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Local Closure under Infinitely Divisible Distribution Roots and Esscher Transform

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Abstract: In this paper, we show that the local distribution class $\mathcal{L}_{loc} \cap \mathcal{OS}_{loc}$ is not closed under infinitely divisible distribution roots, i.e., there is an infinitely divisible distribution which belongs to the class, while the corresponding Lévy distribution does not. Conversely, we give a condition, under which, if an infinitely divisible distribution belongs to the class $\mathcal{L}_{loc} \cap \mathcal{OS}_{loc}$, then so does the Lévy distribution. Furthermore, we find some sufficient conditions that are more concise and intuitive. Using different methods, we also give a corresponding result for another local distribution class, which is larger than the above class. To prove the above results, we study the local closure under random convolution roots. In particular, we obtain a result on the local closure under the convolution root. In these studies, the Esscher transform of distribution plays a key role, which clarifies the relationship between these local distribution classes and related global distribution classes.

Keywords: infinitely divisible distribution roots; Lévy distribution; local distribution class; random convolution roots; closure; Esscher transform

MSC: Primary 60E05; secondary 60F10; 60G50

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1. Preliminary

In this paper, we study the closure under infinitely divisible distribution (I.I.D.) roots for some local distribution classes, also known simply as the local closure under I.I.D. roots. In other words, we discuss the following problem, if an I.I.D. belongs to a local distribution class, does its corresponding Lévy distribution also belong to this class? These results are closely related to some local distribution classes and Esscher transform of distributions. Thus, in order to better illustrate the main results of this paper, we first introduce the above concepts and their basic properties in this section.

Throughout the paper, unless stated otherwise, all limits are taken as x tends to infinity; for two positive functions f and g, $f(x) \sim g(x)$ means $\limsup f(x)/g(x) = 1$, $f(x) \asymp g(x)$ means $0 < \liminf f(x)/g(x) \leq \limsup g(x)/f(x) < \infty$, $f(x) = o(g(x))$ means $\lim f(x)/g(x) = 0$; for a distribution V, let $\overline{V} = 1 - V$ be the tail distribution of V, V^{*k} be the k-fold convolution of V with itself for all integers $k \geq 2$, $V^{*1} = V$ and V^{*0} be the distribution degenerate at zero; and all distributions are supported on $[0, \infty)$.

1.1. Infinitely Divisible Distribution

Let H be an I.D.D. with the Laplace transform

$$\int_0^\infty e^{-\lambda y} H(dy) = \exp\Big\{ - a\lambda - \int_0^\infty (1 - e^{\lambda y}) \upsilon(dy) \Big\}, \tag{1}$$

where $a \geq 0$ is a constant, and υ is a Borel measure on $(0, \infty)$ with the properties $\mu = \upsilon(1, \infty) < \infty$ and $\int_0^\infty \min\{1, y^2\} \upsilon(dy) < \infty$. Let

$$F(x) = \upsilon(0, x] \mathbf{1}_{\{x > 1\}} / \mu, \quad x \in (-\infty, \infty)$$

be the Lévy distribution generated by the measure υ. The distribution H admits the representation $H = H_1 * H_2$, which is reserved for the convolution of two distributions H_1 and H_2 satisfying

$$\overline{H}_1(x) = O(e^{-\beta x}) \quad \text{for each } \beta > 0 \tag{2}$$

and

$$H_2(x) = e^{-\mu} \sum_{k=0}^{\infty} F^{*k}(x)\mu^k/k!, \quad x \in (-\infty, \infty). \tag{3}$$

See, for example, pages 450 and 571 of Feller [1], Embrechts et al. [2] and Chapter 4 of Sato [3].

One of the research topics of I.D.D is the closure under I.D.D. roots for all types of distribution classes. More precisely, we say that a certain distribution class is closed under I.D.D. roots, if an I.D.D. belongs to the class, then its Lévy distribution also belongs to the same one; otherwise, we say that the class is not closed under the I.D.D. roots.

This paper mainly studies the closure of some local distribution classes under the I.D.D. roots, known simply as the local closure under the I.D.D. roots.

1.2. Related Distribution Classes

In this paper, for each $0 < T \le \infty$, we denote

$$V(x + \Delta_T) = V(x, x+T] = \overline{V}(x) - \overline{V}(x+T) \text{ and } V(x+\Delta_\infty) = \overline{V}(x), \quad x \ge 0.$$

For each distribution V and $0 < T < \infty$, we set that there is a $x_0 = x_0(V, T) \ge 0$ such that $V(x + \Delta_T) > 0,\ x \ge x_0$.

We say that a distribution V belongs to the distribution class $\mathcal{L}_{loc}$, if for each $0 < T \le \infty$,

$$V(x - t + \Delta_T) \sim V(x + \Delta_T) \text{ for each } t > 0.$$

We say that a distribution V belongs to the distribution class $\mathcal{S}_{loc}$, if V belongs to the class $\mathcal{L}_{loc}$ and for each $0 < T \le \infty$,

$$V^{*2}(x + \Delta_T) \sim 2V(x + \Delta_T).$$

See, for example, Borokov and Borokov [4].

The classes $\mathcal{L}_{loc}$ and $\mathcal{S}_{loc}$ are included in two new distribution classes $\mathcal{OS}_{loc}$ and $\mathcal{OL}_{loc}$ defined by the following conditions that, for each $0 < T \le \infty$,

$$C^*_{\Delta_T}(V, t) = \limsup V(x - t + \Delta_T)/V(x + \Delta_T) < \infty \quad \text{for each } t > 0;$$

and for each $0 < T \le \infty$,

$$C^*_{\Delta_T}(V) = \limsup V^{*2}(x + \Delta_T)/V(x + \Delta_T) < \infty,$$

respectively.

In the definitions of the above-mentioned local distribution classes, if "for each $0 < T \le \infty$" is replaced by "for some $0 < T \le \infty$", then these classes are successively called local long-tailed distribution class, local subexponential distribution class, O-local long-tailed distribution class and O-local subexponential distribution class, denoted by $\mathcal{L}_{\Delta_T}$, $\mathcal{S}_{\Delta_T}$, $\mathcal{OL}_{\Delta_T}$ with indicator $C^*_{\Delta_T}(V, t)$ for each $0 < t < \infty$ and $\mathcal{OS}_{\Delta_T}$ with indicator $C^*_{\Delta_T}(V)$, respectively. The classes $\mathcal{L}_{\Delta_T}$ and $\mathcal{S}_{\Delta_T}$ for some $0 < T \le \infty$ were introduced by Asmussen et al. [5]. The class $\mathcal{OS}_{\Delta_T}$ for some $0 < T \le \infty$ originates from the work of Wang et al. [6]. Clearly, the inclusion relations $\mathcal{L}_{loc} \subset \mathcal{L}_{\Delta_T}$ and $\mathcal{S}_{loc} \subset \mathcal{S}_{\Delta_T}$ for each $0 < T \le \infty$ are proper.

For research on the local distribution classes, in addition to the above-mentioned references, please refer to Wang et al. [7], Wang et al. [8], Denisov et al. [9], Yang et al. [10], Watanabe [11], etc.

In particular, when $T = \infty$, we get the corresponding global distribution classes $\mathcal{L}$, $\mathcal{S}$, $\mathcal{OL}$ with indicator

$$C^*(V,t) = C^*_{\Delta_\infty}(V,t) = \limsup \overline{V}(x-t)/\overline{V}(x) < \infty \quad \text{for each } t > 0$$

and $\mathcal{OS}$ with indicator

$$C^*(V) = C^*_{\Delta_\infty}(V) = \limsup \overline{V^{*2}}(x)/\overline{V}(x) < \infty,$$

respectively. The classes $\mathcal{L}$ and $\mathcal{S}$ were introduced by Chistyakov [12], and the classes $\mathcal{OL}$ and $\mathcal{OS}$ come from Shimula and Watanabe [13] and Klüppelberg [14], respectively.

Further, Lemma 2 of Chistyakov [12] shows $\mathcal{S} \subset \mathcal{L}$. This inclusion relation is proper, see Section 3 of Embrechts and Goldie [15], and so on. However, with respect to the prerequisite that $V \in \mathcal{L}_{\Delta_T}$ is necessary in the definition of the class $\mathcal{S}_{\Delta_T}$ for some $0 < T < \infty$, see Propositions 3.1 and 3.2 of Chen et al. [16]. Another proper inclusion relation $\mathcal{OS} \subset \mathcal{OL}$ is given by Proposition 2.1 of Shimura and Watanabe [13].

Clearly, the class $\mathcal{OL}$ contains the heavy-tailed distribution classes $\cup_{0<T\le\infty}\mathcal{L}_{\Delta_T}$ and the class $\mathcal{OS}$ contains the heavy-tailed distribution classes $\cup_{0<T\le\infty}\mathcal{S}_{\Delta_T}$. Here, a distribution V is called the heavy-tailed distribution, if $M(V,\alpha) = \int_0^\infty e^{\alpha y} V(dy) = \infty$ for each $\alpha > 0$; otherwise, it is called the light-tailed distribution. Furthermore, for some $\gamma > 0$, the following light-tailed distribution class $\mathcal{L}(\gamma)$ is a subclass of $\mathcal{OL}$, another light-tailed distribution class $\mathcal{S}(\gamma)$ is a subclass of $\mathcal{OS}$. Both classes were introduced by Chover et al. [17,18].

A distribution V belongs to the distribution class $\mathcal{L}(\gamma)$ for some $\gamma > 0$, if

$$\overline{V}(x-t) \sim \overline{V}(x)e^{\gamma t} \quad \text{for each } t > 0.$$

A distribution V belongs to the distribution class $\mathcal{S}(\gamma)$ for some $\gamma > 0$, if $V \in \mathcal{L}(\gamma)$, $M(V,\gamma) = \int_0^\infty e^{\gamma y} V(dy) < \infty$ and

$$\overline{V^{*2}}(x) \sim 2M(V,\gamma)\overline{V}(x).$$

Clearly, here $M(V,\gamma) \ge 1$. In addition, the prerequisite that $V \in \mathcal{L}(\gamma)$ for some $\gamma > 0$ also is necessary in the definition of the class $\mathcal{S}(\gamma)$, because the distribution here is closely related to its local distribution. In fact, if we define two distribution classes $\mathcal{L}_{\Delta_T}(\gamma)$ and $\mathcal{S}_{\Delta_T}(\gamma)$ for some $0 < \gamma, T < \infty$, then we can easily find that $\mathcal{L}_{\Delta_T}(\gamma) = \mathcal{L}(\gamma)$ and $\mathcal{S}_{\Delta_T}(\gamma) = \mathcal{S}(\gamma)$.

In the definition of the class $\mathcal{L}(\gamma)$, if V is a lattice, then x and t should be restricted to values of the lattice span of V, see Bertoin and Doney [19].

In addition, we might also set $\mathcal{L} = \mathcal{L}(0)$ and $\mathcal{S} = \mathcal{S}(0)$.

There are many research results on the distribution classes mentioned above, see Foss et al. [20], Wang [21] and the references therein.

1.3. Esscher Transform

Now, we use the Esscher transform to show the relationship between some heavy-tailed local distribution and the corresponding light-tailed global distribution.

For any distribution V and $\gamma \neq 0$, by $M(V,\gamma) \ge \min\{1, e^{\gamma x}V(x)\}$, $x \ge 0$, we know that $M(V,\gamma) > 0$. Further, if $M(V,\gamma) < \infty$, then we define a distribution V_γ such that

$$V_\gamma(x) = \int_{0-}^{x} e^{\gamma y} V(dy) \mathbf{1}_{[0,\infty)}(x)/M(V,\gamma), \quad x \in (-\infty,\infty), \tag{4}$$

which is called the Esscher transform (or the exponential tilting) of V. Clearly, for $\gamma > 0$, we have

$$0 < M(V, -\gamma) < 1, \ V = (V_{-\gamma})_\gamma \ \text{and} \ M(V, -\gamma)M(V_{-\gamma}, \gamma) = 1, \tag{5}$$

and for all $k \geq 1$,

$$(V^{*k})_\gamma = (V_\gamma)^{*k} = V_\gamma^{*k}, \ (V^{*k})_{-\gamma} = (V_{-\gamma})^{*k} = V_{-\gamma}^{*k} \ \text{and} \ M(V^{*k}, -\gamma) = M^k(V, -\gamma), \tag{6}$$

see Teugels [22], Veraverbeke [23] and Embrechts and Goldie [24] for technical details.

Further, for some $0 < T < \infty$ and $\gamma > 0$, Definitions 1.1 and 1.2 of Wang and Wang [25] define four global distribution classes as follows:

$$\mathcal{TL}_{\Delta_T}(\gamma) = \{V : \ M(V, \gamma) < \infty \text{ and } V_\gamma \in \mathcal{L}_{\Delta_T}\},$$

$$\mathcal{TS}_{\Delta_T}(\gamma) = \{V : \ M(V, \gamma) < \infty \text{ and } V_\gamma \in \mathcal{S}_{\Delta_T}\};$$

$$\mathcal{TL}_{loc}(\gamma) = \{V : \ M(V, \gamma) < \infty \text{ and } V_\gamma \in \mathcal{L}_{loc}\}$$

and

$$\mathcal{TS}_{loc}(\gamma) = \{V : \ M(V, \gamma) < \infty \text{ and } V_\gamma \in \mathcal{S}_{loc}\}.$$

The following proposition reveals the important role of the Esscher transform for the study of local distribution classes, see Propositions 2.1 and 2.2 of Wang and Wang [25]. On the contrary, this result also shows that some local distribution classes give new vitality to the Esscher transform.

