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Article

Admissible Classes of Multivalent Meromorphic Functions Defined by a Linear Operator

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Abstract: The results from this paper are related to the geometric function theory. In order to obtain them, we use the technique based on differential subordination, one of the newest techniques used in the field, also known as the technique of admissible functions. For that, the appropriate classes of admissible functions are first defined. Based on these classes, we obtain some differential subordination and superordination results for multivalent meromorphic functions, analytic in the punctured unit disc, related to a linear operator $\Im^{p}_{\rho,\tau}(\nu)$, for $\tau > 0$, $\nu, \rho \in \mathbb{C}$, such that $Re(\rho - \nu) \geqq 0$, $Re(\nu) > \tau p$, $(p \in \mathbb{N})$. Moreover, taking into account both subordination and superordination results, we derive a sandwich-type theorem. The connection with some other known results and an example are also provided.

Keywords: analytic function; meromorphic univalent function; differential subordination; differential superordination; sandwich-type; admissible class; linear operator

MSC: 30C45; 30C80

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1. Introduction

Let consider $H(U)$, the class of analytic functions defined in the open unit disc $U = \{\xi \in \mathbb{C} : |\xi| < 1\}$, and let $H[a, r]$, $(r \in \mathbb{N} = \{1, 2, 3, \dots\})$ be the subclass of $H(U)$, of functions having the form

$$f(\xi) = a + a_r \xi^r + a_{r+1}\xi^{r+1} + \dots \quad (a \in \mathbb{C}),$$

denote by $H = H[1, 1]$.

Additionally, we denote by Σ_p, the class of multivalent meromorphic functions, analytic in the punctured open unit disc $U^* = U \setminus \{0\}$, of the form:

$$f(\xi) = \xi^{-p} + \sum_{n=1-p}^{\infty} a_n \xi^n \quad (p \in \mathbb{N};\ \xi \in U^*). \tag{1}$$

For $\tau > 0$, $\nu, \rho \in \mathbb{C}$, such that $Re(\rho - \nu) \geqq 0$, $Re(\nu) > \tau p$, $(p \in \mathbb{N})$ and $f \in \Sigma_p$ given by (1) El-Ashwah and Hassan [1] introduced the integral operator $\Im^{p}_{\rho,\tau}(\nu) : \Sigma_p \to \Sigma_p$ given by:

- For $Re(\rho - \nu) > 0$,

$$\Im^{p}_{\rho,\tau}(\nu) f(\xi) = \frac{\Gamma(\rho - \tau p)}{\Gamma(\nu - \tau p)\Gamma(\rho - \nu)} \int_0^1 (1-t)^{\rho-\nu-1} t^{\nu-1} f(\xi t^{\tau}) dt\,;$$

- For $\nu = \rho$,
$$\Im^{p}_{\nu,\tau}(\nu)f(\xi) = f(\xi).$$
It is easily seen that the operator $\Im^{p}_{\rho,\tau}(\nu)f(\xi)$ can be expressed as follows:

$$\Im^{p}_{\rho,\tau}(\nu)f(\xi) = \xi^{-p} + \frac{\Gamma(\rho - \tau p)}{\Gamma(\nu - \tau p)} \sum_{n=1-p}^{\infty} \frac{\Gamma(\nu + n\tau)}{\Gamma(\rho + n\tau)} a_n \xi^n \tag{2}$$

where $\tau > 0$, $\nu, \rho \in \mathbb{C}$, $Re(\rho - \nu) \geqq 0$, $Re(\nu) > \tau p$, $(p \in \mathbb{N})$.

It easily follows from (2) that

$$\xi(\Im^{p}_{\rho,\tau}(\nu)f(\xi))' = (\frac{\nu}{\tau} - p)(\Im^{p}_{\rho,\tau}(\nu+1)f(\xi)) - \frac{\nu}{\tau}(\Im^{p}_{\rho,\tau}(\nu)f(\xi)). \tag{3}$$

The linear operator $\Im^{p}_{\rho,\tau}(\nu)f(\xi)$ is a generalization of some already known operators. In particular, for $f \in \Sigma_p$ we outline the following special cases:

(i) Putting $p = 1$, we obtain the operator $\Im_{\rho,\tau}(\nu)f(\xi)$ studied by El-Ashwah ([2], with $m = 0$);

(ii) Putting $\nu = n + 2p, \rho = p + 1$ and $\tau = 1$, we obtain $D^{n+p-1}f(\xi)$(n is an integer, $n > -p$ and $p \in \mathbb{N}$) which was studied by Aouf [3] (see also [4]);

(iii) Putting $\rho = \nu + 1$ and $\tau = 1$, we obtain $\Im^{\nu}_{p}f(\xi)(Re(\nu) > p;\ p \in \mathbb{N})$ which was studied by Kumar and Shukla [5];

(iv) Putting $\nu = a + p, \rho = c + p$, and $\tau = 1$, we obtain $L_p(a,c)f(\xi)(a \in \mathbb{R},\ c \in \mathbb{R}\backslash\mathbb{Z}_0, \mathbb{Z}_0 = \{0, 1, 2, ..\}, p \in \mathbb{N})$ which was studied by Liu and Srivastava [6];

(v) Putting $\nu = \beta + p, \rho = \alpha + \beta - \gamma + 1 + p$ and $\tau = 1$, we obtain $Q^{p,1}_{\alpha,\beta,\gamma}f(\xi)(\alpha > \gamma - 1, \gamma > 0, \beta > 0, p \in \mathbb{N})$ which was studied by El-Ashwah et al. [7];

(vi) Putting $\nu = \beta + p, \rho = \alpha + \beta + p$ and $\tau = 1$, we obtain $Q^{p}_{\alpha,\beta}f(\xi)(\alpha > 0, \beta > 0, p \in \mathbb{N})$ which was studied by Aqlan et al. [8];

One of the recent techniques used in geometric function theory is that based on differential subordination, also known as the technique based on admissible functions.

