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Review

Polynomial Automorphisms, Deformation Quantization and Some Applications on Noncommutative Algebras

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Abstract: This paper surveys results concerning the quantization approach to the Jacobian Conjecture and related topics on noncommutative algebras. We start with a brief review of the paper and its motivations. The first section deals with the approximation by tame automorphisms and the Belov–Kontsevich Conjecture. The second section provides quantization proof of Bergman's centralizer theorem which has not been revisited for almost 50 years and formulates several related centralizer problems. In the third section, we investigate a free algebra analogue of a classical theorem of Białynicki-Birula's theorem and give a noncommutative version of this famous theorem. Additionally, we consider positive-root torus actions and obtain the linearity property analogous to the Białynicki-Birula theorem. In the last sections, we introduce Feigin's homomorphisms and we see how they help us in proving our main and fundamental theorems on screening operators and in the construction of our lattice W_n-algebras associated with $\mathfrak{sl}_n$, which is by far the simplest known approach concerning constructing such algebras until now.

Keywords: deformation quantization; polynomial automorphisms; generic matrices; centralizers; torus actions; Weyl algebra; Lattice W-algebras; quantum groups; Feigin's homomorphisms

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1. Introduction

The goal of this review is to compile and condense findings related to the quantization approach to the Jacobian Conjecture.

As of the time this text was written, O.-H. Keller's Jacobian Conjecture remains an unresolved and seemingly insurmountable problem. The Jacobian Conjecture has been studied from a variety of perspectives, leading to the accumulation of a sizable amount of literature, while the development of many aspects of modern algebra and algebraic geometry was partially sparked by the search for an appropriate framework in which the Jacobian Conjecture could be investigated. This has led to a situation in which there is circumstantial evidence both for and against this conjecture's validity.

The study of infinite-dimensional algebraic semigroups of polynomial endomorphisms and groups of polynomial automorphisms as well as mappings between them is one of the most well established reasonable approaches to the Jacobian Conjecture. The foundation for this approach was laid by I.R. Shafarevich [1]. During the last several decades, the theory was vastly developed and enriched by the works of Anick, Artamonov, Bass, Bergman, Dicks, Dixmier, Lewin, Makar–Limanov, Czerniakiewicz, Shestakov, Umirbaev, Białynicki-Birula, Asanuma, Kambayashi, Wright and many others. In particular, the outcomes of Anick, Makar–Limanov, Shestakov and Umirbaev connected the combinatorial and geometric properties (stable tameness, approximation) of the spaces of polynomial automorphisms as well as its associative analogues.

Recently, Belov-Kanel and Kontsevich [2] and, independently, Tsuchimoto [3] proved the stable equivalence between the Jacobian Conjecture and the Dixmier conjecture on the endomorphisms of the Weyl algebra. A specific mapping (also known as the anti-quantization map) from the semigroup of Weyl algebra endomorphisms (a quantum object) to the semigroup of the corresponding Poisson algebra's endomorphisms is the basis of this very unexpected property (the appropriate classical object). Given this, it appears logical to assume that research into the quantization of spaces of polynomial mappings and the characteristics of the accompanying quantization morphisms will provide new knowledge.

A series of Kontsevich's conjectures on the equivalence of polynomial symplectomorphisms, holonomic modules over algebras of differential operators and automorphisms of such algebras provide one of the more significant milestones in this field. The relationship between the quantization method and universal algebra is another somewhat crucial feature of it.

We review some of our most recent discoveries with regard to quantization and the Kontsevich Conjecture. We discuss some of our most recent findings on approximation by tame automorphisms and its symplectic version (Section 2), deformation quantization approach on the new proof of Bergman's centralizer theorem (Section 3), torus actions on free associative algebras (Section 4) and Lattice W_n-algebras (Sections 5–7).

The last sections of this review paper are dedicated to the occasion of the 68th birthday anniversary of Boris L. Feigin.

2. Approximation by Tame Automorphisms and the Belov–Kontsevich Conjecture

The lifting problem has its origins in the context of deformation quantization of the affine space and is closely related to several major open problems in algebraic geometry and ring theory. Finally, let φ be a polynomial automorphism of $C[x_1, \ldots, x_n]$ and let $\mathcal{O}_\varphi$ be the local ring generated by the coefficients of φ and with maximal ideal $\mathfrak{m}$. If the sequence of tame automorphisms $\psi_1, \psi_2, \ldots$ converges to φ in the formal power series topology, then the coordinates of ψ_k converge to the coordinates of φ in the $\mathfrak{m}$-adic topology. A similar result is established for the symplectomorphisms of $P_n(\mathbb{C})$.

The n-th Weyl algebra $W_n(\mathbb{K})$ over $\mathbb{K}$ is by definition the quotient of the free associative algebra $\mathbb{K}\langle a_1, \ldots, a_n, b_1, \ldots, b_n\rangle$ by the two-sided ideal generated by elements

$$b_i a_j - a_j b_i - \delta_{ij}, \quad a_i a_j - a_j a_i, \quad b_i b_j - b_j b_i,$$

with $1 \leq i, j \leq n$. One can think of $W_n(\mathbb{K})$ as the algebra $\mathbb{K}[x_1, \ldots, x_n, y_1, \ldots, y_n]$ with two sets of n mutually commuting generators (images of the free generators under the canonical projection) which interact since $[y_i, x_j] = y_i x_j - x_j y_i = \delta_{ij}$. However, unless the context necessitates clarification, we would like to denote the Weyl algebra henceforth by $W_n(\mathbb{K})$ in order to avoid confusion with $\mathbb{K}[X]$—notation reserved for the ring of polynomials in commuting variables.

The polynomial algebra $\mathbb{K}[x_1, \ldots, x_N]$ itself is the quotient of the free associative algebra by congruence that makes all its generators commutative. When $N = 2n$ is even, the algebra A_{2n} carries an additional structure of the Poisson algebra—namely, a bilinear map

$$\{\ ,\ \} : \mathbb{K}[x_1, \ldots, x_N] \otimes \mathbb{K}[x_1, \ldots, x_N] \to \mathbb{K}[x_1, \ldots, x_N]$$

that turns $\mathbb{K}[x_1, \ldots, x_N]$ into a Lie algebra and acts as a derivation with respect to the polynomial multiplication. Under a fixed choice of generators, this map is given by the canonical Poisson bracket

$$\{x_i, x_j\} = \delta_{i,n+j} - \delta_{i+n,j}.$$

We denote the pair $(\mathbb{K}[x_1,\ldots,x_{2n}],\{\ ,\ \})$ by $P_n(\mathbb{K})$. In our discussion, the coefficient ring $\mathbb{K}$ is a field of characteristic zero and for later purposes, we require $\mathbb{K}$ to be algebraically closed. Thus, one may safely assume $\mathbb{K}=\mathbb{C}$ in the sequel.

Throughout, we assume all homomorphisms to be unital and preserve all defining structures carried by the objects in question. Thus, by a Weyl algebra endomorphism, we always mean a $\mathbb{K}$-linear ring homomorphism $W_n(\mathbb{K})$ into itself that maps 1 to 1. Similarly, the set $\operatorname{End}\mathbb{K}[x_1,\ldots,x_n]$ consists of all $\mathbb{K}$-endomorphisms of the polynomial algebra, while $\operatorname{End} P_n$ is the set of polynomial endomorphisms preserving the Poisson structure. We will call elements of the group $\operatorname{Aut} P_n$ *polynomial symplectomorphisms*, due to the fact that they can be identified with polynomial one-to-one mappings $\mathbb{A}^{2n}_{\mathbb{K}}\to\mathbb{A}^{2n}_{\mathbb{K}}$ of the affine space $\mathbb{A}^{2n}_{\mathbb{K}}$ which preserves the symplectic form

$$\omega=\sum_i dp_i\wedge dx_i.$$

Any endomorphism φ of $\mathbb{K}[x_1,\ldots,x_N]$, $P_n(\mathbb{K})$ or $W_n(\mathbb{K})$ can be identified with the ordered set $(\varphi(x_1),\ \varphi(x_2),\ \ldots)$ of images of generators of the corresponding algebra. For $\mathbb{K}[x_1,\ldots,x_N]$ and $P_n(\mathbb{K})$, the polynomials $\varphi(x_i)$ can be decomposed into sums of homogeneous components. This means that the endomorphism φ may be written as a formal sum

$$\varphi=\varphi_0+\varphi_1+\cdots,$$

where φ_k is a string (of length N and $2n$, respectively) whose entries are homogeneous polynomials of total degree k (we set $\deg x_i=1$). Accordingly, the height $\operatorname{ht}(\varphi)$ of the endomorphism is defined as

$$\operatorname{ht}(\varphi)=\inf\{k\mid\varphi_k\neq 0\},\ \ \operatorname{ht}(0)=\infty.$$

This is not to be confused with the degree of endomorphism, which is defined as $\deg(\varphi)=\sup\{k\mid\varphi_k\neq 0\}$ (for W_n the degree is well defined, but the height depends on the ordering of the generators). The height $\operatorname{ht}(f)$ of a polynomial f is defined quite similarly to be the minimal number k such that the homogeneous component f_k is not zero. Obviously, for an endomorphism $\varphi=(\varphi(x_1),\ \ldots,\ \varphi(x_N))$ one has

$$\operatorname{ht}(\varphi)=\inf\{\operatorname{ht}(\varphi(x_i))\mid 1\le i\le N\}.$$

The function

$$d(\varphi,\psi)=\exp(-\operatorname{ht}(\varphi-\psi))$$

is a metric on $\operatorname{End}\mathbb{K}[x_1,\ldots,x_N]$. We will refer to the corresponding topology on End (and on subspaces such as Aut and TAut) as the formal power series topology.

2.1. Tame Automorphisms

We call an automorphism $\varphi\in\operatorname{Aut}\mathbb{K}[x_1,\ldots,x_N]$ *elementary* if it is of the form

$$\varphi=(x_1,\ldots,\ x_{k-1},\ ax_k+f(x_1,\ldots,x_{k-1},\ x_{k+1},\ \ldots,\ x_N),\ x_{k+1},\ \ldots,\ x_N)$$

with $a\in\mathbb{K}^\times$. Observe that linear invertible changes of variables, that is, transformations of the form

$$(x_1,\ \ldots,\ x_N)\mapsto(x_1,\ \ldots,\ x_N)A,\ \ A\in\mathrm{GL}(N,\mathbb{K})$$

are realized as compositions of elementary automorphisms.

The subgroup of $\operatorname{Aut}\mathbb{K}[x_1,\ldots,x_N]$ generated by all elementary automorphisms is the group $\operatorname{TAut}\mathbb{K}[x_1,\ldots,x_N]$ of so-called *tame automorphisms*.

Let $P_n(\mathbb{K}) = \mathbb{K}[x_1, \ldots, x_n, p_1, \ldots, p_n]$ be the polynomial algebra in $2n$ variables with Poisson structure. It is clear that for an elementary automorphism

$$\varphi \in \operatorname{Aut} \mathbb{K}[x_1, \ldots, x_n, p_1, \ldots, p_n]$$

to be a symplectomorphism, it must either be a linear symplectic change of variables, that is, a transformation of the form

$$(x_1, \ldots, x_n, p_1, \ldots, p_n) \mapsto (x_1, \ldots, x_n, p_1, \ldots, p_n)A$$

with $A \in \operatorname{Sp}(2n, \mathbb{K})$ a symplectic matrix, or an elementary transformation of one of the two following types:

$$(x_1, \ldots, x_{k-1}, x_k + f(p_1, \ldots, p_n), x_{k+1}, \ldots, x_n, p_1, \ldots, p_n)$$

or

$$(x_1, \ldots, x_n, p_1, \ldots, p_{k-1}, p_k + g(x_1, \ldots, x_n), p_{k+1}, \ldots, p_n).$$

Note that in both cases we do not include translations of the affine space in our consideration, so we may safely assume the polynomials f and g to be at least of height one.

The subgroup of $\operatorname{Aut} P_n(\mathbb{K})$ generated by all such automorphisms is the group TAut $P_n(\mathbb{K})$ of *tame symplectomorphisms*. One similarly defines the notion of tameness for the Weyl algebra $W_n(\mathbb{K})$, with tame elementary automorphisms having the exact same form as for $P_n(\mathbb{K})$.

The automorphisms which are not tame are called *wild*. It is unknown at the time of writing whether the algebras W_n and P_n have any wild automorphisms in characteristic zero for $n > 1$; however, for $n = 1$ all automorphisms are known to be tame [4–7]. On the other hand, the celebrated example of Nagata

$$(x + (x^2 - yz)x, \; y + 2(x^2 - yz)x + (x^2 - yz)^2 z, \; z)$$

provides a wild automorphism of the polynomial algebra $\mathbb{K}[x, y, z]$.

It is known, due to Kanel-Belov and Kontsevich [2,8], that for $\mathbb{K} = \mathbb{C}$ the groups

$$\operatorname{TAut} W_n(\mathbb{C}) \quad \text{and} \quad \operatorname{TAut} P_n(\mathbb{C})$$

are isomorphic. The homomorphism between the tame subgroups is obtained by means of non-standard analysis and involves certain non-constructible entities, such as free ultrafilters and infinite prime numbers. Recent effort [9,10] has been directed at proving the homomorphism's independence of such auxiliary objects, with limited success.

2.2. Approximation by Tame Automorphisms

A classical theorem in [11] gives that every endomorphism of $\mathbb{K}[x_1, \ldots, x_n]$ with invertible Jacobian is a limit of a sequence of tame automorphisms in the formal power series topology. The first main result of [12] is a slightly modified proof of this theorem with an automorphism with Jacobian equal to 1 as a limit. We study the problem of lifting polynomial symplectomorphisms in characteristic zero to automorphisms of the Weyl algebra by means of approximation by tame automorphisms. We also prove the possibility of lifting a symplectomorphism to an automorphism of the power series completion of the Weyl algebra of the corresponding rank. The use of tame approximation is advantageous due to the fact that tame symplectomorphisms correspond to Weyl algebra automorphisms.

Let $\varphi \in \operatorname{Aut} \mathbb{K}[x_1, \ldots, x_N]$ be a polynomial automorphism. We say that φ is approximated by tame automorphisms if there is a sequence $\psi_1, \psi_2, \ldots, \psi_k, \ldots$ of tame automorphisms such that $\operatorname{ht}((\psi_k^{-1} \circ \varphi)(x_i) - x_i) \geq k$ for $1 \leq i \leq N$ and all k are sufficiently large. Observe that any tame automorphism ψ is approximated by itself—that is, by a stationary sequence $\psi_k = \psi$.

In [12], we get a special case of a classical result of Anick [11] (Anick proved approximation for all étale maps, not just automorphisms), which we reproduce here.

Theorem 1 ([12])**.** *Let $\varphi = (\varphi(x_1),\ \dots,\ \varphi(x_N))$ be an automorphism of the polynomial algebra $\mathbb{K}[x_1, \dots, x_N]$ over a field $\mathbb{K}$ of characteristic zero, such that its Jacobian*

$$\mathrm{J}(\varphi) = \det\left[\frac{\partial\varphi(x_i)}{\partial x_j}\right]$$

is equal to 1*. Then, there exists a sequence $\{\psi_k\} \subset \mathrm{TAut}\,\mathbb{K}[x_1, \dots, x_N]$ of tame automorphisms approximating φ.*

Theorem 2 ([12])**.** *Let $\sigma = (\sigma(x_1),\ \dots,\ \sigma(x_n),\ \sigma(p_1),\ \dots,\ \sigma(p_n))$ be a symplectomorphism of $\mathbb{K}[x_1, \dots, x_n, p_1, \dots, p_n]$ with unit Jacobian. Then there exists a sequence $\{\tau_k\} \subset \mathrm{TAut}\,P_n(\mathbb{K})$ of tame symplectomorphisms approximating σ.*

We give here a slightly simplified sketch of the proof suitable for our context. The second theorem is essential in our approach to the lifting problem in deformation quantization.

The proof of Theorem 1 consists of several steps each of which amounts to composing a given automorphism φ with a tame transformation of a specific type—an operation that allows one to dispose of $\varphi(x_i)$ ($1 \leq i \leq N$) in terms of total degree greater than one and less than a fixed integer k. Thus the approximating sequence of tame automorphisms is constructed. As was mentioned before, we disregard translation automorphisms completely: all automorphisms discussed here are origin-preserving, so the polynomials $\varphi(x_i)$ have zero free parts. This of course leads to no loss of generality.

The process starts with the following straightforward observation [12].

Lemma 1 ([12])**.** *There is a linear transformation $A \in \mathrm{SL}_N(\mathbb{K})$ $(x_1,\ \dots,\ x_N) \mapsto (x_1,\ \dots,\ x_N)A$, such that its composition φ_A with φ fulfills $\mathrm{ht}(\varphi_A(x_i) - x_i) \geq 2$ for all $i \in \{1, \dots, N\}$.*

Using the above lemma, we may replace φ with φ_A (and suppress the A subscript for convenience).