Proposition 1. *(i) For some $0 < T < \infty$ and $\gamma > 0$, a distribution $V \in \mathcal{L}_{\Delta_T}$ (or $\mathcal{S}_{\Delta_T}$) $\Longleftrightarrow$ $V_{-\gamma} \in \mathcal{TL}_{\Delta_T}(\gamma)$ (or $\mathcal{TS}_{\Delta_T}(\gamma)$).*

(ii) A distribution $V \in \mathcal{L}_{loc}$ (or $\mathcal{S}_{loc}$) $\Longleftrightarrow$ $V_{-\gamma} \in \mathcal{L}(\gamma)$ $\big($or $\mathcal{S}(\gamma)\big)$, that is $\mathcal{TL}_{loc}(\gamma) = \mathcal{L}(\gamma)$ $\big($or $\mathcal{TS}_{loc}(\gamma) = \mathcal{S}(\gamma)\big)$. Furthermore, each of them implies that, for each $0 < T < \infty$

$$V(x + \Delta_T) \sim \gamma T e^{\gamma x} \overline{V_{-\gamma}}(x) / M(V_{-\gamma}, \gamma) = M(V, -\gamma)\gamma T e^{\gamma x} \overline{V_{-\gamma}}(x). \tag{7}$$

More results of the Esscher transform can be found in the above references and the others therein.

The paper is organized as follows. In Section 2, we present the main results for Theorems 1–3 related to local closure under I.I.D. roots. In Section 3, we prove the above results. To this end, we study the local closure under random convolution roots. Then in Section 4, we show that the condition (10) of Theorem 3 can be replaced by a more concise and intuitive condition (11). Finally, in Section 5, we briefly introduce some applications of the obtained results and further research problems. As an application of Theorem 2, we give a positive result on the local closure under the convolution root, which represents the local version of common Embrechts and Goldie conjecture.

2. Main Results

Before giving the main results of this paper, we recall some existing results on closure under I.I.D. roots.

For the global distribution classes, the class $\mathcal{S}(\gamma)$ is closed under I.D.D. roots, see Embrechts et al. [2] for the case $\gamma = 0$, Sgibnev [26], Pakes [27] and Watanabe [28] for the case $\gamma > 0$. Recently, Cui et al. [29] proved that the class $\mathcal{L}(\gamma) \cap \mathcal{OS}$ for some $\gamma \geq 0$ is closed under the roots with some restrictive condition.

However, for some global distribution classes without special restrictions, there were some negative results, i.e., there exists an I.D.D. H belonging to some class, while its Lévy distribution F does not belong to the same class; see Theorem 1.1 (iii) of Shimura and Watanabe [13] for the class $\mathcal{OS}$, Theorem 1.2 (3) of Xu et al. [30] for the class $\mathcal{L} \cap \mathcal{OS}$ and $\mathcal{L} \setminus \mathcal{OS}$ and Theorem 1.1 of Xu et al. [31] for the class $\mathcal{L}(\gamma) \cap \mathcal{OS}$ with some $\gamma > 0$.

As previously mentioned, this paper mainly studies the closure of some local distribution classes under the I.D.D. roots. Clearly, if a distribution $V \in \mathcal{L}_{\Delta_T}$ for some $0 < T < \infty$, then $V(x + \Delta_T) = o\big(V(x)\big)$. Therefore, the study of local distribution cannot be replaced by that of global distribution.

One of the difficulties in the study of local distributions is the loss of their almost monotonic decreasing property. Corollary 3.1 of Jiang et al. [32] shows that some local distributions in the class $\mathcal{S}_{loc}$ and the class $\mathcal{L}_{loc} \setminus \mathcal{S}_{loc}$ are not even close to decreasing. Therefore, the study of local distribution is definitely more challenging than that of global distribution. Furthermore, we find hardly any existing results regarding local closure under I.I.D. roots.

Now, we first give a negative conclusion for the class $\mathcal{L}_{loc} \cap \mathcal{OS}_{loc}$.

Theorem 1. *The class $\mathcal{L}_{loc} \cap \mathcal{OS}_{loc}$ is not closed under I.D.D. roots.*

Next, we give two positive conclusions for the class $\mathcal{L}_{loc} \cap \mathcal{OS}_{loc}$ and $\mathcal{TL}_{\Delta_{T_0}}(\gamma) \cap \mathcal{OS}_{\Delta_{T_0}}$ with some $0 < \gamma, T_0 < \infty$, respectively.

Theorem 2. *Let H be an I.D.D. with the Lévy distribution F. Assume that $H \in \mathcal{L}_{loc} \cap \mathcal{OS}_{loc}$, and for all $k \geq 1$,*

$$\liminf \overline{F^{*k}_{-\gamma}}(x - t)/\overline{F^{*k}_{-\gamma}}(x) \geq e^{\gamma t} \quad \textit{for each } t > 0. \tag{8}$$

Then the following two conclusions hold.

(i) *$H_2 \in \mathcal{L}_{loc} \cap \mathcal{OS}_{loc}$ and $H_2(x + \Delta_T) \sim H(x + \Delta_T)$ for each $0 < T \leq \infty$.*

(ii) *There exists an integer $l_0 \geq 1$ such that $F^{*n} \in \mathcal{L}_{loc} \cap \mathcal{OS}_{loc}$ for all $n \geq l_0$ and $F^{*n} \notin \mathcal{L}_{loc} \cap \mathcal{OS}_{loc}$ for all $1 \leq n \leq l_0 - 1$. In particular, if $F \in \mathcal{OS}_{loc}$, then $F^{*n} \in \mathcal{L}_{loc} \cap \mathcal{OS}_{loc}$ for all $n \geq 1$.*

Remark 1. (i) *According to Corollary 1.1 of Cui et al. [29], the condition (8) can be implied by some more concise and convenient conditions that*

$$F_{-\gamma} \in \mathcal{OL}, \quad \lim \overline{F_{-\gamma}}(x)C^*(F_{-\gamma}, x) = 0 \ \textit{ and } (8) \textit{ holds for } k = 1. \tag{9}$$

Therefore, all conclusions of Theorem 2 hold under the conditions (9) and $H \in \mathcal{L}_{loc} \cap \mathcal{OS}_{loc}$. Some related examples can be found in Corollary 1.2 and Example 4.1 of Cui et al. [29].

(ii) *In the proof of Theorem 1, we can find that there exists an I.D.D. H with Lévy distribution F such that $l_0 = 2$. This fact shows that there are many distributions F that satisfy condition (8), but which do not belong to the class $\mathcal{L}(\gamma)$.*

Clearly, the local distribution class $\mathcal{L}_{\Delta_{T_0}} \cap \mathcal{OS}_{\Delta_{T_0}}$ for some $0 < T_0 \leq \infty$ is larger than the class $\mathcal{L}_{loc} \cap \mathcal{OS}_{loc}$. Therefore, it is natural to investigate the corresponding result for the former. To this end, we first consider its corresponding light-tailed global distribution class $\mathcal{TL}_{\Delta_{T_0}}(\gamma) \cap \mathcal{OS}_{\Delta_{T_0}}$ for some $0 < \gamma, T_0 < \infty$, which is larger than the class $\mathcal{L}(\gamma) \cap \mathcal{OS}$. We will find that the research method of the following result is different from that of Theorem 2.

Theorem 3. *Let H be an I.D.D. with the Lévy distribution F. For some $0 < \gamma, T_0 < \infty$, assume that $H \in \mathcal{TL}_{\Delta_{T_0}}(\gamma) \cap \mathcal{OS}_{\Delta_{T_0}}$ and for all $k \geq 1$,*

$$\liminf F^{*k}_{\gamma}(x - t + \Delta_{T_0})/F^{*k}_{\gamma}(x + \Delta_{T_0}) \geq 1 \quad \textit{for each } t > 0. \tag{10}$$

Then the following two conclusions hold.

(i) *$H_2 \in \mathcal{TL}_{\Delta_{T_0}}(\gamma) \cap \mathcal{OS}_{\Delta_{T_0}}$ and $H_2(x + \Delta_{T_0}) \asymp H(x + \Delta_{T_0})$.*

(ii) There is an integer $l_0 \geq 1$ such that $F^{*n} \in \mathcal{TL}_{\Delta_{T_0}}(\gamma) \cap \mathcal{OS}_{\Delta_{T_0}}$ for all $n \geq l_0$ and $F^{*n} \notin \mathcal{TL}_{\Delta_{T_0}}(\gamma) \cap \mathcal{OS}_{\Delta_{T_0}}$ for all $1 \leq n \leq l_0 - 1$. In particular, if $F \in \mathcal{OS}_{\Delta_{T_0}}$, then $F^{*n} \in \mathcal{TL}_{\Delta_{T_0}}(\gamma) \cap \mathcal{OS}_{\Delta_{T_0}}$ for all $n \geq 1$.

Remark 2. *The condition (10) can also be replaced by the following more concise and convenient conditions:*

$$F_\gamma \in \mathcal{OL}_{\Delta_{T_0}},\ \lim F_\gamma(x + \Delta_{T_0}) C^*_{\Delta_{T_0}}(F_\gamma, x) = 0 \text{ and } (10) \text{ holds for } k = 1. \tag{11}$$

See Theorem 6 with $V = F_\gamma$ below.

3. The Proofs of Theorems 1–3

3.1. Proof of Theorem 1

Let $F_{(0)}$ be a heavy-tailed distribution such that

$$\overline{F_{(0)}}(x) = \mathbf{1}_{(-\infty,a_0)}(x) + C\sum_{n=0}^{\infty}\Bigg(\Big(\sum_{i=n}^{\infty}\frac{1}{a_i^\alpha} - \frac{x-a_n}{a_n^{\alpha+1}}\Big)\mathbf{1}_{[a_n,2a_n)}(x) + \Big(\sum_{i=n+1}^{\infty}\frac{1}{a_i^\alpha}\Big)\mathbf{1}_{[2a_n,a_{n+1})}(x)\Bigg) \tag{12}$$

with the density $f_{(0)}(x) = C\sum_{n=0}^{\infty} a_n^{-\alpha-1}\mathbf{1}_{[a_n,2a_n)}(x)$ for all x, where

$$\alpha \in \Big(\frac{3}{2}, \frac{\sqrt{5}+1}{2}\Big),\ a_n = a^{r^n} \text{ for } r = 1 + \frac{1}{\alpha}, \text{ some } a > 8^\alpha \text{ and all } n \geq 1, \text{ and } C = \Big(\sum_{n=0}^{\infty} a_n^{-\alpha}\Big)^{-1}.$$

Let $\mathcal{F}_1(0)$ be the class comprising the above distributions $F_{(0)}$ defined by (12). Further, for some $\gamma > 0$ and distribution $F_{(0)} \in \mathcal{F}_1(0)$, define the light-tailed distribution $F_{(\gamma)}$ in the form

$$\overline{F_{(\gamma)}}(x) = \mathbf{1}_{(-\infty,0)}(x) + e^{-\gamma x}\overline{F_{(0)}}(x)\mathbf{1}_{[0,\infty)}(x) \tag{13}$$

with its density $f_{(\gamma)}$ for all x. Then we can construct a new distribution class

$$\mathcal{F}_1(\gamma) = \{F_{(\gamma)} \text{ defined by } (13) :\ F_{(0)} \in \mathcal{F}_1(0)\}.$$

See the proof of Theorem 1 of Xu et al. [31].

Let $H = H_1 * H_2$ is an I.D.D. with Lévy distribution $F_{(\gamma)} \in \mathcal{F}_1(\gamma)$ for some $\gamma > 0$. Then Proposition 1 and Theorem 1 of Xu et al. [31] show that, H, H_2 and $F^{*k}_{(\gamma)}$ for all $k \geq 2$ belong to the class $\big(\mathcal{L}(\gamma) \cap \mathcal{OS}\big) \setminus \mathcal{S}(\gamma)$, while $F_{(\gamma)}$ with $M(F_{(\gamma)}, \gamma) < \infty$ belongs to the class $\mathcal{OL} \setminus \big(\mathcal{L}(\gamma) \cup \mathcal{OS}\big)$.

Because $M(F_{(\gamma)}, \gamma) < \infty$, $M(H, \gamma) < \infty$, then $H_\gamma = H_{1,\gamma} * H_{2,\gamma}$, as the Esscher transform of H, is defined and is I.D.D. with Lévy distribution $F_{(\gamma),\gamma}$. To reveal the properties of H_γ and $F_{(\gamma),\gamma}$, we need the following result.

