \in A_t$ and $A_t \subseteq [0, \mu(t)]$. Select $s \in [0, \mu(t)]$. We obtain $s \leq \mu(t)$, and taking into account that γ is monotonically increasing (in accordance with Definition 3 (ii)), it follows that $\gamma(s) \leq \gamma(\mu(t))$. Hence, $\gamma(s) \leq \eta(\gamma(t))$, i.e., $s \in A_t$. As a result, $A_t = [0, \mu(t)]$.

(v) From (iii), we obtain $\mu(t) \leq t$ for all $t \in \mathbb{R}_+$. Assume that there is $t > 0$ in order that $\mu(t) = t$. Applying (iv) we obtain $\gamma(t) \leq \eta(\gamma(t))$. On the other side, $t > 0$ implies $\gamma(t) > 0$ (in accordance with to Definition 3 (iii)) and applying hypothesis (3) we obtain $\eta(\gamma(t)) < \gamma(t)$. It results that $\gamma(t) < \gamma(t)$, which contradicts the initial assumption. Therefore, $\mu(t) < t$ for all $t > 0$.

(vi) As $\eta, \gamma : \mathbb{R}_+ \to \mathbb{R}_+$ we deduce $\eta \circ \gamma : \mathbb{R}_+ \to \mathbb{R}_+$. Let $t \in \mathbb{R}_+$ be an arbitrary point. We consider an arbitrary sequence $(t_n)_{n\in\mathbb{N}} \subseteq \mathbb{R}_+$ satisfying $t_n \to t$ as $n \to \infty$, $t_n \geq t$ for all $n \in \mathbb{N}$. Since γ is continuous (in accordance with Definition 3 (i)), we obtain $\gamma(t_n) \to \gamma(t)$ as $n \to \infty$. Because γ is monotonically increasing (by Definition 3 (ii)), we find that $\gamma(t_n) \geq \gamma(t)$ for all $n \in \mathbb{N}$. Therefore, the sequence $(\gamma(t_n))_{n\in\mathbb{N}} \subseteq \mathbb{R}_+$ has the following properties: $\gamma(t_n) \to \gamma(t)$ as $n \to \infty$, $\gamma(t_n) \geq \gamma(t)$ for all $n \in \mathbb{N}$. On the other hand, η is right upper semi-continuous, hence it is right upper semi-continuous at $\gamma(t) \in \mathbb{R}_+$. Applying Theorem 2 (2), it follows that $\limsup_{n\to\infty} \eta(\gamma(t_n)) \leq \eta(\gamma(t))$, i.e.,

$$\limsup_{n\to\infty} (\eta \circ \gamma)(t_n) \leq (\eta \circ \gamma)(t). \tag{4}$$

Since the sequence $(t_n)_{n\in\mathbb{N}} \subseteq \mathbb{R}_+$ satisfying $t_n \to t$ as $n \to \infty$, $t_n \geq t$ for all $n \in \mathbb{N}$, was chosen arbitrarily, from the inequality (4), by using Theorem 2 (2), it results that $\eta \circ \gamma$ is right upper semi-continuous at $t \in \mathbb{R}_+$. Because the point $t \in \mathbb{R}_+$ was arbitrarily selected, we deduce that $\eta \circ \gamma$ is right upper semi-continuous.

(vii) Let $t \in \mathbb{R}_+$ be an arbitrary point. We consider an arbitrary sequence $(t_n)_{n\in\mathbb{N}} \subseteq \mathbb{R}_+$ satisfying $t_n \to t$ as $n \to \infty$, $t_n \geq t$ for all $n \in \mathbb{N}$. Since γ is continuous (in accordance with Definition 3 (i)), we obtain $\gamma(t_n) \to \gamma(t)$ as $n \to \infty$. Because γ is monotonically increasing (by Definition 3 (ii)), we find that $\gamma(t_n) \geq \gamma(t)$ for all $n \in \mathbb{N}$. Therefore, the sequence $(\gamma(t_n))_{n\in\mathbb{N}} \subseteq \mathbb{R}_+$ has the following properties: $\gamma(t_n) \to \gamma(t)$ as $n \to \infty$, $\gamma(t_n) \geq \gamma(t)$ for all $n \in \mathbb{N}$. On the other hand, η is right upper semi-continuous, hence it is right upper semi-continuous at $\gamma(t) \in \mathbb{R}_+$. Applying Theorem 2 (2), it follows that