Lemma 2 ([12])**.** *Suppose that $\varphi \in \mathrm{Aut}\,\mathbb{K}[x_1, \dots, x_N]$ is identity modulo square terms. Then, there exists a set $\rho_1,\ \dots,\ \rho_m$ of tame automorphisms of degree two such that the composition $\rho_m \circ \dots \circ \rho_1 \circ \varphi = \varphi_m$ is identity modulo cubic terms.*

Lemma 3. *Suppose that, given an automorphism φ with unit Jacobian, we can, by composing it on the left with tame automorphisms, transform it into the automorphism φ_m: $\varphi_m(x_i) = x_i + Q_i$, $\mathrm{ht}(Q_i) \geq m+1$. Then we can find a tame automorphism ρ such that $\rho \circ \varphi_m(x_i) = x_i + S_i$, $\mathrm{ht}(S_i) \geq m+2$.*

The last lemma concludes the proof of Theorem 1.

Once the approximation for the case of symplectomorphisms has been established, we can investigate the problem of lifting symplectomorphisms to Weyl algebra automorphisms. More precisely, one has the following

Proposition 1. *Let $\mathbb{K} = \mathbb{C}$ and let $\sigma : P_n(\mathbb{C}) \to P_n(\mathbb{C})$ be a symplectomorphism over complex numbers. Then, there exists a sequence $\psi_1,\ \psi_2,\ \dots,\ \psi_k,\ \dots$ of tame automorphisms of the n-th Weyl algebra $W_n(\mathbb{C})$, such that their images σ_k in $\mathrm{Aut}\,P_n(\mathbb{C})$ approximate σ.*

A few comments are in order. First, the deformation quantization of elementary symplectomorphisms is a very simple procedure: one need only replace the x_i and p_i by their counterparts $\hat{x}_i$ and $\hat{p}_i$ in the Weyl algebra W_n. Because the transvection polynomials f

and g (in the expressions for elementary symplectomorphisms) depend, as has been noted, on one type of generator (resp. p and x), the quantization is well defined.

Second, as the tame automorphism groups TAut $W_n(\mathbb{C})$ and TAut $P_n(\mathbb{C})$ are isomorphic, the correspondence between sequences of tame symplectomorphisms converging to symplectomorphisms and sequences of tame Weyl algebra automorphisms is one to one. The main question is how one may interpret these sequences as endomorphisms of $W_n(\mathbb{C})$.

Our construction shows that these sequences of tame automorphisms may be thought of as (vectors of) power series—that is, elements of $\mathbb{C}[[\hat{x}_1, \ldots, \hat{x}_n, \hat{p}_1, \ldots, \hat{p}_n]]^{2n}$.

The main problem, therefore, consists in verifying that these vectors have polynomial entries in generators—that is, that the limits of lifted tame sequences are Weyl algebra endomorphisms.

We have developed a tame approximation theory for symplectomorphisms in formal power series topology. By virtue of the known correspondence between tame automorphisms of the even-dimensional affine space and tame automorphisms of the Weyl algebra, which is the object corresponding to the affine space in terms of deformation quantization, we have arrived at the lifting property of symplectomorphisms. This line of research may yield new insights into the endomorphisms of the Weyl algebra, the Dixmier Conjecture and the Jacobian Conjecture.

2.3. Jacobian Conjecture, Dixmier Conjecture and Belov–Kontsevich Conjecture

2.3.1. Jacobian Conjecture

One of the most well known unsolved problems in the theory of polynomials in several variables is the so-called *Jacobian Conjecture*, formulated in 1939 by O.-H. Keller [13]. Let $\mathbb{K}$ be the main field and for a fixed positive integer n are given n polynomials

$$f_1(x_1, \ldots, x_n), \ldots, f_n(x_1, \ldots, x_n)$$

of n variables $x_1, \ldots, x_n$. Any such system of polynomials defines a unique image *endomorphism* of the algebra $\mathbb{K}[x_1, \ldots, x_n]$

$$F : \mathbb{K}[x_1, \ldots, x_n] \to \mathbb{K}[x_1, \ldots, x_n],$$
$$F \leftrightarrow (F(x_1), \ldots, F(x_n)) \equiv (f_1(x_1, \ldots, x_n), \ldots, f_n(x_1, \ldots, x_n)).$$

The $\mathbb{K}$-endomorphism F of polynomial algebra is determined by its action on the set of generators. Let $J(F)$ denote *Jacobian* (the determinant of the Jacobi matrix) of the map F:

$$J(F) = \det \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix}$$

The Jacobian Conjecture is as follows.

Conjecture 1 (The Jacobian Conjecture, JC_n)**.** *Let the characteristic of the base field $\mathbb{K}$ be equal to zero. Then, if the Jacobian $J(F)$ of the endomorphism F is equal to a nonzero constant (that is, it belongs to the set $\mathbb{K}^\times$), then F is an automorphism.*

An elementary exercise is to verify the statement that automorphisms of polynomial algebras always have a nonzero Jacobian constant. Conjecture 1 is thus a partially inverse statement of this property. It is also easy to see that if a polynomial endomorphism F is invertible, then the inverse will also be a polynomial endomorphism.

The Jacobian Conjecture is trivial for $n = 1$. On the other hand, when the field $\mathbb{K}$ has a positive characteristic, the Jacobian Conjecture formulated as Conjecture 1 is incorrect even in the case of $n = 1$. Indeed, if $\operatorname{char} \mathbb{K} = p$ and $n = 1$, we can take $\varphi(x) = x - x^p$. The Jacobian of such a mapping is equal to unity, but it is irreversible.

Despite the apparent simplicity of wording and context, the Jacobian Conjecture is one of the most difficult open questions of modern algebraic geometry. This problem has become the subject of numerous studies and has greatly contributed to the development of related fields of algebra, algebraic geometry and mathematical physics, which are also of independent interest.

The literature on the Jacobian Conjecture, its analogues and related problems are extensive. A detailed discussion of the results established in the context of the Jacobian Conjecture is beyond the scope of this work. Below, we give a brief overview of some results directly related to the Jacobian Conjecture (i.e., for the algebra of polynomials in commuting variables). Among studies of topics similar to the Jacobian Conjecture in associative algebra, it is worth noting the work of W. Dicks [14] and Dicks and J. Lewin [15] on an analogue of the Jacobian Conjecture for free associative algebras, the proof by U.U. Umirbaev [16] of an analogue of the Jacobian Conjecture for the free metabelian algebra, as well as the deep and extremely significant work of A.V. Yagzhev [17–20] (see also [21]).

2.3.2. Ind-Schemes and Varieties of Automorphisms

One of the essential areas of algebraic geometry, the development of which was motivated by the Jacobian Conjecture is the theory of infinite-dimensional algebraic groups. The main reference is the seminal article of I.R. Shafarevich [1], in which he defined concepts that allowed one to study questions about some natural infinite-dimensional groups—for example, the group of automorphisms of an algebra of polynomials in several variables—using tools from algebraic geometry. In particular, Shafarevich defines *infinite-dimensional varieties* as inductive limits of directed systems of the form

$$\{X_i, f_{ij},\ i, j \in I\},$$

where X_i are algebraic varieties (more generally, algebraic sets) over a field $\mathbb{K}$ and the morphisms f_{ij} (defined for $i \leq j$) are closed embeddings. The inductive limit of a system of topological spaces carries a natural topology and therefore the natural questions about connectivity and irreducibility arise, which were also studied in [1].

Following generally accepted terminology, we will call the direct limit of systems of varieties and closed embeddings an Ind-*variety* and the corresponding limits of systems of schemes and morphisms of schemes an Ind-*scheme*.

The Jacobian Conjecture has the following elementary connection with Ind-schemes. Since the algebra of polynomials $\mathbb{K}[x_1, \ldots, x_n]$ can be endowed with a natural $\mathbb{Z}$-grading in total degree deg, which is defined as the appropriate monoid homomorphism by the requirement $\deg x_i = 1$, we can define as the degree of endomorphism φ: namely, if $\varphi = (\varphi(x_1), \ldots, \varphi(x_n))$ defined by its action on algebra generators, then the degree $\deg \varphi$ is the maximum value of degree on the polynomials $\varphi(x_1), \ldots, \varphi(x_n)$. It defines an increasing filtration $\operatorname{End}^{\leq N} \mathbb{K}[x_1, \ldots, x_n],\ N \geq 0$ on the set $\operatorname{End} \mathbb{K}[x_1, \ldots, x_n]$ of endomorphisms of the polynomial algebra. Points $\operatorname{End}^{\leq N} \mathbb{K}[x_1, \ldots, x_n]$ are endomorphisms of degree at most N. It is easy to see that the algebraic sets $\operatorname{End}^{\leq N} \mathbb{K}[x_1, \ldots, x_n]$ are isomorphic to affine spaces of appropriate dimension. The coordinates of the point φ are the coefficients of the polynomials $\varphi(x_1), \ldots, \varphi(x_n)$, and for $\operatorname{End} \mathbb{K}[x_1, \ldots, x_n]$ these coordinates are not connected by any relations.

The total degree filtration also enables endowing the sets of automorphisms with the Zariski topology as follows (see also [1]): if φ is a polynomial automorphism, then consider a set of polynomials $(\varphi(x_1), \ldots, \varphi(x_1), \varphi^{-1}(x_1), \ldots, \varphi^{-1}(x_n))$, the images of generators under the action of the automorphism and its inverse. The coefficients of these polynomials serve as coordinates of φ as a point of some affine space.

Define the subsets

$$\operatorname{Aut}^{\leq N} \mathbb{K}[x_1, \ldots, x_n] = \{\varphi \in \operatorname{Aut} \mathbb{K}[x_1, \ldots, x_n] : \deg \varphi, \deg \varphi^{-1} \leq N\}$$

as sets of automorphisms such that all coefficients of polynomials in the presentation above for degrees greater than n are zero.

The sets $\mathrm{Aut}^{\leq N}\,\mathbb{K}[x_1,\ldots,x_n]$ are algebraic sets. Indeed, the identities that define the points $\mathrm{Aut}^{\leq N}$ are derived from the identity $\varphi\circ\varphi^{-1}=\mathrm{Id}$ and, it is easy to see, are specified by polynomials.

Now let $\mathfrak{J}^{\leq N}$ denote a subset of $\mathrm{End}^{\leq N}\,\mathbb{K}[x_1,\ldots,x_n]$, whose points are endomorphisms with a Jacobian equal to a nonzero constant. Then, Conjecture 1 can be clearly reformulated as follows $\forall\varphi\in\mathfrak{J}^{\leq N}\Rightarrow\varphi\in\mathrm{Aut}\,\mathbb{K}[x_1,\ldots,x_n],\ \ \forall N,\ \ \text{for}\ \ \mathrm{char}\,\mathbb{K}=0.$

2.3.3. Dixmier Conjecture and Belov–Kontsevich Conjecture

J. Dixmier [22] in his seminal study of Weyl algebras found a connection between the Jacobian Conjecture and the following Conjecture. Let $W_{n,\mathbb{K}}$ denote the *n-th Weyl algebra* over the field $\mathbb{K}$ defined as the quotient algebra of the free algebra $F_{2n}=\mathbb{K}\langle a_1,\ldots,a_n,b_1,\ldots,b_n\rangle$ of $2n$ generators by the two-sided ideal I_W, generated by polynomials $a_ia_j-a_ja_i$, $b_ib_j-b_jb_i,\quad b_ia_j-a_jb_i-\delta_{ij}\quad(1\leq i,j\leq n)$, where δ_{ij} is the Kronecker symbol. The Dixmier Conjecture states:

Conjecture 2 (Dixmier Conjecture, DC_n)**.** *Let* $\mathrm{char}\,\mathbb{K}=0$*. Then,* $\mathrm{End}\,W_{n,\mathbb{K}}=\mathrm{Aut}\,W_{n,\mathbb{K}}$.

In other words, the Dixmier Conjecture asks whether every endomorphism of the Weyl algebra over a field of characteristic zero is in fact an automorphism.

The Dixmier Conjecture for n variables, DC_n, implies the Jacobian Conjecture JC_n for n variables (see, for example, [23]). Significant progress in recent years in the study of Conjecture 1 has been achieved by Kanel-Belov (Belov) and Kontsevich [2]—and independently by Tsuchimoto [3] (also see [24])—in the form of the following theorem.

Theorem 3 (A.Ya. Kanel-Belov and M.L. Kontsevich [2], Y. Tsuchimoto [3])**.** JC_{2n} *implies* DC_n.

In particular, Theorem 3 implies the stable equivalence of the Jacobian Conjecture and the Dixmier Conjecture—i.e., the equivalence of conjectures JC_∞ and DC_∞, where JC_∞ denotes the conjunction corresponding conjectures for all finite n.

Theorem 3 laid the foundation for the research into the Jacobian Conjecture based on the study of the behavior of varieties of endomorphisms and automorphisms of algebras under deformation quantization. The principal reference in this direction is an article by Kanel-Belov and Kontsevich [8]; in it, several conjectures concerning Ind-varieties of automorphisms of the corresponding algebras are formulated. The main Conjecture is called the Kontsevich Conjecture and is as follows.

Conjecture 3 (Kontsevich Conjecture, [8])**.** *Let* $\mathbb{K}=\mathbb{C}$ *be the field of complex numbers. The automorphism group* $\mathrm{Aut}\,W_{n,\mathbb{C}}$ *of the n-th Weyl algebra over* $\mathbb{C}$ *is isomorphic to the automorphism group* $\mathrm{Aut}\,P_{n,\mathbb{C}}$ *of the so-called n-th (commutative) Poisson algebra* $P_{n,\mathbb{C}}$*:*

$$\mathrm{Aut}\,W_{n,\mathbb{C}}\simeq\mathrm{Aut}\,P_{n,\mathbb{C}}.$$

The algebra $P_{n,\mathbb{C}}$ is by definition the polynomial algebra

$$\mathbb{C}[x_1,\ldots,x_n,p_1\ldots,p_n]$$

of $2n$ variables, equipped with the Poisson bracket—a bilinear operation $\{\ ,\ \}$, which is a Lie bracket satisfying the Leibniz rule and acting on generators of the algebra in the following way:

$$\{x_i,x_j\}=0,\ \ \{p_i,p_j\}=0,\ \ \{p_i,x_j\}=\delta_{ij}.$$

Endomorphisms of the algebra P_n are endomorphisms of the algebra of polynomials that preserve the Poisson bracket (which we sometimes call the Poisson structure). Elements of Aut $P_{n,\mathbb{C}}$ are called polynomial symplectomorphisms. The choice of name is due to the existence of an (anti-) isomorphism between the group Aut $P_{n,\mathbb{C}}$ and the group of polynomial symplectomorphisms of the affine space $\mathbb{A}^{2n}$.

The Kontsevich Conjecture is true for $n = 1$. The proof of this result is a direct description of automorphism groups Aut $P_{1,\mathbb{C}}$ and Aut $W_{1,\mathbb{C}}$, contained in the classical works of Jung [4], Van der Kulk [7], Dixmier [22] and Makar–Limanov [5,6]. Namely, consider the following transformation groups: the group G_1 is a semi-direct product $\mathrm{SL}(2,\mathbb{C}) \rtimes \mathbb{C}^2$, whose elements are called special affine transformations and the group G_2 by definition consists of the following "triangular" substitutions:

$$(x, p) \mapsto (\lambda x + F(p), \lambda^{-1} p), \quad \lambda \in \mathbb{C}^{\times}, \quad F \in \mathbb{C}[t].$$

Then the automorphism group of the algebra $P_{1,\mathbb{C}}$ [4] is isomorphic to the quotient group of the free product of the groups G_1 and G_2 by their intersection. Dixmier [22] and, later, Makar–Limanov [5] showed that if in the description above one replaces the commuting Poisson generators with their quantum (Weyl) analogues, one obtains a description of the group of automorphisms of the first Weyl algebra $W_{1,\mathbb{C}}$.

Remark 1. *The theorems of Jung, van der Kulk, Dixmier and Makar–Limanov also mean that all automorphisms of the polynomial algebra of two variables and the first Weyl algebra W_1 are* tame *(we provide the definition of the concept of tame automorphism, which plays a significant role in this study, in the sequel). Moreover, Makar–Limanov [6] and A. Czerniakiewicz [25,26] proved that all automorphisms of the free algebra $\mathbb{K}\langle x, y\rangle$ are tame.*

In view of these circumstances, the case of two variables is to be considered exceptional. However, the Jacobian Conjecture is a difficult open problem even in this case.

Recently, Kanel-Belov, together with Elishev and Yu, suggested proof of the general case of the Kontsevich Conjecture [10,27]. Independent proof of a closely related result (based on a study of the properties of holonomic $\mathcal{D}$-modules) was proposed by C. Dodd [28].