Lemma 1. *For some $0 < \gamma, T < \infty$, $V_{-\gamma} \in \mathcal{OS}_{\Delta_T} \Longleftrightarrow V \in \mathcal{OS}_{\Delta_T}$. Thus, $V_{-\gamma} \in \mathcal{OS}_{loc} \Longleftrightarrow V \in \mathcal{OS}_{loc}$. Further, if $V_{-\gamma} \in \mathcal{L}(\gamma)$, then $V_{-\gamma} \in \mathcal{OS} \Longleftrightarrow V_{-\gamma} \in \mathcal{OS}_{loc}$. Therefore,*

$$V_{-\gamma} \in \mathcal{L}(\gamma) \cap \mathcal{OS} \Longleftrightarrow V \in \mathcal{L}_{loc} \cap \mathcal{OS}_{loc}.$$

Proof. We now prove the first conclusion. From (2.4) of Wang and Wang [25], we have

$$V_{-\gamma}(x + \Delta_T) = M(V_{-\gamma}, \gamma)e^{-\gamma x}\Big(V(x + \Delta_T) - \gamma\int_0^T e^{-\gamma y}V(x + y, x + T]dy\Big), \quad x \geq 0. \tag{14}$$

Further, we obtain the following inequality,

$$e^{-\gamma T}V(x+\Delta_T)\le e^{\gamma x}V_{-\gamma}(x+\Delta_T)\Big/M(V_{-\gamma},\gamma)\le V(x+\Delta_T),\quad x\ge 0. \tag{15}$$

If $V\in\mathcal{OS}_{\Delta_T}$, then according to Radon–Nikodym Theorem, by (15) and (4), we have

$$\begin{aligned}
&V_{-\gamma}^{*2}(x+\Delta_T)=\int_0^x V_{-\gamma}(x-y+\Delta_T)V_{-\gamma}(dy)+\int_x^{x+T}V_{-\gamma}(0,x-y+T]V_{-\gamma}(dy)\\
\le\;& e^{-\gamma x}M(V_{-\gamma},\gamma)\int_0^x V(x-y+\Delta_T)V(dy)\Big/M(V,-\gamma)+V_{-\gamma}(x+\Delta_T)\\
\le\;& e^{-\gamma x}M(V_{-\gamma},\gamma)V^{*2}(x+\Delta_T)\Big/M(V,-\gamma)+V_{-\gamma}(x+\Delta_T)\\
\le\;& 2C^*_{\Delta_T}(V)M(V_{-\gamma},\gamma)e^{-\gamma x}V(x+\Delta_T)\Big/M(V,-\gamma)+V_{-\gamma}(x+\Delta_T)\\
\le\;& \big(2C^*_{\Delta_T}(V)M(V_{-\gamma},\gamma)e^{\gamma T}\big/M(V,-\gamma)+1\big)V_{-\gamma}(x+\Delta_T)\quad\text{for large enough } x>0,
\end{aligned}$$

that is $V_{-\gamma}\in\mathcal{OS}_{\Delta_T}$. Conversely, if $V_{-\gamma}\in\mathcal{OS}_{\Delta_T}$, then we also get $V\in\mathcal{OS}_{\Delta_T}$ by the same approach.

The second conclusion comes from the arbitrariness of T.

If $V_{-\gamma}\in\mathcal{L}(\gamma)$, then $V_{-\gamma}^{*2}\in\mathcal{L}(\gamma)$. Thus, for each $0<T\le\infty$, by

$$V_{-\gamma}^{*k}(x+\Delta_T)\sim(1-e^{-\gamma T})\overline{V_{-\gamma}^{*k}}(x),\ \ k=1,2,$$

the third conclusion holds.

Proposition 1 and the third conclusion imply the final conclusion. □

Now, we continue to prove the theorem. According to Lemma 1 and Proposition 1, by $H\in\big(\mathcal{L}(\gamma)\cap\mathcal{OS}\big)\setminus\mathcal{S}(\gamma)$ and $F_{(\gamma)}\in\mathcal{OL}\setminus\big(\mathcal{L}(\gamma)\cup\mathcal{OS}\big)$, we know that $H_\gamma\in\big(\mathcal{L}_{loc}\cap\mathcal{OS}_{loc}\big)\setminus\mathcal{S}_{loc}$, while $F_{(\gamma),\gamma}\in\mathcal{OL}_{loc}\setminus\big(\mathcal{L}_{loc}\cup\mathcal{OS}_{loc}\big)$. Therefore, the class $\mathcal{L}_{loc}\cap\mathcal{OS}_{loc}$ is not closed under I.D.D. roots.

3.2. Proof of Theorem 2

To prove this theorem, we give two preliminary results. Firstly, we consider the closure under random convolution roots for the distribution class $\mathcal{L}_{loc}\cap\mathcal{OS}_{loc}$. Clearly, this result and the following Theorem 5 not only play a key role in the proof of Theorems 2 and 3, but also have their own independent value.

Let V be a distribution and let τ be a nonnegative integer-valued random variable with masses $p_k=P(\tau=k)$ for all nonnegative integers k satisfying $\sum_{k=0}^{\infty}p_k=1$. Denoted by $V^{*\tau}$ is the random convolution or compound convolution generated by V and τ, i.e.,

$$V^{*\tau}=\sum_{k=0}^{\infty}p_kV^{*k}.$$

Let $m=\sup\{k:p_k>0\}$. In this paper, we consider the following two cases:

$$\text{Case 1}:\ \ p_k>0 \text{ for all } k\ge 1;\qquad \text{Case 2}:\ 1\le m<\infty \text{ and } p_k>0 \text{ for all } 1\le k\le m.$$

Theorem 4. *Assume that for any $0<\varepsilon<1$ and some $0<T_0<\infty$, there exists an integer $n_0=n_0(V,\varepsilon,\tau,T_0)\ge 1$ such that*

$$\sum_{k=n_0+1}^{\infty}p_kV^{*(k-1)}(x+\Delta_{T_0})\le\varepsilon V^{*\tau}(x+\Delta_{T_0}),\qquad x\ge 0, \tag{16}$$

and for each $k \geq 1$ in Case 1 or $1 \leq k \leq m$ in Case 2,

$$\liminf \overline{V^{*k}_{-\gamma}}(x-t) \Big/ \overline{V^{*k}_{-\gamma}}(x) \geq e^{\gamma t} \qquad \text{for each } t > 0. \tag{17}$$

If $V^{*\tau} \in \mathcal{L}_{loc} \cap \mathcal{OS}_{loc}$, then for the above two cases, there exists an integer $l_0 \geq 1$ in Case 1 or $1 \leq l_0 \leq m$ in Case 2 such that $V^{*n} \in \mathcal{L}_{loc} \cap \mathcal{OS}_{loc}$ for all $l_0 \leq n < m$ and $V^{*n} \notin \mathcal{L}_{loc} \cap \mathcal{OS}_{loc}$ for all $1 \leq n \leq l_0 - 1$. In particular, if $V \in \mathcal{OS}_{loc}$, then $V^{*n} \in \mathcal{L}_{loc} \cap \mathcal{OS}_{loc}$ for all $n \geq 1$ in Case 1 or $1 \leq n \leq m$ in Case 2.

Proof. We first prove the theorem for Case 1 that $m = \infty$.

In Lemma 1, we replace V with $V^{*\tau}$. Then by $V^{*\tau} \in \mathcal{L}_{loc} \cap \mathcal{OS}_{loc}$, we know that

$$(V^{*\tau})_{-\gamma} \in \mathcal{L}(\gamma) \cap \mathcal{OS}.$$

In addition,

$$0 < M(V^{*\tau}, -\gamma) = \sum_{k=0}^{\infty} p_k M^k(V, -\gamma) = EM^{\tau}(V, -\gamma) < 1$$

and

$$\overline{(V^{*\tau})_{-\gamma}}(x) = \sum_{k=1}^{\infty} \frac{p_k M^k(V, -\gamma)}{M(V^{*\tau}, -\gamma)} \overline{V^{*k}_{-\gamma}}(x) = \sum_{k=1}^{\infty} q_k \overline{V^{*k}_{-\gamma}}(x) = \overline{(V_{-\gamma})^{*\sigma}}(x), \qquad x \geq 0, \tag{18}$$

where σ is a random variable such that $P(\sigma = k) = q_k$ for all nonnegative integers k satisfying $\sum_{k=0}^{\infty} q_k = 1$.

For any $0 < \varepsilon < 1$, we denote $\varepsilon_0 = \varepsilon e^{-\gamma T_0}$. By (15), (18) and (16) replaced ε with ε_0, according to Fubini Theorem, for the corresponding $n_0 = n_0(V, \varepsilon_0, \tau, T_0)$ large enough, we have

$$\begin{aligned}
& \sum_{k=n_0+1}^{\infty} q_k \overline{V^{*(k-1)}_{-\gamma}}(x) = \sum_{k=n_0+1}^{\infty} q_k \sum_{m=0}^{\infty} V^{*(k-1)}_{-\gamma}(x + mT_0 + \Delta_{T_0}) \\
\leq & \sum_{k=n_0+1}^{\infty} p_k M^{k-1}(V, -\gamma) M^{k-1}(V_{-\gamma}, \gamma) \sum_{m=0}^{\infty} e^{-\gamma(x+mT_0)} V^{*(k-1)}(x + mT_0 + \Delta_{T_0}) \Big/ M(V^{*\tau}, -\gamma) \\
= & \sum_{k=n_0+1}^{\infty} p_k \sum_{m=0}^{\infty} e^{-\gamma(x+mT_0)} V^{*(k-1)}(x + mT_0 + \Delta_{T_0}) \Big/ M(V^{*\tau}, -\gamma) \\
= & \sum_{m=0}^{\infty} e^{-\gamma(x+mT_0)} \sum_{k=n_0+1}^{\infty} p_k V^{*(k-1)}(x + mT_0 + \Delta_{T_0}) \Big/ M(V^{*\tau}, -\gamma) \\
\leq & \varepsilon_0 \sum_{m=0}^{\infty} e^{-\gamma(x+mT_0)} V^{*\tau}(x + mT_0 + \Delta_{T_0}) \Big/ M(V^{*\tau}, -\gamma) \\
\leq & \varepsilon_0 e^{\gamma T_0} \sum_{m=0}^{\infty} (V^{*\tau})_{-\gamma}(x + mT_0 + \Delta_{T_0}) \\
= & \varepsilon \overline{(V^{*\tau})_{-\gamma}}(x) \\
= & \varepsilon \overline{(V_{-\gamma})^{*\sigma}}(x), \qquad x \geq 0.
\end{aligned} \tag{19}$$

Since $(V^{*\tau})_{-\gamma} \in \mathcal{L}(\gamma) \cap \mathcal{OS}$, according to Theorem 2.1 with $\gamma > 0$ of Cui et al. [29], by (19) and (17), we have $V^{*n}_{-\gamma} \in \mathcal{L}(\gamma) \cap \mathcal{OS}$ for all $n \geq n_0$. Thus, according to Lemma 1, $V^{*n} \in \mathcal{L}_{loc} \cap \mathcal{OS}_{loc}$ for all $n \geq n_0$.

Let $l_0 = \min\{n : V^{*n} \in \mathcal{L}_{loc} \cap \mathcal{OS}_{loc}\}$. Then $1 \leq l_0 \leq n_0$. According to Lemma 1, by $V^{*l_0} \in \mathcal{L}_{loc} \cap \mathcal{OS}_{loc}$, we know that $V^{*l_0}_{-\gamma} \in \mathcal{L}(\gamma) \cap \mathcal{OS}$. Furthermore, according to Theorem 3 of Embrechts and Goldie [15] and Proposition 2.6 of Shimura and Watanabe [13], we have $V_{-\gamma}^{*n} \in \mathcal{L}(\gamma) \cap \mathcal{OS}$ for all $n \geq l_0$. Therefore, $V^{*n} \in \mathcal{L}_{loc} \cap \mathcal{OS}_{loc}$ for all $n \geq l_0$ after using Lemma 1 again.

Similarly, we can prove $V^{*n} \notin \mathcal{L}_{loc} \cap \mathcal{OS}_{loc}$ for all $1 \leq n \leq l_0 - 1$.

In particular, if $V \in \mathcal{OS}_{loc}$, then $V_{-\gamma} \in \mathcal{OS}$. Thus, according to Theorem 2.1 with $\gamma > 0$ of Cui et al. [29], we have $V_{-\gamma} \in \mathcal{L}(\gamma)$, which implies $V \in \mathcal{L}_{loc}$. Therefore, $l_0 = 1$, that is $V^{*n} \in \mathcal{L}_{loc} \cap \mathcal{OS}_{loc}$ for all $n \geq 1$.

Next, we prove the theorem for the Case 2 that $1 \leq m < \infty$ and $p_m > 0$.