$$\limsup_{n\to\infty} \eta(\gamma(t_n)) \leq \eta(\gamma(t)). \tag{5}$$

Taking into account Theorem 4, from the relation (5) we deduce that there exists a number $n_0 \in \mathbb{N}$ such that

$$\eta(\gamma(t_n)) \leq \eta(\gamma(t)) \text{ for all } n \geq n_0. \tag{6}$$

From the relation (6) we obtain

$$\{s \in \mathbb{R}_+ \mid \gamma(s) \leq \eta(\gamma(t_n))\} \subseteq \{s \in \mathbb{R}_+ \mid \gamma(s) \leq \eta(\gamma(t))\} \text{ for all } n \geq n_0,$$

hence

$$\sup\{s \in \mathbb{R}_+ \mid \gamma(s) \leq \eta(\gamma(t_n))\} \leq \sup\{s \in \mathbb{R}_+ \mid \gamma(s) \leq \eta(\gamma(t))\} \text{ for all } n \geq n_0,$$

and considering the definition of the function μ (the relation (2)) we find

$$\mu(t_n) \leq \mu(t) \text{ for all } n \geq n_0. \tag{7}$$

Using Theorem 4, the inequality (7) implies

$$\limsup_{n\to\infty} \mu(t_n) \leq \mu(t). \tag{8}$$

Since the sequence $(t_n)_{n\in\mathbb{N}} \subseteq \mathbb{R}_+$ satisfying $t_n \to t$ as $n \to \infty$, $t_n \geq t$ for all $n \in \mathbb{N}$, was chosen arbitrarily, from the inequality (8), by using Theorem 2 (2), it results that μ is right upper semi-continuous at $t \in \mathbb{R}_+$. Because the point $t \in \mathbb{R}_+$ was arbitrarily selected, we deduce that μ is right upper semi-continuous. □

Lemma 3. *Let $\eta : \mathbb{R}_+^5 \to \mathbb{R}_+$ be a function under the following hypothesis:*

(1) *$t \to \eta(t,t,t,t,t) \in \mathbb{R}_+$ is increasing and right upper semi-continuous;*
(2) *$\eta(t_1,t_2,t_3,t_4,t_5) < \max\{t_1,t_2,t_3,t_4,t_5\}$, for all $(t_1,t_2,t_3,t_4,t_5) \in \mathbb{R}_+^5 \setminus \{(0,0,0,0,0)\}$;*
(3) *η is increasing with respect to each variable*

and a function $\gamma \in \gamma$. We define the functions $\mu : \mathbb{R}_+^5 \to \mathbb{R}_+$ and $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ by

$$\mu(t_1,t_2,t_3,t_4,t_5) = \sup\{s \in \mathbb{R}_+ \mid \gamma(s) \leq \eta(\gamma(t_1),\gamma(t_2),\gamma(t_3),\gamma(t_4),\gamma(t_5))\} \tag{9}$$

and

$$\alpha(t) := \mu(t,t,t,t,t). \tag{10}$$

Then, the following statements are true:

(i) *μ is well defined and increasing with respect to each variable;*
(ii) *α is well defined and increasing;*
(iii) *$\alpha(t) < t$ for all $t > 0$;*
(iv) *α is right upper semicontinuous;*
(v) *for every $t > 0$, the iterates sequence $\{\alpha^n(t)\}_{n\in\mathbf{N}}$ converges to zero as $n \to \infty$;*
(vi) *μ is a comparison function.*

Proof.