Let be the functions $f, g \in H(U)$, we say that the function $f(\xi)$ is subordinate to $g(\xi)$ or the function $g(\xi)$ is superordinate to $f(\xi)$, if we can find a Schwarz function $w(\xi)$, analytic in U with $w(0) = 0$ and $|w(\xi)| < 1$, $(\xi \in U)$, such that $f(\xi) = g(w(\xi))$ and we write $f(\xi) \prec g(\xi)$. For the case when the function $g(\xi)$ is univalent in U, we have $f(\xi) \prec g(\xi)$ if, and only if, $f(0) = g(0)$ and $f(U) \subset g(U)$. (cf., e.g., [9]; see also [10], p. 4, [11]).

The theory of differential subordinations and the references to its numerous applications to the univalent function theory are thoroughly presented in the monograph by Miller and Mocanu [10]. Earlier, Miller and Mocanu [12] approached the dual theory of differential superordination, and some developments on the subject are presented in the monograph by Bulboaca [9]. Additionally, general subordination problems for analytic functions defined in connection with linear operators were studied by Ali et al. [13–15], Aghalary et al. [16], Aouf and Hosssen [17], and Kim and Srivastava [18] through the appropriate classes of admissible functions. Additionally, for meromorphic functions, some subordination properties were investigated in [2,19–34].

In that follows, we denote by $\wp$ the set of the functions χ that are holomorphic and univalent on $\overline{U} \setminus E(\chi)$, where

$$E(\chi) = \{\varsigma : \varsigma \in \partial U \ \ : \ \lim_{\xi \to \varsigma} \chi(\xi) = \infty\},$$

and satisfy the condition $\chi'(\varsigma) \neq 0$ for $\varsigma \in \partial U \setminus E(\chi)$. Additionally, we denote by $\wp(a)$, the subclass of $\wp$ for which $\chi(0) = a$, and $\wp(1) = \wp_1$.

In this paper, we find the sufficient conditions for some admissible classes associated with $\Im^{p}_{\rho,\tau}(\nu)$ on meromorphically multivalent functions so that

$$\chi_1(\xi) \prec \left[\xi^p \Im^{p}_{\rho,\tau}(\nu) f(\xi)\right]^{\delta} \prec \chi_2(\xi)$$

where the functions $\chi_1(\xi)$ and $\chi_2(\xi)$ are given univalent in U with $\chi_1(0) = \chi_2(0) = 1$.

The rest of the paper is organized as follows: two known classes of admissible functions and some results related to these classes are presented in the Section 2, Preliminaries; the next section, entitled Results, contains the main results of the paper and it is divided into two subsections, first one presenting some subordination results, involving the operator $\Im^{p}_{\rho,\tau}(\nu)$ and the second one investigating some similar results but in the superordination framework, a sandwich type theorem being also obtained; conclusions are outlined in the last section.

2. Preliminaries

In order to state and prove our main results, the following known definitions and lemmas are needed.

Definition 1. *([10], Definition 2.3.a, p. 27). Let Ω be a set from $\mathbb{C}$, $\chi \in \wp$ and n, a positive integer. The class of admissible functions $\Psi_n[\Omega,\chi]$ consists of the functions $\psi : \mathbb{C}^3 \times U \to \mathbb{C}$ that satisfy the admissibility condition $\psi(r,s,t;\xi) \notin \Omega$ whenever $r = \chi(\varsigma)$, $s = k\varsigma\chi'(\varsigma)$, and*

$$Re\left\{\frac{t}{s} + 1\right\} \geq kRe\left\{1 + \frac{\varsigma\chi''(\varsigma)}{\chi'(\varsigma)}\right\},$$

where $\xi \in U$, $\varsigma \in \partial U \backslash E(\chi)$ and $k \geq n$. We denote $\Psi_1[\Omega,\chi]$ by $\Psi[\Omega,\chi]$.

In the particular case when

$$\chi(\xi) = M\frac{M\xi + a}{M + \overline{a}\xi}, \quad (M > 0,\ |a| < M),$$

we have $\chi(U) = U_M = \{w : |w| < M\}$, $\chi(0) = a$, $E(\chi) = \phi$ and $\chi \in \wp(a)$. In this case, we write $\Psi_n[\Omega, M, a] = \Psi_n[\Omega,\chi]$, and in the special case when $\Omega = U_M$, we use the following denotation: $\Psi_n[M,a]$.

Definition 2. *([12], Definition 3, p. 817). Let Ω be a set in $\mathbb{C}$, $\chi \in H[a,n]$ with $\chi'(\xi) \neq 0$. The class of admissible functions $\Psi'_n[\Omega,\chi]$ consists of the functions $\psi : \mathbb{C}^3 \times \overline{U} \to \mathbb{C}$ that satisfy the admissibility condition $\psi(r,s,t;\varsigma) \in \Omega$ whenever $r = \chi(\xi)$, $s = \frac{\xi\chi'(\xi)}{m}$, and*

$$Re\left\{\frac{t}{s} + 1\right\} \leq \frac{1}{m}Re\left\{1 + \frac{\xi\chi''(\xi)}{\chi'(\xi)}\right\},$$

where $\xi \in U$, $\varsigma \in \partial U$ and $m \geq n \geq 1$. In particular, we denote $\Psi'_1[\Omega,\chi]$ by $\Psi'[\Omega,\chi]$.

Lemma 1. *([10], Theorem 2.3.b, p.28). Let $\psi \in \Psi_n[\Omega,\chi]$ with $\chi(0) = a$. If the analytic function $\omega(\xi) = a + a_n\xi^n + a_{n+1}\xi^{n+1} + \ldots$ satisfies*

$$\psi(\omega(\xi), \xi\omega'(\xi), \xi^2\omega''(\xi); \xi) \in \Omega,$$

then $\omega(\xi) \prec \chi(\xi)$.

