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Article

On the Fundamental Analyses of Solutions to Nonlinear Integro-Differential Equations of the Second Order

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Abstract: In this article, a scalar nonlinear integro-differential equation of second order and a nonlinear system of integro-differential equations with infinite delays are considered. Qualitative properties of solutions called the global asymptotic stability, integrability and boundedness of solutions of the second-order scalar nonlinear integro-differential equation and the nonlinear system of nonlinear integro-differential equations with infinite delays are discussed. In the article, new explicit qualitative conditions are presented for solutions of both the second-order scalar nonlinear integro-differential equations with infinite delay and the nonlinear system of integro-differential equations with infinite delay. The proofs of the main results of the article are based on two new Lyapunov–Krasovskiĭ functionals. In particular cases, the results of the article are illustrated with three numerical examples, and connections to known tests are discussed. The main novelty and originality of this article are that the considered integro-differential equation and system of integro-differential equations with infinite delays are new mathematical models, the main six qualitative results given are also new.

Keywords: integro-differential equation; integro-differential system; first order; second order; infinite delay; global asymptotic stability; boundedness; integrability; LKF

MSC: 34C11; 34D05; 34D20

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1. Introduction

In the relevant literature, the global asymptotic stability, boundedness, integrability, etc., of linear and nonlinear integro-differential equations (IDEs) of the first order without delay, scalar nonlinear delay integro-differential equations (DIDEs), nonlinear delay systems of IDEs of the first order, functional differential equations (FDEs), etc., have attracted a lot of attention from researchers. For a comprehensive treatment of the subject on the stability, boundedness, integrability, etc., of solutions of first-order IDEs without delay, see, Alahmadi et al. [1], Burton [2], Furumochi and Matsuoka [3], Grimmer and Seifert [4], Jordan [5], Lakshmikantham and Rama Mohana Rao [6], Mohana Rao and Srinivas [7], Murakami [8], Rama Mohana Rao and Raghavendra [9], Sedova [10], and the bibliographies therein.

We would now like to outline some qualitative results on IDEs without delay.

In the book of Burton [2], which can be considered as a reference book of integral equations and IDEs, using the second Lyapunov method and the Lyapunov–Krasovskiĭ functional (LKF) approach, various kind of stabilities of zero solutions, integrabilities of solutions, as well as boundedness of solutions when $F(t) \neq 0$ are discussed for the systems of IDEs given by:

$$x' = A(t)x + \int_0^t C(t,s)x(s)ds,$$

$$x' = Ax + \int_0^t C(t,s)x(s)ds + F(t),$$

$$x' = Ax + \int_0^t B(t-s)x(s)ds,$$

$$x' = Ax + \int_0^t D(t-s)x(s)ds + F(t),$$

$$x' = A(t)x + \int_0^t C_1(t,s)x(s)ds + \int_0^t C_2(t,s)x(s)ds,$$

$$x' = Ax + f(t,x) + \int_0^t C(t,s)x(s)ds.$$

Next, the book of Lakshmikantham and Rama Mohana Rao [6] is also considered as a reference book of the qualitative theory of IDEs. In the book of Lakshmikantham and Rama Mohana Rao [6], using the second Lyapunov method, various qualitative behaviors of solutions such as stability, uniform stability, asymptotic stability, uniform asymptotic stability of zero solutions, as well as the integrability and boundedness of nonzero solutions when $f(t,x) \neq 0$ and $g(t,y) \neq 0$, are discussed, and some interesting results are obtained for the scalar or systems of IDEs given by:

$$u' = \alpha u + \int_0^t a(t-s)u(s)ds,$$

$$u' = \alpha(t)u + \int_0^t a(t,s)u(s)ds,$$

$$x' = A(t)x + \int_0^t K(t,s)x(s)ds,$$

$$x' = Ax + \int_0^t K(t,s)x(s)ds,$$

$$x' = A(t)x + \int_{-\infty}^t K(t,s)x(s)ds + f(t,x),$$

$$x' = Ax + \int_0^t C(t-s)x(s)ds,$$

$$y' = A(t)y + \int_0^t C(t-s)y(s)ds + g(t,y).$$

Sedova [10] considered the nonlinear system of IDEs

$$x' = G(t,x) + \int_0^t H(t,s,x(s))ds.$$

In [10], sufficient conditions for uniform asymptotic stability of the zero solution of this system are obtained using the Razumikhin method. Similar qualitative results can be found in the other sources mentioned above.