Because

$$\overline{(V^{*\tau})_{-\gamma}}(x) \geq q_m \overline{V_{-\gamma}^{*m}}(x) \geq q_m \overline{V_{-\gamma}^{*k}}(x), \;\; 1 \leq k \leq m-1,$$

$\overline{(V^{*\tau})_{-\gamma}}(x) \asymp \overline{V_{-\gamma}^{*m}}(x)$. Then by $(V^{*\tau})_{-\gamma} \in \mathcal{OS}$, we immediately get $V_{-\gamma}^{*m} \in \mathcal{OS}$. Consequently, there is an integer $l_0 = \min\{1 \leq n \leq n_0 : V_{-\gamma}^{*n} \in \mathcal{OS}\}$ such that $1 \leq l_0 \leq m$ and $V_{-\gamma}^{*l_0} \in \mathcal{OS}$. According to Proposition 2.6 of Shimura and Watanabe [13], $V_{-\gamma}^{*n} \in \mathcal{OS}$ and $\overline{(V^{*\tau})_{-\gamma}}(x) \asymp \overline{V_{-\gamma}^{*n}}(x)$ for all $l_0 \leq n \leq m$. Thus, for each $n \geq l_0$, there is a constant $D_n = D_n(V, \tau) > 0$ such that

$$\limsup \overline{(V^{*\tau})_{-\gamma}}(x) \Big/ \overline{V_{-\gamma}^{*n}}(x) = D_n < \infty.$$

Further, we prove $V_{-\gamma}^{*n} \in \mathcal{L}(\gamma)$ for each $l_0 \leq n \leq m$. Since $(V^{*\tau})_{-\gamma} \in \mathcal{L}(\gamma)$, for any $0 < \varepsilon < 1$ and each $t > 0$, there is a constant $x_1 = x_1(V_{-\gamma}, \tau, \varepsilon, t) > t$ such that, for all $x > x_1$,

$$\begin{aligned}
& \varepsilon\overline{(V^{*\tau})_{-\gamma}}(x) \geq \overline{(V^{*\tau})_{-\gamma}}(x-t) - e^{\gamma t}\overline{(V^{*\tau})_{-\gamma}}(x) \\
= \;& \Big(\sum_{1 \leq k \neq n \leq m} + \sum_{k=n} \Big) p_k \big(\overline{V_{-\gamma}^{*k}}(x-t) - e^{\gamma t}\overline{V_{-\gamma}^{*k}}(x) \big) \\
\geq \;& -\varepsilon e^{\gamma t} \sum_{1 \leq k \neq n \leq n_0} p_k \overline{V_{-\gamma}^{*k}}(x) + p_n \big(\overline{V_{-\gamma}^{*n}}(x-t) - e^{\gamma t}\overline{V_{-\gamma}^{*n}}(x) \big) \\
\geq \;& p_n \big(\overline{V_{-\gamma}^{*n}}(x-t) - e^{\gamma t}\overline{V_{-\gamma}^{*n}}(x) \big) - \varepsilon e^{\gamma t}\overline{(V^{*\tau})_{-\gamma}}(x),
\end{aligned}$$

which implies that for all $x > x_1$,

$$\overline{V_{-\gamma}^{*n}}(x-t) \leq e^{\gamma t}\overline{V_{-\gamma}^{*n}}(x) + (1 + e^{\gamma t})\varepsilon\overline{(V^{*\tau})_{-\gamma}}(x) \Big/ p_n.$$

Hence,

$$\limsup \overline{V_{-\gamma}^{*n}}(x-t) \Big/ \overline{V_{-\gamma}^{*n}}(x) \leq e^{\gamma t} + (1 + 2e^{\gamma t})\varepsilon D_n \Big/ p_n. \tag{20}$$

Clearly, the fixed integer n is independent of ε. Thus, combined with the arbitrariness of ε, (20) and (17) lead to $V_{-\gamma}^{*n} \in \mathcal{L}(\gamma)$.

In particular, if $V_{-\gamma} \in \mathcal{OS}$, then by the same method, we can get $V_{-\gamma}^{*n} \in \mathcal{L}(\bigcirc) \cap \mathcal{OS}$ for all $1 \leq n \leq m$. $\square$

Secondly, we consider the closure under convolution roots for the distribution class $\mathcal{L}_{loc} \cap \mathcal{OS}_{loc}$.

Lemma 2. *Let G_1 be a distribution, $G_2 = V^{*\tau}$ as above and $G = G_1 * G_2$. Assume that for any $0 < \varepsilon < 1$, there exists an integer $n_0 = n_0(V, \varepsilon, \tau) \geq 1$ such that*

$$\sum_{k=n_0+1}^{\infty} p_k \overline{V_{-\gamma}^{*(k-1)}}(x) \leq \varepsilon\overline{(V^{*\tau})_{-\gamma}}(x), \qquad x \geq 0. \tag{21}$$

Further, suppose that (17) is satisfied for all $k \geq 1$ and

$$\overline{G_{1,-\gamma}}(x) = o\big(\overline{G_{2,-\gamma}}(x)\big) \tag{22}$$

If $G \in \mathcal{L}_{loc} \cap \mathcal{OS}_{loc}$, then

$$G_2 \in \mathcal{L}_{loc} \cap \mathcal{OS}_{loc} \quad \textit{and} \quad G_2(x + \Delta_{T_0}) \sim G(x + \Delta_{T_0}).$$

Proof. According to Lemma 1, by $G \in \mathcal{L}_{loc} \cap \mathcal{OS}_{loc}$, we know that

$$G_{-\gamma} = G_{1,-\gamma} * G_{2,-\gamma} \in \mathcal{L}(\gamma) \cap \mathcal{OS}.$$

Thus, according to Lemma 3.1 of Cui et al. [29], by (17) for all $k \geq 1$, (21) for any given $0 < \varepsilon < 1$ and (22), we have

$$G_{2,-\gamma} \in \mathcal{L}(\gamma) \cap \mathcal{OS} \ \text{ and } \ \overline{G_{-\gamma}}(x) \sim M(G_{1,-\gamma}, \gamma)\overline{G_{2,-\gamma}}(x).$$

Therefore, according to Lemma 1 and Proposition 1, by (7), we can prove the lemma. □

Now, we prove Theorem 2.

(i) Firstly, we prove

$$\overline{H_{1,-\gamma}}(x) = o\big(\overline{H_{2,-\gamma}}(x)\big). \tag{23}$$

To the end, we denote

$$\overline{H_{-\gamma}}(x) = \overline{H_{-\gamma}}(\ln e^x) = \overline{H_{-\gamma}}(\ln y) = f_{-\gamma}(y).$$

According to Lemma 1, by $H \in \mathcal{L}_{loc} \cap \mathcal{OS}_{loc}$, we have $H_{-\gamma} \in \mathcal{L}(\gamma) \cap \mathcal{OS}$. Thus, $f_{-\gamma}(\cdot)$ is a regular variation function with index γ, which implies

$$e^{\beta x}\overline{H_{-\gamma}}(x) \to \infty \quad \text{for each } \ \beta > \gamma. \tag{24}$$

By $\overline{H_1}(x) = O\big(e^{-\beta x}\big)$ for each $\beta > 0$, we have

$$e^{\beta x}\overline{H_{1,-\gamma}}(x) \leq e^{\beta x}\overline{H_1}(x)\Big/M(H_1, \gamma) \to 0 \quad \text{for each } \ \beta > 0. \tag{25}$$

For $i = 1, 2$, let X_i be a random variable with distribution $H_{i,-\gamma}$. Then

$$\overline{H_{-\gamma}}(x) = \overline{H_{1,-\gamma} * H_{2,-\gamma}}(x) \leq P\big(\max\{X_1, X_2\} > x/2\big) \leq \overline{H_{1,-\gamma}}(x/2) + \overline{H_{2,-\gamma}}(x/2).$$

Thus, by (24) and (25), we know that

$$e^{\beta x}\overline{H_{2,-\gamma}}(x) = e^{(2^{-1}\beta)2x}\overline{H_{2,-\gamma}}(2x/2) \to \infty \quad \text{for each } \ \beta > 2\gamma. \tag{26}$$

Combining with (25) and (26), we know that (23) holds.

Secondly, by (18), according to Proposition 6.1 of Watanabe and Yamamuro [33], we have

$$q_k = p_k M^k(F_{-\gamma}, -\gamma)\Big/M(H_2, -\gamma) = e^{-\mu}\mu^k M^k(F_{-\gamma}, -\gamma)\Big/\big(M(H_2, -\gamma)k!\big) \quad \text{for all } \ k \geq 0.$$

Thus, for any $0 < \varepsilon < 1$, there exists an integer $n_0 = n_0(F_{-\gamma}, H_{2,-\gamma}, \varepsilon) \geq 1$ such that

$$\sum_{k=n_0+1}^{\infty} q_k \overline{F_{-\gamma}^{*(k-1)}}(x) \leq \varepsilon\overline{(F^{*\tau})_{-\gamma}}(x) = \varepsilon\overline{H_{2,-\gamma}}(x), \qquad x \geq 0. \tag{27}$$

Finally, according to Lemma 2 replaced G_i with H_i, $i = 1, 2$, combining with (8), (23) and (27), by $H \in \mathcal{L}_{loc} \cap \mathcal{OS}_{loc}$, we know that

$$H_2 \in \mathcal{L}_{loc} \cap \mathcal{OS}_{loc} \quad \text{and} \quad H_2(x + \Delta_{T_0}) \sim H(x + \Delta_{T_0}).$$

(ii) In Theorem 4, we take $V = F$ and $V^{*\tau} = H_2$. Clearly, (16) holds for each distribution $V = F$, any $0 < \varepsilon < 1$ and some $n_0 \geq 1$, see, for example, Watanabe and Yamamuro [33]. Further, according to Theorem 4, by $H_2 \in \mathcal{L}_{loc} \cap \mathcal{OS}_{loc}$, combined with (8) and (16), we obtain all the conclusions.

3.3. *Proof of Theorem 3*

In order to prove the theorem, we need the following two results. The first result is the local version of the half of Lemma 2.1 of Cui et al. [29].

Lemma 3. *Let $V^{*\tau}$ be a random convolution defined as above.*

(i) If $p_{k-1} \geq p_k > 0$ for all $k \geq 2$, then the following proposition (B) implies the proposition (A) for some $0 < T < \infty$.

(A) For any $0 < \varepsilon < 1$, there exists an integer $n_0 = n_0(V, \tau, \varepsilon, T) \geq 1$ such that

$$\sum_{k=n_0+1}^{\infty} p_k V^{*k}(x + \Delta_T) \leq \varepsilon V^{*\tau}(x + \Delta_T), \quad x \geq 0. \tag{28}$$

(B) For any $0 < \varepsilon < 1$, there exists an integer $n_0 = n_0(V, \tau, \varepsilon, T) \geq 1$ such that (16) holds.

(ii) If $V^{*\tau} \in \mathcal{OS}_{\Delta_T}$ for some $0 < T < \infty$ with $p_1 > 0$, then the proposition (B) implies the proposition (A) replaced $x \geq 0$ by $x \geq x_1$ for some $x_1 \geq x_0$.

Remark 3. *(i) In particular, if τ obeys a Poisson distribution, then for any $0 < \varepsilon < 1$, (16) holds for some $n_0 \geq 1$. Further, because $p_{k-1} \geq p_k > 0$ for all $k \geq 2$, (28) holds for the same ε and n_0.*

(ii) The condition $0 < p_k \leq p_{k-1}$ for all $k \geq 2$ can be relaxed to the condition that $0 < p_k \leq C p_{k-1}$ for some $C > 0$ and all $k \geq 2$.

Proof. (i) If (16) holds, then by $p_{k-1} \geq p_k > 0$ for all $k \geq 2$, we know that for any $n_0 \geq 1$,

$$\sum_{k=n_0+1}^{\infty} p_k V^{*k}(x + \Delta_T) \leq \sum_{k=n_0+1}^{\infty} p_{k-1} V^{*k}(x + \Delta_T), \quad x \geq 0.$$

Therefore, (28) is implied by (16).

(ii) Clearly, we only need to prove the lemma for Case 1. Because $V^{*\tau} \in \mathcal{OS}_{\Delta_T}$ for some $0 < T < \infty$, there exists a constant $x_1 = x_1(V, \tau, T) \geq x_0$ such that

$$D^*(V^{*\tau}, T) = \sup_{x \geq x_1} (V^{*\tau})^{*2}(x + \Delta_T) \Big/ V^{*\tau}(x + \Delta_T) < \infty.$$

For any $0 < \varepsilon < 1$, we take

$$\varepsilon_0 = p_1 \varepsilon \Big/ \big(1 + D^*(V^{*\tau}, T)\big),$$

then $0 < \varepsilon_0 < 1$.