Lemma 2. *([12], Theorem 1, p. 818). Let $\psi \in \Psi_n^{'}[\Omega,\chi]$ with $\chi(0)=a$. If $\omega(\xi) \in \wp(a)$ and $\psi(\omega(\xi),\ \xi\omega^{'}(\xi),\ \xi^2\omega^{''}(\xi);\xi)$ is univalent in U, then*

$$\Omega \subset \left\{\psi(\omega(\xi),\xi\omega^{'}(\xi),\xi^2\omega^{''}(\xi);\xi) : \xi \in U\right\}$$

implies $\chi(\xi) \prec \omega(\xi)$.

3. Results

3.1. Subordination Results Based on the Operator $\Im^{p}_{\rho,\tau}(\nu)$

Throughout this paper, unless otherwise mentioned, we suppose that $\delta > 0$, $\tau > 0$, $\nu,\rho \in \mathbb{R}$, $(\rho-\nu) \geqq 0$, $\nu > \tau p$, $(p \in \mathbb{N})$, $\xi \in U$ and all powers are principal ones.

Definition 3. *Let Ω be a set in $\mathbb{C}$ and $\chi \in \wp_1 \cap H$. The class of admissible functions $\Phi_H[\Omega,\chi,\delta]$ contains the functions $\varphi : \mathbb{C}^3 \times U \to \mathbb{C}$ that satisfy the admissibility condition*

$$\varphi(u,v,w;\xi) \notin \Omega$$

for

$$u=\chi(\varsigma),\ v=\frac{k\zeta\chi^{'}(\varsigma)+\delta\left(\frac{\nu-\tau p}{\tau}\right)\chi(\varsigma)}{\delta\left(\frac{\nu-\tau p}{\tau}\right)},$$

$$Re\left\{\frac{(\nu-\tau p)w-(2\delta(\nu-\tau p)+1)v+\delta(\nu-\tau p)u}{\tau(v-u)}\right\} \geq kRe\left\{1+\frac{\varsigma\chi^{''}(\varsigma)}{\chi^{'}(\varsigma)}\right\} \tag{4}$$

where $\xi \in U$, $\varsigma \in \partial U \backslash E(\chi)$ and $k \geq 1$.

Theorem 1. *Let $\varphi \in \Phi_H[\Omega,\chi,\delta]$. If $f(\xi) \in \Sigma_p$ satisfies*

$$\begin{aligned}&\left\{\varphi\left(\left[\xi^p\Im^{p}_{\rho,\tau}(\nu)f(\xi)\right]^{\delta},\ \left[\xi^p\Im^{p}_{\rho,\tau}(\nu)f(\xi)\right]^{\delta}\frac{\Im^{p}_{\rho,\tau}(\nu+1)f(\xi)}{\Im^{p}_{\rho,\tau}(\nu)f(\xi)},\ \left[\xi^p\Im^{p}_{\rho,\tau}(\nu)f(\xi)\right]^{\delta}\right.\right.\\ &\left.\left.\times\left[(\delta-1)\left(\frac{\Im^{p}_{\rho,\tau}(\nu+1)f(\xi)}{\Im^{p}_{\rho,\tau}(\nu)f(\xi)}\right)^2+\frac{\nu+1-\tau p}{\nu-\tau p}\frac{\Im^{p}_{\rho,\tau}(\nu+2)f(\xi)}{\Im^{p}_{\rho,\tau}(\nu)f(\xi)}\right];\ \xi\right) : \xi \in U\right\}\\ &\subset\ \Omega\end{aligned} \tag{5}$$

then

$$\left[\xi^p\Im^{p}_{\rho,\tau}(\nu)f(\xi)\right]^{\delta} \prec \chi(\xi)\,.$$

Proof. Suppose that

$$\omega(\xi)=\left[\xi^p\Im^{p}_{\rho,\tau}(\nu)f(\xi)\right]^{\delta}. \tag{6}$$

Differentiating (6) with respect to ξ and using the identity (3), we obtain

$$\left[\xi^p\Im^{p}_{\rho,\tau}(\nu)f(\xi)\right]^{\delta}\frac{\Im^{p}_{\rho,\tau}(\nu+1)f(\xi)}{\Im^{p}_{\rho,\tau}(\nu)f(\xi)}=\frac{\xi\omega^{'}(\xi)+\delta\left(\frac{\nu-\tau p}{\tau}\right)\omega(\xi)}{\delta\left(\frac{\nu-\tau p}{\tau}\right)}. \tag{7}$$

Further computations show that

$$\begin{aligned}&\left[\xi^p \Im^p_{\rho,\tau}(\nu) f(\xi)\right]^\delta \left[(\delta-1)\left(\frac{\Im^p_{\rho,\tau}(\nu+1) f(\xi)}{\Im^p_{\rho,\tau}(\nu) f(\xi)}\right)^2 + \frac{\nu+1-\tau p}{\nu-\tau p}\frac{\Im^p_{\rho,\tau}(\nu+2) f(\xi)}{\Im^p_{\rho,\tau}(\nu) f(\xi)}\right] \\ = &\frac{\xi^2\omega''(\xi) + \left(1+\frac{2\delta(\nu-\tau p)+1}{\tau}\right)\xi\omega'(\xi) + \frac{(\delta(\nu-\tau p)+1)\delta(\nu-\tau p)}{\tau^2}\omega(\xi)}{\delta\left(\frac{\nu-\tau p}{\tau}\right)^2}. \end{aligned} \tag{8}$$

Define the transformations $\varphi : \mathbb{C}^3 \times U \to \mathbb{C}$

$$\begin{aligned} u(r,s,t) &= r\ ,\ v(r,s,t) = \frac{s+\delta\left(\frac{\nu-\tau p}{\tau}\right)r}{\delta\left(\frac{\nu-\tau p}{\tau}\right)}, \\ w(r,s,t) &= \frac{t+\left(1+\frac{2\delta(\nu-\tau p)+1}{\tau}\right)s + \frac{(\delta(\nu-\tau p)+1)\delta(\nu-\tau p)}{\tau^2}r}{\delta\left(\frac{\nu-\tau p}{\tau}\right)^2}. \end{aligned} \tag{9}$$