(i) For every $(t_1,t_2,t_3,t_4,t_5) \in \mathbb{R}_+^5$ we define the set

$$A_{(t_1,t_2,t_3,t_4,t_5)} := \{s \in \mathbb{R}_+ \mid \gamma(s) \leq \eta(\gamma(t_1),\gamma(t_2),\gamma(t_3),\gamma(t_4),\gamma(t_5))\}. \tag{11}$$

Since $\gamma(0) = 0$ and $\eta(\gamma(t_1),\gamma(t_2),\gamma(t_3),\gamma(t_4),\gamma(t_5)) \in \mathbb{R}_+$, we obtain that

$$\gamma(0) \leq \eta(\gamma(t_1),\gamma(t_2),\gamma(t_3),\gamma(t_4),\gamma(t_5)),$$

hence $0 \in A_{(t_1,t_2,t_3,t_4,t_5)}$ and thus $A_{(t_1,t_2,t_3,t_4,t_5)}$ is a non-empty set. On the other hand the hypothesis (1) leads us to the fact that α is increasing on $\mathbb{R}_+$ and taking into account that $\alpha(\mathbb{R}_+) \subseteq \mathbb{R}_+$ one has $\alpha(0) = \mu(0,0,0,0,0) = 0$. Further, let us consider $(t_1,t_2,t_3,t_4,t_5) \in \mathbb{R}_+^5$. Then, for every $s \in A_{(t_1,t_2,t_3,t_4,t_5)}$, we have

$$\gamma(s) \leq \eta(\gamma(t_1),\gamma(t_2),\gamma(t_3),\gamma(t_4),\gamma(t_5)) \leq \max\{\gamma(t_1),\gamma(t_2),\gamma(t_3),\gamma(t_4),\gamma(t_5)\}.$$

Therefore, there exists $i_0 \in \{1,2,3,4,5\}$ such that $s \leq t_{i_0} \leq \max\{t_1, t_2, t_3, t_4, t_5\}$. Thus,

$$A_{(t_1,t_2,t_3,t_4,t_5)} \subseteq [0, \max\{t_1, t_2, t_3, t_4, t_5\})$$

and consequently

$$\mu(t_1, t_2, t_3, t_4, t_5) \leq \max\{t_1, t_2, t_3, t_4, t_5\}.$$

From here, we find that $\alpha(t) \leq t$ for each $t \geq 0$. Finally, by using the hypothesis (3) and definition of $A_{(t_1,t_2,t_3,t_4,t_5)}$, we find that μ is increasing with respect to each variable.

(ii) It follows from (i).

(iii) Let us assume that there is $t_0 > 0$ in order that

$$t_0 = \alpha(t_0) = \mu(t_0, t_0, t_0, t_0, t_0) = \sup A_{(t_0,t_0,t_0,t_0,t_0)}.$$

Then, there exists a sequence $\{s_n\}_{n\in\mathbb{N}} \subseteq A_{(t_0,t_0,t_0,t_0,t_0)}$ such that $s_n \nearrow \alpha(t_0)$ as $n \to \infty$. Therefore, for all $n \in \mathbb{N}$, we have that $\gamma(s_n) \leq \eta(\gamma(t_0), \gamma(t_0), \gamma(t_0), \gamma(t_0), \gamma(t_0))$ and taking into consideration that γ is continuous, we find that

$$\gamma(t_0) = \gamma(\alpha(t_0)) \leq \eta(\gamma(t_0), \gamma(t_0), \gamma(t_0), \gamma(t_0), \gamma(t_0)) < \gamma(t_0),$$

which is a contradiction.

(iv) Let us consider $t \in \mathbb{R}_+$ and $\{t_n\}_{n\in\mathbb{N}} \subseteq \mathbb{R}_+$ such that $t_n \searrow t$ as $n \to \infty$. Then $\gamma(t_n) \searrow \gamma(t)$ as $n \to \infty$ and by considering the hypothesis (1) we find that

$$\limsup_{n\to\infty} \eta(\gamma(t_n), \gamma(t_n), \gamma(t_n), \gamma(t_n), \gamma(t_n)) \leq \eta(\gamma(t), \gamma(t), \gamma(t), \gamma(t), \gamma(t)).$$

From here, by using Theorem 4, we deduce that there exists a number $n_0 \in \mathbb{N}$ such that

$$\eta(\gamma(t_n), \gamma(t_n), \gamma(t_n), \gamma(t_n), \gamma(t_n)) \leq \eta(\gamma(t), \gamma(t), \gamma(t), \gamma(t), \gamma(t))$$

for all $n \geq n_0$. Hence,

$$A_{(t_n,t_n,t_n,t_n,t_n)} \subseteq A_{(t,t,t,t,t)},$$

which implies that

$$\alpha(t_n) = \mu(t_n, t_n, t_n, t_n, t_n) \leq \mu(t, t, t, t, t) = \alpha(t),$$

for all $n \geq n_0$. By passing to the limit as $n \to \infty$ one has that

$$\limsup_{n\to\infty} \alpha(t_n) \leq \alpha(t),$$

i.e., that α is right upper semi-continuous on $\mathbb{R}_+$.