In contrast to the Jacobian Conjecture, which is an extremely difficult problem, in the study of the Kontsevich Conjecture, there are several possible approaches. First of all, in [8], Kanel-Belov and Kontsevich formulated several generalizations of Conjecture 3. In [2,3], which is devoted to the proof of Theorem 3, the construction of homomorphisms

$$\varphi : \mathrm{Aut}\, W_{n,\mathbb{C}} \to \mathrm{Aut}\, P_{n,\mathbb{C}}$$

and

$$\varphi : \mathrm{End}\, W_{n,\mathbb{C}} \to \mathrm{End}\, P_{n,\mathbb{C}}$$

involved in the construction, from a counterexample to DC_n, of an irreversible endomorphism with a single Jacobian, was presented. A straightforward strengthening of Conjecture 3 is the statement that the homomorphism φ realizes the isomorphism of the Kontsevich Conjecture. Moreover, namely, in Chapter 8 of [8], an approach to solve the problem of the lifting of polynomial symplectomorphisms to automorphisms of the Weyl algebra (i.e., constructing a homomorphism inverse to φ) was discussed. Conjecture 5 of [8], along with Conjecture 6, which is a weaker form of Conjecture 3, make up the essential contents of the construction proposed in [8]. To solve the problem of the lifting symplectomorphisms in the sense of these conjectures, it is necessary to study the properties of $\mathcal{D}$-modules, (left) modules over the Weyl algebra. The work of Dodd [28] is based on this approach.

2.4. Tame Automorphisms and the Kontsevich Conjecture

Dodd's constructions are deep in content and, apparently, prove the Kontsevich Conjecture on the correspondence between Lagrangian varieties and holonomic modules (more precisely, its essential part). On the other hand, starting from the Theorem of Dodd, we cannot immediately arrive at the general case of Conjecture 3. The proof of Conjecture 1 of [8] requires a solution to the lifting problem of symplectomorphisms to automorphisms of the corresponding Weyl algebra.

One of the main results of [8] was the proof of the following homomorphism properties

$$\varphi : \operatorname{Aut} W_{n,\mathbb{C}} \to \operatorname{Aut} P_{n,\mathbb{C}}$$

constructed in [8] and [3]. First, let φ be an automorphism of the polynomial algebra $\mathbb{K}[x_1, \ldots, x_n]$. We call φ *elementary* if it has the form

$$\varphi = (x_1, \ldots, x_{k-1}, ax_k + f(x_1, \ldots, x_{k-1}, x_{k+1}, \ldots, x_n), x_{k+1}, \ldots, x_n).$$

In particular, automorphisms given by linear substitutions of generators are elementary. Denote by $\operatorname{TAut} \mathbb{K}[x_1, \ldots, x_n]$ the subgroup generated by all elementary automorphisms. Elements of this subgroup are called *tame automorphisms* of the algebra of polynomials. Non-tame automorphisms are called *wild automorphisms*.

Tame automorphisms of the algebra $P_{n,\mathbb{K}}$ are, by definition, compositions of those tame elementary automorphisms which preserve the Poisson bracket. Tame automorphisms of the Weyl algebra are defined $W_{n,\mathbb{K}}$ similarly.

The following theorem is proved in [8].

Theorem 4 (A. Kanel-Belov and M.L. Kontsevich, [8])**.** *The homomorphism*

$$\varphi : \operatorname{Aut} W_{n,\mathbb{C}} \to \operatorname{Aut} P_{n,\mathbb{C}}$$

restricts to the isomorphism

$$\varphi_{|\,\mathrm{TAut}} : \operatorname{TAut} W_{n,\mathbb{C}} \to \operatorname{TAut} P_{n,\mathbb{C}}$$

between subgroups of tame automorphisms.

In particular, due to the tame nature of automorphism groups of Weyl and Poisson algebras for $n = 1$, the homomorphism φ gives an isomorphism of the Kontsevich Conjecture between $\operatorname{Aut} W_{1,\mathbb{C}}$ and $\operatorname{Aut} P_{1,\mathbb{C}}$.

It is not known whether all automorphisms of the Poisson and Weyl algebras are tame for $n > 1$, or even stably tame (an automorphism is called *stably tame* if it becomes tame after adding dummy variables and extending the action on them by means of the identity automorphism). For the algebra of polynomials in three variables, the Nagata automorphism

$$(x, y, z) \mapsto (x - 2(xz + y^2)y - (xz + y^2)^2 z, y + (xz + y^2)z, z)$$

is wild (the famous result due to I.P. Shestakov and Umirbaev [29,30]).

Nevertheless, tame automorphisms turn out to play a significant role in the context of the Kontsevich Conjecture and the Jacobian Conjecture, due to the following reason. Anick [11] showed that the set of tame automorphisms of the algebra of polynomials $\mathbb{K}[x_1, \ldots, x_n]$ ($n \geq 2$) is dense in the topology of formal power series in the space $\mathfrak{J}$ of polynomial endomorphisms with nonzero constant Jacobian. In particular, for any automorphism of a polynomial algebra there exists a sequence of tame automorphisms converging to it in this topology—in other words, Anick's theorem implies the existence of *approximations* of automorphisms, or *approximations* by tame automorphisms (and in general, endomorphisms with nonzero constant Jacobian). In view of Anick's theorem, the Jacobian Conjecture can be formulated as a problem of invertibility of limits of sequences of tame automorphisms (this is discussed in the conclusion of [11]). This formulation of the Jacobian Conjecture can be directly generalized to the case of a field of arbitrary characteristic, see more below as well as in [31].

Anick's results, together with Theorem 4, suggest the idea of solving the lifting problem of polynomial symplectomorphisms to automorphisms of the Weyl algebra (an alternative construction to that proposed in [8]). Namely, if there is a symplectic analogue of Anick's theorem—that is, if there is an approximation of polynomial symplectomorphisms by tame symplectomorphisms—then, taking a sequence of tame symplectomorphisms converging to a given point, we can take the sequence of their pre-images under the isomorphism $\varphi_{|\,\mathrm{TAut}}$ and try to prove that its limit exists and is an automorphism of the Weyl algebra. The symplectic analogue of Anick's theorem was proved in [12]. The application of approximation theory to the lifting problem constitutes the main idea of the proof of Conjecture 3 in [27].

However, the direct application of the main result of [12] to the solution of the lifting problem does not achieve the desired result, since the homomorphism φ does not preserve the topology of formal power series (due to commutation relations in the Weyl algebra). In this connection, the naive approximation approach needs some modification. It turns out that such a modification is possible (see [27]). The nature of this modification is significant and is connected with the geometric properties of Ind-schemes of automorphisms of the corresponding algebras. *Therefore, the study of the geometry of* Ind*-schemes of automorphisms is justified in the framework of the Kontsevich Conjecture.*

3. Quantization Proof of Bergman's Centralizer Theorem

This section is a relatively independent part of the thesis and only sketches proofs with classical tools, while the following sections will focus on the new proof of Bergman's centralizer theorem.

We demonstrate the direct relationship between Bergman's theorem and Kontsevich's quantization in a classic way. The proof of the theorem has been broken into multiple steps.

A generic matrix is a matrix whose entries are distinct commutative indeterminates and the so-called algebra of generic matrices of order m is generated by associative generic $m\times$ matrices. In [32], we use the fact that when we come to the quantization of generic matrices, those matrices are allowed to commute but have no other relations.

It is shown that any two commuting elements in the free associative algebra also commute in some algebras of generic matrices. They also prove that if A is a free-associative algebra, then there is no commutative subalgebra with a transcendent degree greater than or equal to 2 of A. It is seen that two commuting generic matrices f, g with $\mathrm{tr.deg}(f, g) = 2$ do not commute after quantization.

Let X be a set of noncommuting variables, which may or may not be finite and $\mathbb{F}$ be a field. Let X^* denote the free monoid generated by X. An element of X (resp. X^*) is also called a letter (resp. word) and X is called an alphabet. Let $\mathbb{F}\langle\langle X\rangle\rangle$ and $\mathbb{F}\langle X\rangle$ denote the $\mathbb{F}$-algebra of formal series and polynomials in X, respectively. So an element of $\mathbb{F}\langle\langle X\rangle\rangle$ is in the form $a = \sum_{\omega\in X^*} a_\omega \omega$, where $a_\omega \in \mathbb{F}$ is the coefficient of the word ω in a. The length $\mid \omega \mid$ of $\omega \in X^*$ is the number of letters appearing in ω. For example, if $X = \{x_i\}$ and $\omega = x_1 x_2^2 x_1 x_3$, then $\mid \omega \mid = 5$. Now, we define the valuation

$$\nu : \mathbb{F}\langle\langle X\rangle\rangle \to \mathbb{Z}_{\geqslant 0} \cup \{\infty\}$$

as follows: $\nu = \infty$ and if $a = \sum_{\omega\in X^*} a_\omega \omega \neq 0$, then $\nu(a) = \min\{\mid \omega \mid : a_\omega \neq 0\}$. Note that if w is constant, then $\nu(\omega) = 0$ and $\nu(ab) = \nu(a) + \nu(b)$ for all $a, b \in \mathbb{F}\langle\langle X\rangle\rangle$. The following fact is easy to prove.

Lemma 4 (Levi's Lemma, [33])**.** *Let $\omega_1, \omega_2, \omega_3, \omega_4 \in X^*$ be nonzero with $|\omega_2| \geq |\omega_4|$. If $\omega_1\omega_2 = \omega_3\omega_4$, then $\omega_2 = \omega\omega_4$ for some $\omega \in X^*$.*

The proof is trivial by backward induction on $|\omega_2|$ since ω_2 has the same last letter as ω_4. The next lemma extends Levi's lemma to $k\langle\langle X\rangle\rangle$ and we state the result as follows.

Lemma 5 ([34], Lemma 9.1.2). *Let $a, b, c, d \in k\langle\langle X\rangle\rangle$ be nonzero. If $\nu(a) \geq \nu(c)$ and $ab = cd$, then $a = cq$ for some $q \in k\langle\langle X\rangle\rangle$.*

An interesting consequence of Lemma 5 is the following result:

Corollary 2. *Let $a \in \langle\langle X\rangle\rangle$. Then, $b \in C(a; \mathbb{F}\langle\langle X\rangle\rangle)$ if and only if a, b are not free, i.e., $f(a, b) = 0$ for some nonzero series $f \in \mathbb{F}\langle\langle x, y\rangle\rangle$.*

Lemma 6. *Suppose that the constant term of an element $a \in \mathbb{F}\langle\langle X\rangle\rangle$ is zero and $b, c \in C(a; \mathbb{F}\langle\langle X\rangle\rangle)\ \{0\}$. If $\nu(c) \geqslant \nu(b)$, then $c = bd$ for some $d \in C(a; \mathbb{F}\langle\langle X\rangle\rangle)$.*

3.1. Centralizer Theorems

With the help of the preceding lemmas, we can state and prove this well known centralizer theorem of k-algebra of formal series by Cohn.

Theorem 5 (Cohn's Centralizer Theorem, [35]). *If $a \in k\langle\langle X\rangle\rangle$ is not a constant, then the centralizer $C(a; k\langle\langle X\rangle\rangle) \cong k[[x]]$, where $k[[x]]$ is the algebra of formal power series in the variable x.*

Now since $k\langle X\rangle \subset K\langle\langle X\rangle\rangle$, it follows from the above theorem that if $a \in k\langle X\rangle$ is not constant, then $C(a; k\langle X\rangle)$ is commutative because $C(a; k\langle\langle X\rangle\rangle)$ is commutative. The next theorem is our main goal which shows that there is a similar result for $C(a; k\langle X\rangle)$.

Theorem 6 (Bergman's Centralizer Theorem, [36]). *If $a \in k\langle X\rangle$ is not constant, then the centralizer $C(a; k\langle X\rangle) \cong k[x]$, where $k[x]$ is the polynomial algebra in one variable x.*

We will not fully restate the original proof of Bergman's centralizer theorem since this is not the main idea here. However, we do use a result in his original proof [36] which helps us to finish the proof of the fact that the centralizer is integrally closed. This will be shown in Section 3.4.

First of all, we need to emphasize that the proof of Cohn's centralizer theorem is included in the proof. Here is a sketch of the proof.

For simplicity, we denote by $C := C(a; k\langle X\rangle)$ the centralizer of a which from now on is not a constant. Recall that the centralizer C is also commutative. Moreover, C is finitely generated, as a module over $k[a]$ or as algebra. Then, since $k\langle X\rangle$ is a 2-fir (free ideal ring, cf. [36], Lemma 1.5) and the center of a 2-fir is integrally closed, we obtain that the centralizer of a is integrally closed in its field of fractions after using the lifting to $k\langle X\rangle \otimes k(x)$ (where x is a free variable). Then, our aim is to show that C is a polynomial ring over k. In order to obtain this fact, we shall study homomorphisms of C in polynomial rings. By using "infinite" words, we obtained an embedding from C into polynomial rings by lexicographically ordered semigroup algebras, which completes this sketch of the proof. Indeed, any subalgebra not equal to k of a polynomial algebra $k[x]$ that is integrally closed in its own field of fractions is of form $k[y]$ (by Lüroth's theorem).

An analogue of Bergman's centralizer theorem for free group algebras was considered by N. Miasnikov in his very interesting paper (cf. [37]). He proved

Theorem 7 ([37]). *Suppose that* $\mathbf{u} \in A$ *is* not *supported on a cyclic group. Let C be the centralizer of* $\mathbf{u}$. *Then, C is the affine coordinate ring of the complement of a k-point in a proper nonsingular curve over k.*

We finally restate a conjecture by Miasnikov [37] as follows:

Conjecture 4. *Does every finitely generated subalgebra $R \neq k$ of a free group algebra A over k admit a k-algebra homomorphism $f : R \to k[t, t^{-1}]$ into the ring of Laurent polynomials of a single variable $k[t, t^{-1}] \cong k[\mathbb{Z}]$ which is nontrivial in the sense that $f(R) \neq k$?*

We conclude this section by pointing out that the method of "infinite" words inspires us to find a possibility to prove Bergman's centralizer theorem by deformation quantization. In the next section, we will establish this new approach of quantization for generic matrices.

3.2. Reduction to Generic Matrix

In this section, we will establish an important theorem that gives a relation between commutative subalgebras in free associative algebra and the algebra of generic matrices. Let $k\langle X\rangle$ be the free associative algebra over a field k generated by a finite set $X = \{x_1, \ldots, x_s\}$ of s indeterminates and let $k\langle X_1, \ldots, X_s\rangle$ be the algebra of $n \times n$ generic matrices generated by the matrices X_ν. The canonical homomorphism $\pi : k\langle x_1, \ldots, x_s\rangle \to k\langle X_1, \ldots, X_s\rangle$ is shown in the last section.

We claim that if we have a commutative subalgebra of rank two in the free associative algebra $k\langle X\rangle$, then we also have a commutative subalgebra of rank two if we consider a reduction to generic matrices of big enough order n. We call two elements of a free algebra *algebraically independent* if the subalgebra generated by these two elements is a free algebra of rank two. Otherwise, we will call them *algebraically dependent*.

In other words, if we have a commutative subalgebra $k[f, g]$ of rank two in the free associative algebra, then we have to prove that its projection to generic matrices of some order also has rank two; i.e., $\pi(f), \pi(g)$ do not have any relations.

We need the following theorem:

Theorem 8 ([38]). *Let $k\langle X\rangle$ be the free associative algebra over a field k generated by a finite set X of indeterminates. If $k\langle X\rangle$ has a commutative subalgebra with two algebraically independent generators $f, g \in k\langle X\rangle$, then the subalgebra of n by n generic matrices generated by reduction of f and g in $k\langle X_1, \ldots, X_s\rangle$ also has rank two for big enough n.*

Recall that the centralizer $C := C(a; k\langle X\rangle)$ of $a \in k\langle X\rangle \setminus k$ is a commutative subalgebra of $k\langle X\rangle$. So from the above theorem, we conclude that if the centralizer is a subalgebra in $k\langle X\rangle$ of rank two then the π-image subalgebra of C has also rank two.

However, we prefer discussing this general case of subalgebras instead of just considering a centralizer subalgebra. Furthermore, we want to prove that there are no commutative subalgebras of the free associative algebra $k\langle X\rangle$ of rank greater than or equal to two.

3.3. Quantization Proof of Rank One

By the opinion of most specialists, including E. Rips, there are no new proofs of Bergman's centralizer theorem [36] for almost fifty years. We use a method of deformation quantization presented by Kontsevich to give an alternative proof of Bergman's centralizer theorem. In this section, we get that the centralizer is a commutative domain of transcendence degree one (see [32]).

Let $k\langle X\rangle$ be the free associative algebra over a field k generated by s free variables $X = \{x_1, \ldots, x_s\}$. Now, we concentrate our proof on the fact that there are no commutative subalgebras of rank greater than or equal to two. From the homomorphism $\pi : k\langle x_1, \ldots, x_s\rangle \to k\langle X_1, \ldots, X_s\rangle$ and Theorem 8, we change our objective from the elements of $k\langle X\rangle$ to the algebra of generic matrices $k\langle X_1, \ldots, X_s\rangle$ and we consider the quantization of this algebra and its subalgebras.

Lemma 7 ([38]). *Let $\hat{A} \equiv A_0 + \mathfrak{h}A_1 (\operatorname{mod} \mathfrak{h}^2)$ be the quantized image of a generic matrix $A \in k\langle X_1, \ldots, X_s\rangle$, where A_0 is diagonal with distinct eigenvalues. Then, the quantized images $\hat{A}$ can be diagonalized over some finite extension of $k[x_{ij}^{(\nu)}]$.*

Let A, B be two commuting generic matrices in $k\langle X_1, \ldots, X_s\rangle$ which are algebraically independent, i.e., $\operatorname{rank} k\langle A, B\rangle = 2$. We have the following theorem.