Let

$$\psi(r,s,t;\xi) = \varphi(u,v,w;\xi)$$

$$= \varphi\left(r, \frac{s+\delta\left(\frac{\nu-\tau p}{\tau}\right)r}{\delta\left(\frac{\nu-\tau p}{\tau}\right)}, \frac{t+\left(1+\frac{2\delta(\nu-\tau p)+1}{\tau}\right)s + \frac{(\delta(\nu-\tau p)+1)\delta(\nu-\tau p)}{\tau^2}r}{\delta\left(\frac{\nu-\tau p}{\tau}\right)^2}; \xi\right). \tag{10}$$

By using Equations (6)–(10), we obtain

$$\psi(\omega(\xi), \xi\omega'(\xi), \xi^2\omega''(\xi); \xi)$$

$$\begin{aligned} = \ &\varphi\Bigg(\left[\xi^p \Im^p_{\rho,\tau}(\nu) f(\xi)\right]^\delta, \left[\xi^p \Im^p_{\rho,\tau}(\nu) f(\xi)\right]^\delta \frac{\Im^p_{\rho,\tau}(\nu+1) f(\xi)}{\Im^p_{\rho,\tau}(\nu) f(\xi)}, \left[\xi^p \Im^p_{\rho,\tau}(\nu) f(\xi)\right]^\delta \\ &\times\left[(\delta-1)\left(\frac{\Im^p_{\rho,\tau}(\nu+1) f(\xi)}{\Im^p_{\rho,\tau}(\nu) f(\xi)}\right)^2 + \frac{\nu+1-\tau p}{\nu-\tau p}\frac{\Im^p_{\rho,\tau}(\nu+2) f(\xi)}{\Im^p_{\rho,\tau}(\nu) f(\xi)}\right];\ \xi\Bigg). \end{aligned} \tag{11}$$

Hence, Equation (6) becomes

$$\psi(\omega(\xi), \xi\omega'(\xi), \xi^2\omega''(\xi); \xi) \in \Omega\,.$$

In order to complete the proof, we have to prove that the admissibility condition for $\varphi \in \Phi_H[\Omega, \chi, \delta]$ is equivalent to the admissibility condition for ψ as given in Definition 1. We have that

$$\frac{t}{s}+1 = \frac{(\nu-\tau p)w - (2\delta(\nu-\tau p)+1)v + \delta(\nu-\tau p)u}{\tau(v-u)},$$

and, hence, $\psi \in \Psi[\Omega, \chi]$. From Lemma 1, we get

$$\omega(\xi) \prec \chi(\xi) \quad or \quad \left[\xi^p \Im^p_{\rho,\tau}(\nu) f(\xi)\right]^\delta \prec \chi(\xi).$$

□

For a simply connected domain, $\Omega \neq \mathbb{C}$ we have $\Omega = h(U)$ for some conformal mapping $h(\xi)$ of U onto Ω. We denote the class $\Phi_H[h(U), \chi, \delta]$ by $\Phi_H[h, \chi, \delta]$.

The following result can be easily obtain as a direct consequence of Theorem 1.

Theorem 2. *Let $\varphi \in \Phi_H[h,\chi,\delta]$. If $f(\xi) \in \Sigma_p$ satisfies*

$$\varphi\Bigg(\left[\xi^p \Im^p_{\rho,\tau}(\nu)f(\xi)\right]^\delta,\ \left[\xi^p \Im^p_{\rho,\tau}(\nu)f(\xi)\right]^\delta \frac{\Im^p_{\rho,\tau}(\nu+1)f(\xi)}{\Im^p_{\rho,\tau}(\nu)f(\xi)},\ \left[\xi^p \Im^p_{\rho,\tau}(\nu)f(\xi)\right]^\delta \times\left[(\delta-1)\left(\frac{\Im^p_{\rho,\tau}(\nu+1)f(\xi)}{\Im^p_{\rho,\tau}(\nu)f(\xi)}\right)^2 + \frac{\nu+1-\tau p}{\nu-\tau p}\frac{\Im^p_{\rho,\tau}(\nu+2)f(\xi)}{\Im^p_{\rho,\tau}(\nu)f(\xi)}\right];\ \xi\Bigg) \prec h(\xi), \tag{12}$$

then

$$\left[\xi^p \Im^p_{\rho,\tau}(\nu)f(\xi)\right]^\delta \prec \chi(\xi)\,.$$

In what follows, we obtain the best dominant of the differential subordination (12).

Theorem 3. *Let be $h(\xi)$ univalent in U, and $\varphi : \mathbb{C}^3 \times U \to \mathbb{C}$. If the second order differential equation*

$$\varphi(\omega(\xi), \xi\omega'(\xi), \xi^2\omega''(\xi); \xi) = h(\xi) \tag{13}$$

has a solution $\chi(\xi)$ with $\chi(0) = 1$ that satisfies one of the following conditions:
(1) $\chi(\xi) \in \wp_1$ and $\varphi \in \Phi_H[h,\chi,\delta]$,
(2) $\chi(\xi)$ is univalent in U and $\varphi \in \Phi_H[h,\chi_\rho,\delta]$, for some $\rho \in (0,1)$,
(3) $\chi(\xi)$ is univalent in U and there exists $\rho_0 \in (0,1)$ such that $\varphi \in \Phi_H[h_\rho,\chi_\rho,\delta]$, for all $\rho \in (\rho_0,1)$,
and $f(\xi) \in \Sigma_p$ satisfies (12), then

$$\left[\xi^p \Im^p_{\rho,\tau}(\nu)f(\xi)\right]^\delta \prec \chi(\xi)\,,$$

and $\chi(\xi)$ is the best dominant.