For the above ε_0, according to proposition (B), by $m = \infty$, there exists an integer $n_0 = n_0(V, \tau, \varepsilon_0, T) \geq 1$ such that $0 < a_{n_0} = \sum\limits_{k=n_0+1}^{\infty} p_k < \varepsilon_0$ and (16) holds, in which ε is replaced with ε_0 . Then $\sum\limits_{k=n_0+1}^{\infty} p_k V^{*(k-1)} \Big/ a_{n_0}$ can be considered as a distribution. Therefore, by $p_1 > 0$ and $V^{*\tau}(x+\Delta_T) \geq p_1 V(x+\Delta_T)$ for all $x \geq 0$, we have

$$
\begin{aligned}
\sum_{k=n_0+1}^{\infty} p_k V^{*k}(x+\Delta_T) &= a_{n_0} V * \Big(\sum_{k=n_0+1}^{\infty} p_k V^{*(k-1)} \Big/ a_{n_0} \Big)(x+\Delta_T) \\
&\leq a_{n_0} \int_{0-}^{x} \Big(\sum_{k=n_0+1}^{\infty} p_k V^{*(k-1)} \Big/ a_{n_0} \Big)(x-y+\Delta_T) V(dy) + a_{n_0} V(x+\Delta_T) \\
&\leq \varepsilon_0 \Big(\int_{0-}^{x} V^{*\tau}(x-y+\Delta_T) V^{*\tau}(dy) + V^{*\tau}(x+\Delta_T) \Big) \Big/ p_1 \\
&\leq \varepsilon_0 \big(V^{*2\tau}(x+\Delta_T) + V^{*\tau}(x+\Delta_T) \big) \Big/ p_1 \\
&\leq \varepsilon_0 \big(1 + D^*(V^{*\tau}, T) \big) V^{*\tau}(x+\Delta_T) \Big/ p_1 \\
&= \varepsilon V^{*\tau}(x+\Delta_T), \qquad x \geq x_1,
\end{aligned}
$$

that is (28) holds for any $0 < \varepsilon < 1$, all $x \geq x_1$ and some $n_0 \geq 1$. □

Theorem 5. *Assume that $V^{*\tau} \in \mathcal{TL}_{\Delta_{T_0}}(\gamma) \cap \mathcal{OS}_{\Delta_{T_0}}$ for some $0 < \gamma, T_0 < \infty$ with $p_k > 0$ for all $k \geq 1$ in Case 1 or $1 \leq k \leq m$ in Case 2. If for any $0 < \varepsilon < 1$, there exists an integer $n_0 = n_0(V, \tau, \varepsilon, T_0) \geq 1$ such that (16) holds, and for each the above k,*

$$\liminf V_\gamma^{*k}(x-t+\Delta_{T_0}) \Big/ V_\gamma^{*k}(x+\Delta_{T_0}) \geq 1 \quad \textit{for each } t > 0, \tag{29}$$

then there exists an integer $l_0 \geq 1$ in Case 1 or $1 \leq l_0 \leq m$ in Case 2 such that $V^{*n} \in \mathcal{TL}_{\Delta_{T_0}} \cap \mathcal{OS}_{\Delta_{T_0}}$ for all $n \geq l_0$ in Case 1 or $l_0 \leq n \leq m$ in Case 2 and $V^{*n} \notin \mathcal{TL}_{\Delta_{T_0}} \cap \mathcal{OS}_{\Delta_{T_0}}$ for all $1 \leq n \leq l_0 - 1$. In particular, if $V \in \mathcal{OS}_{\Delta_{T_0}}$, then $V^{*n} \in \mathcal{TL}_{\Delta_{T_0}} \cap \mathcal{OS}_{\Delta_{T_0}}$ for all $n \geq 1$ in Case 1 or $1 \leq n \leq m$ in Case 2.

Proof. For case 1, we first prove $V^{*n} \in \mathcal{OS}_{\Delta_{T_0}}$ for all $n \geq n_0$, where n_0 fixed in (16). Because $V^{*\tau} \in \mathcal{TL}_{\Delta_{T_0}}(\gamma)$, $M(V^{*\tau}, \gamma) < \infty$. Thus, $M(V^{*n}, \gamma) < \infty$ for all $n \geq 1$. By (14), it holds that,

$$V^{*\theta}(x+\Delta_{T_0}) = \frac{M(V^{*\theta}, \gamma)}{e^{\gamma x}} \Big((V^{*\theta})_\gamma(x+\Delta_{T_0}) - \gamma \int_0^{T_0} \frac{(V^{*\theta})_\gamma(x+y, x+T_0]}{e^{\gamma y}} dy \Big), \ \ x \geq 0, \tag{30}$$

where $\theta = k$ for each $k \geq 1$ or $\theta = \tau$. Thus, similar to (15), we have

$$e^{-\gamma T_0} (V^{*\theta})_\gamma(x+\Delta_{T_0}) \leq e^{\gamma x} V^{*\theta}(x+\Delta_{T_0}) \Big/ M(V^{*\theta}, \gamma) \leq (V^{*\theta})_\gamma(x+\Delta_{T_0}), \ \ x \geq 0. \tag{31}$$

When $\theta = \tau$, just as (18), we denote

$$(V^{*\tau})_\gamma(x+\Delta_{T_0}) = \sum_{k=1}^{\infty} q_k V_\gamma^{*k}(x+\Delta_{T_0}) = (V_\gamma)^{*\sigma}(x+\Delta_{T_0}), \qquad x \geq 0, \tag{32}$$

where $q_k = p_k M^k(V,\gamma)/M(V^{*\tau},\gamma)$, $k \geq 1$. According to Proposition 1 and Lemma 1, since $V^{*\tau} \in \mathcal{TL}_{\Delta_{T_0}}(\gamma) \cap \mathcal{OS}_{\Delta_{T_0}}$, so $(V^{*\tau})_\gamma \in \mathcal{L}_{\Delta_{T_0}} \cap \mathcal{OS}_{\Delta_{T_0}}$. Furthermore, according to Lemma 3, by (31), (28) with $x_1 \geq x_0$ and (32), for $0 < \varepsilon < 1$ and n_0 in (16), we have

$$\begin{aligned} \sum_{k=1}^{n_0} q_k V_\gamma^{*k}(x+\Delta_{T_0}) &\geq e^{\gamma x} \sum_{k=1}^{n_0} p_k V^{*k}(x+\Delta_{T_0})/M(V^{*\tau},\gamma) \\ &\geq (1-\varepsilon)e^{\gamma x} V^{*\tau}(x+\Delta_{T_0})\Big/M(V^{*\tau},\gamma) \\ &\geq (1-\varepsilon)e^{-\gamma T_0}(V^{*\tau})_\gamma(x+\Delta_{T_0}), \qquad x \geq x_1. \end{aligned} \tag{33}$$

Further, for each $n \geq 1$, using Fatou lemma, by (29), we have

$$\liminf \frac{V_\gamma^{*(n+1)}(x+\Delta_{T_0})}{V_\gamma^{*n}(x+\Delta_{T_0})} \geq \int_0^\infty \liminf \frac{V_\gamma^{*n}(x-y+\Delta_{T_0})}{V_\gamma^{*n}(x+\Delta_{T_0})} \mathbf{1}_{[0,x]}(y) V_\gamma(dy) \geq 1. \tag{34}$$

Combining with (33), (34) and $(V^{*\tau})_\gamma \in \mathcal{OS}_{\Delta_{T_0}}$, we know that

$$V_\gamma^{*n}(x+\Delta_{T_0}) \asymp (V^{*\tau})_\gamma(x+\Delta_{T_0}) \quad \text{and} \quad V_\gamma^{*n} \in \mathcal{OS}_{\Delta_{T_0}} \quad \text{for all } n \geq n_0. \tag{35}$$

Using Lemma 1 again, by (31), we have

$$V^{*n}(x+\Delta_{T_0}) \asymp V^{*\tau}(x+\Delta_{T_0}).$$

Therefore, by $V^{*\tau} \in \mathcal{OS}_{\Delta_{T_0}}$, we know that $V^{*n} \in \mathcal{OS}_{\Delta_{T_0}}$ for all $n \geq n_0$.

Next, we prove that $V^{*n} \in \mathcal{TL}_{\Delta_{T_0}}(\gamma)$ for each $n \geq n_0$. According to Lemma 3 (ii), by (31), (16) and (32), for the above $0 < \varepsilon < 1$, n_0, $x_1 \geq x_0$ and each $n \geq n_0$, there exists an integer $m_0 = m_0(F,\tau,\varepsilon,T_0,\gamma) \geq n$ such that

$$\begin{aligned} \sum_{k=m_0+1}^{\infty} q_k V_\gamma^{*k}(x+\Delta_{T_0}) &\leq e^{\gamma(T_0+x)} \sum_{k=m_0+1}^{\infty} p_k V^{*k}(x+\Delta_{T_0})\Big/M(V^{*\tau},\gamma) \\ &\leq e^{\gamma(T_0+x)}\varepsilon e^{-\gamma T_0} V^{*\tau}(x+\Delta_{T_0})\Big/M(V^{*\tau},\gamma) \\ &\leq \varepsilon(V^{*\tau})_\gamma(x+\Delta_{T_0}) \\ &= \varepsilon(V_\gamma)^{*\sigma}(x+\Delta_{T_0}) \qquad \text{for all } x \geq x_1. \end{aligned} \tag{36}$$

Further, by $(V^{*\tau})_\gamma \in \mathcal{L}_{\Delta_{T_0}} \cap \mathcal{OS}_{\Delta_{T_0}}$, (29) and (36), for each $t > 0$, there exists a constant $x_2 = x_2(V,\tau,\varepsilon,t,m_0,\gamma) \geq x_1$ such that, for all $x \geq x_2$,

$$\begin{aligned} &\varepsilon(V^{*\tau})_\gamma(x+\Delta_{T_0}) \geq (V^{*\tau})_\gamma(x-t+\Delta_{T_0}) - (V^{*\tau})_\gamma(x+\Delta_{T_0}) \\ = &\Big(\sum_{1\leq k\neq n\leq m_0} + \sum_{k=n} + \sum_{k\geq m_0+1}\Big) q_k\big(V_\gamma^{*k}(x-t+\Delta_{T_0}) - V_\gamma^{*k}(x+\Delta_{T_0})\big) \\ \geq &-\varepsilon \sum_{1\leq k\leq m_0} q_k V_\gamma^{*k}(x+\Delta_{T_0}) + q_n\big(V_\gamma^{*n}(x-t+\Delta_{T_0}) - V_\gamma^{*n}(x+\Delta_{T_0})\big) - \varepsilon(V^{*\tau})_\gamma(x+\Delta_{T_0}), \end{aligned}$$

which implies that

$$V_\gamma^{*n}(x-t+\Delta_{T_0}) \leq V_\gamma^{*n}(x+\Delta_{T_0}) + 3\varepsilon(V^{*\tau})_\gamma(x+\Delta_{T_0})/q_n, \quad x \geq x_2.$$

Hence, by $(V^{*\tau})_\gamma(x+\Delta_{T_0}) \asymp V_\gamma^{*n}(x+\Delta_{T_0})$ and the arbitrariness of ε, we can get

$$\limsup V_\gamma^{*n}(x-t+\Delta_{T_0})\Big/V_\gamma^{*n}(x+\Delta_{T_0}) \leq 1. \tag{37}$$

Combined with (29) and (37), $V_\gamma^{*n} \in \mathcal{L}_{\Delta_{T_0}}$. Therefore, $V^{*n} \in \mathcal{TL}_{\Delta_{T_0}(\gamma)}$, for all $n \geq n_0$.

Similar to the proof of Theorem 4, the theorem can be proved.

For Case 2, by (34), we have $V_\gamma^{*m}(x+\Delta_{T_0}) \asymp (V^{*\tau})_\gamma(x+\Delta_{T_0})$. Then, it is easy to get that $V_\gamma^{*m} \in \mathcal{OS}_{\Delta_{T_0}}$.

Next, we prove that $V^{*m} \in \mathcal{TL}_{\Delta_{T_0}}(\gamma)$. For any $0<\varepsilon<1$ and x large enough, by (29) for $1 \le k \le m$, we can get

$$\begin{aligned} & \varepsilon(V^{*\tau})_\gamma(x+\Delta_{T_0}) \ge (V^{*\tau})_\gamma(x-t+\Delta_{T_0}) - (V^{*\tau})_\gamma(x+\Delta_{T_0}) \\ = & \Big(\sum_{1\le k<m} + \sum_{k=m}\Big) q_k\big(V_\gamma^{*k}(x-t+\Delta_{T_0}) - V_\gamma^{*k}(x+\Delta_{T_0})\big) \\ \ge & -\varepsilon \sum_{1\le k<m} q_k V_\gamma^{*k}(x+\Delta_{T_0}) + q_m\big(V_\gamma^{*m}(x-t+\Delta_{T_0}) - V_\gamma^{*m}(x+\Delta_{T_0})\big) \qquad \text{for each } t>0. \end{aligned}$$

After the same simplification, we have

$$V_\gamma^{*m}(x-t+\Delta_{T_0}) \le V_\gamma^{*m}(x+\Delta_{T_0}) + 2\varepsilon(V^{*\tau})_\gamma(x+\Delta_{T_0})\Big/q_m \qquad \text{for each } t>0.$$

Hence, we can obtain the same conclusion as (37) for $n=m$ which implies $V^{*m} \in \mathcal{TL}_{\Delta_{T_0}}(\gamma)$.

We omit the proof of the remaining conclusion, which is similar to that of Theorem 4. □

Now, we prove Theorem 3.