(v) From (ii) and (iii), we obtain

$$0 \leq \alpha^{n+1}(t) \leq \alpha^n(t) \leq \alpha(t),$$

for all $t > 0$. Then, there is $l \geq 0$ in order that $\alpha^n(t) \searrow l$ as $n \to \infty$. If $l > 0$, then from (iii) and (iv), we find that $l = \alpha(l) < l$, which is a contradiction. Thus, $l = 0$.

(vi) By taking into consideration (i) and (v), we find that the function μ fulfills the Definition 2 i.e., it is a comparison function.

□

Example 2. *Let us consider* $\eta : \mathbb{R}_+ \to \mathbb{R}_+$ *defined:*

$$\eta(t) = \begin{cases} \frac{t}{t+1}, & t \in [0,1] \\ \frac{t}{2\cdot t+1}, & t \in (1, \infty). \end{cases}$$

Then,

(i) *η verifies the condition of Lemma 2;*
(ii) *η is not right continuous at $t = 1$;*
(iii) *for every $\alpha \in (0,1)$ there exists $t_0 > 0$ such that $\alpha \cdot t_0 < \eta(t_0)$.*

Proof.
(i) It is obvious that $\eta(0) = 0$ and $\eta(t) < t$ for each $t > 0$. On the other hand, we observe that for every $\varepsilon > 0$ we have $\eta(t) \leq \eta(1) + \varepsilon$ for each $t \in (1, \infty)$. Thus, η is right upper semicontinuous.
(ii) Since $\lim\limits_{t \searrow 1} \eta(t) = \frac{1}{3} \neq \frac{1}{2} = \eta(1)$, it follows that η is not right continuous at $t = 1$.
(iii) Let us consider $\alpha \in (0,1)$. We distinguish the following cases:
Case 1: $\alpha \geq \frac{1}{2}$. Then, there exists $0 < t_0 < \frac{1-\alpha}{\alpha} \leq 1$ such that $\alpha \cdot t_0 < \eta(t_0)$.
Case 2: $\alpha < \frac{1}{3}$. Then, there exists $1 < t_0 < \frac{1-\alpha}{2\cdot\alpha}$ such that $\alpha \cdot t_0 < \eta(t_0)$.
Case 3: $\frac{1}{3} \leq \alpha < \frac{1}{2}$. Then, there exists $1 < t_0$ such that $\alpha \cdot t_0 < \eta(t_0)$.

□

We aim to analyze the existence and uniqueness of fixed points for operators described on spaces endowed with such altering metrics. In the following part, we set up some fixed point results on spaces with altering metrics.

Theorem 7. *Let $\gamma \in \Gamma$ and $\eta : \mathbb{R}_+ \to \mathbb{R}_+$ be such that:*
(1) $\eta(0) = 0$;
(2) *η is right upper semi-continuous;*
(3) *$\eta(t) < t$ for all $t > 0$.*
If (X, d) is a complete metric space and $T : X \to X$ is an operator such that:

$$\gamma(d(T(x), T(y))) \leq \eta(\gamma(d(x,y))),\ \forall x, y \in X, \tag{12}$$

then the following statements are true:
(i) *$\mu(0) = 0$, μ is right upper semi-continuous and $\mu(t) < t$ for all $t > 0$, where the function $\mu : \mathbb{R}_+ \to \mathbb{R}_+$ is defined by the relation (2);*
(ii) *T verifies the inequality*

$$d(T(x), T(y)) \leq \mu(d(x,y)),\ \forall x, y \in X. \tag{13}$$

(iii) *T has a unique fixed point $x^* \in X$ and the sequence $T^m(x_0) \to x^*$ as $m \to \infty$, for any arbitrary point $x_0 \in X$.*