Proof. From Theorems 1 and 2, using the Technique in ([10], Theorem 2.3e, p. 31), we obtain that $\chi(\xi)$ is a dominant. Since a solution $\chi(\xi)$ of (13) it is also a solution of (12) it means that $\chi(\xi)$ will be dominated by all dominants. Hence, we obtain that $\chi(\xi)$ is the best dominant. □

Having in view the Definition 3, in the particular case $\chi(\xi) = 1 + M\xi,\ M > 0$, we describe the class of admissible functions $\Phi_H[\Omega,\chi,\delta]$, denoted by $\Phi_H[\Omega,M,\delta]$ as follows.

Definition 4. *Let be Ω a set in $\mathbb{C}$ and $M > 0$. We define the class of admissible functions $\Phi_H[\Omega,M,\delta]$ as the set of functions $\varphi : \mathbb{C}^3 \times U \to \mathbb{C}$ satisfying*

$$\varphi\left(1+Me^{i\theta}, 1+\frac{k+\delta\left(\frac{\nu-\tau p}{\tau}\right)}{\delta\left(\frac{\nu-\tau p}{\tau}\right)}Me^{i\theta}, 1+\frac{L+\left[\left(1+\frac{2\delta(\nu-\tau p)+1}{\tau}\right)k+\frac{(\delta(\nu-\tau p)+1)\delta(\nu-\tau p)}{\tau^2}\right]Me^{i\theta}}{\delta\left(\frac{\nu-\tau p}{\tau}\right)^2}; \xi\right) \notin \Omega \tag{14}$$

whenever $\xi \in U,\ \theta \in \mathbb{R},\ Re\left(Le^{-i\theta}\right) \geq (k-1)kM$ for all real θ and $k \geq 1$.

Corollary 1. *Let be $\varphi \in \Phi_H[\Omega,M,\delta]$. If we have that $f(\xi) \in \Sigma_p$ satisfies*

$$\varphi\Bigg(\left[\xi^p \Im^p_{\rho,\tau}(\nu)f(\xi)\right]^\delta,\ \left[\xi^p \Im^p_{\rho,\tau}(\nu)f(\xi)\right]^\delta \frac{\Im^p_{\rho,\tau}(\nu+1)f(\xi)}{\Im^p_{\rho,\tau}(\nu)f(\xi)},\ \left[\xi^p \Im^p_{\rho,\tau}(\nu)f(\xi)\right]^\delta \times\left[(\delta-1)\left(\frac{\Im^p_{\rho,\tau}(\nu+1)f(\xi)}{\Im^p_{\rho,\tau}(\nu)f(\xi)}\right)^2 + \frac{\nu+1-\tau p}{\nu-\tau p}\frac{\Im^p_{\rho,\tau}(\nu+2)f(\xi)}{\Im^p_{\rho,\tau}(\nu)f(\xi)}\right];\ \xi\Bigg) \in \Omega,$$

then

$$\left|\left[\xi^p \Im^p_{\rho,\tau}(\nu)f(\xi)\right]^\delta - 1\right| < M\,.$$

In the special case $\Omega = \chi(U) = \{\vartheta \in \mathbb{C} : |\vartheta - 1| < M\}$, we denote the class $\Phi_H[\Omega, M, \delta]$ by $\Phi_H[M, \delta]$. Now we can write the Corollary 1 as:

Corollary 2. *Let be $\varphi \in \Phi_H[M, \delta]$. If we have that $f(\xi) \in \Sigma_p$ satisfies*

$$\left|\varphi\left(\left[\xi^p \Im^p_{\rho,\tau}(\nu) f(\xi)\right]^\delta, \left[\xi^p \Im^p_{\rho,\tau}(\nu) f(\xi)\right]^\delta \frac{\Im^p_{\rho,\tau}(\nu+1) f(\xi)}{\Im^p_{\rho,\tau}(\nu) f(\xi)}, \left[\xi^p \Im^p_{\rho,\tau}(\nu) f(\xi)\right]^\delta \right.\right.$$
$$\left.\left. \times \left[(\delta - 1)\left(\frac{\Im^p_{\rho,\tau}(\nu+1) f(\xi)}{\Im^p_{\rho,\tau}(\nu) f(\xi)}\right)^2 + \frac{\nu + 1 - \tau p}{\nu - \tau p} \frac{\Im^p_{\rho,\tau}(\nu+2) f(\xi)}{\Im^p_{\rho,\tau}(\nu) f(\xi)}\right]; \xi\right) - 1\right| < M,$$

then

$$\left|\left[\xi^p \Im^p_{\rho,\tau}(\nu) f(\xi)\right]^\delta - 1\right| < M\,.$$

Corollary 3. *If $M > 0$ and $f(\xi) \in \Sigma_p$ satisfies*

$$\left|\left[\xi^p \Im^p_{\rho,\tau}(\nu) f(\xi)\right]^\delta \frac{\Im^p_{\rho,\tau}(\nu+1) f(\xi)}{\Im^p_{\rho,\tau}(\nu) f(\xi)} - 1\right| < M\,,$$

then

$$\left|\left[\xi^p \Im^p_{\rho,\tau}(\nu) f(\xi)\right]^\delta - 1\right| < M\,.$$

Proof. If we take $\varphi(u, v, w; \xi) = v$ in the Corollary 2, the proof is complete. □

Corollary 4. *Let be $M > 0$. If $f(z) \in \Sigma_p$ satisfies*

$$\left|\left[\xi^p \Im^p_{\rho,\tau}(\nu) f(\xi)\right]^\delta \frac{\Im^p_{\rho,\tau}(\nu+1) f(\xi)}{\Im^p_{\rho,\tau}(\nu) f(\xi)} - \left[\xi^p \Im^p_{\rho,\tau}(\nu) f(\xi)\right]^\delta\right| < \frac{M}{\delta\left(\frac{\nu - \tau p}{\tau}\right)}\,,$$

then

$$\left|\left[\xi^p \Im^p_{\rho,\tau}(\nu) f(\xi)\right]^\delta - 1\right| < M\,.$$