(i) Firstly, we prove that

$$H_{1,\gamma}(x+\Delta_{T_0}) = o\big(H_{2,\gamma}(x+\Delta_{T_0})\big). \tag{38}$$

Its proof is slightly more difficult than that of (23). For this, we denote

$$H_\gamma(x+\Delta_{T_0}) = H_\gamma(\ln e^x+\Delta_{T_0}) = H_\gamma(\ln y+\Delta_{T_0}) = f_\gamma(y), \quad x\ge 0.$$

According to Lemma 1, by $H \in \mathcal{TL}_{\Delta_{T_0}}(\gamma) \cap \mathcal{OS}_{\Delta_{T_0}}$, we have $H_\gamma \in \mathcal{L}_{\Delta_{T_0}} \cap \mathcal{OS}_{\Delta_{T_0}}$. Thus, $f_\gamma(\cdot)$ is a regular variation function with index 0, which implies

$$e^{\beta x} H_\gamma(x+\Delta_{T_0}) \to \infty \quad \text{for each } \ \beta>0. \tag{39}$$

By $\overline{H_1}(x) = O\big(e^{-\beta x}\big)$ for each $\beta>0$ and (15) with $V=H_{1,\gamma}$ and $T=T_0$, we have

$$e^{\beta x} H_{1,\gamma}(x+\Delta_{T_0}) \le e^{\gamma T_0} M^{-1}(H_1,\gamma) e^{(\beta+\gamma)x} H_1(x+\Delta_{T_0}) \to 0 \ \text{ for each } \ \beta>0. \tag{40}$$

Then by (39) and (40), we know that

$$H_{1,\gamma}(x+\Delta_{T_0}) = o\big(H_\gamma(x+\Delta_{T_0})\big). \tag{41}$$

Furthermore, by (10), for each pair $m,k\ge 1$, we have

$$\liminf F_\gamma^{*k}(x-jT_0+\Delta_{T_0})\Big/F_\gamma^{*k}(x+\Delta_{T_0}) \ge 1 \qquad \text{for each } 1\le j\le m. \tag{42}$$

In addition, there exists an integer $n_1 = n_1(H_{1,\gamma},T_0)$ large enough such that $H_{1,\gamma}(0,n_1T_0]>0$. Then by (41) and $H_\gamma \in \mathcal{L}_{\Delta_{T_0}} \cap \mathcal{OS}_{\Delta_{T_0}}$, for any

$$0<\varepsilon<H_{1,\gamma}(0,n_1T_0]\Big/\big(2(n_1+1)C^*_{\Delta_{T_0}}(H_\gamma)\big),$$

there exists an integer $m_1 = m_1(H_1,H_2,\varepsilon,T_0,\gamma)$ and a constant $x_3 \ge x_2$ such that

$$H_{1,\gamma}(x+\Delta_{T_0}) < \varepsilon H_\gamma(x+\Delta_{T_0}), \quad x\ge m_1T_0, \tag{43}$$

$$\sum_{j=0}^{n_1} H_\gamma^{*2}(x + jT_0 + \Delta_{T_0}) \le 2(n_1+1)C^*_{\Delta_{T_0}}(H_\gamma)H_\gamma(x+\Delta_{T_0}), \quad x \ge m_1T_0 \tag{44}$$

and

$$\sum_{j=0}^{m_1} F_\gamma^{*n_0}(x - jT_0 + \Delta_{T_0}) \le 2(m_1+1)F_\gamma^{*n_0}(x - m_1T_0 + \Delta_{T_0}), \quad x \ge x_3, \tag{45}$$

where the final inequality stems from (42) with $k = n_0$ in (35) and (16). In addition, by (35) with $V = F$, we know that for the above m_1 and n_0, there are

$$0 < C_1 = C_1(H_{2,\gamma}, T_0, m_1, n_0) < C_2 = C_2(H_{2,\gamma}, T_0, m_1, n_0) < \infty$$

and $x_4 = x_4(H_{2,\gamma}, T_0, m_1, n_0) \ge x_3$ such that, for all $1 \le j \le m_1$,

$$C_1H_{2,\gamma}(x - jT_0 + \Delta_{T_0}) \le F_\gamma^{*n_0}(x - jT_0 + \Delta_{T_0}) \le C_2H_{2,\gamma}(x - jT_0 + \Delta_{T_0}), \quad x \ge x_4. \tag{46}$$

For $i = 1, 2$, let X_i be a random variable with distribution $H_{i,\gamma}$. Assume that X_1 is independent of X_2 and (X_1^*, X_2^*) is an independent copy of (X_1, X_2). Further, denote $A_0 = \{X_1 + X_2 \in x + \Delta_{T_0}\}$ for all $x \ge 0$. We then divide $H_\gamma(x + \Delta_{T_0}) = P(A_0)$ as follows:

$$\begin{aligned} & P(A_0) = P(A_0, 0 \le X_2 \le x - m_1T_0) + P(A_0, x - m_1T_0 < X_2 \le x + T_0) \\ = \ & P_1(x) + P_2(x), \qquad x \ge 0. \end{aligned} \tag{47}$$

For $P_1(x)$, by (43), (44) and $H_\gamma \in \mathcal{L}_{\Delta_{T_0}} \cap \mathcal{OS}_{\Delta_{T_0}}$, we have

$$\begin{aligned} & P_1(x) = \int_{0-}^{x-m_1T_0} H_{1,\gamma}(x - y + \Delta_{T_0})H_{2,\gamma}(dy) \\ \le \ & \varepsilon \int_{0-}^{x-m_1T_0} H_\gamma(x - y + \Delta_{T_0})H_{2,\gamma}(dy) \\ = \ & \varepsilon P(X_1^* + X_2^* + X_2 \in x + \Delta_{T_0}, 0 \le X_2 \le x - m_1T_0) \\ \le \ & \varepsilon P(x < X_1^* + X_2^* + X_2 \le x + T_0, 0 \le X_1 \le n_2T_0)/H_{1,\gamma}((0, n_2T_0]) \\ \le \ & \varepsilon P(x < X_1 + X_2 + X_1^* + X_2^* \le x + T_0 + n_2T_0)/H_{1,\gamma}(0, n_2T_0] \\ = \ & \varepsilon \sum_{j=0}^{n_2} H_\gamma^{*2}(x + jT_0 + \Delta_{T_0}) \Big/ H_{1,\gamma}(0, n_2T_0] \\ \le \ & 2\varepsilon(n_2+1)C^*_{\Delta_{T_0}}(H_\gamma)H_\gamma(x + \Delta_{T_0}) \Big/ H_{1,\gamma}(0, n_2T_0], \quad x \ge m_1T_0. \end{aligned} \tag{48}$$

For $P_2(x)$, by (45) and (46), we have

$$\begin{aligned} & P_2(x) \le P(x - x_0 < X_2 \le x + T_0) = \sum_{j=0}^{m_1} H_{2,\gamma}(x - jT_0 + \Delta_{T_0}) \\ \le \ & \sum_{j=0}^{m_1} F_\gamma^{*n_0}(x - jT_0 + \Delta_{T_0})/C_1 \\ \le \ & 2(m_1+1)F_\gamma^{*n_0}(x - m_1T_0 + \Delta_{T_0}) \Big/ C_1 \\ \le \ & 2C_2(m_1+1)H_{2,\gamma}(x - m_1T_0 + \Delta_{T_0}) \Big/ C_1, \qquad x \ge x_4. \end{aligned} \tag{49}$$

Combined with (47), (48) and (49), we have

$$\left(1 - \frac{2\varepsilon(n_2+1)C^*_{\Delta_{T_0}}(H_\gamma)}{H_{1,\gamma}(0, n_2T_0]}\right)H_\gamma(x + \Delta_{T_0}) \le \frac{2C_2(m_1+1)}{C_1}H_{2,\gamma}(x - m_1T_0 + \Delta_{T_0}) \tag{50}$$

for $x \geq \max\{m_1T_0, x_4\}$. Furthermore, by (50), (39) and $2\varepsilon(n_2+1)C^*(H_\gamma)/H_{1,\gamma}(0, n_2T_0] < 1$, we know that

$$e^{\beta(x-m_1T_0)}H_{2,\gamma}(x - m_1T_0 + \Delta_{T_0}) = e^{-\beta m_1T_0}e^{\beta x}H_{2,\gamma}(x - m_1T_0 + \Delta_{T_0}) \to \infty \quad \text{for each} \ \ \beta > 0,$$

that is

$$e^{\beta x}H_{2,\gamma}(x + \Delta_{T_0}) \to \infty \quad \text{for each} \ \ \beta > 0. \tag{51}$$

Then by (15), (40) and (51), it holds that

$$H_{1,\gamma}(x + \Delta_{T_0})\Big/H_{2,\gamma}(x + \Delta_{T_0}) = e^{\beta x}H_{1,\gamma}(x + \Delta_{T_0})\Big/e^{\beta x}H_{2,\gamma}(x + \Delta_{T_0}) \to 0 \quad \text{for each} \ \ \beta > 0,$$

and thus (38) holds.

Secondly, by $H_\gamma \in \mathcal{L}_{\Delta_{T_0}} \cap \mathcal{OS}_{\Delta_{T_0}}$ and (50), we know that

$$H_{2,\gamma}(x + \Delta_{T_0}) \asymp H_\gamma(x + \Delta_{T_0}) \quad \text{and} \quad H_{2,\gamma} \in \mathcal{OS}_{\Delta_{T_0}}.$$

Finally, we prove that $H_{2,\gamma} \in \mathcal{L}_{\Delta_{T_0}}$. On one hand, for any $0 < \varepsilon < 1/2$, take n_0 in (33) and (16) with $V_\gamma = F_\gamma$, by (32) and (10), according to Lemma 3 (i), for each $t > 0$, there is a constant $x_5 = x_5(F, \varepsilon, t, \gamma) \geq x_4$ such that

$$\begin{aligned} H_{2,\gamma}(x - t + \Delta_{T_0}) &\geq \sum_{k=1}^{n_0} q_kF_\gamma^{*k}(x - t + \Delta_{T_0}) \\ &\geq (1-\varepsilon)\sum_{k=1}^{n_0} q_kF_\gamma^{*k}(x + \Delta_{T_0}) \\ &\geq (1-\varepsilon)\sum_{k=1}^{\infty} q_kF_\gamma^{*k}(x + \Delta_{T_0}) - \sum_{k=n_0+1}^{\infty} q_kF_\gamma^{*k}(x + \Delta_{T_0}) \\ &\geq (1-2\varepsilon)H_{2,\gamma}(x + \Delta_{T_0}), \qquad x \geq x_5. \end{aligned} \tag{52}$$

On the other hand, for any $0 < \varepsilon < 1$, each $t > 0$ and n_0 in (33) with $V_\gamma = F_\gamma$, by (36) and (10) for all $k \geq 1$, there is a constant $x_6 = x_6(F, \varepsilon, t, \gamma) \geq x_5$ such that, when $x \geq x_6$,

$$\frac{H_{2,\gamma}(x + \Delta_{T_0}) - H_{2,\gamma}(x - t + \Delta_{T_0})}{H_{2,\gamma}(x - t + \Delta_{T_0})} \leq \sum_{k=1}^{n_0}\left(\frac{F_\gamma^{*k}(x + \Delta_{T_0})}{F_\gamma^{*k}(x - t + \Delta_{T_0})} - 1\right) + \varepsilon \leq \varepsilon(n_0 + 1). \tag{53}$$

Combining (52) and (53), with the arbitrariness of ε, we know that $H_{2,\gamma} \in \mathcal{L}_{\Delta_{T_0}}$. Then (i) holds by Lemma 1 and Proposition 1.

(ii) In Theorem 5, we take $V = F$, $G = H$, $G_1 = H_1$ and $G_2 = H_2$. Because $p_k = e^{-\mu}\mu^k/k!\ k \geq 0$, according to Remark 3 (i), (16) holds for each distribution V, thus for F. Therefore, since $H_2 \in \mathcal{TL}_{\Delta_{T_0}}(\gamma) \cap \mathcal{OS}_{\Delta_{T_0}}$, according to Theorem 5, by (16) for F and (10), we obtain all the results.

4. On the Condition (10)

In this section, we give some concise and convenient conditions to replace condition (10), see the following Theorem 6. To this end, we require three lemmas.

Lemma 4. *If a distribution $V \in \mathcal{OL}_{\Delta_T}$ for some $0 < T < \infty$ satisfying*

$$\liminf V(x - t + \Delta_T)\Big/V(x + \Delta_T) \geq 1 \qquad \textit{for each} \ \ t > 0, \tag{54}$$

then

$$V(x-t+\Delta_T) \asymp V(x+\Delta_T) \asymp V(x+t+\Delta_T) \quad \textit{for each } t>0 \tag{55}$$

and

$$V(x+\Delta_{T_1}) = O\big(V(x+\Delta_T)\big) \quad \textit{for each pair } 0<T_1 \neq T<\infty. \tag{56}$$

Proof. Firstly, by (54) and $V \in \mathcal{OL}_{\Delta_T}$, we know that, for each $t>0$,

$$V(x+\Delta_T) \lesssim V(x-t+\Delta_T) \lesssim C_{\Delta_T}(V,t)V(x+\Delta_T),$$

that is $V(x-t+\Delta_T) \asymp V(x+\Delta_T)$. Thus

$$V(x+\Delta_T) = V(x+t-t+\Delta_T) \asymp V(x+t+\Delta_T).$$

Therefore, (55) holds.