Next, for numerous results in relation to the stability, boundedness, integrability, etc., of solutions of scalar and vector DIDEs of the first order and DIDEs of fractional order, see Berezansky and Braverman [11], Berezansky et al. [12], Du [13], Tunç and Tunç [14], Funakubo et al. [15], Tunç and Tunç [16–18], Tunç [19], Tunç et al. [20], Xu [21], Wang [22], Wang [23], Wang et al. [24], and the bibliographies therein.

We would now like to outline some of these qualitative results in relation to delay integro-differential equations.

In Berezansky and Braverman [11], new explicit exponential stability conditions are obtained for the non-autonomous scalar linear DIDE:

$$x'(t) = \sum_{k=1}^{m} a_k(t)x(h_k(t)) + \int_{g(t)}^{t} K(t,s)x(s)ds,$$

$$t \in [0,\infty), x \in \mathbb{R}.$$

The proofs in the article of Berezansky and Braverman [11] are based on establishing the boundedness of solutions and exponential dichotomy.

Next, in Berezansky et al. [12], uniform exponential stability of the linear delayed integro-differential vector equation

$$x'(t) = \sum_{k=1}^{m} A_k(t)x(h_k(t)) + \sum_{k=1}^{l} \int_{g_k(t)}^{t} P_k(t,s)x(s)ds, t \in [0,\infty), x \in \mathbb{R}^n,$$

is studied. In [12], the main technique of the proofs is splitting the linear expressions in the equation (both with points and with distributed delays) into a "dominant" and a "remainder" part, which can be achieved in a number of different ways, thus providing a number of different criteria. The next important ingredient is the use of a Bohl–Perron-type result stating that a linear equation is exponentially stable if the solutions of the inhomogeneous counterpart of that equation are bounded.

In 1995, Du [13] considered the following system of linear DIDEs:

$$\frac{dx}{dt} = Ax + Bx(t-\tau(t)) + \int_{t-\tau(t)}^{t} \Omega(t,s)x(s)ds,$$

where $x \in \mathbb{R}^n$, $t \in [0,\infty)$, τ is a non-negative and differentiable variable delay, $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times n}$ and $\Omega(t,s) \in C(\mathbb{R}^{n\times n},\mathbb{R}^{n\times n})$. Du [13] is interested in constructing an LFK for this system of DIDEs, which yields uniform asymptotic stability of zero solutions of this system.

Tunç and Tunç [14] considered the nonlinear system of IDDEs with the constant time delay:

$$\dot{x}(t) = -A(t)x(t) - A_d G(x(t-h)) + C\int_{t-h}^{t} F(x(s))ds + Q(t,x(t),x(t-h)),$$

where $x \in \mathbb{R}^n$ is the sate vector, $t \in [0,\infty)$, and h is a positive constant, that is, the constant time delay. The authors [14] investigated the uniform asymptotic stability and integrability of solutions when $Q = 0$ and boundedness of solutions when $Q \neq 0$, based on the LKF approach. Similar qualitative results for the IDDEs of integer and fractional order have been obtained in [15–24].

We now outline some papers in relation to the results of this article. Additionally, for several classes of nonlinear scalar DIDEs of second order, linear and nonlinear two-dimensional systems and nonlinear n-dimensional systems, numerous qualitative results can be seen in the literature, see, e.g., Becker and Burton [25], Dishen [26], Hale and Kato [27], Berezansky and Domoshnitsky [28], Crisci et al. [29], Gözen and Tunç [30], Graef and Tunç [31], and the references of these sources. In particular, there is a scarcity of qualitative results for both scalar DIDEs of second order and system of DIDEs of first order with infinite delays, which are considered in this article.

