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Article

The Analytical Solutions of the Stochastic mKdV Equation via the Mapping Method

Wael W. Mohammed [1,2,*], Farah M. Al-Askar [3] and Clemente Cesarano [4]

1 Department of Mathematics, Collage of Science, University of Ha'il, Ha'il 2440, Saudi Arabia
2 Department of Mathematics, Faculty of Science, Mansoura University, Mansoura 35516, Egypt
3 Department of Mathematical Science, Collage of Science, Princess Nourah bint Abdulrahman University, P.O. Box 84428, Riyadh 11671, Saudi Arabia
4 Section of Mathematics, International Telematic University Uninettuno, Corso, Vittorio, Emanuele II 39, 00186 Roma, Italy
* Correspondence: wael.mohammed@mans.edu.eg

Abstract: Here, we analyze the (2+1)-dimensional stochastic modified Kordeweg–de Vries (SmKdV) equation perturbed by multiplicative white noise in the Stratonovich sense. We apply the mapping method to obtain new trigonometric, elliptic, and rational stochastic fractional solutions. Because of the importance of the KdV equation in characterizing the behavior of waves in shallow water, the obtained solutions are beneficial in interpreting certain fascinating physical phenomena. We plot our figures in MATLAB and show several 3D and 2D graphical representations to show how the multiplicative white noise affects the solutions of the SmKdV. We show that the white noise around zero stabilizes SmKdV solutions.

Keywords: stochastic mKdV; the mapping method; exact solutions; stability by noise

MSC: 35A20; 60H10; 60H15; 35Q51; 83C15

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1. Introduction

The development of innovative traveling wave solutions for non-linear partial differential equations (NLPDEs) is crucial and important from many viewpoints for the most physical mathematical phenomena.The non-linear wave phenomenon occurs in a variety of disciplines of science and engineering, including meteorology, geology, solid-state physics, biology, chemical kinematics, fluid-mechanics, ocean engineering, and chemical physics [1–4]. In non-linear wave equations, the non-linear wave phenomena of convection, diffusion, dispersion, response, and dissipation are very significant. Consequently, one of the central issues of interest in physics and mathematics has been the study of exact solutions to those equations. Numerous techniques, such as the generalized Kudryashov method [5], sine-Gordon expansion [6,7], Exp-function [8], perturbation [9,10], Lie symmetry [11], Ricatti equation expansion [12], sn-ns method [13], Bernoulli sub-equation function [14], improved $\tan(\varphi/2)$-expansion [15], tanh-sech [16–18], and (G'/G)-expansion [19], have been explored, and some of them have been created in the process of looking for exact solutions to those equations.

One of the most familiar models for NLPDEs is the Korteweg–de Vries (KdV) equation:

$$\varphi_t + 6\varphi\varphi_x + \varphi_{xxx} = 0. \quad (1)$$

The KdV equation explains ion acoustic-waves in plasma, acoustic-waves on a crystal lattice, and long internal waves in a density-stratified ocean and in weakly interacting shallow-water waves. Numerous researchers have explored various forms of the KdV equation using different methodologies and approaches from a variety of perspectives (see, for example, Refs. [20–27] and the references therein).

On the other hand, the modified Kordeweg–de Vries (mKdV) equation

$$\varphi_t + 6\varphi^2\varphi_x + \varphi_{xxx} = 0, \quad (2)$$

has had a significant impact on the history of soliton theory. Additionally, it was employed to create an unlimited number of conservation laws for the KdV equation [28], that led to the identification of the Lax pair for the KdV equation and the invention of the inverse scattering transform [29]. Many authors have addressed exact solutions using different methods, such as the Exp-function [30], first integral [31], (G'/G)-expansion [32], and tanh methods [33], Bifurcation [34], etc.

A new (2+1)-dimensional mKdV equation that depends on the extended Lax equation was established and reported in 2019 [35] in the following form

$$\varphi_t + 6\varphi^2(\varphi_y - \varphi_x) - \varphi_{xxx} + \varphi_{yyy} + 3\varphi_{xxy} - 3\varphi_{xyy} = 0. \quad (3)$$

It is worth noting that if φ does not rely on x, i.e., $\varphi = \varphi(y,t)$, then Equation (3) turns out the mKdV equation

$$\varphi_t + 6\varphi^2\varphi_y + \varphi_{yyy} = 0. \quad (4)$$

If we rewrite the variable y as x, we have the mKdV Equation (2). The mKdV equation has multiple uses, including fluid mechanics [36], the dynamics of traffic flow [37], and the study of waves propagating in plasma [38]. Additionally, it is utilized in non-linear optics to explain pulses made up of a few optical cycles [39].