Theorem 9 ([32,38])**.** *Let A, B be two commuting generic matrices in $k\langle X_1, \ldots, X_s\rangle$ with* rank $k\langle A, B\rangle = 2$ *and let $\hat{A}$ and $\hat{B}$ be quantized images (by sending multiplications to star products by means of Kontsevich's formal quantization) of A and B, respectively, by considering lifting A and B in $k\langle X_1, \ldots, X_s\rangle[[\mathfrak{h}]]$. Then, $\hat{A}$ and $\hat{B}$ do not commute. Moreover,*

$$\frac{1}{\mathfrak{h}}[\hat{A}, \hat{B}]_\star \equiv \begin{pmatrix} \frac{1}{\mathfrak{h}}\{\lambda_1, \mu_1\} & & 0 \\ & \ddots & \\ 0 & & \frac{1}{\mathfrak{h}}\{\lambda_n, \mu_n\} \end{pmatrix} \mod \mathfrak{h}, \tag{1}$$

where λ_i and μ_i are eigenvalues(weights) of A and B, respectively.

To prove this theorem, we need some preparation. It is not easy to directly compute two such generic matrices with order n. However, if we can diagonalize those matrices, then the computation will be easier. So first of all, we should show the possibilities. Without loss of generality, we may assume that one of the generic matrices B is diagonal if we have a proper choice of basis of the algebra of generic matrices. Now consider the other generic matrix A which we mentioned above.

Remark 3. *The generic matrix A may not be diagonalizable over $k[x_{ij}^{(\nu)}]$, but it can be diagonalized over some integral extension of the algebra $k[x_{ij}^{(\nu)}]$ with $i, j = 1, \ldots, n; \nu = 1, \ldots, s$.*

Remark 4. *Any non-scalar element A of the algebra of generic matrices must have distinct eigenvalues. In fact, by Amitsur's Theorem [39], the algebra of generic matrices is a domain. If the minimal polynomial is not a central polynomial, then the algebra can be embedded in a skew field. Hence, the minimal polynomial is irreducible and the eigenvalues are pairwise different.*

Remark 5. *Suppose λ_i and δ_i, $i = 1, \ldots, n$ are algebraically dependent. Then, there are polynomials P_i in two variables such that $P_i(\lambda_i, \delta_i) = 0$. Put*

$$P(x, y) = \prod_{i=1}^{n} P_i(x, y).$$

Then, $P(A, B)$ is a diagonal matrix having zeros on the main diagonal, i.e., $P(A, B) = 0$. This means that if rank $k\langle A, B\rangle = 2$, then λ_i, δ_i are algebraically independent for some i.

Let us conclude this section by explaining the whole process of this proof. Recall that we have the free associative algebra $k\langle X\rangle$ over a field k; if we have a commutative subalgebra of rank two generated by $a, b \in k\langle X\rangle$, then we may have a commutative subalgebra of the algebra of generic matrices $k\langle X_1, \ldots, X_s\rangle$ of rank two generated by A, B (they are images of a homomorphism $\pi : k\langle X\rangle \to k\langle X_1, \ldots, X_s\rangle$). Consider the element $0 = [a, b]$ of the free associative algebra $k\langle X\rangle$, homomorphism π and canonical quantization homomorphism q sending multiplications to star products. Then, we obtain that

$$0 = q\pi([a, b]) = q[A, B] = [\hat{A}, \hat{B}]_\star.$$

This leads to a contradiction to Theorem 9 which shows that $[\hat{A}, \hat{B}]_\star \neq 0$. So we obtain the following result.

Theorem 10. *There are no commutative subalgebras of rank ≥ 2 in the free associative algebra $k\langle X\rangle$.*

The centralizer ring is commutative from our discussion in Section 3.1 and from the above theorem, it is of rank 1. So it is a commutative subalgebra with form $k[x]$ for some $x \in k\langle X\rangle \setminus k$. We will show it implies Bergman's centralizer Theorem 6 in the next section.

3.4. Centralizers Are Integrally Closed

We have shown that the centralizer C is a commutative domain of transcendence degree one. For us, it was the most interesting part of the proof of Bergman's centralizer theorem. However, we have to prove the fact that C is integrally closed in order to complete the proof of Bergman's Centralizer Theorem. We showed above that centralizers are commutative domains of transcendence degree one over the ground field of characteristic zero.

However, in this section, our proofs are characteristic-free instead of the deep noncommutative divisibility theorems of Cohn and Bergman.

In our proof, despite the deep noncommutative divisibility theorems of Cohn and Bergman, the method which we use is characteristic-free. We use generic matrices reduction, invariant theory for characteristic zero by C. Procesi [40–42] and for positive characteristic due to A. N. Zubkov [43,44] and S. Donkin [45,46].

By transferring centralizers of non-scalar elements in the free associative algebra onto the algebra of generic matrices, we first consider the localization of the algebra of generic matrices.

Theorem 11 ([47])**.** *The algebra of generic matrices is a domain. Its localization as a skew field coincides with the localization of algebra of generic matrices with traces (for positive characteristic—with forms).*

We then prove the integrally closedness of the algebra of generic matrices.

Theorem 12 ([47])**.** *The algebra of generic matrices with characteristic coefficients is integrally closed.*

3.5. Proof of Bergman's Centralizer Theorem

Now we can sketch out the proof of Bergman's centralizer theorem. Consider the homomorphism from the following Proposition given by Bergman.

Proposition 2 (Bergman, [36])**.** *For $C \neq K$, a finitely generated subalgebra of $K\langle X\rangle$, there is a homomorphism f of C into the polynomial algebra over K in one variable, such that $f(C) \neq K$.*

Because C is the centralizer of $K\langle X\rangle$, it has transcendence degree 1. Consider the homomorphism ρ which sends C to the ring of polynomials. The homomorphism has kernel zero; otherwise, $\rho(C)$ will have a smaller transcendence degree. Note that C is integrally closed and finitely generated; therefore, it can be embedded into the polynomial ring in one indeterminate. Since C is integrally closed, it is isomorphic to the polynomial ring of one indeterminate.

Consider the set of system of C_ℓ, ℓ-generated subring of C such that $C = \cup_\ell C_\ell$. Let $\overline{C_\ell}$ be the integral closure of C_ℓ. Consider the set of embedding of C_ℓ to ring of polynomial; then, $\overline{C_\ell}$ are integral closure of those images, i.e., $\overline{C_\ell} = K[z_\ell]$, where z_ℓ belongs to the integral closure of C_ℓ. Consider the sequence of z_ℓ. Because $K[z_\ell] \subseteq K[z_{\ell+1}]$ and degree of $z_{\ell+1}$ is strictly less than the degree of z_ℓ, this sequence stabilizes for some element x and it shows that $K[z]$ is the needed centralizer.

4. Noncommutative Białynicki-Birula Theorem

The study of algebraic group actions on varieties and coordinate algebras is a major area of research in algebraic geometry and ring theory. The subject has its connections with the theory of polynomial mappings, tame and wild automorphisms and the Jacobian Conjecture of O.-H. Keller, infinite-dimensional varieties according to Shafarevich, the cancellation problem (together with various cancellation-type problems) and the theory of locally nilpotent derivations, among other topics. One of the central problems in the theory of algebraic group actions has been the linearization problem, formulated and studied in the work of T. Kambayashi and P. Russell [48], which states that any algebraic torus action on an affine space is always linear with respect to some coordinate system. The linearization conjecture was inspired by the classical and well known result of A. Bialynicki–Birula; it states that every effective regular torus action of maximal dimension on the affine space over an algebraically closed field is linearizable. Although the linearization conjecture has turned out negative in its full generality, according to, among other results, the positive-characteristic counterexamples of T. Asanuma, the Bialynicki–Birula has remained an important milestone of the theory thanks to its connection to the theory of polynomial automorphisms. Recent progress in the latter area has stimulated the search for various noncommutative analogues of the Bialynicki–Birula theorem. In [49], we obtain the linearization theorem for effective maximal torus actions by automorphisms of the free associative algebra, which is the free analogue of the Bialynicki–Birula theorem. This statement is the free algebra analogue of a classical theorem of A. Białynicki-Birula.

4.1. Actions of Algebraic Tori

In [49], we consider algebraic torus actions on the affine space, according to Białynicki-Birula and formulate certain noncommutative generalizations.

We begin by recalling a few basic definitions. Let $\mathbb{K}$ be an algebraically closed field.

Definition 1. *An algebraic group is a variety G equipped with the structure of a group, such that the multiplication map $m : G \times G \to G : (g_1, g_2) \mapsto g_1 g_2$ and the inverse map $\iota : G \to G : g \mapsto g^{-1}$ are morphisms of varieties.*

Definition 2. *A G-variety is a variety equipped with an action of the algebraic group G,*

$$\alpha : G \times X \to X : (g, x) \mapsto g \cdot x,$$

which is also a morphism of varieties. We then say that α is an algebraic G-action.

Let $\mathbb{K}$ be our ground field, which is assumed to be algebraically closed. Let $Z = \{z_1, z_2, \dots\} = \{z_i : i \in I\}$ be a finite or a countable set of variables (where $I = \{1, 2, \dots\}$ is an index set), and let Z^* denote the free semigroup generated by Z, $Z^+ = Z^* \setminus \{1\}$. Moreover, let $F_I(\mathbb{K}) = \mathbb{K}\langle Z\rangle$ be the free-associative $\mathbb{K}$-algebra and $\hat{F}_I(\mathbb{K}) = \mathbb{K}\langle\langle Z\rangle\rangle$ be the algebra of formal power series in free variables.

Denote by $\mathcal{W} = \langle Z\rangle$ the free monoid of words over the alphabet Z (with 1 as the empty word) such that $|\mathcal{W}| \geqslant 1$, for $|\mathcal{W}|$ the length of the word $\mathcal{W} \in Z^+$.

For an alphabet Z, the free-associative $\mathbb{K}$-algebra on Z is

$$\mathbb{K}\langle Z\rangle := \oplus_{\mathcal{W} \in Z^*} \mathbb{K}\mathcal{W},$$

where the multiplication is $\mathbb{K}$-bilinear extension of the concatenation on words, Z^* denotes the free monoid on Z and $\mathbb{K}\mathcal{W}$ denotes the free $\mathbb{K}$-module on one element, the word $\mathcal{W}$. Any element of $\mathbb{K}\langle Z\rangle$ can thus be written uniquely in the form

$$\sum_{k=0}^{\infty} \sum_{i_1,\dots,i_k \in I} a_{i_1,i_2,\dots,i_k} z_{i_1} z_{i_2} \dots z_{i_k},$$

where the coefficients $a_{i_1,i_2,\dots,i_k}$ are elements of the field $\mathbb{K}$ and all but finitely many of these elements are zero.

In our context, the alphabet Z is the same as the set of algebra generators; therefore, the terms “monomial” and “word” will be used interchangeably.

In the sequel, we employ a (slightly ambiguous) short-hand notation for a free algebra monomial. For an element z, its powers are defined as usual. Any monomial $z_{i_1} z_{i_2} \dots z_{i_k}$ can then be written in a reduced form with subwords $zz \dots z$ replaced by powers.

We then write

$$z^I = z_{j_1}^{i_1} z_{j_2}^{i_2} \dots z_{j_k}^{i_k}$$

where by I we mean an assignment of i_k to j_k in the word z^I. Sometimes we refer to I as a multi-index, although the term is not entirely accurate. If I is such a multi-index, its absolute value $|I|$ is defined as the sum $i_1 + \cdots + i_k$.

For a field $\mathbb{K}$, let $\mathbb{K}^\times = \mathbb{K} \backslash \{0\}$ denote the multiplicative group of its non-zero elements viewed as an algebraic $\mathbb{K}$-group.

It is usually denoted by G_m and is the affine algebraic group $\mathrm{Spec}(\mathbb{K}[t, t^{-1}])$. An n-dimensional algebraic torus over $\mathbb{K}$ is an algebraic group $\mathbb{T}_n$ isomorphic to a finite direct product $\mathbb{K}^\times \times \ldots \times \mathbb{K}^\times$ which is a type of commutative affine algebraic group.

Definition 3. *An n-dimensional algebraic $\mathbb{K}$-torus is a group*

$$\mathbb{T}_n \simeq (\mathbb{K}^\times)^n$$

(with obvious multiplication).

Denote by $\mathbb{A}^n$ the affine space of dimension n over $\mathbb{K}$.

Definition 4. *A (left) torus action is a morphism*

$$\sigma : \mathbb{T}_n \times \mathbb{A}^n \to \mathbb{A}^n.$$

that fulfills the usual axioms (identity and compatibility):

$$\sigma(1, x) = x, \ \ \sigma(t_1, \sigma(t_2, x)) = \sigma(t_1 t_2, x).$$

An action σ is effective if for every $t \neq 1$ there is an element $x \in \mathbb{A}^n$ such that $\sigma(t, x) \neq x$.

Let us first restrict ourselves to the situation in which an r-dimensional torus $\mathbb{T}_r = (\mathbb{G}_m)^r \simeq (\mathbb{F}^*)^r$ acts on an affine n-space $\mathbb{A}^n := Spec\mathbb{F}[x_1, \cdots, x_n]$, where $\mathbb{F}[x_1, \cdots, x_n] = \mathbb{F}^{[n]}$ is an n-variable polynomial ring over $\mathbb{F}$. Since the quotient of a torus by any subgroup is again a torus, we may assume that $\mathbb{T}_r$ acts effectively, i.e., that no proper subgroup of $\mathbb{T}_r$ acts neutrally on $\mathbb{A}^n$.

In [50,51], Białynicki-Birula proved the following results, for $\mathbb{F}$ algebraically closed.

Theorem 13 ([50])**.** *Any regular action of $\mathbb{T}_n$ on $\mathbb{A}^n$ has a fixed point.*

Theorem 14 ([51])**.** *Any effective and regular action of $\mathbb{T}_n$ on $\mathbb{A}^n$ is a representation in some coordinate system.*

Theorem 15 ([51])**.** *The action of $\mathbb{T}_r$ on $\mathbb{A}^n$ is linearizable in the cases of $r = n$ or $r = n - 1$, which means that one can find isobaric elements $y_1, \cdots, y_n$ in $\mathbb{F}^{[n]} = \mathbb{F}[x_1, \cdots, x_n]$ such that $\mathbb{F}^{[n]} = \mathbb{F}[y_1, \cdots, y_n]$.*

The term "regular" is to be understood here as in the algebro-geometric context of regular function (Białynicki-Birula also considered birational actions). The last theorem says that any effective regular maximal torus action on the affine space is conjugate to a linear action, or, as it is sometimes called, *linearizable*.

An algebraic group action on $\mathbb{A}^n$ is the same as an action by automorphisms on the algebra

$$\mathbb{K}[x_1, \ldots, x_n]$$

of global sections. In other words, it is a homomorphism

$$\sigma : \mathbb{T}_n \to \mathrm{Aut}\, \mathbb{K}[x_1, \ldots, x_n].$$

An action is effective iff $\mathrm{Ker}\, \sigma = \{1\}$.

The polynomial algebra is a quotient of the free associative algebra

$$F_n = \mathbb{K}\langle z_1, \ldots, z_n \rangle$$

by the commutator ideal I (it is the two-sided ideal generated by all elements of the form $fg - gf$). From the standpoint of noncommutative geometry, the algebra $\Gamma(X, \mathcal{O}_X)$ of global sections (along with the category of f.g. projective modules) contains all the relevant topological data of X and various non-commutative algebras (PI-algebras included) may be thought of as global function algebras over "noncommutative spaces". Therefore, a noncommutative analogue of the Białynicki-Birula theorem is a subject of legitimate interest.

In [49] we establish the free algebra version of the Białynicki-Birula theorem. The latter is formulated as follows.

Theorem 16. *Suppose given an action σ of the algebraic n-torus $\mathbb{T}_n$ on the free algebra F_n. If σ is effective, then it is linearizable.*

The linearization problem, as it has become known since Kambayashi, asks whether all (effective, regular) actions of a given type of algebraic group on the affine space of a given dimension are conjugate to representations. According to Theorem 16, the linearization problem extends to the noncommutative category. Several known results concerning the (commutative) linearization problem are summarized below.

1. Any effective regular torus action on $\mathbb{A}^2$ is linearizable (Gutwirth [52]).
2. Any effective regular torus action on $\mathbb{A}^n$ has a fixed point (Bialynicki-Birula [50]).
3. Any effective regular action of $\mathbb{T}_{n-1}$ on $\mathbb{A}^n$ is linearizable (Bialynicki-Birula [51]).
4. Any (effective, regular) one-dimensional torus action (i.e., action of $\mathbb{K}^\times$) on $\mathbb{A}^3$ is linearizable (Koras and Russell [53]).
5. If the ground field is not algebraically closed, then a torus action on $\mathbb{A}^n$ need not be linearizable. In [54], Asanuma proved that over any field $\mathbb{K}$, if there exists a non-rectifiable closed embedding from $\mathbb{A}^m$ into $\mathbb{A}^n$, then there exist non-linearizable effective actions of $(\mathbb{K}^\times)^r$ on $\mathbb{A}^{n+m+1}$ for $1 \leqslant r \leqslant 1+m$.
6. When $\mathbb{K}$ is infinite and has a positive characteristic, there are examples of non-linearizable torus actions on $\mathbb{A}^n$ (Asanuma [54]).