Secondly, for each $0<T_1 \neq T<\infty$, there exists an integer $m \geq 1$ such that $(m-1)T < T_1 \leq mT$. Further, by $V \in \mathcal{OL}_{\Delta_T}$ and $V(x+\Delta_T) \asymp V(x+t+\Delta_T)$ for each $t>0$, we have

$$V(x+\Delta_{T_1}) \leq V(x+\Delta_{mT}) = \sum_{k=0}^{m-1} V(x+kT+\Delta_T) \lesssim \sum_{k=0}^{m-1} C^*_{\Delta_T}(V,kT)V(x+\Delta_T),$$

that is (56) holds. □

Lemma 5. *For $i=1,2$, let V_i be a distribution such that $V_i \in \mathcal{OL}_{\Delta_T}$ for some $0<T<\infty$ and*

$$\liminf V_i(x-t+\Delta_T)\big/V_i(x+\Delta_T) \geq 1 \quad \textit{for each } t>0. \tag{57}$$

(i) *Then*

$$V_i(x+\Delta_T) = O\big(V_1 * V_2(x+\Delta_T)\big), \quad i=1,2. \tag{58}$$

(ii) *If*

$$\lim C^*_{\Delta_T}(V_1,x)V_2(x+\Delta_T) = 0, \tag{59}$$

then $V_1 * V_2 \in \mathcal{OL}_{\Delta_T}$ and

$$C^*_{\Delta_T}(V_1*V_2,t) \leq \max\{C^*_{\Delta_T}(V_1,t), C^*_{\Delta_T}(V_2,t)\} \quad \textit{for each } t \geq 0. \tag{60}$$

Proof. (i) For any $0<A<\infty$, according to Fatou lemma, by (57), we have

$$\liminf \frac{V_1*V_2(x+\Delta_T)}{V_i(x+\Delta_T)} \geq \int_{0-}^{A} \liminf \frac{V_i(x-y+\Delta_T)}{V_i(x+\Delta_T)} V_j(dy) \geq V_j([0,A]) \to 1, \text{ as } A \to \infty,$$

for all $1 \leq i \neq j \leq 2$, that is (58) holds.

(ii) In order to prove (60), we perform some preparatory work.

For each $t>0$, any $0<\varepsilon<1$ and $i=1,2$, by $V_i \in \mathcal{OL}_{\Delta_T}$, there exists $x_i = x_i(V_i,\varepsilon,T,t)>0$ such that

$$V_i(x-t+\Delta_T) \leq (1+\varepsilon)C^*_{\Delta_T}(V_i,t)V_i(x+\Delta_T) \quad \text{for all } x \geq x_i. \tag{61}$$

For the above ε, by (59), there exists $x_3 = x_3(V_i,\varepsilon,T)>0$ such that when $x \geq x_3$,

$$C^*_{\Delta_T}(V_1,x)V_2(x+\Delta_T) < \varepsilon. \tag{62}$$

For above $t > 0$, by (56), (61), (62) and (58), we know that there is $x_0 = x_0(V_1, V_2, \varepsilon, T, t) \geq \max\{x_1, x_2, x_3\}$, $0 < K_i = K_i(V_1, V_2, T) < \infty, i = 1, 2$ and $m = m(V_1, V_2, T, \varepsilon, t) \geq 2$ such that, when $x \geq mx_0$,

$$\begin{aligned} & V_1(x - x_0 - t - T + \Delta_{2T})V_2(x_0 + \Delta_T) \leq K_1 V_1(x - x_0 - t - T + \Delta_T)V_2(x_0 + \Delta_T) \\ \leq\ & K_1(1+\varepsilon)^3 C^*_{\Delta_T}(V_1, T)C^*_{\Delta_T}(V_1, t)C^*_{\Delta_T}(V_1, x_0)V_1(x + \Delta_T)V_2(x_0 + \Delta_T) \\ \leq\ & K_1 K_2(1+\varepsilon)^3 C^*_{\Delta_T}(V_1, T)C^*_{\Delta_T}(V_1, t)C^*_{\Delta_T}(V_1, x_0)V_2(x_0 + \Delta_T)V_1 * V_2(x + \Delta_T) \\ <\ & \varepsilon(1+\varepsilon)^3 K V_1 * V_2(x + \Delta_T), \end{aligned} \tag{63}$$

where $K = K_1 K_2 C^*_{\Delta_T}(V_1, T)C^*_{\Delta_T}(V_1, t)$. In addition, let X and Y be the two random variables with corresponding distributions V_1 and V_2. Suppose that X is independent of Y. Denote

$$A_t = \{X + Y \in x - t + \Delta_T\} \quad \text{for } t \geq 0.$$

In the following, we deal with $V_1 * V_2(x - t + \Delta_T)$ in two cases where $T \leq t < \infty$ and $0 < t < T$. For $T \leq t < \infty$, by (61) and $x \geq mx_0 + T$, we have

$$\begin{aligned} & V_1 * V_2(x - t + \Delta_T) = P(A_t, 0 \leq X \leq x - t - x_0) + P(A_t, x - t - x_0 < X \leq x - t + T) \\ \leq\ & P(A_t, 0 \leq X \leq x - t - x_0) + P(A_t, 0 < Y \leq T + x_0) \\ =\ & \int_{0-}^{x-t-x_0} V_2(x - t - y + \Delta_T)V_1(dy) + \int_0^{T+x_0} V_1(x - t - y + \Delta_T)V_2(dy) \\ \leq\ & (1+\varepsilon)\Big(C^*_{\Delta_T}(V_2, t)\int_{0-}^{x-t-x_0} V_2(x - y + \Delta_T)V_1(dy) \\ & \qquad + C^*_{\Delta_T}(V_1, t)\int_0^{T+x_0} V_1(x - y + \Delta_T)V_2(dy)\Big) \\ \leq\ & (1+\varepsilon)\max\{C^*_{\Delta_T}(V_1, t), C^*_{\Delta_T}(V_2, t)\}V_1 * V_2(x + \Delta_T). \end{aligned} \tag{64}$$

For $0 < t < T$, we give a segmentation for $V_1 * V_2(x - t + \Delta_T)$ which is different from (64) as follows. Further, by (61), (63) and $x \geq mx_0 + T$, we have

$$\begin{aligned} & V_1 * V_2(x - t + \Delta_T) \leq P(A_t, 0 \leq X \leq x - t - x_0) + P(A_t, 0 < Y \leq x_0) \\ & \qquad + P(A_t, x_0 < Y \leq T + x_0) \\ \leq\ & \int_{0-}^{x-t-x_0} V_2(x - t - y + \Delta_T)V_1(dy) + \int_0^{x_0} V_1(x - t - y + \Delta_T)V_2(dy) \\ & \qquad + V_1(x - x_0 - t - T + \Delta_{2T})V_2(x_0 + \Delta_T) \\ \leq\ & (1+\varepsilon)\Big(C^*_{\Delta_T}(V_2, t)\int_{0-}^{x-t-x_0} V_2(x - y + \Delta_T)V_1(dy) + C^*_{\Delta_T}(V_1, t)\int_0^{x_0} V_1(x - y + \Delta_T)V_2(dy)\Big) \\ & \qquad + \varepsilon(1+\varepsilon)^2 K V_1 * V_2(x + \Delta_T) \\ \leq\ & (1+\varepsilon)\big(\max\{C^*_{\Delta_T}(V_1, t), C^*_{\Delta_T}(V_2, t)\} + \varepsilon(1+\varepsilon)^3 K\big)V_1 * V_2(x + \Delta_T). \end{aligned} \tag{65}$$

Therefore, $V_1 * V_2 \in \mathcal{OL}_{\Delta_T}$ and (60) holds by (64), (65) and the arbitrariness of ε. □

Lemma 6. *Let V_1 and V_2 be the two distributions belonging to the class $\mathcal{OL}_{\Delta_T}$ for some $0 < T < \infty$. If conditions (57) and (59) are satisfied, then for each $t > 0$,*

$$\liminf V_1 * V_2(x - t + \Delta_T)\Big/V_1 * V_2(x + \Delta_T) \geq 1. \tag{66}$$

Proof. In order to prove (66), we carry out some preparatory work. For each $t > 0$, by (57) and $V_2 \in \mathcal{OL}_{\Delta_T}$, we know that, for any $0 < \varepsilon < 1$, there exists $x_1 = x_1(V_1, V_2, \varepsilon, T, t) > 0$ such that, when $x \geq x_1$,

$$(1-\varepsilon)V_i(x+\Delta_T) \leq V_i(x-t+\Delta_T) \leq (1+\varepsilon)C^*_{\Delta_T}(V_i,t)V_i(x+\Delta_T), \quad i=1,2. \tag{67}$$

Furthermore, according to Lemma 5, there exists $x_2 = x_2(V_1, V_2, T, t) > 0$ and $C > 0$ such that, when $x \geq x_2$,

$$V_1(x+\Delta_T) \leq CV_1 * V_2(x+\Delta_T) \ \text{ and } \ V_{1\ (\text{or } 2)}\big(x+\Delta_t\ (\text{or } \Delta_{t+T})\big) \leq CV_1(x+\Delta_T). \tag{68}$$

Further, we denote $x_0 = \max\{x_1, x_2\} + t + T$ and $A_t = \{X + Y \in x - t + \Delta_T\}$, where X and Y are two random variables defined in Lemma 5.

Now, we prove (66) for each $t > 0$. When $x \geq x_0$, by (67), we have

$$\begin{aligned}
& V_1 * V_2(x-t+\Delta_T) = P(A_t, 0 \leq X \leq x-t-x_0) + P(A_t, x-t-x_0 < X \leq x-t+T) \\
= \ & \int_{0-}^{x-t-x_0} V_2(x-t-y+\Delta_T)V_1(dy) + P(A_t, x-t-x_0 < X \leq x-t+T) \\
\geq \ & (1-\varepsilon)\int_{0-}^{x-t-x_0} V_2(x-y+\Delta_T)V_1(dy) + P(A_t, x-t-x_0 < X \leq x-t+T) \\
\geq \ & (1-\varepsilon)\Big(V_1 * V_2(x+\Delta_T) - \int_{x-t-x_0}^{x-x_0} V_2(x-y+\Delta_T)V_1(dy) \\
& \quad -P(A_0, x-x_0 < X \leq x+T) + P(A_t, x-t-x_0 < X \leq x-t+T)\Big) \\
= \ & (1-\varepsilon)\big(V_1 * V_2(x+\Delta_T) - P_1(x) - P_2(x) + P_3(x)\big).
\end{aligned} \tag{69}$$

Firstly, we estimate $P_1(x)$. When $x - t - x_0 < y \leq x - x_0$,

$$V_2(x-y+\Delta_T) \leq P(x_0 < Y \leq x_0+t+T).$$

Then, by (68), (67) and (59), we know that

$$\begin{aligned}
& P_1(x) \leq V_2(x_0+\Delta_{t+T})V_1(x-x_0-t+\Delta_t) \\
\leq \ & C^2 V_2(x_0+\Delta_T)V_1(x-x_0-t+\Delta_T) \\
\leq \ & (1+\varepsilon)^2 C^2 V_1(x+\Delta_T)C^*_{\Delta_T}(V_1,t)C^*_{\Delta_T}(V_1,x_0)V_2(x_0+\Delta_T) \\
\leq \ & (1+\varepsilon)^2 C^3 V_1 * V_2(x+\Delta_T)C^*_{\Delta_T}(V_1,t)C^*_{\Delta_T}(V_1,x_0)V_2(x_0+\Delta_T) \\
= \ & o\big(V_1 * V_2(x+\Delta_T)\big) \qquad \text{as } \ x_0 \to \infty.
\end{aligned} \tag{70}$$

Secondly, we estimate $P_3(x) - P_2(x)$.

$$\begin{aligned}
& P_3(x) - P_2(x) = P(A_t, x-t-x_0 < X \leq x-t+T, 0 < Y < x_0+T) \\
& \qquad -P(A_0, x-x_0 < X \leq x+T, 0 \leq Y < x_0+T) \\
= \ & \int_{0-}^{x_0} \big(V_1(x-t-y+\Delta_T) - V_1(x-y+\Delta_T)\big)V_2(dy) \\
& \qquad + \int_{x_0}^{x_0+T} P(X \in x-t-y+\Delta_T, x-t-x_0 < X \leq x-t+T)V_2(dy) \\
& \qquad - \int_{x_0}^{x_0+T} P(X \in x-y+\Delta_T, x-x_0 < X \leq x+T)V_2(dy) \\
= \ & P_{11}(x) + P_{12}(x) - P_{13}(x).
\end{aligned} \tag{71}$$

By (67), we have

$$\frac{P_{11}(x)}{V_1 * V_2(x+\Delta_T)} \ \geq \ \frac{-\varepsilon}{V_1 * V_2(x+\Delta_T)}\int_{0-}^{x_0} V_1(x-y+\Delta_T)V_2(dy) \geq -\varepsilon. \tag{72}$$

Using the proof method of (70), we can get that

$$\begin{aligned} P_{12}(x) &= \int_{x_0}^{x_0+T} P(x-t-x_0 < X \le x-t-y+T)V_2(dy) \\ &\le V_1(x-t-x_0+\Delta_T)V_2(x_0+\Delta_T) \\ &= o\big(V_1 * V_2(x+\Delta_T)\big), \quad as \ \ x_0 \to \infty. \end{aligned} \tag{73}$$

Similarly, we have

$$\begin{aligned} P_{13}(x) &= \int_{x_0}^{x_0+T} P(x-x_0 < X \le x-y+T)V_2(dy) \\ &\le V_1(x-x_0+\Delta_T)V_2(x_0+\Delta_T) \\ &= o\big(V_1 * V_2(x+\Delta_T)\big), \quad as \ \ x_0 \to \infty. \end{aligned} \tag{74}$$

Combining with (69)–(74), we know that (66) holds. □

Theorem 6. *Suppose that $V \in \mathcal{OL}_{\Delta_T}$ for some $0 < T < \infty$. If conditions (57) and (59) are satisfied for $V_1 = V_2 = V$, then for all $k \ge 2$, $V^{*k} \in \mathcal{OL}_{\Delta_T}$,*

$$C^*_{\Delta_T}(V^{*k}, t) \le C^*_{\Delta_T}(V, t) \qquad for\ each\ t \ge 0. \tag{75}$$

and

$$\liminf V^{*k}(x-t+\Delta_T)\Big/V^{*k}(x+\Delta_T) \ge 1 \qquad for\ each\ t \ge 0. \tag{76}$$

Proof. We use mathematical induction to prove the result.