In recent years, the realization of adding random effects when predicting, modeling, simulating, and evaluating complex systems has been extensively appreciated in telecommunications, cryptography, biology, computer science, signal processing, climatic dynamics, physics, chemistry, geophysics, neuroscience, ecology, and other domains. Therefore, here, we consider Equation (3), which is derived by multiplicative noise, in the following form:

$$d\varphi + [6\varphi^2(\varphi_y - \varphi_x) - \varphi_{xxx} + \varphi_{yyy} + 3\varphi_{xxy} - 3\varphi_{xyy}]dt = \sigma\varphi \circ d\beta. \quad (5)$$

where $\beta(t)$ is the white noise and σ is the intensity of noise.

If we evaluate the stochastic integral in the middle, the integral $\int_0^t \varphi(\tau)d\beta(\tau)$ is called the Stratonovich stochastic integral (denoted by $\int_0^t \varphi(\tau) \circ d\beta(\tau)$). If we evaluate it at the left end, the integral $\int_0^t \varphi(\tau)d\beta(\tau)$ is called the Itô stochastic integral (denoted by $\int_0^t \varphi(\tau)d\beta(\tau)$) [40]. The Itô integral and Stratonovich integral have the following relationship:

$$\int_0^t \sigma\varphi(\tau)d\beta(\tau) = \int_0^t \sigma\varphi(\tau) \circ d\beta(\tau) - \frac{\sigma^2}{2}\int_0^t \varphi(\tau)d\tau. \quad (6)$$

Our objective is to apply the mapping method to determine the exact stochastic solutions of the SmKdV Equation (5). The solutions provided would be tremendously helpful to physicists in characterizing some important physical phenomena. Additionally, we explore the influence of noise on the analytical solutions of the SmKdV Equation (5) by introducing several figures through the use of MATLAB software.

The following is the order of the article: In Section 2, we employ wave transformation to attain the wave equation for the stochastic SmKdV Equation (5). In Section 3, we describe the mapping method, which we use in this article. In Section 4, the mapping method is used to ensure the exact stochastic solution of the SmKdV Equation (5). Next, in Section 5, we can see the influence of white noise on the acquired solutions of the SmKdV equation. Finally, the article's conclusions are provided.

2. Traveling Wave Equation for SmKdV

The wave equation for the SmKdV Equation (5) is obtained using the transformation:

$$\varphi(x,y,t) = \psi(\mu)e^{[\sigma\beta(t) - \sigma^2 t]}, \quad \mu = \mu_1 x + \mu_2 y + \mu_3 t, \quad (7)$$

where ψ is a real deterministic function and $\mu_1,\ \mu_2,\ \mu_3$ are non-zero constants. We note that

$$\begin{aligned} d\varphi &= [\mu_3\psi' dt + \sigma\psi d\beta - \frac{1}{2}\sigma^2\psi dt]e^{[\sigma\beta(t)-\sigma^2 t]} \\ &= [\mu_3\psi' dt + \sigma\psi \circ d\beta]e^{[\sigma\beta(t)-\sigma^2 t]}, \end{aligned} \tag{8}$$

and

$$\begin{aligned} \varphi_x &= \mu_1\psi' e^{[\sigma\beta(t)-\sigma^2 t]},\ \varphi_{xxx} = \mu_1^3\psi''' e^{[\sigma\beta(t)-\sigma^2 t]}, \\ \varphi_y &= \mu_2\psi' e^{[\sigma\beta(t)-\sigma^2 t]},\ \varphi_{yyy} = \mu_2^3\psi''' e^{[\sigma\beta(t)-\sigma^2 t]} \\ \varphi_{xxy} &= \mu_1^2\mu_2\psi''' e^{[\sigma\beta(t)-\sigma^2 t]},\ \varphi_{xyy} = \mu_1\mu_2^2\psi''' e^{[\sigma\beta(t)-\sigma^2 t]}. \end{aligned} \tag{9}$$

Inserting Equation (7) into (5) and utilizing (8) and (9), we obtain

$$(\mu_2-\mu_1)^3\psi''' + \mu_3\psi' + 6(\mu_2-\mu_1)\psi^2\psi' e^{[2\sigma\beta(t)-2\sigma^2 t]} = 0. \tag{10}$$