Proof.
(i) We notice that the functions η, γ satisfy the hypotheses of Lemma 2. It results that, we can take into consideration the function $\mu : \mathbb{R}_+ \to \mathbb{R}_+$ defined by the relation (2), which has the properties: $\mu(0) = 0$ (by Lemma 2 (ii)), μ is right upper semicontinuous (in accordance with Lemma 2 (vii)) and $\mu(t) < t$ for all $t > 0$ (by Lemma 2 (v)).
(ii) Let $x, y \in X$ be arbitrary elements. Considering that the operator $T : X \to X$ fulfills the inequality (12), we obtain

$$d(T(x), T(y)) \in \{s \in \mathbb{R}_+ \mid \gamma(s) \leq \eta(\gamma(d(x,y)))\},$$

hence,

$$d(T(x), T(y)) \leq \sup\{s \in \mathbb{R}_+ \mid \gamma(s) \leq \eta(\gamma(d(x,y)))\} = \mu(d(x,y)).$$

As the elements $x, y \in X$ are chosen arbitrarily, from the previous relation we deduce that T verifies the inequality (13).
(iii) $\mu : \mathbb{R}_+ \to \mathbb{R}_+$ is right upper semi-continuous (by (i)), $\mu(t) < t$ for all $t > 0$ (from (i)), (X, d) is a complete metric space (in accordance with the hypothesis) and $T : X \to X$

is an operator verifying the inequality (13) (by (ii)). Applying Theorem 5, we find that T has a unique fixed point $x^* \in X$ and the sequence $T^m(x_0) \to x^*$ as $m \to \infty$, for any arbitrary point $x_0 \in X$.

□

Theorem 8. *Let us consider $\eta : \mathbb{R}_+^5 \to \mathbb{R}_+$, $\gamma \in \Gamma$ under hypothesis of Lemma 3, (X, d) a complete metric space and $T : X \to X$ an operator such that:*

$$\gamma(d(T(x), T(y))) \leq \eta(\gamma(d(x, y)), \gamma(d(x, T(x))), \gamma(d(y, T(y))), \gamma(d(x, T(y))), \gamma(d(y, T(x)))),$$

$\forall x, y \in X$. Then:

(i) T verifies the inequality

$$d(T(x), T(y)) \leq \mu(d(x, y), d(x, T(x)), d(y, T(y)), d(x, T(y)), d(y, T(x))),\ \forall x, y \in X,$$

where the function $\mu : \mathbb{R}_+ \to \mathbb{R}_+$ is defined by the relation (9) from Lemma 3.

(ii) T has a unique fixed point $x^* \in X$ and the sequence $T^m(x_0) \to x^*$ as $m \to \infty$, for any arbitrary point $x_0 \in X$.

Proof.

(i) Let $x, y \in X$ be arbitrary elements. Then, for all $x, y \in X$ we have that

$$d(T(x), T(y)) \in$$
$$\{s \in \mathbb{R}_+ \mid \gamma(s) \leq \eta(\gamma(d(x, y)), \gamma(d(x, T(x))), \gamma(d(y, T(y))), \gamma(d(x, T(y))), \gamma(d(y, T(x))))\},$$

hence,

$$d(T(x), T(y)) \leq$$
$$\sup\{s \in \mathbb{R}_+ \mid \gamma(s) \leq \eta(\gamma(d(x, y)), \gamma(d(x, T(x))), \gamma(d(y, T(y))), \gamma(d(x, T(y))), \gamma(d(y, T(x))))\}$$
$$= \mu(d(x, y), d(x, T(x)), d(y, T(y)), d(x, T(y)), d(y, T(x))).$$

(ii) From Lemma 3 (vi), we have that μ defined by Equation (9) is a comparison function. Now, the conclusion follows by taking into account (i) and by applying Theorem 6 to operator T.

□

Corollary 1. *Let (X, d) be a complete metric space $\gamma \in \Gamma$, $a, b, c \in \mathbb{R}_+$, $a + b + c < 1$ and $T : X \to X$ be an operator such that:*

$$\gamma(d(T(x), T(y))) \leq a \cdot \gamma(d(x, y)) + b \cdot \gamma(d(x, T(x))) + c \cdot \gamma(d(y, T(y))),$$

for all $x, y \in X$. Then, T has a unique fixed point $x^* \in X$ and the sequence $T^m(x_0) \to x^*$ as $m \to \infty$, for any arbitrary point $x_0 \in X$.