Proof. Let be $\varphi(u, v, w; \xi) = v - u$ and $\Omega = h(U)$, where

$$h(z) = \frac{M\xi}{\delta\left(\frac{\nu - \tau p}{\tau}\right)},\; M > 0.$$

Having in mind using Corollary 1, we prove that $\varphi \in \Phi_H[\Omega, M]$, which means that the admissible condition (14) is satisfied. Since we have

$$\left|\varphi\left(1 + Me^{i\theta}, 1 + \frac{k + \delta\left(\frac{\nu - \tau p}{\tau}\right)}{\delta\left(\frac{\nu - \tau p}{\tau}\right)} Me^{i\theta}, 1 + \frac{L + \left[\left(1 + \frac{2\delta(\nu - \tau p) + 1}{\tau}\right)k + \frac{(\delta(\nu - \tau p) + 1)\delta(\nu - \tau p)}{\tau^2}\right] Me^{i\theta}}{\delta\left(\frac{\nu - \tau p}{\tau}\right)^2}; \xi\right)\right|$$
$$= \frac{kM}{\delta\left(\frac{\nu - \tau p}{\tau}\right)} \geq \frac{M}{\delta\left(\frac{\nu - \tau p}{\tau}\right)},$$

where $\xi \in U,\ \theta \in \mathbb{R}$, and $k \geq 1$, by applying Corollary 1, the proof is complete.

Moreover, the result is sharp, based on the Theorem 3. We can notice that the differential equation

$$\frac{\xi\chi'(\xi)}{\delta\left(\frac{\nu-\tau p}{\tau}\right)} = \frac{M}{\delta\left(\frac{\nu-\tau p}{\tau}\right)}\xi \qquad \left(\left|\frac{\delta(\nu-\tau p)}{\tau}\right| < M\right)$$

has a univalent solution $\chi(\xi) = 1 + M\xi$. It follows from Theorem 3 that $\chi(\xi) = 1 + M\xi$ is the best dominant. □

Example 1. *For $\rho = \nu = p+1$ and $\delta = \tau = 1$ then*

(i) $\Im^{p}_{\rho+1,1}(p+1)f(\xi) = f(\xi)$;

(ii) $\Im^{p}_{\rho+1,1}(p+2)f(\xi) = \xi f'(\xi) + (p+1)f(\xi)$;

(iii) $\Im^{p}_{\rho+1,1}(p+3)f(\xi) = \frac{1}{2}\left[\xi^2 f''(\xi) + 2(p+2)\xi f'(\xi) + (p+1)(p+2)f(\xi)\right]$.

Substituting in Corollary 4 with (i) and (ii) the above result shows that for $f(\xi) \in \Sigma_p$, if

$$\xi^p\left[\xi f'(\xi) + pf(\xi)\right] \prec M\xi,$$

then

$$\xi^p f(\xi) \prec 1 + M\xi.$$

Remark 1. *We note that the result in Example 1 was obtained by Ali et al. ([14] at $\ell = 2, m = 1$ and $\alpha_1 = \alpha_2 = \beta_1 = 1$ in Corollary 2.5).*

3.2. Superordination and Sandwich Results Based on the Operator $\Im^{p}_{\rho,\tau}(\nu)$

In this section, we extend the study to differential superordination and also we prove a sandwich-type theorem for the linear operator $\Im^{p}_{\rho,\tau}(\nu)$. Here, we define the following class $\Phi'_H[\Omega, \chi, \delta]$ of admissible functions:

Definition 5. *Let be Ω a set in $\mathbb{C}$ and $\chi(\xi) \in H$ with $\xi\chi'(\xi) \neq 0$. We define the class of admissible functions, $\Phi'_H[\Omega, \chi, \delta]$, as the set of functions $\varphi : \mathbb{C}^3 \times \overline{U} \to \mathbb{C}$ satisfying the admissibility condition*

$$\varphi(u, v, w; \varsigma) \in \Omega$$

whenever

$$u = \chi(\xi), v = \frac{\xi\chi'(\xi) + m\delta\left(\frac{\nu-\tau p}{\tau}\right)\chi(\xi)}{m\delta\left(\frac{\nu-\tau p}{\tau}\right)},$$

$$Re\left\{\frac{(\nu-\tau p)w - (2\delta(\nu-\tau p)+1)v + \delta(\nu-\tau p)u}{\tau(v-u)}\right\} \leq \frac{1}{m}Re\left\{1 + \frac{\xi\chi''(\xi)}{\chi'(\xi)}\right\}, \tag{15}$$

where $\xi \in U$, $\varsigma \in \partial U$ and $m \geq 1$.

Theorem 4. *Let $\varphi \in \Phi'_H[\Omega, \chi, \delta]$. If $f(\xi) \in \Sigma_p$, $\left[\xi^p \Im^{p}_{\rho,\tau}(\nu)f(\xi)\right]^{\delta} \in \wp_1$ and*

$$\varphi\left(\left[\xi^p \Im^{p}_{\rho,\tau}(\nu)f(\xi)\right]^{\delta}, \left[\xi^p \Im^{p}_{\rho,\tau}(\nu)f(\xi)\right]^{\delta}\frac{\Im^{p}_{\rho,\tau}(\nu+1)f(\xi)}{\Im^{p}_{\rho,\tau}(\nu)f(\xi)}, \left[\xi^p \Im^{p}_{\rho,\tau}(\nu)f(\xi)\right]^{\delta}\right.$$
$$\left.\times\left[(\delta-1)\left(\frac{\Im^{p}_{\rho,\tau}(\nu+1)f(\xi)}{\Im^{p}_{\rho,\tau}(\nu)f(\xi)}\right)^2 + \frac{\nu+1-\tau p}{\nu-\tau p}\frac{\Im^{p}_{\rho,\tau}(\nu+2)f(\xi)}{\Im^{p}_{\rho,\tau}(\nu)f(\xi)}\right]; \xi\right),$$