Considering the expectations on both sides, we attain

$$(\mu_2-\mu_1)^3\psi''' + \mu_3\psi' + 6(\mu_2-\mu_1)\psi^2\psi' e^{-2\sigma^2 t}\mathbb{E}e^{[2\sigma\beta(t)]} = 0. \tag{11}$$

Since $\beta(t)$ is the normal process, then $E(e^{2\sigma\beta(t)}) = e^{2\sigma^2 t}$. Therefore, Equation (11) becomes

$$(\mu_2-\mu_1)^3\psi''' + \mu_3\psi' + 6(\mu_2-\mu_1)\psi^2\psi' = 0. \tag{12}$$

Integrating Equation (12) once and setting the constant of integration equal to zero, we have the following Duffing equation

$$\psi'' + \ell_1\psi^3 + \ell_2\psi = 0, \tag{13}$$

where

$$\ell_1 = \frac{2}{(\mu_2-\mu_1)^2} \text{ and } \ell_2 = \frac{\mu_3}{(\mu_2-\mu_1)^3}.$$

3. The Description of Mapping Method

Let us now explain the mapping method stated in Ref. [41]. Supposing that the solutions to Equation (13) are

$$\psi(\mu) = \sum_{i=0}^{m} \hbar_i F^i(\mu), \tag{14}$$

where $\hbar_i$, for $i = 1, 2, \ldots \hbar_m$, are undefined constants to be evaluated and F fulfills the first type of elliptic equation

$$F' = \sqrt{r + qF^2 + pF^4}, \tag{15}$$

where the parameters r, q, and p are real.

We see that Equation (15) has several solutions based on $r,\ q$, and p as following Table 1:

Table 1. All solutions for Equation (15) for various $r,\ q$, and p values.

Case	p	q	r	$F(\mu)$
1	κ^2	$-(1+\kappa^2)$	1	$sn(\mu)$
2	1	$2\kappa^2-1$	$-\kappa^2(1-\kappa^2)$	$ds(\mu)$
3	1	$2-\kappa^2$	$(1-\kappa^2)$	$cs(\mu)$
4	$-\kappa^2$	$2\kappa^2-1$	$(1-\kappa^2)$	$cn(\mu)$
5	-1	$2-\kappa^2$	(κ^2-1)	$dn(\mu)$
6	$\frac{\kappa^2}{4}$	$\frac{(\kappa^2-2)}{2}$	$\frac{1}{4}$	$\frac{sn(\mu)}{1\pm dn(\mu)}$

Table 1. *Cont.*

Case	p	q	r	$F(\mu)$
7	$\frac{\kappa^2}{4}$	$\frac{(\kappa^2-2)}{2}$	$\frac{\kappa^2}{4}$	$\frac{sn(\mu)}{1\pm dn(\mu)}$
8	$\frac{-1}{4}$	$\frac{(\kappa^2+1)}{2}$	$\frac{-(1-\kappa^2)^2}{4}$	$\kappa cn(\mu) \pm dn(\mu)$
9	$\frac{\kappa^2-1}{4}$	$\frac{(\kappa^2+1)}{2}$	$\frac{(\kappa^2-1)}{4}$	$\frac{dn(\mu)}{1\pm sn(\mu)}$
10	$\frac{1-\kappa^2}{4}$	$\frac{(1-\kappa^2)}{2}$	$\frac{(1-\kappa^2)}{4}$	$\frac{cn(\mu)}{1\pm sn(\mu)}$
11	$\frac{(1-\kappa^2)^2}{4}$	$\frac{(1-\kappa^2)^2}{2}$	$\frac{1}{4}$	$\frac{sn(\mu)}{dn\pm cn(\mu)}$
12	1	0	0	$\frac{c}{\mu}$
13	0	1	0	ce^{μ}

Where $cn(\mu) = cn(\mu,\kappa), sn(\mu) = sn(\mu,\kappa), dn(\mu) = dn(\mu,\kappa)$, for $0 < \kappa < 1$ are the Jacobi elliptic functions (JEFs). If $\kappa \to 1$, then JEFs are transformed into the following hyperbolic functions:

$$\begin{aligned} cs(\mu) &\to \operatorname{csch}(\mu), sn(\mu) \to \tanh(\mu),\ \ cn(\mu) \to \operatorname{sech}(\mu), \\ dn(\mu) &\to \operatorname{sech}(\mu),\ \ ds \to \operatorname{csch}(\mu). \end{aligned}$$