In [25], Becker and Burton obtained a number of results on uniform stability and equi-asymptotic stability of the zero solution of the FDE:

$$x'(t) = f(t, x_t), (t \geq 0),$$

where $f : \mathbb{R} \times C \to \mathbb{R}^n$, $\mathbb{R} = (-\infty, \infty)$ is a continuous mapping with $f(t, 0) = 0$, and f takes bounded sets into bounded sets. For some $h > 0$, $C = C([-h, 0], \mathbb{R}^n)$ denotes the space of continuous functions $\phi : [-h, 0] \to \mathbb{R}^n$. For any $a \geq 0$, some $t_0 \geq 0$, and $x \in C([t_0 - h, t_0 + a], \mathbb{R}^n)$, it is assumed that $x_t = x(t + s)$ for $s \in [-h, 0]$ and $t \geq t_0$. They also found results on the uniform stability of the Volterra functional equation:

$$x'(t) = F(t, x(s); \alpha \leq s \leq t), (t \geq t_*),$$

where, for $-\infty \leq \alpha \leq t_*$, the right-hand side of this equation is a Volterra functional whose value in $\mathbb{R}^n$ is determined by $t \geq t_*$ and the values of $x(s)$ for $\alpha \leq s \leq t$. It is assumed that F is continuous in t and x for $t \geq t_*$ whenever $x \in C([\alpha, \infty), \mathbb{R}^n)$ is bounded (see [25]).

In Becker and Burton [25], the investigations are based on the Lyapunov's direct method and Jensen's inequality. Some results of Becker and Burton [25] are well illustrated by examples, including the DIDEs of second order with infinite delay. In [25], as the first application, the following DIDEs of second order with infinite delay is considered:

$$x'' + tx' + \int_{-\infty}^{t} a \exp(-\ (t - s))x(s)ds = 0, a > 1. \tag{1}$$

Next, in Becker and Burton [25], depending upon suitable Lyapunov–Krasovskiĭ functionals (LKFs),

$$V(t) = V(t, x(.), y(.)) = y^2 + ax^2 + \int_{-\infty}^{t} a \exp(-(t - s))y^2(s)ds, \tag{2}$$

the authors proved that the zero solution of DIDE (1) is uniformly stable for $t \geq t_*$.

In addition, in the same paper of Becker and Burton [25], as the second application, the authors considered the following non-linear DIDE of second order with infinite delay:

$$x'' + tf(x)x' + \int_{-\infty}^{t} \gamma(t - s)g(x(s))ds = 0. \tag{3}$$

Becker and Burton [25], using the following two multi-functional approaches of the LKFs:

$$\begin{aligned} U(t) =& U(t, x(.), z(.)) = \left[z - \int_{-\infty}^{t} T(t - s)g(x(s))ds \right]^2 - 2\int_{0}^{x} \tilde{g}(s)ds \\ &+ K \int_{-\infty}^{t} \int_{t-s}^{\infty} |T(u)|du\, g^2(x(s))\mathrm{ds}, \end{aligned} \tag{4}$$

and

$$V(t) = V(t, x(.), z(.)) = y^2 + 2T(0)g(x) - 2\int_0^x \tilde{g}(s)ds + D\int_{-\infty}^{t}\int_{t-s}^{\infty} |T(u)|du\, y^2(s)ds, \quad (5)$$

where $T(t) = \int_t^\infty \gamma(u)du$, $F(x) = \int_0^x f(u)du$, $\tilde{g}(x) = T(0)g(x) - F(x)$, K and D are positive constants such that $|g'(x)| \le D$ for $|x| < \delta$, $\delta > 0$, $\delta \in \mathbb{R}$, obtained sufficient conditions for both the uniform and equi-asymptotic stability of zero solution of DIDE (3).

We should note that the first reference paper for this research is the paper of Becker and Burton [25]. Motivated by Becker and Burton [25], in this article, first, we are concerned with the nonlinear DIDE of second order with infinite delay:

$$x'' + a(t)F(t, x, x') + b(t)G(x, x') + c(t)H(x') + d(t)Q(x) + \int_{-\infty}^{t} \exp(-\,(t - s))U(s, x'(s))ds = E(t, x, x'), \quad (6)$$