Proof. Let us consider $\eta : {\mathbb{R}_+}^5 \to \mathbb{R}_+$ defined by

$$\eta(t_1, t_2, t_3, t_4, t_5) = a \cdot t_1 + b \cdot t_2 + c \cdot t_3.$$

We remark that η fulfills the conditions from Theorem 8 and the conclusion follows from it. □

Corollary 2. *Let (X, d) be a complete metric space, $\gamma \in \Gamma$, $a, b, c : \mathbb{R}_+ \setminus \{0\} \to \mathbb{R}_+$ and $T : X \to X$ be an operator such that:*

(1) a, b, c are increasing;

(2) $a(t) + 2 \cdot b(t) + c(t) < 1$ for every $t > 0$;

(3) the function $t \to a(t) + 2 \cdot b(t) + c(t) \in \mathbb{R}_+$ is right upper semi-continuous;

(4) *for all $x, y \in X$, $x \neq y$ we have:*

$$\gamma(d(T(x), T(y))) \leq$$
$$a(d(x,y)) \cdot \gamma((d(x,y)) + b(d(x,y)) \cdot \{\gamma(d(x, T(x))) + \gamma(d(y, T(y)))\} +$$
$$c(d(x,y)) \cdot \min\{\gamma(d(x, T(y))), \gamma(d(y, T(x)))\}.$$

Then, T has a unique fixed point $x^* \in X$ and the sequence $T^m(x_0) \to x^*$ as $m \to \infty$, for any arbitrary point $x_0 \in X$.

Proof. Let us consider $\eta : {\mathbb{R}_+}^5 \to \mathbb{R}_+$ described by:

$$\eta(t_1, t_2, t_3, t_4, t_5) = a(t_1) \cdot t_1 + b(t_1) \cdot (t_2 + t_3) + c(t_1) \cdot \min\{t_4, t_5\}.$$

We remark that η fulfills the conditions from Theorem 8 and the conclusion follows from it. □

Further, Theorem 7 will be applied to continuous data dependence of the fixed points of Picard operators defined on spaces with altering metrics.

Let us consider a function $\mu : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying the conditions: $\mu(0) = 0$, μ is right upper semi-continuous and $\mu(t) < t$ for all $t > 0$. According with [21], if

$$s - \mu(s) \to \infty \text{ as } s \to \infty, \tag{14}$$

we can define the function

$$\theta_\mu : \mathbb{R}_+ \to \mathbb{R}_+,\ \theta_\mu(t) = \sup\{s \in \mathbb{R}_+ \mid s - \mu(s) \leq t\}. \tag{15}$$

We notice that θ_μ is monotonically increasing and $\theta_\mu(t) \to 0$ as $t \to 0$. The function θ_μ appears when we analyze the data dependence of the fixed points.

Theorem 9. *Let $\gamma \in \Gamma$ and $\eta : \mathbb{R}_+ \to \mathbb{R}_+$ under the following hypothesis:*

(1) $\eta(0) = 0$;
(2) *η is right upper semi-continuous;*
(3) *$\eta(t) < t$ for all $t > 0$.*

If (X, d) is a complete metric space and $T : X \to X$ is an operator such that:

$$\gamma(d(T(x), T(y))) \leq \eta(\gamma(d(x,y))),\ \forall x, y \in X, \tag{16}$$

then the statements are true:

(i) *T has a unique fixed point $x^* \in X$;*
(ii) *$d(x, x^*) \leq \theta_\mu(d(x, T(x)))$, $\forall x \in X$;*
(iii) *if $\{y_n\}_{n \in \mathbb{N}}$ is a sequence in X such that $d(y_n, T(y_n)) \to 0$ as $n \to \infty$ then $y_n \to x^*$ as $n \to \infty$, i.e., T has the Ostrowski property;*
(iv) *if the function $\mu : \mathbb{R}_+ \to \mathbb{R}_+$ described by the relation (2) satisfies the hypothesis (14) and $U : X \to X$ is an operator verifying the conditions:*
    (a) *F_U, the fixed point set of operator U is not empty,*
    (b) *there is $\eta > 0$ in order that $d(U(x), T(x)) \leq \eta$, $\forall x \in X$,*
    then $d(y^*, x^*) \leq \theta_\mu(\eta)$, $\forall y^* \in F_U$.