4. Exact Solutions of mKdV

To find the parameter m, let us equalize ψ'' with ψ^3 in Equation (13) as

$$m + 2 = 3m \Longrightarrow m = 1.$$

Rewriting Equation (15) with $m = 1$ as

$$\psi(\mu) = \hbar_0 + \hbar_1 F(\mu). \tag{16}$$

Differentiating Equation (16) twice and using (15), we obtain

$$\psi'' = \hbar_1 qF + \hbar_1 pF^3. \tag{17}$$

Plugging Equations (16) and (17) into Equation (13) we have

$$(\hbar_1 p + \ell_1\hbar_1^3)F^3 + 3\hbar_0\hbar_1^2\ell_1 F^2 + (\hbar_1 q + 3\ell_1\hbar_0^2\hbar_1 + \ell_2\hbar_1)F + (\ell_1\hbar_0^3 + \ell_2\hbar_0) = 0.$$

Comparing each coefficient of F^j with zero, we attain

$$\hbar_1 p + \ell_1\hbar_1^3 = 0,$$

$$3\hbar_0\hbar_1^2\ell_1 = 0,$$

$$\hbar_1 q + 3\ell_1\hbar_0^2\hbar_1 + \ell_2\hbar_1 = 0,$$

and

$$\ell_1\hbar_0^3 + \ell_2\hbar_0 = 0.$$

When we solve these equations, we obtain

$$\hbar_0 = 0,\ \ \hbar_1 = \pm\sqrt{\frac{-p}{\ell_1}},\ \ q = -\ell_2.$$

Therefore, Equation (13) has the solution:

$$\psi(\mu) = \pm\sqrt{\frac{-p}{\ell_1}}F(\mu), \tag{18}$$

for $p < 0$ where $\ell_1 = \frac{2}{(\mu_2-\mu_1)^2} > 0$. There are many cases for the solutions $\psi(\mu)$ of Equation (13) relying on p as shown in Table 1, as following Table 2:

Table 2. All solutions for wave Equation (13) when $p < 0$.

Case	p	q	r	$F(\mu)$	$\psi(\mu)$
1	$-\kappa^2$	$2\kappa^2-1$	$(1-\kappa^2)$	$cn(\mu)$	$\pm\sqrt{\frac{-p}{\ell_1}}cn(\mu)$
2	-1	$2-\kappa^2$	(κ^2-1)	$dn(\mu)$	$\pm\sqrt{\frac{-p}{\ell_1}}dn(\mu)$
3	$\frac{-1}{4}$	$\frac{(\kappa^2+1)}{2}$	$\frac{-(1-\kappa^2)^2}{4}$	$\kappa cn(\mu) \pm dn(\mu)$	$\pm\sqrt{\frac{-p}{\ell_1}}[\kappa cn(\mu) \pm dn(\mu)]$
4	$\frac{\kappa^2-1}{4}$	$\frac{(\kappa^2+1)}{2}$	$\frac{(\kappa^2-1)}{4}$	$\frac{dn(\mu)}{1\pm sn(\mu)}$	$\pm\sqrt{\frac{-p}{\ell_1}}\frac{dn(\mu)}{1\pm sn(\mu)}$

If $\kappa \to 1$, then Table 2 degenerates to Table 3.

Table 3. All solutions for wave Equation (13) when $\kappa \to 1$ and $p < 0$.

Case	p	q	r	$F(\mu)$	$\psi(\mu)$
1	-1	1	0	$sech(\mu)$	$\pm\sqrt{\frac{1}{\ell_1}}sech(\mu)$
2	$\frac{-1}{4}$	2	0	$2sech(\mu)$	$\pm\sqrt{\frac{1}{\ell_1}}sech(\mu)$

Now, by utilizing Table 2 (or Table 3 when $\kappa \to 1$), we can have the exact solutions of the SmKdV Equation (5) as follows:

$$\varphi(x,y,t) = \psi(\mu)e^{[\sigma\beta(t)-\sigma^2 t]}. \tag{19}$$

Remark 1. *We can use various methods, including the Adomian decomposition, $exp(-\varphi)$-expansion method, improved $tan(\frac{\phi(\rho)}{2})$expansion, extended tanh method, Exp-function, Hirota bilinear, Weierstrass elliptic function, extended trial equation, complex hyperbolic function, etc., to obtain various solutions.*