where $x \in \mathbb{R}$, $\mathbb{R} = (-\infty, \infty)$, $x(t) = \phi(t)$ on $(-\infty, 0]$, $s,\ t \in \mathbb{R}$, $t \ge s$. We suppose that $F,\ E \in C(\mathbb{R}^+ \times \mathbb{R}^2, \mathbb{R})$, $\mathbb{R}^+ = [0, \infty)$, $G,\ U \in C(\mathbb{R} \times \mathbb{R}, \mathbb{R})$, $H,\ Q \in C(\mathbb{R}, \mathbb{R})$, $F(t, x, 0) = 0$, $G(x, 0) = 0$, $H(0) = 0$, $Q(0) = 0$, $U(s, 0) = 0$, $a,\ b,\ c \in C(\mathbb{R}^+,\ (0, \infty))$ and $d \in C^1(\mathbb{R}^+, \mathbb{R}^+)$, where $C(\mathbb{R}^+,\ (0, \infty))$ is the space of functions defined and continuous on $\mathbb{R}^+$, taking values in $(0, \infty)$, and $C^1(\mathbb{R}^+, \mathbb{R}^+)$ is the space of functions defined and continuously differentiable on $\mathbb{R}^+$, taking values in $\mathbb{R}^+$.

We convert DIDE (6) to the following system:

$$\begin{aligned} x' &= y, \\ y' &= -\ a(t)F(t, x, y) - b(t)G(x, y) - c(t)H(y) - d(t)Q(x) \\ &\quad - \int_{-\infty}^{t} \exp(-(t - s))U(s, y(s))ds + E(t, x, y). \end{aligned} \quad (7)$$

As for our next reference paper, Dishen [26] deals with the following linear system of DIDEs with infinite delay:

$$\begin{cases} x' = A(t)x + \int_{-\infty}^{t} C(t, s)ds + f(t), \\ y' = A(t)y + \int_{-\infty}^{t} C(t, s)ds + f(t), \end{cases} \quad (8)$$

and the author investigates the properties of this system such as the boundedness of solutions as well as the h-stability of this system. These properties of solutions are studied by using a phase space and the space C_h (which is somewhat different from the traditional phase space for infinite delay, in the sense of Hale and Kato [27]). In [26], the LKF

$$V(t, x_t, y_t) = |x(t) - y(t)| + \int_{-\infty}^{t}\int_{t}^{\infty} h(s - u)|x(s) - y(s)|duds. \quad (9)$$

In this article, secondly, motivated from the results of Dishen [26], instead of the linear system of DIDEs (8), we investigate the following non-linear system of DIDEs with infinite delay:

$$\begin{cases} x' = -A_1(t)f_1(x) + \int_{-\infty}^{t} C_1(t,s)g_1(x(s))ds + \ell_1(t,x), \\ y' = A_2(t)f_2(y) + \int_{-\infty}^{t} C_2(t,s)g_2(y(s))ds + \ell_2(t,y), \end{cases} \tag{10}$$

where $x,\ y,\ s,\ t \in \mathbb{R}$, $x(t) = \phi(t)$ on $(-\infty, 0]$, $s,\ t \in \mathbb{R}$, $t \geq s$. We suppose that $A_1,\ A_2 \in C(\mathbb{R}, (0,\infty))$, $C_1,\ C_2 \in C(\mathbb{R} \times \mathbb{R},\ \mathbb{R})$, $f_1,\ f_2,\ g_1,\ g_2 \in C(\mathbb{R}, \mathbb{R})$, $f_1(0) = 0$, $f_2(0) = 0$, $g_1(0) = 0$, $g_2(0) = 0$, $\ell_1,\ \ell_2 \in C(\mathbb{R} \times \mathbb{R},\ \mathbb{R})$, $\ell_1(t,0) = 0$ and $\ell_2(t,0) = 0$.

In this article, we construct new sufficient qualitative conditions on the global asymptotic stability, boundedness, and integrability of solutions for both the scalar nonlinear DIDE (6) of second order and the non-linear system of DIDEs (10) with infinite delays. Defining and then using these two new LKFs, the main results of this article are proved. In special cases of (6) and (10), three examples are given as numerical applications to illustrate and verify our results. We aim to provide some new contributions to qualitative theory of FDEs and some known results in the relevant literature.

Scientific interest in both of these kinds of FDEs with infinite delays is not purely theoretical. Indeed, there are numerous and very interesting real-world applications for these kinds of FDEs with infinite delays. For example, for various real-world applications of such scalar FDEs of second order and two-dimensional systems of FDEs with infinite delays, we refer the readers to look at the books of Fridman [32], Gopalsamy [33], Hale and Verduyn Lunel [34], Hsu [35], Kolmanovskii and Myshkis [36], Rihan [37], Smith [38], Yoshizawa [39], and the bibliographies therein.