Clearly, (75) and (76) hold for $k = 1$. Assume that $V^{*k} \in \mathcal{OL}_{\Delta_T}$, (75) and (76) hold for some $k \ge 2$. Set $V_1 = V^{*k}$ and $V_2 = V$ in Theorem 6. By (59) and (75), we have $V^{*(k+1)} = V_1 * V_2 \in \mathcal{OL}_{\Delta_T}$ and (76) holds for $k+1$. Thus, (75) holds for $k+1$, too. □

In particular, we take $V = F_\gamma$ and $T = T_0$ in (76), then we obtain (10) in Theorem 3 of this paper.

5. Conclusions and Future Work

In this paper, we prove that the class $\mathcal{L}_{loc} \cap \mathcal{OS}_{loc}$, in addition to $\mathcal{TL}_{\Delta_{T_0}}(\gamma) \cap \mathcal{OS}_{\Delta_{T_0}}$ for some $0 < \gamma, T_0 < \infty$ are not closed under the I.I.D. root. However, by adding certain conditions, the two classes become closed under the I.I.D. root. At the same time, we also provide the corresponding results under the random convolution roots.

In this section, we briefly introduce the theoretical significance and application value of the above results reported herein, in addition to some unresolved problems.

5.1. Theoretical Significance and Application Value

In complex practice, F is often in a "black box", that is, it is unknown or partially unknown. For example, in Theorem 2, we only know that F has property (8) or (9), but we do not know whether it has property $F^{*k} \in \mathcal{L}_{loc} \cap \mathcal{OS}_{loc}$ for some $k \ge 1$. Furthermore, the properties of H, as the external expression of F, can be estimated by some statistical methods. Therefore, it is of great theoretical significance and application value to use known H to estimate unknown F. This presents the research purpose of this paper.

In the following, we provide some specific examples to illustrate applications of the research findings herein.

Firstly, it is well known that the distribution of components of the Lévy process is I.I.D. Therefore, research on I.I.D. H is beneficial to the Lévy process.

Secondly, in the Cramér–Lundeberg risk model, the distributions F, H_2 and H_1 satisfying the conditions (2) and (3) can be regarded as the distributions of the claim, the total claim amount and the perturbation to the total claim amount, respectively, see Sub-Section 1.3.3 of Embrechts et al. [34]. If the disturbed distribution of total claim amount $H = H_1 * H_2$ is an I.I.D. and $H \in \mathcal{L}_{loc} \cap \mathcal{OS}_{loc}$, then according to Theorem 2, we have $H_2 \in \mathcal{L}_{loc} \cap \mathcal{OS}_{loc}$ and $H_2(x + \Delta_T) \sim H(x + \Delta_T)$ for each $0 < T \leq \infty$ under condition (8) or (9). Interestingly, F does not have to belong to class $\mathcal{L}_{loc} \cap \mathcal{OS}_{loc}$, but F^{*k} belongs to that class for all $k \geq 2$, see Theorems 1 and 2 mentioned in this paper.

There are many similar examples, such as H_2 which is the distribution of proportional reinsurance or the claim in Poisson model, see Example 5.2 (i) of Klüppelberg and Mikosch [35] and the main theorems of Veraverbeke [36].

Therefore, the results of this paper undoubtedly play an important role in risk theory and other fields.

Finally, the results of this paper can offer a more complete and profound answer to the famous Embrechts–Goldie conjecture, see Section 5.2 below for details.

5.2. On the Embrechts–Goldie Conjecture

Let $\mathcal{X}$ be a distribution class, and let V be a distribution. If $V^{*2} \in \mathcal{X}$ implies $V \in \mathcal{X}$, then we say that the class $\mathcal{X}$ is closed under convolution roots. Clearly, the closure under I.D.D roots is the natural extension of the closure under convolution roots for some distribution class.

Theorem 2 of Embrechts et al. [2] shows that the class $\mathcal{S}$ is closed under convolution roots. The same conclusion also holds for the class $\mathcal{S}(\gamma)$ for some $\gamma > 0$ if the distribution $V \in \mathcal{L}(\gamma)$, see Theorem 2.10 of Embrechts and Goldie [24]. Therefore, Embrechts and Goldie [15,24] put forward a famous conjecture:

$$\text{If } V^{*k} \in \mathcal{L}(\gamma) \text{ for some (even for all) } k \geq 2 \text{ and } \gamma \geq 0, \ \text{ then } V \in \mathcal{L}(\gamma).$$

Many positive or negative conclusions related to the conjecture are then proposed. Some positive results can be found in Theorem 1.2 of Watanabe [11] for the class $\mathcal{S}(\gamma)$ for some $\gamma > 0$, Theorem 6 of Xu et al. [31] for the classes $\mathcal{L}(\gamma)$ and $\mathcal{L}(\gamma) \cap \mathcal{OS}$. Of course, these outcomes are valid under certain restrictive conditions.

The following references provide us with the negative results.

Theorem 1.1 of Watanabe [11] shows that the class $\mathcal{S}(\gamma)$ for some $\gamma > 0$ is not closed under the convolution roots in general.

Earlier, Shimura and Watanabe [37] showed that there is a distribution V such that $V^{*2} \in \mathcal{L}(\gamma) \setminus \mathcal{OS}$ for some $\gamma \geq 0$, while $V \in \mathcal{OL} \setminus \big(\cup_{\gamma \geq 0} \mathcal{L}(\gamma) \cup \mathcal{OS}\big)$ and $\overline{V}(x) = o\big(\overline{V^{*2}}(x)\big)$.

Further, Theorem 1.1 of Xu et al. [31] points out that there is a distribution $V \in \mathcal{OL} \setminus \big(\cup_{\gamma \geq 0} \mathcal{L}(\gamma) \cup \mathcal{OS}\big)$ and $\overline{V}(x) \neq o\big(\overline{V^{*2}}(x)\big)$ such that $V^{*2} \in \big(\mathcal{L}(\gamma) \cap \mathcal{OS}\big) \setminus \mathcal{S}(\gamma)$ for each $\gamma > 0$.

For $\gamma = 0$, Theorem 2.2 (1) of Xu et al. [30] shows that there is a distribution V such that $V \in \mathcal{OL} \setminus \big(\mathcal{L} \cup \mathcal{OS}\big)$ and $\overline{V}(x) \neq o\big(\overline{V^{*2}}(x)\big)$, while $V^{*k} \in \big(\mathcal{L}(\gamma) \cap \mathcal{OS}\big) \setminus \mathcal{S}$ for all $k \geq 2$. Then, Proposition 2.2 of Xu et al. [30] points out that there are two distributions V_1 and V_2 such that $V_1, V_2 \notin \mathcal{OL}$, while $V_1^{*k} \in \big(\mathcal{L} \cap \mathcal{OS}\big) \setminus \mathcal{S}$ and $V_2^{*k} \in \mathcal{L} \setminus \mathcal{OS}$ for all $k \geq 2$.

This result reveals a surprising phenomenon that, although the properties of a distribution V are very poor, its convolution, and even its random convolution and the corresponding I.I.D., bear good properties.

Therefore, the Embrechts–Goldie conjecture has been denied for the class $\mathcal{L}(\gamma)$ and its subclasses $\mathcal{S}(\gamma) \setminus \mathcal{S}$, $(\mathcal{L}(\gamma) \cap \mathcal{OS}) \setminus \mathcal{S}(\gamma)$ and $\mathcal{L}(\gamma) \setminus \mathcal{OS}$ for each $\gamma \geq 0$, where the corresponding distribution $V \in \mathcal{OL} \setminus \big(\cup_{\gamma \geq 0} \mathcal{L}(\gamma) \cup \mathcal{OS}\big)$, and even $V \notin \mathcal{OL}$.

In this subsection, we mainly focus on the local closure under the convolution root.

For negative conclusions, Corollary 1.1 of Watanabe [11] shows that the classes $\mathcal{S}_{loc}$, $\mathcal{L}_{loc}$, $\mathcal{S}_{\Delta_T}$ and $\mathcal{L}_{\Delta_T}$ for some $0 < T < \infty$ are not closed under convolution roots. Further, Theorem 1 of the paper and its proof show that the class $\mathcal{L}_{loc} \cap \mathcal{OS}_{loc}$ is not closed either.

In addition, Theorem 1.1 and Corollary 1.1 of Watanabe and Yamamuro [38] and Theorem 1.1 and Corollary of Watanabe [39] obtain some results corresponding to Corollary 1.1 of Watanabe [11] for the subexponential density classes and the subexponential lattice distribution classes, respectively. Clearly, the lattice distribution is a special local distribution, and the density is closely related to its local distribution.

As positive conclusions, Theorem 2.1 of Watanabe [39] shows that the subexponential lattice distribution classes are closed under convolution roots with a condition. However, other positive conclusions about the local closure in non-lattice cases are rare.

In this paper, according to Theorem 6 of Xu et al. [31], Proposition 1 and Lemma 1 of the paper, using the Esscher transform, we give a corresponding positive result for the class $\mathcal{L}_{loc} \cap \mathcal{OS}_{loc}$ and omit the proof details.

Theorem 7. *Let V be a distribution, and let γ and T be two positive and finite constants.*

(i) Assume that $V \in \mathcal{OS}_{loc}$ and

$$\liminf \overline{V_{-\gamma}}(x-t)\Big/\overline{V_{-\gamma}}(x) \geq e^{\gamma t} \text{ for each } t > 0 \tag{77}$$

or

$$\overline{V_{-\gamma}}(x) = o\big(\overline{V_{-\gamma}^{*2}}(x)\big). \tag{78}$$

If $V^{*2} \in \mathcal{L}_{loc}$, then $V \in \mathcal{L}_{loc}$.

(ii) Assume that $V \in \mathcal{L}_{loc}$ with the mean $\mu_V < \infty$, the condition (77) is satisfied and

$$C^*(V_{-\gamma}^{*2}) < 6M(V_{-\gamma}^{*2}, \gamma). \tag{79}$$

If $V^{*2} \in \mathcal{L}_{loc} \cap \mathcal{OS}_{loc}$, then $V \in \mathcal{L}_{loc} \cap \mathcal{OS}_{loc}$.

Using the Esscher transform, by (15), we can replace the condition (78) with a more immediate condition.

Proposition 2. *If $V^{*2} \in \mathcal{L}_{loc}$, then (78) is implied by the following condition:*

$$V(x + \Delta_T) = o\big(V^{*2}(x + \Delta_T)\big).$$

5.3. Some Unresolved Problems

Clearly, for other local distribution classes, such as the class $\mathcal{L}_{loc} \setminus \mathcal{OS}_{loc}$ and the class $\mathcal{TL}_{\Delta_{T_0}}(\gamma) \setminus \mathcal{OS}_{\Delta_{T_0}}$ for some $0 < \gamma, T_0 < \infty$, the following corresponding questions arise:

Are they closed under the I.I.D. root? If not, under what conditions are they closed under the I.I.D. root?

Perhaps we can first solve the corresponding problem of the global distribution class $\mathcal{L}(\gamma) \setminus \mathcal{OS}$ with some $\gamma > 0$. In addition, the existing results, apart from Proposition 2.1 of Xu et al. [30], often assume that $F \in \mathcal{OL}$. Then, if $F \notin \mathcal{OL}$, what will we get?

Further, if F does not belong to the class $\mathcal{L}_{loc} \cap \mathcal{OS}_{loc}$, $\mathcal{L}_{loc} \setminus \mathcal{OS}_{loc}$ or $\mathcal{TL}_{\Delta_{T_0}}(\gamma) \setminus \mathcal{OS}_{\Delta_{T_0}}$ for some $0 < \gamma, T_0 < \infty$, what kind of F can make F^{*k} for all $k \geq l_0$ and some $l_0 \geq 2$, H_2 and H belong to the same class? Even if $F \notin \mathcal{OL}_{loc}$, what will we get?

In our opinion, these questions are both interesting and difficult to solve. The theory will become more complete following the provision of solutions to these questions.

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