Proof. We notice that the hypotheses of Theorem 7 are satisfied.

(i) Applying Theorem 7 (iii), we obtain that T has a unique fixed point $x^* \in X$.
(ii) By using Theorem 7 (ii), we obtain that T verifies the inequality

$$d(T(x), T(y)) \leq \mu(d(x,y)),\ \forall x, y \in X.$$

Let us consider $x \in X$ an arbitrary selected element. Taking into account the properties of the metric d and the previous inequality we obtain

$$d(x, x^*) \leq d(x, T(x)) + d(T(x), x^*)$$

$$= d(x, T(x)) + d(T(x), T(x^*)) \leq d(x, T(x)) + \mu(d(x, x^*)),$$

hence,

$$d(x, x^*) - \mu(d(x, x^*)) \leq d(x, T(x)),$$

thus,

$$d(x, x^*) \in \{s \in \mathbb{R}_+ \mid s - \mu(s) \leq d(x, T(x))\}.$$

Considering the definition of the function θ_μ (by relation (15)), from the previous relation we deduce

$$d(x, x^*) \leq \sup\{s \in \mathbb{R}_+ \mid s - \mu(s) \leq d(x, T(x))\} = \theta_\mu(d(x, T(x))).$$

(iii) Let us consider $\{y_n\}_{n\in\mathbb{N}}$ a sequence in X such that $d(y_n, f(y_n)) \to 0$ as $n \to \infty$. Taking into account (ii) one has $d(y_n, x^*) \leq \theta_\mu(d(y_n, f(y_n))) \to 0$ as $n \to \infty$ and thus $y_n \to x^*$ as $n \to \infty$.

(iv) Let us consider $y^* \in F_U$ an arbitrary-selected fixed point of the operator U. From (ii), using the condition (b) and the fact that θ_μ is monotonically increasing, it results that

$$d(y^*, x^*) \leq \theta_\mu(d(y^*, T(y^*))) = \theta_\mu(d(U(y^*), T(y^*))) \leq \theta_\mu(\eta).$$

□

The following examples represent applications of our main results (Theorems 7 and 8) to the existence and uniqueness of fixed point for certain operators.

Example 3. *Let us consider $\gamma, \eta : \mathbb{R}_+ \to \mathbb{R}_+$ defined as in Example 1, respectively, Example 2 and the integral equation*

$$x(t) = \int_0^t K(t, s, x(s))ds + g(t),\ t \in [0, 1], \tag{17}$$

under the following conditions:

(H_0) $K \in C([0, 1] \times [0, 1] \times \mathbb{R}, \mathbb{R})$, $g \in C([0, 1], \mathbb{R})$;

(H_1) $|K(t, s, u) - K(t, s, v)|^4 \leq \eta(|u - v|^4)$ for all $t, s \in [0, 1]$ and $u, v \in \mathbb{R}$.

Then, the Equation (17) has a unique solution in $C([0, 1], \mathbb{R})$ (the class of continuous functions $x : [0, 1] \to \mathbb{R}$).

Proof. Let us consider $C([0, 1], \mathbb{R})$ endowed with $\|x\|_\infty = \sup_{t\in[0,1]} |x(t)|$, and let

$$T : C([0, 1], \mathbb{R}) \to C([0, 1], \mathbb{R}),$$

defined by

$$Tx(t) = \int_0^t K(t, s, x(s))ds + g(t).$$

Then, for each $x, y \in C([0, 1], \mathbb{R})$ and $t \in [0, 1]$, we have

$$\gamma(|Tx(t) - Ty(t)|) = |Tx(t) - Ty(t)|^4 \leq$$

$$\Big(\int_0^t |K(t,s.x(s))-K(t,s,y(s))|ds\Big)^4 \leq \int_0^t |K(t,s.x(s))-K(t,s,y(s))|^4 ds \leq$$

$$\int_0^t \eta(|x(s)-y(s)|^4)ds \leq \eta(\|x-y\|_\infty^4) = \eta(\gamma(\|x-y\|_\infty)).$$