Remark 6. *A closed embedding $\iota : \mathbb{A}^m \to \mathbb{A}^n$ is said to be rectifiable if it is conjugate to a linear embedding by an automorphism of $\mathbb{A}^n$.*

As can be inferred from the review above, the context of the linearization problem is rather broad, even in the case of torus actions. The regulating parameters are the dimensions of the torus and the affine space. This situation is due to the fact that the general form of the linearization conjecture (i.e., the conjecture that states that any effective regular torus action on any affine space is linearizable) has a negative answer.

Transition to the noncommutative geometry presents the inquirer with an even broader context: one now may vary the dimensions as well as impose restrictions on the action in the form of preservation of the PI identities. Caution is well advised. Some of the results are generalized in a straightforward manner—the main theorem of this paper being the typical example, others requiring more subtlety and effort. Of some note to us, given our ongoing work in deformation quantization (see, for instance, [12]), is the following instance of the linearization problem, which we formulate as a conjecture.

Conjecture 5. *For $n \geqslant 1$, let P_n denote the commutative Poisson algebra, i.e., the polynomial $\mathbb{K}[z_1, \ldots, z_{2n}]$ equipped with the Poisson bracket defined by $\{z_i, z_j\} = \delta_{i,n+j} - \delta_{i+n,j}$. Then, any effective regular action of $\mathbb{T}_n$ by automorphisms of P_n is linearizable.*

It is interesting to note that the context of Conjecture 1 admits a vague analogy in the real transcendental category (with P_n replaced by an appropriate algebra of smooth functions, cf., for instance, the work of Zung [55]). Although the instances of the linearization problem we consider in [49], as well as the original theorem of Białynicki-Birula, are essential for the complex algebraic nature, it may be worthwhile to search for analytic analogues of the real transcendental linearization (however, whether this will give a feasible approach to Conjecture 5 is unclear, the hurdles being significant and fairly obvious).

4.2. Non-Linearizable Torus Actions and Problems

The noncommutative toric action linearization theorem that we have proved has several useful applications. In reference [31], it is used to investigate the properties of the group $\operatorname{Aut} F_n$ of automorphisms of the free algebra. As a corollary of Theorem 16, one gets

Corollary 7. *Let θ denote the standard action of $\mathbb{T}_n$ on $K[x_1, \ldots, x_n]$—i.e., the action*

$$\theta_t : (x_1, \ldots, x_n) \mapsto (t_1 x_1, \ldots, t_n x_n).$$

Let $\tilde{\theta}$ denote its lifting to an action on the free associative algebra F_n. Then $\tilde{\theta}$ is also given by the standard torus action.

This statement plays a part, along with a number of results concerning the induced formal power series topology on $\operatorname{Aut} F_n$, in the establishment of the following proposition (cf. [31]).

Proposition 3. *When $n \geq 3$, any* Ind*-scheme automorphism φ of* $\operatorname{Aut}(K\langle x_1, \ldots, x_n \rangle)$ *is inner.*

One could try and generalize the free algebra version of Białynicki-Birula's theorem to other noncommutative situations. Another method of generalization lies in changing the dimension of the torus. It is nonetheless possible that the free analogue of the main result of [51] exists. We have then the following conjecture.

Conjecture 6. *Any effective regular action of $\mathbb{T}_{n-1}$ on the free algebra $F_n(\mathbb{F})$ is linearizable, provided that $\mathbb{F}$ is algebraically closed.*

According to the above results, we are now able to state and prove one of our main results:

Theorem 17 ([56])**.** *Let $\mathbb{F}$ be algebraically closed. Any effective regular action of (the one-dimensional torus) $\mathbb{F}^*$ on the free algebra $\mathbb{F}\langle z_1, z_2 \rangle$ is linearizable.*

Next, we consider positive-root torus actions and prove the linearity property analogous to the Białynicki-Birula theorem.

Theorem 18 ([56])**.** *Any effective positive-root action of $\mathbb{T}_r$ on $\mathbb{F}[x_1, \ldots, x_n]$ is linearizable.*

In order to prove the free-associative version of this theorem, we devise a way to reduce the positive-root case to the commutative one. To that end, we introduce the generic matrices and induce the action on the rings of coefficients.

More precisely, we have the following.

Theorem 19 ([56])**.** *Let $\sigma : \mathbb{T}_r \times F_n \to F_n$ be a regular torus action with positive roots. Then, it is linearizable.*

Next, we study non-linearizable torus actions. The examples of non-linearizable torus actions, as well as a way to study them, were developed by Asanuma [54]. It is not difficult to observe that most of Asanuma's techniques can be carried to the free-associative case without loss of generality. As in Asanuma's case, the existence of non-linearizable torus actions is tied to the existence of so-called non-rectifiable ideals in the appropriate algebras. One rather remarkable feature of Asanuma's technique is the fact that modulo minor details and replacements may be repeated almost verbatim in the associative category—a situation similar to the one we have observed in the Białynicki-Birula theorem concerning the action of the maximal torus.

Definition 5. *Two (regular) $\mathbb{T}_r$-actions, φ and ψ, respectively, on A and B are equivalent if there exists a $\mathbb{F}$-homomorphism $\sigma : A \to B$ such that the diagram*

$$\begin{tikzcd} A[T_1, \ldots, T_r, T_1^{-1}, \ldots, T_r^{-1}] \arrow[r, "\sigma\otimes Id"] & B[T_1, \ldots, T_r, T_1^{-1}, \ldots, T_r^{-1}] \\ A \arrow[u, "\varphi"] \arrow[r, "\sigma"] & B \arrow[u, "\psi"] \end{tikzcd}$$

commutes.

The main problem of interest is the free-associative analogue of the so-called Cancellation Conjecture, as formulated by V. Drensky and Yu [57]:

Conjecture 7. *Let R be a $\mathbb{F}$-algebra. If*

$$R * \mathbb{F}\langle y \rangle \simeq_{\mathbb{F}} \mathbb{F}\langle x_1, \ldots, x_n \rangle,$$

then

$$R \simeq_{\mathbb{F}} \mathbb{F}\langle x_1, \ldots, x_{n-1} \rangle.$$

Asanuma's results on the Rees algebras allow us to establish a version of the Cancellation Conjecture for co-products over a (commutative) $\mathbb{F}$-algebra D. The following statement holds.

Theorem 20 ([56])**.** *Let D be an integral domain which is a $\mathbb{F}$-algebra and let x be an indeterminate over D. Assume a non-zero element $t \in D$ and monic polynomials $f(x)$ and $g(x)$ in $\mathbb{F}[x]$ of degree greater than 1. Set $A = D[x, t^{-1}f(x)]$ and $B = D[x, t^{-1}g(x)]$. If*

$$\mathbb{F}[x]/(f(x)) \simeq_{\mathbb{F}} \mathbb{F}[x]/(g(x)),$$

then

$$A *_D \mathbb{F}\langle y \rangle \simeq_D B *_D \mathbb{F}\langle y \rangle,$$

where the product $R *_D S$ is the quotient of the free product $R * S$ over $\mathbb{F}$ by the ideal generated by all elements of the form

$$r * (ds) - d(r * s).$$

We have the following conjectures:

One notable example is that we expect the free-associative analogue of the second Białynicki-Birula theorem to hold and formulate it here as a conjecture.

Conjecture 8. *Any effective action of $\mathbb{T}_{n-1}$ on F_n is linearizable.*

Also of independent interest is the following instance of the linearity problem.

Conjecture 9. *For $n \geq 1$, let P_n denote the commutative Poisson algebra, i.e., the polynomial algebra $\mathbb{F}[z_1, \ldots, z_{2n}]$ equipped with the Poisson bracket defined by*

$$\{z_i, z_j\} = \delta_{i,n+j} - \delta_{i+n,j}.$$

Then, any effective regular action of $\mathbb{T}_n$ by automorphisms of P_n is linearizable.

This problem is loosely analogous to the Białynicki-Birula theorem, in the sense of maximality of torus with respect to the dimension of the configurations space (spanned by x_i).

On the other hand, there is little reason to expect this statement to hold with further lowering of the torus dimension. In fact, even in the commutative case the conjecture that any effective toric action is linearizable, in spite of considerable effort (see [48]), proved negative (counterexamples in positive characteristic due to Asanuma [54]).

Another direction would be to replace $\mathbb{T}$ by an arbitrary reductive algebraic group; however, the commutative case also does not hold even in characteristic zero (cf. [58]).

It is also a problem of legitimate interest to obtain the proof of Conjecture 5—i.e., to resolve the linearization problem of the regular action of the n-dimensional torus on the group $\mathrm{Sympl}(k^{2n})$ of polynomial symplectomorphisms of the $2n$-dimensional affine space (k is a field of characteristic zero). One could hope to utilize the latter result in order to obtain a description of the space of Ind-scheme automorphisms of $\mathrm{Sympl}(k^{2n})$ along the lines of [31]. This space plays a prominent role in the study of quantization of symplectomorphisms, initiated by Kanel-Belov and Kontsevich [8], where the characteristic zero isomorphisms between the group of automorphisms of the n-th Poisson and Weyl algebras has been posed as the main conjecture (Kontsevich Conjecture). Recently, the first, the second and the fourth named authors have proposed a proof of this conjecture [10,27].

5. Feigin's Conjecture and the Lattice W-Algebras

Feigin's homomorphisms were born in Feigin's new formulation of quantum Gelfand–Kirillov Conjecture, which came into public view at RIMS in 1992 for the nilpotent part $U_q(\mathfrak{n})$, and are now known as "Feigin's Conjecture" [59].

In the mentioned talk, Feigin proposed the existence of a family of homomorphisms from a quantized enveloping algebra to rings of skew-polynomials. These *homomorphisms* are becoming very useful tools for studying the fraction field of quantized enveloping algebra.

5.1. Feigin's Homomorphisms on $U_q(\mathfrak{n})$

Here, we will attempt to succinctly explain what Feigin's homomorphisms are and how they will help us arrive at and demonstrate that the screening operators in quantum Serre relations are satisfactory.

Let C be an arbitrary symmetrizable Cartan matrix of rank r and $\mathfrak{n} = \mathfrak{n}_+$ the standard maximal nilpotent sub-algebra in the Kac–Moody algebra associated with C (thus, $\mathfrak{n}$ is generated by the elements $e_1, \ldots, e_r$, satisfying the Serre relations). As usual, $U_q(\mathfrak{n})$ will be considered the quantized enveloping algebra of $\mathfrak{n}$. $A = (A_{ij}) = (d_i c_{ij})$ will be assumed the symmetric matrix corresponding to C for non-zero relatively prime integers $d_1, \ldots, d_n$ such that $d_i a_{ij} = d_j a_{ji}$ for all i, j. Set $\mathfrak{g}$ as the Kac–Moody Lie algebra attached to A, on generators $e_i, f_i, h_i, 1 \leq i \leq n$.

For root lattice Λ, let A_1 and A_2 be Λ-graded associative algebras and define a q-twisted tensor product as the algebra $A_1 \bar{\otimes} A_2$ isomorphic with $A_1 \otimes A_2$ as a linear space with multiplication given by $(a_1 \otimes a_2) \cdot (a'_1 \otimes a'_2) := q^{\langle \alpha'_1, \alpha_2 \rangle} a_1 a'_1 \otimes a_2 a'_2$, where $\alpha'_1 = deg(a'_1)$ and $\alpha_2 = deg(a_2)$ and invariant bilinear form $\langle , \rangle : \Lambda \times \Lambda \to \mathbb{Z}$ defined by $\langle \alpha_i, \alpha_j \rangle = d_i a_{ij}$. By this definition $A_1 \bar{\otimes} A_2$ become a Λ-graded algebra.

Proposition 4 ([60]). *Let $\mathfrak{g}$ be an arbitrary Kac–Moody algebra; then, the map*

$$\bar{\Delta} : U_q^{\pm}(\mathfrak{g}) \to U_q^{\pm}(\mathfrak{g}) \bar{\otimes} U_q^{\pm}(\mathfrak{g}) \tag{2}$$

Such that

$$\begin{cases} \bar{\Delta}(1) := 1 \otimes 1 \\ \bar{\Delta}(E_i) := E_i \otimes 1 + 1 \otimes E_i \\ \bar{\Delta}(F_i) := F_i \otimes 1 + 1 \otimes F_i \end{cases}$$

for $1 \leqslant i \leqslant n$, is a homomorphism of associative algebras.

Remark 1 ([60]). *There are no such maps as $U_q^{\pm}(\mathfrak{g}) \to U_q^{\pm}(\mathfrak{g}) \bar{\otimes} U_q^{\pm}(\mathfrak{g})$ in the case where g is an associative algebra.*

As always, after defining a co-multiplication $\bar{\Delta}$, we extend it by an iteration on the sequence of maps as follows [61]

$$\bar{\Delta}^n : U_q^-(\mathfrak{g}) \to U_q^-(\mathfrak{g})^{\otimes n}, \quad n = 2, 3, \ldots \tag{3}$$

determined by $\bar{\Delta}^2 = \bar{\Delta}$, $\bar{\Delta}^n = (\bar{\Delta} \otimes id) \circ \bar{\Delta}^{n-1}$.

Now, let $\mathbb{C}[X_i]$ be the ring of polynomials in one variable. Then, by equipping it with the grading structure $degX_i = \alpha_i$ for any simple root α_i, we can regard it as a Λ-graded.

By this grading, there will be a morphism of Λ-graded associative algebras

$$\varphi_i : U_q^-(\mathfrak{g}) \to \mathbb{C}[X_i] : F_j \mapsto \delta_{ij} X_i. \tag{4}$$

By following this construction for any sequence of simple roots $\beta_{i_1}, \ldots, \beta_{i_k}$, there will be a morphism of Λ-graded associative algebras

$$(\varphi_{i_1} \otimes \varphi_{i_k}) \circ \bar{\Delta}^k : U_q^-(\mathfrak{g}) \to \mathbb{C}[X_{1i_1}] \bar{\otimes} \ldots \bar{\otimes} \mathbb{C}[X_{ki_k}] \tag{5}$$

(here, the reason for double indexation is the appearance of i_js more than once in the sequence).

Finally, $\mathbb{C}[X_{1i_1}] \bar{\otimes} \cdots \bar{\otimes} \mathbb{C}[X_{ki_k}]$ is an algebra of skew polynomials $\mathbb{C}[X_{1i_1}, \ldots, X_{ki_k}]$, with Λ-grading $X_{si_s} X_{ti_t} = q^{\langle \alpha_{i_s}, \alpha_{i_t} \rangle} X_{ti_t} X_{si_s}$, for $s > t$. However, let us simplify it as $X_i X_j = q^{\langle degX_i, degX_j \rangle} X_j X_i$, which we will use after now more frequently.

So, very briefly, we constructed the family of Feigin's homomorphisms from $U_q^-(\mathfrak{g})$ (the maximal nilpotent sub-algebra of a quantum group associated with an arbitrary Kac–Moody algebra [62]) to the algebra of skew polynomials and now we can continue to our construction.

5.2. The Quantum Serre Relations and the Screening Operators

Here in this subsection, we will rediscover the relation between the screening operators and the quantum Serre relations.

Theorem 21 ([63]). *Let $Q = q^2$ and points $x_1, \cdots, x_n$ such that $x_i x_j = Q x_j x_i$ for $i < j$. Set $\Sigma^x = x_1 + \cdots + x_n$. If $Q^N = 1$ and $x_i^N = 0$ for some natural number N; then, we claim that $(\Sigma^x)^N = 0$.*

This is straightforward; it just requires the use of q-calculation.

$\mathfrak{sl}(3)$ Case

As we know, $M_2 = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$ is the generalized Cartan matrix for $\mathfrak{sl}(3)$. Set $M_{q_2} = \begin{bmatrix} q^2 & q^{-1} \\ q^{-1} & q^2 \end{bmatrix}$ and call it Cartan type matrix related to M_2.