The paper is organized as follows. Section 2 contains four new qualitative results on the global asymptotic stability, the integrability of solutions of (6) and (10), and a numerical application for the particular case of (6). In Section 3, we obtain two new theorems on the bounded solutions of (6) and (10), and in particular cases for (6) and (10), two examples are given as numerical applications of these results. In Section 4, we compare qualitative results of the present paper with known ones, as well as discuss some open problems for future research.

2. Stability and Integrability

As we know from the relevant literature according to the LKF approach, to investigate the qualitative behaviors of solutions of FDEs, it is needed to construct suitable LKFs for the problems under study. The construction of LKFs for linear and nonlinear FDE still remains as an open problem in literature by this time. There is no general method to construct LKFs. When an LKF is defined or constructed, the essential question is whether the LKF has to be positive definite and its time derivative along solutions of the considered FDE has to be negative semidefinite or negative definite such that the stability or asymptotic stability of the solutions can be guaranteed, respectively. In this section, we take into consideration these facts and define two new LKFs to achieve the aim of this paper.

We define two new LKFs, $\mathrm{L} = \mathrm{L}(t, x_t, y_t)$ and $\mathrm{W} = \mathrm{W}(t, x_t, y_t)$, which are given by:

$$\mathrm{L}(t, x_t, y_t) = d(t)\int_{0}^{x} Q(\xi)d\xi + \frac{1}{2}y^2 + \gamma\int_{-\infty}^{t} \exp(-(t-s))U^2(s, y(s))ds, \tag{11}$$

and

$$W(t,x_t,y_t) = |x(t)| + |y(t)| + \rho_1 \int_{-\infty}^{t}\int_{t}^{\infty} |C(u,s)|\,|g_1(x(s))|duds$$
$$+ \rho_2 \int_{-\infty}^{t}\int_{t}^{\infty} |C(u,s)|\,|g_2(x(s))|duds, \tag{12}$$

where γ , ρ_1 and ρ_2 are positive constants, and they will be chosen in the coming proofs. LKF (11) and LKF (12) are our basic tools in the proofs of the new results: Theorems 1, 3, 5 and Theorems 2, 4, 6, of this paper, respectively.

We now give the stability, integrability, and boundedness results of solutions of DIDE (6) and prove them using the LKF approach. At the first, we present the fundamental assumptions, called (A1)–(A4), of the main results of Theorems 1, 3, and 5 for DIDE (6):

(A1) There are positive constants F_0, G_0, H_0 and Q_0 such that:

$$a(t) \geq 1, b(t) \geq 1, c(t) \geq 1, d(t) \geq 1, d'(t) \geq 0, \forall t \in \mathbb{R}^+,$$
$$F(t,x,0) = 0, yF(t,x,y) \geq F_0 y^2, \forall t \in \mathbb{R}^+, \forall y \neq 0 \text{ as } x,\ y \in \mathbb{R},$$
$$G(x,0) = 0, yG(x,y) \geq G_0 y^2, \forall y \neq 0 \text{ as } x,\ y \in \mathbb{R},$$
$$H(0) = 0, yH(y \geq H_0 y^2, \forall y \neq 0 \text{ as } y \in \mathbb{R},$$
$$Q(0) = 0, xQ(x) \geq Q_0 x^2, \forall x \neq 0 \text{ as } x \in \mathbb{R}.$$

(A2) There is a positive constant U_0 such that:

$$U(t,0) = 0, U^2(t,y) \leq U_0^2 y^2, \forall t, y \in \mathbb{R}.$$

(A3) There are positive constants F_0, G_0, H_0 from (A1) and U_0 from (A2) and ℓ_0 such that:

$$F_0 a(t) + G_0 b(t) + H_0 c(t) - 2^{-1}U_0^2 - 2^{-1} \geq \ell_0, \forall t \in \mathbb{R}^+.$$

(A4) Let λ be a continuous function such that:

$$|E(t,x,y)| \leq |\lambda(t)|\,|y|, \forall t \in \mathbb{R}^+, \forall x,\ y \in \mathbb{R},$$

and there are positive constants F_0, G_0, H_0 from (A1) and U_0 from (A2) and $\hbar_0$ such that:

$$F_0 a(t) + G_0 b(t) + H_0 c(t) - |\lambda(t)| - 2^{-1}U_0^2 - 2^{-1} \geq \hbar_0, \forall t \in \mathbb{R}^+.$$