Since γ is increasing, we find that $\gamma(\|Tx-Ty\|_\infty) \leq \eta(\gamma(\|x-y\|_\infty))$ for each $x,y \in C([0,1],\mathbb{R})$ The conclusion now follows from Theorem 7 applied to operator T. □

Example 4. *Let us consider*

(a) $X=\{1,2,3,4\}$ *and* $d: X\times X \to \mathbb{R}_+$ *described by:*

$$d(1,1)=d(2,2)=d(3,3)=d(4,4)=0,$$

$$d(1,2)=d(2,1)=\frac{2}{6},\ d(1,3)=d(3,1)=\frac{1}{6},\ d(1,4)=d(4,1)=\frac{4}{6},$$

$$d(2,3)=d(3,2)=\frac{2}{6},\ d(2,4)=d(4,2)=1,\ d(3,4)=d(4,3)=\frac{\sqrt{2}}{2};$$

(b) $T: X\to X$ *described by:*

$$T(1)=T(3)=T(4)=1,\ T(2)=4.$$

Then, T has a unique fixed point.

Proof. It results from Corollary 1 applied for $\gamma:\mathbb{R}_+\to\mathbb{R}_+, \gamma(t)=t^2$ and $\eta:\mathbb{R}_+{}^5\to\mathbb{R}_+$, $\eta(t_1,t_2,t_3,t_4,t_5)=\frac{1}{36}\cdot t_1+\frac{1}{2}\cdot t_2+\frac{4}{9}\cdot t_3$. □

4. Conclusions

In this paper, we have extended the results from [14] by considering for an operator $T: X\to X$ a general contractive condition. First, we proved that for a given control function $\gamma:\mathbb{R}_+\to\mathbb{R}_+$ and a contractive condition of type

$$\gamma(d(T(x),T(y))) \leq \eta(\gamma(d(x,y))), \forall x,y\in X,$$

we can build a function $\mu:\mathbb{R}_+\to\mathbb{R}_+$ such that

$$d(T(x),T(y)) \leq \mu(d(x,y)), \forall x,y\in X.$$

Further, we built Example 2, where we gave an example of function $\eta:\mathbb{R}_+\to\mathbb{R}_+$, which satisfies Lemma 2, but does not satisfy the setup from [14]. Next, we provided an existence and uniqueness result and a data dependence result for fixed point of operator T and we showed additionally that it has the Ostrowski property. The paper is completed by Example 3 as an application of Theorem 7 to an integral equation. Next, we considered a more general contractive condition of type

$$\gamma(d(T(x),T(y))) \leq \eta(\gamma(d(x,y)),\gamma(d(x,T(x))),\gamma(d(y,T(y))),\gamma(d(x,T(y))),\gamma(d(y,T(x)))),$$

$\forall x,y\in X$. Corollary 1 showed us that Theorem 1 from [3] is obtained as a particular case of Theorem 8, and additionally we obtained in Corollary 2 a similar result as in Theorem 2 from [3], but imposing different condition to the functions a,b,c. Moreover, for $\gamma(t)=t$ in Theorem 7 we get Theorem 5. As future research direction we would like to point the following ones:

- To extend the main results to common fixed point theory;

- To generalize the above results to the setup of general metric spaces, e.g., fuzzy, orthogonal or partially ordered metric spaces.

Author Contributions: Conceptualization, I.M.O. and A.N.B.; methodology, I.M.O. and A.N.B.; formal analysis, I.M.O. and A.N.B.; writing—original draft preparation, I.M.O. and A.N.B.; writing—review and editing, I.M.O. and A.N.B.; funding acquisition, I.M.O. and A.N.B. All authors have read and agreed to the published version of the manuscript.

Funding: Project financed by Lucian Blaga University of Sibiu through the research grant LBUS-IRG-2022-08.

Data Availability Statement: Not applicable.

Acknowledgments: The authors thank the anonymous reviewers for their valuable comments and suggestions which helped us to improve the content of this paper.

Conflicts of Interest: The authors declare no conflict of interest.

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