Theorem 22 ([63])**.** *Suppose we have two different types of points x_i, namely, $(x_{2i-1})_i$, which we will call of type 1 and $(x_{2i})_i$, of type 2 for $i \in I = \{1,2\}$ and the following q-commutative relations:*

$$\begin{cases} x_j x_{j'} = q^2 x_{j'} x_j & \text{if } j < j' \text{ and } j, j' \in \{1,3\} \text{ and } j = j' \\ x_i x_{i'} = q^2 x_{i'} x_i & \text{if } i < i' \text{ and } i, i' \in \{2,4\} \text{ and } i = i' \\ x_i x_j = q^{-1} x_j x_i & \text{if } i < j \text{ and } i, j \in \{1,2,3,4\} \end{cases}$$

Set $\Sigma_1^x = \Sigma_{i \in I} x_{2i+1}$ and $\Sigma_2^x = \Sigma_{i \in I} x_{2i}$. We will call these sums screening operators. Then Σ_1^x and Σ_2^x satisfy quantum Serre relations:

$$(\Sigma_1^x)^2 \Sigma_2^x - [2]_q \Sigma_1^x \Sigma_2^x \Sigma_1^x + \Sigma_2^x (\Sigma_1^x)^2 = 0 \tag{6}$$

$$(\Sigma_2^x)^2 \Sigma_1^x - [2]_q \Sigma_2^x \Sigma_1^x \Sigma_2^x + \Sigma_1^x (\Sigma_2^x)^2 = 0$$

This is also straightforward; it just requires the use of q-calculation. The following theorem can be proved by induction on k.

Theorem 23 ([63])**.** *Use notations of Theorem 22 in a general case; i.e., set points $X_i \in \{X_1, \cdots, X_n\}$ and $Y_i \in \{Y_1, \cdots, Y_n\}$ with the following relations:*

$$\begin{cases} X_i X_j = q^2 X_j X_i & \text{if } i < j \\ Y_i Y_j = q^2 Y_j Y_i & \text{if } i < j \\ X_i Y_j = q^{-1} Y_j X_i & \text{if } i < j \end{cases}$$

and the screening operators $\Sigma_1^x = \Sigma_{i=1}^k X_i$ and $\Sigma_1^y = \Sigma_{j=1}^k Y_j$.

Then Σ_1^x and Σ_1^y satisfy quantum Serre relations.

6. Lattice Virasoro Algebra

In this section we are interested in solutions Σ_{1_x} of the system of difference equation

$$\begin{cases} X_i X_j = q X_j X_i \\ deg(\Sigma_{1_x}) = 0 \\ [\Sigma_{-\infty}^{+\infty} X_i, \Sigma_{1_x}]_q = 0, \end{cases}$$

which will be a generator of the lattice Virasoro algebra and if that happens then we can extend it to another generator by using the following shift operators:

$$\Sigma_{2_x} = \Sigma_{1_x}[x_1 \to x_2, x_2 \to x_3, x_3 \to x_4, \cdots] \tag{7}$$

$$\Sigma_{3_x} = \Sigma_2^x[x_2 \to x_3, x_3 \to x_4, x_4 \to x_5, \cdots]$$

$$\vdots$$

for $\Sigma_{1_x} = \Sigma_{1_x}(x_1, x_2, \cdots, x_k)$.

6.1. Lattice Virasoro Algebra Associated with $\mathfrak{sl}_2$

Here, as always, we have the q-commutation relation $X_iX_j = qX_jX_i,\ i < j$ between the points in $\mathfrak{sl}_2$. Let us try to find three-point invariants. What this means is that we need to solve the following system of difference equations:

$$\begin{cases} X_iX_j = qX_jX_i \\ deg(\Sigma_{1_x}) = 0 \\ (X_1 + X_2 + X_3)\Sigma_{1_x}(X_1, X_2, X_3) = \Sigma_{1_x}(X_1, X_2, X_3)(X_1 + X_2 + X_3). \end{cases}$$

One can easily find the trivial solutions to the second equation as follows:

$$\Sigma_{11_x}(X_1, X_2, X_3) = X_1 + X_2 + X_3\Sigma_{12_x}(X_1, X_2, X_3) = X_1X_2^{-1}X_3.$$

However, as is clear, none of the above solutions have zero grading. So we need to find another solution.

By just continuing to look at them for a while, we can see that by multiplying these kinds of solutions, one can find a zero grading expression, but there is a problem that does not satisfy the other two ones in the above set of relations.

Again, we note that for a solution, its inverse is again a solution, so by this remark, the option which we have is to inverse Σ_{11_x} or Σ_{12_x} and then multiply it with the other one. Which will result in the same set of generators except in the first case (inverse of Σ_{11_x}). Hence, we have that the lattice Virasoro algebra in the first case will be generated by elements of the form $\Sigma_{i_x} = X_iX_{i+1}^{-1}X_{i+2}(X_i + X_{i+1} + X_{i+2})^{-1}$ and in the second case (inverse of Σ_{12_x}) will be generated by elements of the form $\Sigma_{i_x} = (X_i + X_{i+1} + X_{i+2})X_i^{-1}X_{i+1}X_{i+2}^{-1}$.

Noting that our working space is closed under multiplication, these new recently found generators are the trivial solutions for our system of difference equations. By using the shift operators (7), we will obtain the set of generators for our lattice Virasoro algebra associated with $\mathfrak{sl}_2$.

Now, the claim is that the following generators are the solutions of the above set of the system of difference equations and hence generate the lattice Virasoro algebra coming from the two-dimensional representation of $\mathfrak{sl}_2$.

Lemma 8 ([63])**.** *The following equations satisfy:*

1. *We have* $[\Sigma_{j=-\infty}^{j=+\infty}x_j, (x_3 + x_4)^{-1}x_4x_3(x_2 + x_3)^{-1}] = 0.$
2. $[\Sigma_{j=-\infty}^{j=+\infty}x_j, (x_2 + x_3 + x_4)^{-1}(x_3 + x_4)x_2(x_1 + x_2)^{-1}] = 0.$
3. $[\Sigma_{j=-\infty}^{j=+\infty}x_j, (x_2 + \cdots + x_k)^{-1}(x_3 + \cdots + x_k)x_2(x_1 + x_2)^{-1}] = 0.$

Then, by using the shift operators (7), we will have the set of all generators.

6.2. Generators of Lattice Virasoro Algebra Coming from 3 and 4-Dimensional Representation of sl_2

Let us suppose the following three-dimensional representation of $\mathfrak{sl}_2$. The process of defining this representation is the same as the two-dimensional one. Define:

$$\begin{cases} F = \partial_{(U_+ - X_3)} \\ H = U_+\partial_{U_+} + X_1\partial_{X_1} + X_2\partial_{X_2} + X_3\partial_{X_3} \\ E = (U_+ - X_3)^2\partial_{(U_+ - X_3)} + (X_1^2 + X_1X_2 + X_1X_3 + X_1(U_+ - X_3))\partial_{X_1} \\ \quad +(X_2^2 + X_2X_3 + X_2(U_+ - X_3))\partial_{X_2} + (X_3^2 + X_3(U_+ - X_3))\partial_{X_3}, \end{cases}$$

where $U_+ = \Sigma_{i=3}^{+\infty}X_i$. In what comes, for a detailed discussion on the method of construction, we refer the interested reader to [63].

It is clear that the following generator, which we call $\rho_{1,5}$, satisfies the system of difference equation in Section 6.1 associated with the above representation.

$$\rho_{1,5} = X_1^{\frac{1}{2}} X_2^{-\frac{1}{2}} (X_2 + X_3)^{-\frac{1}{2}} X_3^{\frac{1}{2}} X_4^{-\frac{1}{2}} (X_4 + X_5)^{-\frac{1}{2}} \tag{8}$$

This is a four-point generator (invariant) of lattice Virasoro algebra, but due to its complexity we are less interested in it and we still have to look for the simplest one of type $ABCD$ (it is good to remark that the type $ABCD$ consists of four different parts and it is just restricted to this paper and it does not exist in the literature related to lattice W-algebras).

So let us define another such generator as proposed and experienced before. For this reason, we propose the following lemma and interested readers are encouraged to review [63] for more details on how to construct these kinds of generators.

Lemma 9. *The claim is that*

$$\rho_{1,4,5} = (X_4 + X_5)^{-\frac{1}{2}} X_4^{\frac{1}{2}} X_2^{\frac{1}{2}} (X_3 + X_4)^{-\frac{1}{2}} \tag{9}$$

should give us the desired four-point invariant for the lattice Virasoro algebra. $\rho_{1,4,5}$ has degree zero.

Now, to find five-point invariants, we need to consider another representation. For this reason, suppose the following representation of $\mathfrak{sl}_2$. Define:

$$\begin{cases} F = \partial_{(U_+ - X_3 - X_4)} \\ H = (U_+ - X_3 - X_4)\partial_{(U_+ - X_3 - X_4)} + X_1\partial_{X_1} + X_2\partial_{X_2} + X_3\partial_{X_3}, \\ E = (U_+ - X_3 - X_4)^2\partial_{(U_+ - X_3 - X_4)} + (X_1^2 + X_1X_2 + X_1X_3 + X_1(U_+ - X_3 - X_4))\partial_{X_1} \\ +(X_2^2 + X_2X_3 + X_2(U_+ - X_3 - X_4))\partial_{X_2} + (X_3^2 + X_3(U_+ - X_3 - X_4))\partial_{X_3}, \end{cases}$$

for U_+ as before.

Claim 1. *Then the claim is that*

$$\rho_{1,5,6} = (X_4 + X_5 + X_6)^{-\frac{1}{2}} X_4^{\frac{1}{2}} X_2^{\frac{1}{2}} (X_3 + X_4 + X_5)^{-\frac{1}{2}} \tag{10}$$

should give us the desired five-point invariant for the lattice Virasoro algebra. $\rho_{1,5,6}$ has degree zero.

6.3. Conclusions

The four-point invariant that comes from the three-dimensional representation of $\mathfrak{sl}_2$ is:

$$[\Sigma_{-\infty}^{+\infty} X_i, (X_4 + X_5)^{-1} X_4 X_2 (X_3 + X_4)^{-1}]_q = 0$$

The five-point invariant that comes from the four-dimensional representation of $\mathfrak{sl}_2$ is:

$$[\Sigma_{-\infty}^{+\infty} X_i, (X_4 + X_5 + X_6)^{-1} X_4 X_2 (X_3 + X_4 + X_5)^{-1}]_q = 0$$

Claim 2. *We have the following n-point invariant that comes from the n-dimensional representation.*

$$(X_4 + \cdots + X_n)^{-1} X_4 X_2 (X_3 + \cdots + X_{n-1})^{-1}$$

Then, by using the shift operators (7), we will have the space of all nontrivial generators of lattice Virasoro algebra.

We call these kinds of generators, which are the only nontrivial ones:

Generators of type "ABCD"

These (new lattice) algebras are so important and may in principle lead to a new integrable chain equation that people can hardly provide.

7. Weak Faddeev–Takhtajan–Volkov Algebras; Lattice W_n Algebras

This section is based on the paper [64]. As mentioned at the beginning of Section 5, there is an old problem that was considered and introduced by Boris Feigin in 1992. It was born in its new formulation, concerning quantum Gelfand–Kirillov Conjecture, in a public talk at RIMS in 1992 based on the nilpotent part of $U_q(\mathfrak{g})$, i.e., $U_q(\mathfrak{n})$ for $\mathfrak{g}$—a simple Lie algebra—and now this problem is known as "Feigin's Conjecture" [59].

In the mentioned talk, Feigin proposed the existence of a certain family of homomorphisms on the quantized enveloping algebra $U_q(\mathfrak{g})$, which will lead us to a definition of lattice W-algebras.

These "homomorphisms" have been turned into a very useful tool for studying the fraction field of quantized enveloping algebras.

There have been many attempts to construct lattice W-algebras in Feigin's sense, which ensures the simplicity of the construction process of lattice W-algebra. For example, the best-known articles on the subject have been written by Kazuhiro Hikami and Rei Inoue, who tried to obtain the algebra structure by using lax operators and generalized R-matrices [65,66], or Alexander Belov and Alexander Antonov and Karen Chaltikian, who first tried to follow Feigin's construction but finally also solved part of the conjecture by getting the help of lax operators and because of its construction, that made it very difficult to follow their publication [67,68].

However, here, in [64], we proceeded and introduced the simplest way of constructing such kinds of algebras by just employing Feigin's homomorphisms and screening operators by defining a Poisson bracket on our variables just based on our Cartan matrix [63,69].

We have to note that in [69], Yaroslav Pugai constructed lattice W_3 algebras already, but here we will introduce its weaker version based on our newly defined Poisson bracket, constructed just based on the Cartan matrix A_n, which will make our job easier and more elegant.

As before, to do this, let us set C as an arbitrary symmetrizable Cartan matrix of rank r and let $\mathfrak{n} = \mathfrak{n}_+$ be the standard maximal nilpotent sub-algebra of the Kac–Moody algebra associated with C. So $\mathfrak{n}$ is generated by elements $e_1, \ldots, e_r$ which satisfy Serre relations [59], where r stands for $\mathrm{rank}(C)$.

In Section 5.1, we proved that screening operators $S_{X_i^{ji}} = \sum_{\substack{j\in\mathbb{Z} \\ \text{for } i \text{ fixed}}}^{n} X_i^{ji}$; for X_i^{ji} generators of the q-commutative ring

$$\mathbb{C}_q[X_i^{ji}] := \frac{\mathbb{C}[X_i^{ji}]}{\left\langle X_i^{ji} X_k^{jk} - q^{\langle \alpha_i, \alpha_j \rangle} X_k^{jk} X_i^{ji} \right\rangle}$$

and for $\langle \alpha_i, \alpha_j \rangle = a_{ij}$ the ij's components of our Cartan matrix C satisfy quantum Serre relations $\mathrm{ad}_q(X_i)^{1-a_{ij}}(X_j)$.

Here again, as in Section 5.1, we can define

$$U_q(\mathfrak{n}) := \langle S_{X_i^{ji}}, S_{X_k^{jk}} \mid (\mathrm{ad}_q(S_{X_i^{ji}}))^2(S_{X_k^{jk}}) = 0 \rangle,$$

and for $\mathbb{C}_q[X]$ the quantum polynomial ring in one variable and twisted tensor product $\bar{\otimes}$, we can define

$$\begin{aligned} U_q(\mathfrak{n}) \bar{\otimes} \mathbb{C}_q[X_l^{jl}] := \langle S_{X_i^{ji}}, S_{X_k^{jk}}, X_l^{jl} \mid (\mathrm{ad}_q(S_{X_i^{ji}}))^2(S_{X_k^{jk}}) = 0, \\ S_{X_i^{ji}} X_l^{jl} = q^2 X_l^{jl} S_{X_i^{ji}}, S_{X_k^{jk}} X_l^{jl} = q^{-1} X_l^{jl} S_{X_k^{jk}} \rangle \end{aligned}$$

such that we have the following embedding

$$U_q(\mathfrak{n}) \hookrightarrow U_q(\mathfrak{n}) \bar{\otimes} \mathbb{C}_q[X_l^{jl}] \hookrightarrow U_q(n) \bar{\otimes} \mathbb{C}_q[X_l^{jl}] \bar{\otimes} \mathbb{C}_q[X_m^{jm}]$$

where $\mathbb{C}_q[X_l^{jl}] \bar{\otimes} \mathbb{C}_q[X_m^{jm}] = \mathbb{C}\langle X_l^{jl}, X_m^{jm} \mid X_l^{jl} X_m^{jm} = q^{a_{lm}} X_m^{jm} X_l^{jl}$ [63].

Which will ensure the well definedness of our definition of lattice W-algebras.

7.1. Weak Faddeev–Takhtajan–Volkov Algebras

In 1985, the first example of W_3 algebras was introduced by Alexander Zamolodchikov in the investigation of the possibility of the existence of new additional infinite symmetries in the context of two-dimensional Conformal Field Theory [70] and as the possible extension of Virasoro algebra.

Vladimir Fateev and Zamolodchikov [71] found the bosonic representation for W_3 algebras and noted some connection with $\mathfrak{sl}_3$ Lie algebra. In a series of articles [72–74], Fateev and Lukyanov have shown that there exist W-algebras associated with every simple Lie algebra and found the bosonic representation of generators in W-algebras. They discovered that free bosonic representation of W-algebras is given by quantum Miura transformation, a classical analogue of which was well known in the theory of integrable non-linear evaluation of Korteweg–de Vries type [75]. In the spirit of reference [76], Virasoro algebra should commute (in the Feigin–Fuchs representation) with screening operators. As a matter of fact, this property was given in the works [70,74,77] as the main mathematical background of such a definition of W-algebras was developed in references [77–79], where it was shown that W-algebras are the result of quantum Drinfeld–Sokolov reduction of K-M. algebras. It has been shown that screening operators satisfy the quantum Serre relation, i.e., they constitute the nilpotent part of quantum groups. So mathematically speaking we have

$$W \simeq \mathrm{Inv} U_q(\mathfrak{n}_+), \tag{11}$$

where $g = \mathfrak{n}_+ \oplus h \oplus \mathfrak{n}_-$ is the Lie algebra associated with W-algebra. In our work, we describe some variant of lattice analogue of W-algebras, given by Definition 11. The first example of classical lattice W_2 algebra (lattice Virasoro algebra) was found by Faddeev and Takhtajan in reference [80] in their study of the Liouville model on the lattice. The Quantum analogue of Faddeev–Takhtajan algebra was obtained by Volkov in 1992. Boris L. Feigin noticed that the lattice "bosonization" rule for Virasoro algebra can be obtained from the solution of some kind of difference equations in one unknown f with non-commutative coefficients composed of functions of n independent variables $x_1, x_2, \cdots, x_n$ which do not contain the unknown function f. At the time of publishing the work by Yaroslav Pugai [69], no one knew any way to solve similar equations for W-algebras associated with other simple Lie algebras, but in [63,64] we have shown concerning the examples how the classical limit consideration can help in finding the right solution. To do this, we defined a new Poisson bracket based on the Cartan matrix A_n of $\mathfrak{sl}_n$. For example, in the case of $\mathfrak{sl}_2$, we define our Poisson bracket as in the following process.