As for the next step, we introduce the basic assumptions, called (C1)–(C3) of the main results, Theorems 2, 4, and 6 for the system of DIDEs (10) with infinite delay:

(C1) There are positive constants f_{10}, g_{10}, f_{20}, g_{20} and functions α_0, $\beta_0 \in C(\mathbb{R}^+,\ (0,\infty))$ such that

$$f_1(0) = 0, xf_1(x) \geq f_{10}x^2, g_1(0) = 0, |g_1(x)| \leq g_{10}|x|, \forall x \neq 0 \text{ as } x \in \mathbb{R},$$
$$f_2(0) = 0, yf_2(y) \geq f_{20}y^2, g_2(0) = 0, |g_2(y)| \leq g_{20}|y|, \forall y \neq 0 \text{ as } y \in \mathbb{R},$$

$$\ell_1(t,0) = 0, |\ell_1(t,x)| \leq \alpha_0(t)|x|,\ \ell_2(t,0) = 0, |\ell_2(t,y)| \leq \beta_0(t)|y|,$$
$$\forall t \in \mathbb{R}^+, \forall x,\ y \neq 0 \text{ as } x,\ y \in \mathbb{R}.$$

(C2)

$$\int_t^\infty |C_1(u,t)|du < \infty, \int_t^\infty |C_2(u,t)|du < \infty.$$

(C3) There are positive constants f_{10}, g_{10}, f_{20}, and g_{20} from (C1) and h_0, h_1 such that:

$$f_{10}A_1(t) - \alpha_0(t) - g_{10}\int_t^\infty |C_1(u,t)|du \geq h_0$$

and

$$f_{20}A_2(t) - \beta_0(t) - g_{20}\int_t^\infty |C_2(u,t)|du \geq h_1, \forall t \in \mathbb{R}^+, \mathbb{R}^+ = [0,\infty).$$

First, we give the following new global asymptotic stability theorem of (6), which is equivalent to system (7).

Theorem 1. *If (A1)–(A3) hold and $E(t,x,y) \equiv 0$, then the trivial solution of (7) is global asymptotic stable.*

Proof. We consider the LKF $\mathrm{L}(t,x_t,y_t)$ of (11). Hence, it is obvious that

$$\mathrm{L}(t,x_t,y_t) = 0 \text{ iff } x = y = 0.$$

By virtue of (A1), we obtain:

$$\begin{aligned}\mathrm{L}(t,x_t,y_t) &= d(t)\int_0^x \frac{Q(\xi)}{\xi}\xi d\xi + \frac{1}{2}y^2 + \gamma\int_{-\infty}^t \exp(-(t-s))U^2(s,y(s))ds \\ &\geq d(t)\int_0^x \frac{Q(\xi)}{\xi}\xi d\xi + \frac{1}{2}y^2 \\ &\geq \frac{1}{2}Q_0x^2 + \frac{1}{2}y^2,\end{aligned}$$

i.e., we obtain:

$$\mathrm{L}(t,x_t,y_t) \geq \frac{1}{2}Q_0x^2 + \frac{1}{2}y^2. \tag{13}$$

By the time derivative of the LKF (11) along solutions of system (7), we obtain:

$$\begin{aligned}\frac{d}{dt}\mathrm{L}(t,x_t,y_t) =& -a(t)yF(t,x,y) - b(t)yG(x,y) - c(t)yH(y) \\ &- d'(t)\int_0^x H(\xi)d\xi - y\int_{-\infty}^t \exp(-(t-s))U(s,y(s))ds \\ &+ \gamma U^2(t,y) - \gamma\int_{-\infty}^t \exp(-(t-s))U^2(s,y(s))ds.\end{aligned}$$