is univalent in U, then

$$\Omega \subset \left\{ \varphi\left(\left[\xi^p \Im^p_{\rho,\tau}(\nu)f(\xi)\right]^\delta, \left[\xi^p \Im^p_{\rho,\tau}(\nu)f(\xi)\right]^\delta \frac{\Im^p_{\rho,\tau}(\nu+1)f(\xi)}{\Im^p_{\rho,\tau}(\nu)f(\xi)}, \left[\xi^p \Im^p_{\rho,\tau}(\nu)f(\xi)\right]^\delta \right.\right.$$
$$\left.\left. \times \left[(\delta-1)\left(\frac{\Im^p_{\rho,\tau}(\nu+1)f(\xi)}{\Im^p_{\rho,\tau}(\nu)f(\xi)}\right)^2 + \frac{\nu+1-\tau p}{\nu-\tau p}\frac{\Im^p_{\rho,\tau}(\nu+2)f(\xi)}{\Im^p_{\rho,\tau}(\nu)f(\xi)} \right]; \xi \right) : \xi \in U \right\}, \tag{16}$$

implies

$$\chi(\xi) \prec \left[\xi^p \Im^p_{\rho,\tau}(\nu)f(\xi)\right]^\delta.$$

Proof. Let be $\omega(\xi)$ defined as in (6) and $\psi(\xi)$ as in (11). Taking into account that $\varphi \in \Phi'_H[\Omega, \chi, \delta]$, based on (11) and (16), we get

$$\Omega \subset \left\{ \psi(\omega(\xi), \xi\omega'(\xi), \xi^2\omega''(\xi); \xi) : \xi \in U \right\}.$$

From (10), we see that the admissibility condition for the function $\varphi \in \Phi'_H[\Omega, \chi, \delta]$ is equivalent to the admissibility condition for the function ψ as given in Definition 2. Consequently, $\psi \in \Psi'[\Omega, \chi]$, and further, from Lemma 2, we get

$$\chi(\xi) \prec \omega(\xi) \quad \text{or} \quad \chi(\xi) \prec \left[\xi^p \Im^p_{\rho,\tau}(\nu)f(\xi)\right]^\delta.$$

□

For $\Omega \neq C$, a simply connected domain, we have that $\Omega = h(U)$ for some conformal mapping $h(\xi)$ for U onto Ω. In this case, we denote the class $\Phi'_H[h(U), \chi, \delta]$ by $\Phi'_H[h, \chi, \delta]$.

Using the same procedure as in Section 3.1, we get the following result as a direct consequence of Theorem 4.

Theorem 5. *Let be $\chi(\xi) \in H$, $h(\xi)$ analytic on U and $\varphi \in \Phi'_H[h, \chi, \delta]$.*

If $f(\xi) \in \Sigma_p$, $\left[\xi^p \Im^p_{\rho,\tau}(\nu)f(\xi)\right]^\delta \in \wp_1$ and

$$\varphi\left(\left[\xi^p \Im^p_{\rho,\tau}(\nu)f(\xi)\right]^\delta, \left[\xi^p \Im^p_{\rho,\tau}(\nu)f(\xi)\right]^\delta \frac{\Im^p_{\rho,\tau}(\nu+1)f(\xi)}{\Im^p_{\rho,\tau}(\nu)f(\xi)}, \left[\xi^p \Im^p_{\rho,\tau}(\nu)f(\xi)\right]^\delta \right.$$
$$\left. \times \left[(\delta-1)\left(\frac{\Im^p_{\rho,\tau}(\nu+1)f(\xi)}{\Im^p_{\rho,\tau}(\nu)f(\xi)}\right)^2 + \frac{\nu+1-\tau p}{\nu-\tau p}\frac{\Im^p_{\rho,\tau}(\nu+2)f(\xi)}{\Im^p_{\rho,\tau}(\nu)f(\xi)} \right]; \xi \right),$$

is univalent in U, then

$$h(\xi) \prec \varphi\left(\left[\xi^p \Im^p_{\rho,\tau}(\nu)f(\xi)\right]^\delta, \left[\xi^p \Im^p_{\rho,\tau}(\nu)f(\xi)\right]^\delta \frac{\Im^p_{\rho,\tau}(\nu+1)f(\xi)}{\Im^p_{\rho,\tau}(\nu)f(\xi)}, \left[\xi^p \Im^p_{\rho,\tau}(\nu)f(\xi)\right]^\delta \right.$$
$$\left. \times \left[(\delta-1)\left(\frac{\Im^p_{\rho,\tau}(\nu+1)f(\xi)}{\Im^p_{\rho,\tau}(\nu)f(\xi)}\right)^2 + \frac{\nu+1-\tau p}{\nu-\tau p}\frac{\Im^p_{\rho,\tau}(\nu+2)f(\xi)}{\Im^p_{\rho,\tau}(\nu)f(\xi)} \right]; \xi \right), \tag{17}$$

implies

$$\chi(\xi) \prec \left[\xi^p \Im^p_{\rho,\tau}(\nu)f(\xi)\right]^\delta.$$

One can use the Theorem 4 and Theorem 5 only to get subordinants of differential superordination of the form (16) or (17). The following result states for the existence of the best subordinant of (17) for certain φ.