As mentioned already in Section 5.2, the main tools which we will use are difference equations, screening operators, Feigin's homomorphisms, adjoint actions, partial differential equations and Cartan matrices.

We know that from an abstract view $g = \mathfrak{sl}_{m+1}$ is an algebra related to the Cartan matrix $(a_{ij})_{i,j}$, for $a_{ij} = \begin{cases} 2 & \text{if } i = j \\ -1 & \text{if } |i-j| = 1 \\ 0 & \text{if } |i-j| > 1 \end{cases}$ and so for $\mathfrak{sl}_2$ it will consist of just one row and one column, i.e., we have $A_1 = (2)$ and let us denote by $C\langle X \rangle$ the skew polynomial ring on generators $X = (X_i)_i$ labeled by $i \in \{-\infty, \cdots -1, 0, 1, \cdots, +\infty\}$ and the defining q-commutation relations $X_i X_j = q^2 X_j X_i$ for if $i \leq j$ all with the same color.

Definition 6. *Let us define our Poisson bracket as follows in the case of* $\mathfrak{sl}_2$*:*

$$\begin{cases} \{X_i, X_j\} := 2X_i X_j & \text{if } i < j \\ \{X_i, X_i\} := 0. \end{cases} \tag{12}$$

The main problem is to find solutions to the system of difference equations from an infinite number of non-commutative variables in the quantum case and commutative variables in the classical case. It is significant that commutation relations (12) depend just on the sign of the difference $(i-j)$ and are based on our Cartan matrix. We should try to find all solutions to the system:

$$\begin{cases} \mathfrak{D}_x^{(n)} \triangleleft \tau_1 = 0 \\ H_x^{(n)} \triangleleft \tau_1 = 0. \end{cases} \tag{13}$$

Let us define our system of variables as follows

$$\begin{array}{cccccc} \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \cdots & X_1^{(11)} & X_1^{(21)} & X_1^{(31)} & X_1^{(41)} & \cdots \\ \cdots & X_2^{(12)} & X_2^{(22)} & X_2^{(32)} & X_2^{(42)} & \cdots \\ \cdots & X_3^{(13)} & X_3^{(23)} & X_3^{(33)} & X_3^{(43)} & \cdots \\ \cdots & X_4^{(14)} & X_4^{(24)} & X_4^{(34)} & X_4^{(44)} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{array}$$

Let us equip this system of variables with lexicographic ordering, i.e., $j_{k_m}i < j_{k_n}i$ if $j_{k_m} < j_{k_n}$ and $ji_{k_m} < ji_{k_n}$ if $i_{k_m} < i_{k_n}$. We need this kind of order because we have different kinds of sets of variables with proper coloring such that each set has its own color different from its neighbors. We have $\tau_1 := \tau_1[\cdots, X_1^{(11)}, X_1^{(21)}, X_1^{(31)}, \cdots, X_2^{(12)}, X_2^{(22)}, X_2^{(32)}, \cdots]$, a multi-variable function dependent on $\{X_i^{(ji)}\}$'s for $i, j \in \{-\infty, \cdots, 1, \cdots, n, \cdots, +\infty\}$ and $\mathfrak{D}_x^{(n)}$ comes from

$$\{S_{X_i^{ji}}, \tau_1\}_p = S_{X_i^{ji}}\tau_1 - p^{\deg \tau_1 \langle \alpha_i, \alpha_j \rangle} \tau_1 S_{X_i^{ji}} \tag{14}$$

where $\langle \alpha_i, \alpha_j \rangle = a_{ij}$ which is related to our Cartan matrix and $S_{X_i^{ji}}$ is the screening operator on one of our variable sets, i.e., $S_{X_i^{ji}} = \sum_{j \in \mathbb{Z}} X_i^{ji}$. Then, we will obtain the whole set of solutions by using the following shift operator:

$$\begin{aligned} \tau_2 &= \tau_1[X_1^{(11)} \to X_1^{(21)}, X_1^{(21)} \to X_1^{(31)}, \cdots], \\ \tau_3 &= \tau_2[X_1^{(21)} \to X_1^{(31)}, X_1^{(31)} \to X_1^{(41)}, \cdots] \\ &\vdots \end{aligned} \tag{15}$$

Definition 7. *Let us define our lattice W-algebra based on its generators according to [63,69].*

Generators of lattice W-algebra associated with simple Lie algebra g constitute the functional basis of the space of invariants

$$\tau_i := Inv_{U_q(\mathfrak{n}_+)}(\mathbb{C}_q[X_i^{ji} | i \in \mathbb{Z}]) \tag{16}$$

with additional requirements

$$H_{X_i^{ji}}(\tau_i) = 0 \qquad \text{and} \qquad D_{X_i^{ji}}(\tau_i) = 0 \tag{17}$$

where $H_{X_i^{ji}}$ and $D_{X_i^{ji}}$ will be specified later.

Equation (15) means that the generators have to satisfy quantum Serre relations and the first equation in (17) means that they should have zero degrees.

Here in this paper, we work on the case where $g = \mathfrak{sl}_n$ and we use $\tau_i^{(n)}$ instead of τ_i, where (n) stands for n in $\mathfrak{sl}_n$.

Lattice W_2 Algebra

Let us first consider the $\mathfrak{sl}_2$ case and to simplify the notations, let us consider our set of variables as $X_i := X_i^{ji}$. As shown in [63], it is enough just to work with $S_{X_i^{ji}} =: S_{X_i} = \sum_{i=1}^{3} X_i$, because the other parts for $i > 3$ and $i < 1$ will tend to zero. By setting $q = e^{-\mathfrak{h}}$, for the Planck constant $\mathfrak{h}$, we will try to find generators of our lattice W_2-algebra in the case of $\mathfrak{sl}_2$.

– *First step:* find $D_X^{(2)}$.
To do this and for simplicity, we will skip details here in this review paper and will refer the interested reader to [64]. We have

$$D_X^{(2)} = X_1(X_1 + 2X_2 + 2X_3)\frac{\partial}{\partial X_1} + X_2(X_2 + 2X_3)\frac{\partial}{\partial X_2} + X_3^2\frac{\partial}{\partial X_3}. \tag{18}$$

– *Second step:* find $H_X^{(2)}$.
To find $H_X^{(2)}$, we note that it resembles the degree of our polynomial function. So if, for example, $H_X^{(2)}$ acts on $X_1^n X_2^m X_3^l$, then we should get $(n + m + l)$. So let us define:

$$H_X^{(2)} := \sum_i X_i \frac{\partial}{\partial X_i} \tag{19}$$

and then we have:

$$\begin{aligned} H_X^{(2)}(X_1^n X_2^m X_3^l) &= (\textstyle\sum_i X_i \frac{\partial}{\partial_{X_i}})(X_1^n X_2^m X_3^l) \\ &= \textstyle\sum_i X_i \frac{\partial X_1^n X_2^m X_3^l}{\partial_{X_i}} \\ &= X_1 \frac{\partial X_1^n X_2^m X_3^l}{\partial_{X_1}} + X_2 \frac{\partial X_1^n X_2^m X_3^l}{\partial_{X_2}} + X_3 \frac{\partial X_1^n X_2^m X_3^l}{\partial_{X_3}} \\ &= nX_1^n X_2^m X_3^l + mX_1^n X_2^m X_3^l + lX_1^n X_2^m X_3^l \\ &= (n+m+l)X_1^n X_2^m X_3^l. \end{aligned}$$

Which gives us

$$H_X^{(2)}(X_1^n X_2^m X_3^l) = (n+m+l)X_1^n X_2^m X_3^l$$

and on the other side, we have

$$\begin{aligned} (n+m+l)X_1^n X_2^m X_3^l &= nX_1X_1^{n-1}X_2^m X_3^l + mX_1^n X_2 X_2^{m-1} X_3^l + lX_1^n X_2^m x_3 X_3^{l-1} \\ &= X_1 \frac{X_2^m X_3^l \partial X_1^n}{\partial_{X_1}} + X_2 \frac{X_1^n X_3^l \partial X_2^m}{\partial_{X_2}} + X_3 \frac{X_1^n X_2^m \partial X_3^l}{\partial_{X_3}} \\ &= X_1 \frac{\partial}{\partial_{X_1}} + X_2 \frac{\partial}{\partial_{X_2}} + X_3 \frac{\partial}{\partial_{X_3}}. \end{aligned}$$

Which gives us

$$(n+m+l)X_1^n X_2^m X_3^l = \sum_i X_i \frac{\partial}{\partial X_i}.$$

This shows that (19) is well defined.

Now the only thing that remains is to find the solutions to the following system of two-linear homogeneous equations in one unknown $\tau_1[\cdots, X_1, X_2, X_3, \cdots]$:

$$\begin{cases} (X_1(X_1 + 2X_2 + 2X_3)\frac{\partial}{\partial X_1} + X_2(X_2 + 2X_3)\frac{\partial}{\partial X_2} + X_3^2 \frac{\partial}{\partial X_3})\tau_1[\cdots, X_1, X_2, X_3, \\ \quad \cdots] = 0, \\ (X_1 \frac{\partial}{\partial_{X_1}} + X_2 \frac{\partial}{\partial_{X_2}} + X_3 \frac{\partial}{\partial_{X_3}})\tau_1[\cdots, X_1, X_2, X_3, \cdots] = 0. \end{cases} \tag{20}$$

The second equation ensures that the solution has degree 0 and also the partial differentials will give us a multi-variable function dependent on just X_1, X_2, X_3. The system of PDEs (20) can be solved using the procedure described in Chapter V, Section IV of [81].

After doing some calculations in Mathematica it becomes clear that system (20) has only one functional dependent nontrivial solution:

$$\tau_1^{(2)}[X_1, X_2, X_3] = \frac{(X1+X2)(X2+X3)}{X2(X1+X2+X3)} = \frac{(\sum_{1\le i_1\le 2} X_{i_1}^{(1)})(\sum_{1\le i_1\le 2} X_{i_1+1}^{(1)})}{X_2^{(1)}(\sum_{1\le i_1\le 3} X_{i_1}^{(1)})}. \tag{21}$$

Again, as before, (2) goes back to 2 in $\mathfrak{sl}_2$ and 1 is a default index that will be used later to employ the shifting operator. According to the number of variables, we will have two shifts and then everything will be in a loop. So here in the $\mathfrak{sl}_2$ case we have three solutions for our system of linear equations (20) which belong to the fraction ring of polynomial functions:

$$\begin{cases} \tau_1^{(2)}[X_1, X_2, X_3] = \frac{(\sum_{1\le i_1\le 2} X_{i_1}^{(1)})(\sum_{1\le i_1\le 2} X_{i_1+1}^{(1)})}{X_2^{(1)}(\sum_{1\le i_1\le 3} X_{i_1}^{(1)})}; \\ \tau_2^{(2)}[X_2, X_3, X_4] = \frac{(\sum_{2\le i_1\le 3} X_{i_1}^{(1)})(\sum_{2\le i_1\le 3} X_{i_1+1}^{(1)})}{X_2^{(1)}(\sum_{2\le i_1\le 4} X_{i_1}^{(1)})}; \\ \tau_3^{(2)}[X_3, X_4, X_5] = \frac{(\sum_{3\le i_1\le 4} X_{i_1}^{(1)})(\sum_{3\le i_1\le 4} X_{i_1+1}^{(1)})}{X_2^{(1)}(\sum_{3\le i_1\le 5} X_{i_1}^{(1)})}. \end{cases} \tag{22}$$

We define our non-commutative Poisson algebra according to the definition of Poisson brackets given by Poisson himself [82] with the difference that here we work on the q-commutative ring $\frac{\mathbb{C}[X_i^{ji}]}{X_i^{ji}X_k^{jk} - q^{\langle \alpha_i, \alpha_k \rangle} X_k^{jk} X_i^{ji}}$, based on the generators which are the solutions of the PDE system (13).

To do this we use the following bracket:

$$F_j^{(n)} := \{\tau_i^{(n)}, \tau_j^{(n)}\} = \sum_i \frac{\partial \tau_i^{(n)}}{\partial X_i} \sum_j \frac{\partial \tau_j^{(n)}}{\partial X_j} \{X_i, X_j\}, \tag{23}$$

where $\{X_i, X_j\}$ is our previously defined Poisson bracket on our set of variables. For instance, in the case of $\mathfrak{sl}_2$ we have

$$\begin{aligned} \{\tau_1^{(2)}, \tau_2^{(2)}\} &= \Big(\frac{\partial \tau_1^{(2)}}{\partial X_1}\Big)\Big(\frac{\partial \tau_2^{(2)}}{\partial X_2}\{X_1, X_2\} + \frac{\partial \tau_2^{(2)}}{\partial X_3}\{X_1, X_3\} + \frac{\partial \tau_2^{(2)}}{\partial X_2}\{X_1, X_4\}\Big) \\ &+ \Big(\frac{\partial \tau_1^{(2)}}{\partial X_2}\Big)\Big(\frac{\partial \tau_2^{(2)}}{\partial X_2}\{X_2, X_2\} + \frac{\partial \tau_2^{(2)}}{\partial X_3}\{X_2, X_3\} + \frac{\partial \tau_2^{(2)}}{\partial X_2}\{X_2, X_4\}\Big) \\ &+ \Big(\frac{\partial \tau_1^{(2)}}{\partial X_3}\Big)\Big(\frac{\partial \tau_2^{(2)}}{\partial X_2}\{X_3, X_2\} + \frac{\partial \tau_2^{(2)}}{\partial X_3}\{X_3, X_3\} + \frac{\partial \tau_2^{(2)}}{\partial X_2}\{X_3, X_4\}\Big) \\ &= \Big(\frac{\partial \tau_1^{(2)}}{\partial X_1}\Big)\Big(\frac{\partial \tau_2^{(2)}}{\partial X_2}(2X_1X_2) + \frac{\partial \tau_2^{(2)}}{\partial X_3}(2X_1X_3) + \frac{\partial \tau_2^{(2)}}{\partial X_2}(2X_1X_4)\Big) \\ &+ \Big(\frac{\partial \tau_1^{(2)}}{\partial X_2}\Big)\Big(\frac{\partial \tau_2^{(2)}}{\partial X_2}(0) + \frac{\partial \tau_2^{(2)}}{\partial X_3}(2X_2X_3) + \frac{\partial \tau_2^{(2)}}{\partial X_2}(2X_2X_4)\Big) \\ &+ \Big(\frac{\partial \tau_1^{(2)}}{\partial X_3}\Big)\Big(\frac{\partial \tau_2^{(2)}}{\partial X_2}(-2X_3X_2) + \frac{\partial \tau_2^{(2)}}{\partial X_3}(0) + \frac{\partial \tau_2^{(2)}}{\partial X_2}(2X_3X_4)\Big) \\ &= 2\frac{X_1X_2^2X_3^2X_4(X_1+X_2+X_3+X_4)}{(X_1+X_2)^2(X_2+X_3)^3(X_3+X_4)^2}. \end{aligned}$$

So we have

$$F_2^{(2)} = \{\tau_1^{(2)}, \tau_2^{(2)}\} = \frac{2X_1X_2^2X_3^2X_4(X_1+X_2+X_3+X_4)}{(X_1+X_2)^2(X_2+X_3)^3(X_3+X_4)^2}. \tag{24}$$

It is enough to find our brackets just based on the first generator because after that we are able to find other brackets based on the other generators; so for $\tau_3^{(2)}$ in almost the same process, we have:

$$\begin{aligned}
F_3^{(2)} &= \{\tau_1^{(2)}, \tau_3^{(2)}\} \\
&= \Big(\frac{\partial \tau_1^{(2)}}{\partial X_1}\Big)\Big(\frac{\partial \tau_3^{(2)}}{\partial X_3}\{X_1, X_3\} + \frac{\partial \tau_3^{(2)}}{\partial X_4}\{X_1, X_4\} + \frac{\partial \tau_3^{(2)}}{\partial X_5}\{X_1, X_5\}\Big) \\
&+\Big(\frac{\partial \tau_1^{(2)}}{\partial X_2}\Big)\Big(\frac{\partial \tau_3^{(2)}}{\partial X_3}\{X_2, X_3\} + \frac{\partial \tau_3^{(2)}}{\partial X_4}\{X_2, X_4\} + \frac{\partial \tau_3^{(2)}}{\partial X_5}\{X_2, X_5\}\Big) \\
&+\Big(\frac{\partial \tau_1^{(2)}}{\partial X_3}\Big)\Big(\frac{\partial \tau_3^{(2)}}{\partial X_3}\{X_3, X_3\} + \frac{\partial \tau_3^{(2)}}{\partial X_4}\{X_3, X_4\} + \frac{\partial \tau_3^{(2)}}{\partial X_5}\{X_3, X_5\}\Big) \\
&= \Big(\frac{\partial \tau_1^{(2)}}{\partial X_1}\Big)\Big(\frac{\partial \tau_3^{(2)}}{\partial X_3}(2X_1X_3) + \frac{\partial \tau_3^{(2)}}{\partial X_4}(2X_1X_4) + \frac{\partial \tau_3^{(2)}}{\partial X_5}(2X_1X_5)\Big) \\
&+\Big(\frac{\partial \tau_1^{(2)}}{\partial X_2}\Big)\Big(\frac{\partial \tau_3^{(2)}}{\partial X_3}(2X_2X_3) + \frac{\partial \tau_3^{(2)}}{\partial X_4}(2X_2X_4) + \frac{\partial \tau_3^{(2)}}{\partial X_5}(2X_2X_5)\Big) \\
&+\Big(\frac{\partial \tau_1^{(2)}}{\partial X_3}\Big)\Big(\frac{\partial \tau_3^{(2)}}{\partial X_3}(0) + \frac{\partial \tau_3^{(2)}}{\partial X_4}(2X_3X_4) + \frac{\partial \tau_3^{(2)}}{\partial X_5}(2X_3X_5)\Big) \\
&= \frac{-2X_1X_2X_3^2X_4X_5}{(X_1+X_2)(X_2+X_3)^2(X_3+X_4)^2(X_4+X_5)}.
\end{aligned} \tag{25}$$

We have to note that we are almost done with our Poisson algebra in the $\mathfrak{sl}_2$ case, but for further plans, i.e., to find our Volterra system, the differential-difference chain of non-linear equations

$$\begin{cases} H = \sum_i [\ln(\tau_i)]; \\ \dot{\tau}_j = \{\tau_j, H\} = \tau_j \times \sum_i \Gamma_i; \end{cases} \tag{26}$$

where Γ_i stands for $\frac{\tau_1,\tau_i}{\tau_1\tau_i}$ [69], we have to write down the brackets $\{\tau_1, \tau_i\}$ in terms of their decompositions to τ_j's for $1 \le j \le i$. So we need to write it as the decomposition of our generators and this will be done by using the Mathematica coding which is presented in Appendix A [64] (please see the extended version of these results in [64]).