5. The Effect of Noise on SmKdV Solutions

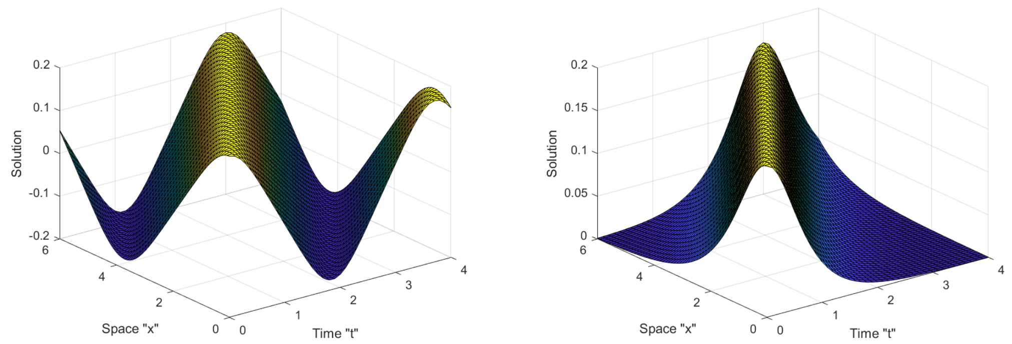
Here, we address the effect of white noise on the analytical solutions of the SmKdV Equation (5). We give various figures to describe the behavior of these solutions. For various σ (noise strength), we simulate some figures for obtained solutions, such as

$$\varphi(x,y,t) = \frac{\kappa}{\sqrt{2}}|\mu_2-\mu_1|cn(\mu_1 x+\mu_2 y+\mu_3 t)e^{[\sigma\beta(t)-\sigma^2 t]}, \tag{20}$$

and

$$\varphi(x,y,t) = \frac{1}{\sqrt{2}}|\mu_2-\mu_1|sech(\mu_1 x+\mu_2 y+\mu_3 t)e^{[\sigma\beta(t)-\sigma^2 t]}. \tag{21}$$

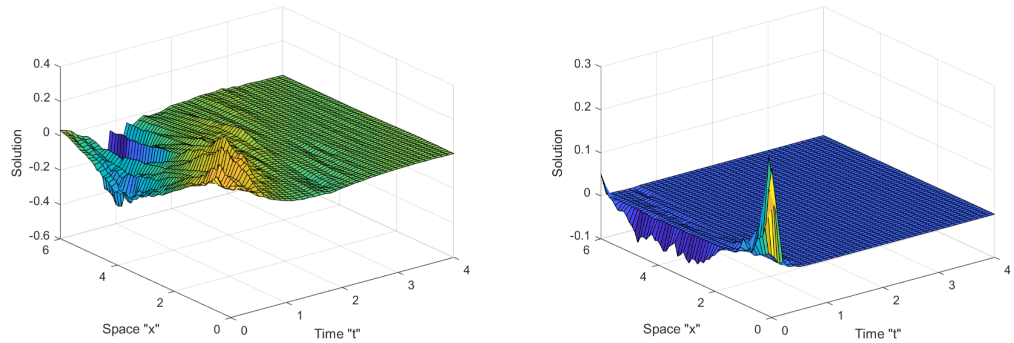
Let us first fix the parameters μ_1, μ_2, and μ_3 as follows: $\mu_1 = 1$, $\mu_2 = 0.5$, $\mu_3 = -2$, and $\kappa = 0.5$. Additionally, let $y = 0$, $x \in [0,5]$ and $t \in [0,5]$. In the next Figure 1, when there is no noise (i.e., $\sigma = 0$), we observe that the surface fluctuates