Theorem 6. *Let $h(\xi)$ be analytic in U and $\varphi : \mathbb{C}^3 \times \overline{U} \to \mathbb{C}$. If the differential equation*

$$\varphi\Big(\chi(\xi), \xi\chi^{'}(\xi), \xi^2\chi^{''}(\xi); \xi\Big) = h(\xi),$$

has a solution $\chi(\xi) \in \wp_1$, and $\varphi \in \Phi^{'}_H[h, \chi, \delta]$, $f(\xi) \in \Sigma_p$, $\Big[\xi^p \Im^p_{\rho,\tau}(\nu) f(\xi)\Big]^\delta \in \wp_1$ with

$$\varphi\Bigg(\Big[\xi^p\Im^p_{\rho,\tau}(\nu)f(\xi)\Big]^\delta, \Big[\xi^p\Im^p_{\rho,\tau}(\nu)f(\xi)\Big]^\delta \frac{\Im^p_{\rho,\tau}(\nu+1)f(\xi)}{\Im^p_{\rho,\tau}(\nu)f(\xi)}, \Big[\xi^p\Im^p_{\rho,\tau}(\nu)f(\xi)\Big]^\delta \times\left[(\delta-1)\left(\frac{\Im^p_{\rho,\tau}(\nu+1)f(\xi)}{\Im^p_{\rho,\tau}(\nu)f(\xi)}\right)^2 + \frac{\nu+1-\tau p}{\nu-\tau p}\frac{\Im^p_{\rho,\tau}(\nu+2)f(\xi)}{\Im^p_{\rho,\tau}(\nu)f(\xi)}\right]; \xi\Bigg),$$

univalent in U, then

$$h(\xi) \prec \varphi\Bigg(\Big[\xi^p\Im^p_{\rho,\tau}(\nu)f(\xi)\Big]^\delta, \Big[\xi^p\Im^p_{\rho,\tau}(\nu)f(\xi)\Big]^\delta \frac{\Im^p_{\rho,\tau}(\nu+1)f(\xi)}{\Im^p_{\rho,\tau}(\nu)f(\xi)}, \Big[\xi^p\Im^p_{\rho,\tau}(\nu)f(\xi)\Big]^\delta \times\left[(\delta-1)\left(\frac{\Im^p_{\rho,\tau}(\nu+1)f(\xi)}{\Im^p_{\rho,\tau}(\nu)f(\xi)}\right)^2 + \frac{\nu+1-\tau p}{\nu-\tau p}\frac{\Im^p_{\rho,\tau}(\nu+2)f(\xi)}{\Im^p_{\rho,\tau}(\nu)f(\xi)}\right]; \xi\Bigg),$$

implies

$$\chi(\xi) \prec \Big[\xi^p\Im^p_{\rho,\tau}(\nu)f(\xi)\Big]^\delta$$

and $\chi(\xi)$ is the best subordinant.

Proof. One can notice that the proof is similar to the proof of Theorem 3. □

If we combine the Theorems 2 and 5, we get the following sandwich-type result.

Corollary 5. *Let be $h_1(\xi)$ and $\chi_1(\xi)$ analytic functions in U, $h_2(\xi)$ univalent function in U, $\chi_2(\xi) \in \wp_1$ with $\chi_1(0) = \chi_2(0) = 1$ and $\varphi \in \Phi_H[h_2, \chi_2, \delta] \cap \Phi^{'}_H[h_1, \chi_1, \delta]$. If $f(\xi) \in \Sigma_p$, $\Big[\xi^p\Im^p_{\rho,\tau}(\nu)f(\xi)\Big]^\delta \in \wp_1 \cap H$ and*

$$\varphi\Bigg(\Big[\xi^p\Im^p_{\rho,\tau}(\nu)f(\xi)\Big]^\delta, \Big[\xi^p\Im^p_{\rho,\tau}(\nu)f(\xi)\Big]^\delta \frac{\Im^p_{\rho,\tau}(\nu+1)f(\xi)}{\Im^p_{\rho,\tau}(\nu)f(\xi)}, \Big[\xi^p\Im^p_{\rho,\tau}(\nu)f(\xi)\Big]^\delta \times\left[(\delta-1)\left(\frac{\Im^p_{\rho,\tau}(\nu+1)f(\xi)}{\Im^p_{\rho,\tau}(\nu)f(\xi)}\right)^2 + \frac{\nu+1-\tau p}{\nu-\tau p}\frac{\Im^p_{\rho,\tau}(\nu+2)f(\xi)}{\Im^p_{\rho,\tau}(\nu)f(\xi)}\right]; \xi\Bigg)$$

is univalent in U, then

$$h_1(\xi) \prec \varphi\Bigg(\Big[\xi^p\Im^p_{\rho,\tau}(\nu)f(\xi)\Big]^\delta, \Big[\xi^p\Im^p_{\rho,\tau}(\nu)f(\xi)\Big]^\delta \frac{\Im^p_{\rho,\tau}(\nu+1)f(\xi)}{\Im^p_{\rho,\tau}(\nu)f(\xi)}, \Big[\xi^p\Im^p_{\rho,\tau}(\nu)f(\xi)\Big]^\delta \times\left[(\delta-1)\left(\frac{\Im^p_{\rho,\tau}(\nu+1)f(\xi)}{\Im^p_{\rho,\tau}(\nu)f(\xi)}\right)^2 + \frac{\nu+1-\tau p}{\nu-\tau p}\frac{\Im^p_{\rho,\tau}(\nu+2)f(\xi)}{\Im^p_{\rho,\tau}(\nu)f(\xi)}\right]; \xi\Bigg) \prec h_2(\xi)$$

implies

$$\chi_1(\xi) \prec \Big[\xi^p\Im^p_{\rho,\tau}(\nu)f(\xi)\Big]^\delta \prec \chi_2(\xi).$$

4. Conclusions

By using the linear operator $\Im^{p}_{\rho,\tau}(\nu) : \Sigma_p \to \Sigma_p$ introduced by El-Ashwah and Hassan [1] we derive some differential subordination and superordination results for certain classes of admissible functions $\Phi_H[\Omega, \chi, \delta]$ and $\Phi'_H[\Omega, \chi, \delta]$ associated with the operator $\Im^{p}_{\rho,\tau}(\nu)$.

The first section contains subordination results for class of admissible functions $\Phi_H[\Omega, \chi, \delta]$, then, in the next section, we investigate differential superordination and sandwich-type theorem for class of admissible functions $\Phi'_H[\Omega, \chi, \delta]$ involving the linear operator $\Im^{p}_{\rho,\tau}(\nu)$.

The results we obtained are new and could help the researchers in the field of Geometric Function Theory to obtain other new results in this field.

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