The result is as follows:

$$\begin{cases} F_2^{(2)} = \{\tau_1^{(2)}, \tau_2^{(2)}\} = 2(1-\tau_1^{(2)})(1-\tau_2^{(2)})(-1+\tau_1^{(2)}+\tau_2^{(2)}); \\ F_3^{(2)} = \{\tau_1^{(2)}, \tau_3^{(2)}\} = -2(1-\tau_1^{(2)})(1-\tau_2^{(2)})(1-\tau_3^{(2)}); \\ F_i^{(2)} = \{\tau_1^{(2)}, \tau_i^{(2)}\} = 0 \qquad \text{for } |i-1| \ge 3. \end{cases} \tag{27}$$

This result is weaker than the Faddeev–Takhtajan–Volkov algebra which was mentioned in [69] and if we continue this for $\mathfrak{sl}_3$, then we will again have a weaker version of what was mentioned in [69].

7.2. Lattice W_3 Algebra

In this case, we will use the following defined Poisson bracket based on the Cartan matrix $A_2 = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$. However, to do this according to our previous ordering and list of variables, let us for simplicity set our variables as follows:

Set $X_i^{(1i)} := X_i$ and $X_i^{(2i)} := Y_i$.

Definition 8. *Let us define our Poisson bracket as follows in the case of $\mathfrak{sl}_3$:*

$$\begin{cases} \{X_i, X_j\} := 2X_iX_j & \text{if } i < j; \\ \{Y_i, Y_j\} := 2Y_iY_j & \text{if } i < j; \\ \{X_i, X_i\} := 0; \\ \{Y_i, Y_i\} := 0; \\ \{X_i, Y_j\} := X_iY_j & \text{if } i > j; \\ \{X_i, Y_j\} := -X_iY_j & \text{if } i \leq j. \end{cases} \tag{28}$$

Instead of (12), we have the following $q-$commutation relations

$$\begin{cases} X_iX_j = q^2X_jX_i & \text{if } i \leq j; \\ Y_iY_j = q^2Y_jY_i & \text{if } i \leq j; \\ X_iY_j = q^{-1}Y_jX_i & \text{if } i \leq j. \end{cases} \tag{29}$$

We obtain the following equations in the same manner as in $\mathfrak{sl}_2$:
Therefore as in (20) we have the following system of PDEs

$$\begin{cases} (X_1(X_1+2X_2+2X_3)\frac{\partial\tau_1^{(3)}}{\partial X_1} + X_2(X_2+2X_3)\frac{\partial\tau_1^{(3)}}{\partial X_2} + X_3^2\frac{\partial\tau_1^{(3)}}{\partial X_2} \\ -Y_1(X_1+X_2+X_3)\frac{\partial\tau_1^{(3)}}{\partial Y_1} - Y_2(X_2+X_3)\frac{\partial f}{\partial Y_2} - Y_3X_3\frac{\partial\tau_1^{(3)}}{\partial Y_3}) = 0; \\ (2X_1\frac{\partial\tau_1^{(3)}}{\partial X_1} + 2X_2\frac{\partial\tau_1^{(3)}}{\partial X_2} + 2X_3\frac{\partial\tau_1^{(3)}}{\partial X_3} - Y_1\frac{\partial\tau_1^{(3)}}{\partial Y_1} - Y_2\frac{\partial\tau_1^{(3)}}{\partial Y_2} - Y_3\frac{\partial\tau_1^{(3)}}{\partial Y_3}) = 0; \\ D_Y^{(3)} = (Y_1(Y_1+2Y_2+2Y_3)\frac{\partial\tau_1^{(3)}}{\partial Y_1} + Y_2(Y_2+2Y_3)\frac{\partial\tau_1^{(3)}}{\partial Y_2} + Y_3^2\frac{\partial\tau_1^{(3)}}{\partial Y_2} \\ -Y_1(X_1+X_2+X_3)\frac{\partial\tau_1^{(3)}}{\partial Y_1} - Y_2(X_2+X_3)\frac{\partial\tau_1^{(3)}}{\partial Y_2} - Y_3X_3\frac{\partial\tau_1^{(3)}}{\partial Y_3}) = 0; \\ (2X_1\frac{\partial\tau_1^{(3)}}{\partial X_1} + 2X_2\frac{\partial\tau_1^{(3)}}{\partial X_2} + 2X_3\frac{\partial\tau_1^{(3)}}{\partial X_3} - Y_1\frac{\partial\tau_1^{(3)}}{\partial Y_1} - Y_2\frac{\partial\tau_1^{(3)}}{\partial Y_2} - Y_3\frac{\partial\tau_1^{(3)}}{\partial Y_3}) = 0. \end{cases} \tag{30}$$

In accordance with appendix A (please see the extended version of these results in [64]), we have the following functional dependent nontrivial solution for the whole system of PDEs (30)

$$\tau_1^{(3)} = \frac{(\Sigma_{1\leq i\leq j\leq 2} \quad X_iY_j)(\Sigma_{1\leq i\leq j\leq 2} \quad X_{i+1}Y_{j+1})}{X_2Y_2(\Sigma_{1\leq i\leq j\leq 3} \quad X_iY_j)}. \tag{31}$$

Again, as before, (3) goes back to 3 in $\mathfrak{sl}_3$ and 1 is a default index which later we will use to employ our shifting operators. In accordance with the number of variables, we will have six shifts and then after that, it will be in a loop. So here in the $\mathfrak{sl}_3$ case, we have six solutions that belong to the fraction ring of polynomial functions.

$$\begin{cases} \tau_1^{(3)}[X_1, Y_1, X_2, Y_2, X_3, Y_3] = \frac{X_2Y_2(X_3Y_3+X_2(Y_2+Y_3)+X_1(Y_1+Y_2+Y_3))}{(X_2Y_2+X_1(Y_1+Y_2))(X_3Y_3+X_2(Y_2+Y_3))}; \\ \tau_2^{(3)}[Y_1, X_2, Y_2, X_3, Y_3, X_4] = \frac{X_3Y_2(X_2Y_1+(X_3+X_4)(Y_1+Y_2)+X_4Y_3)}{(X_2Y_1+X_3(Y_1+Y_2))(X_3Y_2+X_4(Y_2+Y_3))}; \\ \tau_3^{(3)}[X_2, Y_2, X_3, Y_3, X_4, Y_4] = \frac{X_3y_3(X_4Y_4+X_3(Y_3+Y_4)+X_2(Y_2+Y_3+Y_4))}{(X_3Y_3+X_2(Y_2+Y_3))(X_4Y_4+X_3(Y_3+Y_4))}; \\ \tau_4^{(3)}[Y_2, X_3, Y_3, X_4, Y_4, X_5] = \frac{X_4Y_3(X_3Y_2+(X_4+X_5)(Y_2+Y_3)+X_5Y_4)}{(X_3Y_2+X_4(Y_2+Y_3))(X_4Y_3+X_5(Y_3+Y_4))}; \\ \tau_5^{(3)}[X_3, Y_3, X_4, Y_4, X_5, Y_5] = \frac{X_4Y_4(X_5Y_5+X_4(Y_4+Y_5)+X_3(Y_3+Y_4+Y_5))}{(X_4Y_4+X_3(Y_3+Y_4))(X_5Y_5+X_4(Y_4+Y_5))}; \\ \tau_6^{(3)}[Y_3, X_4, Y_4, X_5, Y_5, X_6] = \frac{X_5Y_4(X_4Y_3+(X_5+X_6)(Y_3+Y_4)+X_6Y_5)}{(X_4Y_3+X_5(Y_3+Y_4))(X_5Y_4+X_6(Y_4+Y_5))}. \end{cases} \tag{32}$$

where $\tau_1^{(3)} := \tau_1^{(3)}[\cdots, X_1, Y_1, X_2, Y_2, X_3, Y_3 \cdots]$. Again, by setting $X_i^{(1i)} := X_i$ and $X_i^{(2i)} := Y_i$ and $X_i^{(3i)} := Z_i$ and according to (26), we have to write down the following brackets as a composition of $\tau_i^{(3)}$s because of the algebra structure, and this will be done by using Mathematica coding in appendix A (please see the extended version of these results in [64]).

$$\begin{cases} F_2^{(3)} = \{\tau_1^{(3)}, \tau_2^{(3)}\} = -(1-\tau_1^{(3)})(1-\tau_2^{(3)})(\tau_1^{(3)}\tau_2^{(3)}); \\ F_3^{(3)} = \{\tau_1^{(3)}, \tau_3^{(3)}\} = (1-\tau_1^{(3)})(1-\tau_3^{(3)})(\tau_1^{(3)}\tau_2^{(3)} + \tau_2^{(3)}\tau_3^{(3)} - \tau_2^{(3)}); \\ F_4^{(3)} = \{\tau_1^{(3)}, \tau_4^{(3)}\} = -(1-\tau_1^{(3)})(1-\tau_4^{(3)}) \\ \qquad\qquad (\tau_1^{(3)}\tau_2^{(3)} + \tau_2^{(3)}\tau_3^{(3)} + \tau_3^{(3)}\tau_4^{(3)} - \tau_1^{(3)} - \tau_2^{(3)} - \tau_3^{(3)} - \tau_4^{(3)} + 1); \\ F_5^{(3)} = \{\tau_1^{(3)}, \tau_5^{(3)}\} = (1-\tau_1^{(3)})(1-\tau_5^{(3)})(\tau_2^{(3)}\tau_3^{(3)} + \tau_3^{(3)}\tau_4^{(3)} - \tau_2^{(3)} \\ \qquad\qquad -\tau_3^{(3)} - \tau_4^{(3)} + 1); \\ F_6^{(3)} = \{\tau_1^{(3)}, \tau_6^{(3)}\} = -(1-\tau_1^{(3)})(1-\tau_6^{(3)})(\tau_3^{(3)}\tau_4^{(3)} - \tau_4^{(3)} - \tau_3^{(3)} + 1); \\ F_i^{(3)} = \{\tau_1^{(3)}, \tau_i^{(3)}\} = 0 \qquad\qquad \text{for } |i-1| \geq 6. \end{cases} \tag{33}$$

Lattice W_n Algebra; Main Generator

Here, for $\mathfrak{sl}_n$, we skip writing down all steps which we have done in the previous sections and we just write down the main generator of the lattice W_n algebra. The functional dependent nontrivial solution for the whole system of the first order partial differential equations are as follows:

$$\tau_1^{(n)} = \frac{(\Sigma_{1\leq i_1\leq i_2\cdots\leq i_{n-1}\leq 2} \quad x_{i_1}^{(1)} x_{i_2}^{(2)} \cdots x_{i_{n-1}}^{(n-1)})(\Sigma_{1\leq i_1\leq i_2\cdots\leq i_{n-1}\leq 2} \quad x_{i_1+1}^{(1)} x_{i_2+1}^{(2)} \cdots x_{i_{n-1}+1}^{(n-1)})}{x_2^{(1)} \cdots x_2^{(n-1)} (\Sigma_{1\leq i_1\leq i_2\cdots\leq i_{n-1}\leq 3} \quad x_{i_1}^{(1)} x_{i_2}^{(2)} \cdots x_{i_{n-1}}^{(n-1)})}. \tag{34}$$

We should note that $x_{i_j}^{(j)}$s are different to each other for any $j \in \{1, \cdots n-1\}$

8. Concluding Remarks and Some Open Directions

Recall that for a given Cartan matrix $C \in M_{I\times I}(\mathbb{Z})$, we have three $\mathbb{N}I$-graded bialgebras: the quantum enveloping algebra $U_+ \equiv U_q(\mathfrak{n}_+)$ (which has a bialgebra structure), the quantum shuffle algebra $\mathcal{F}^*$ and the dual Ringel–Hall algebra $\mathcal{H}^*(Q)$ [83]. In this paper, we only considered the first one. For a fixed word $w \in W$, there is a quantum polynomial algebra $\mathbb{P}_w$ associated with the Cartan matrix C and the word w, which is the $\mathbb{Q}(v)$-algebra generated by $t_1, \ldots, t_m$ subject to the relation $t_\ell t_k = v^{(\alpha_{i_k}, \alpha_{i_\ell})} t_k t_\ell$ for $k < \ell$. There are various homomorphisms of algebras between these four algebras which have been studied extensively due to their connection to quantum groups and cluster algebras (cf. [84–88]). However, in this paper, in particular, we are interested in the following.

Theorem 24. *Fix a word $w = (i_1 i_2 \ldots i_m) \in W$. Then, the linear map*

$$\begin{aligned} F_w : U_+ &\to \mathbb{P}_w \\ E_j &\mapsto \sum_{1\leqslant k\leqslant m, i_k = j} t_k \end{aligned} \tag{35}$$

is a homomorphism of algebras.

Morphism (35) is called Feigin's map of type w, and was first proposed by B. Feigin as a tool for studying the skew-field of fractions of $(U_+)^*$ and it is the main object of study in this paper.

Very recently, in a paper published in ArXiv, Anton Izosimov and Gloria Marí Beffa, questioned the reason behind the creation of the lattice W-algebras in [89]. They studied the discretization criteria of the Adler–Gelfand–Dickey bracket, both from the multiplicative structure on the algebra of formal pseudo-difference operators (which can also be interpreted as a Poisson–Lie structure on the extended group of such operators [90]) or by performing the so-called Drinfeld–Sokolov reduction to a specific symplectic leaf in the dual of an affine (Kac–Moody) Lie algebra [75]. They showed that both constructions admit a discretization (a lattice version) and hence there is a well defined notion of a lattice W_m-algebra. In the process, recovering familiar structures, such as for $m = 2$, they obtain the lattice Virasoro algebra of Faddeev–Takhtajan–Volkov [80,91] (known as a cubic Poisson structure associated with the Volterra lattice [92]) and for $m = 3$ the construction gives the lattice W_3-algebra of Belov-Chaltikian [68] and finally, they formulate their main theorem on the coincidence of two definitions of lattice W_m-algebras.

We could follow the opposite direction to that proposed in [89] to find some approaches to making continuous W_m-algebras based on our lattice W-algebras, which we think will be a very interesting project and may result in a new Adler–Gelfand–Dickey bracket.

On the other hand, we could do the same constructions done in the A_n case, for other classes of Cartan matrices B_n, C_n and D_n, which we believe could open a new direction in defining integrable chain systems, something which has never been done.

Moreover, we have to remark that our study regarding the lattice W_n-algebras is not completed and still needs much work to be done. In accordance with (26), we need to write down the brackets $\{\tau_1, \tau_i\}$ in terms of their decompositions to τ_j's for $1 \leqslant j \leqslant i$, for i such that $\mid i-1 \mid \geqslant n$ in the case of lattice W_n-algebras and obtain the final algebra structure and we have to note that this can only be done by using Mathematica or any other computing software, with the exactly same approach as for lattice W_2 and W_3-algebras.

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