$\sigma = 0$ $\sigma = 0$

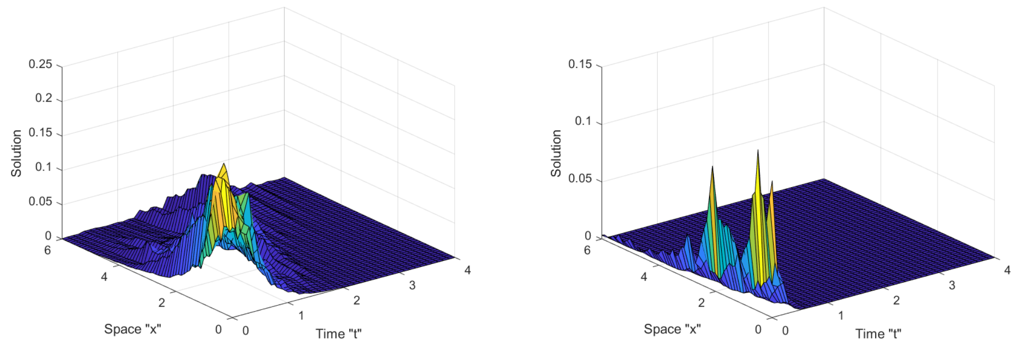
Figure 1. 3D-diagram of solution $\varphi(x, y, t)$ in Equations (20) and (21).

While we see that in Figures 2 and 3, after minor transit behaviors, the surface turn into more planar:



$\sigma = 1$ $\sigma = 2$

Figure 2. 3D-diagram of solution $\varphi(x, y, t)$ in Equation (20) for different $\sigma = 1, 2$.



$\sigma = 1$ $\sigma = 2$

Figure 3. 3D-diagram of solution $\varphi(x, y, t)$ in Equation (21) for different $\sigma = 1, 2$.

In Figures 4 and 5, we draw a two-dimensional graph representing the solution $\varphi(x, y, t)$ in Equations (20) and (21) to illustrate our previous results as follows:

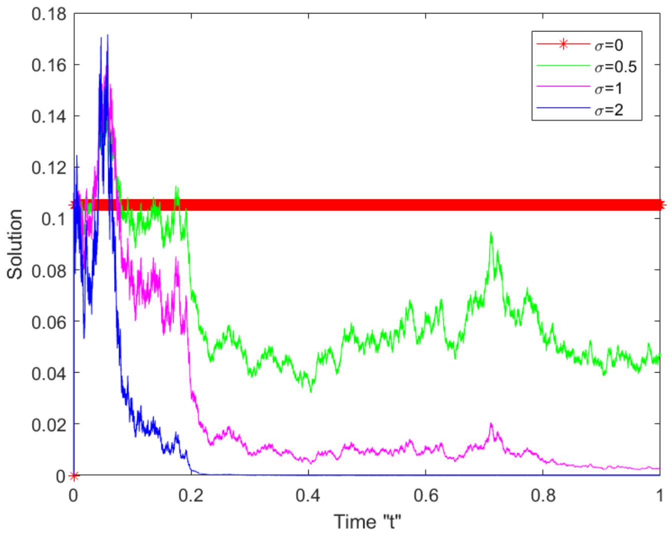


Figure 4. 2D-diagram of solution $\varphi(x, y, t)$ in Equation (20).

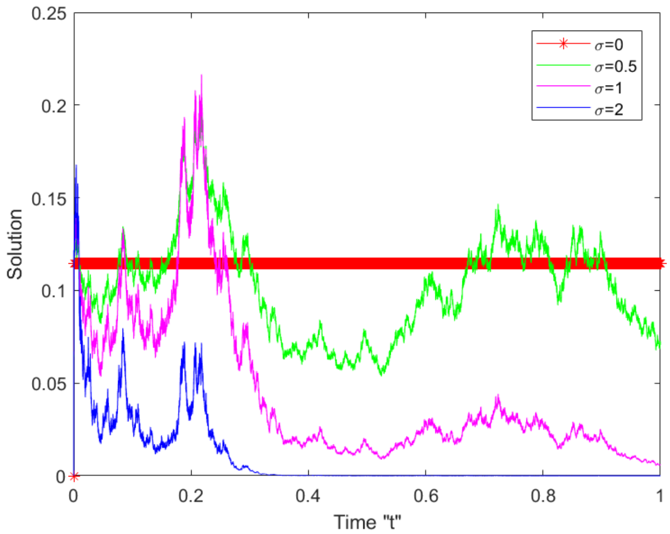


Figure 5. 2D-diagram of solution $\varphi(x, y, t)$ in Equation (21).

6. Conclusions

In this paper, we took into account the stochastic mKdV equation, which was created in the Stratonovich sense by multiplicative white noise. Utilizing the mapping method, we were able to obtain exact solutions. These solutions play a vital role in describing a number of interesting and complicated physical phenomena. Finally, the MATLAB package was used to demonstrate the effect of multiplicative white noise on the exact solution of the SmKdV equation.

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