Hence, according to (A1) and (A2), we have:

$$\frac{d}{dt}\mathrm{L}(t,x_t,y_t) \leq -F_0a(t)y^2 - G_0b(t)y^2 - H_0c(t)y^2 - d'(t)\int_0^x H(\xi)d\xi$$

$$
\begin{aligned}
&+\frac{1}{2}\int_{-\infty}^{t}\exp(-(t-s))\Big[y^2(t)+U^2(s,y(s))\Big]ds\\
&+\gamma U^2(t,y)-\gamma\int_{-\infty}^{t}\exp(-(t-s))U^2(s,y(s))ds\\
\le&-\Big[F_0a(t)+G_0b(t)+H_0c(t)-2^{-1}\Big]y^2\\
&+\frac{1}{2}\int_{-\infty}^{t}\exp(-(t-s))U^2(s,y(s))ds\\
&+\gamma U^2(t,y)-\gamma\int_{-\infty}^{t}\exp(-(t-s))U^2(s,y(s))ds\\
\le&-\Big[F_0a(t)+G_0b(t)+H_0c(t)-2^{-1}\Big]y^2\\
&+\frac{1}{2}\int_{-\infty}^{t}\exp(-(t-s))U^2(s,y(s))ds\\
&+\Big(\gamma U_0^2\Big)y^2-\gamma\int_{-\infty}^{t}\exp(-(t-s))U^2(s,y(s))ds. \qquad (14)
\end{aligned}
$$

Let $\gamma = \frac{1}{2}$. Then, according to (A3), we obtain from (14) that:

$$
\begin{aligned}
\frac{d}{dt}\mathrm{L}(t,x_t,y_t)\le&-\Big[F_0a(t)+G_0b(t)+H_0c(t)-2^{-1}U_0^2-2^{-1}\Big]y^2\\
\le&-\ (\ell_0)y^2\le 0. \qquad (15)
\end{aligned}
$$

The inequalities (13) and (15) together imply that the trivial solution of system (7) is stable, when $E(t,x,y)\equiv 0$. Next, $\frac{d}{dt}\mathrm{L}(t,x_t,y_t)=0$ if $y=0$. Since $y=\frac{dx}{dt}$, then $\frac{dx}{dt}=0$. Hence, integrating this term, we have $x(t)=\xi$, $\xi\in\mathbb{R}$, say $\xi\neq 0$. When we take $x(t)=\xi$ and $y(t)=0$ into system (7), we derive that $Q(\xi)=0$. It is obvious that $Q(\xi)=0$ if $\xi=0$. Consequently, the largest invariant set is $\{(0,0)\}$. Thus, the trivial solution of system (7) is global asymptotic stable. This is the end of proof. □

Second, we present the following new global asymptotic stable theorem of (10).

Theorem 2. *If (C1)–(C3) hold, then the trivial solution of (10) is global asymptotic stable.*

Proof. According to the LKF of (12), we derive that:

$$
\mathrm{W}(t,0,0)=0\ \text{ and } \mathrm{W}(t,x_t,y_t)\ge|x(t)|+|y(t)|.
$$

From the LKF (12) and system (10), by the virtue of (C1)–(C3), we obtain:

$$
\begin{aligned}
\frac{d}{dt}\mathrm{W}(t,x_t,y_t)\le&-A_1(t)|f_1(x)|+\int_{-\infty}^{t}|C_1(t,s)|\,|g_1(x(s))|ds+|\ell_1(t,x)|\\
&-A_2(t)|f_2(y)|+\int_{-\infty}^{t}|C_2(t,s)|\,|g_2(y(s))|ds+|\ell_2(t,y)|
\end{aligned}
$$

$$+\rho_1 \int_t^{\infty} |C_1(u,t)|\, |g_1(x(t))| du - \rho_1 \int_{-\infty}^{t} |C_1(t,s)||g_1(x(s))| ds$$
$$+\rho_2 \int_t^{\infty} |C_2(u,t)|\, |g_2(y(t))| du - \rho_2 \int_{-\infty}^{t} |C_2(t,s)|\, |g_2(y(s))| ds. \quad (16)$$

Let $\rho_1 = \rho_2 = 1$. Then, (16) implies that:

$$\begin{aligned}\frac{d}{dt}\mathrm{W}(t,x_t,y_t) \le &- A_1(t)|f_1(x)| + |\ell_1(t,x)| - A_2(t)|f_2(y)| + |\ell_2(t,y)| \\ &+ \int_t^{\infty} |C_1(u,t)|\, |g_1(x(t))| du + \int_t^{\infty} |C_2(u,t)|\, |g_2(y(t))| du. \end{aligned} \quad (17)$$

According to (C1)–(C3), from (17), we obtain:

$$\begin{aligned}\frac{d}{dt}\mathrm{W}(t,x_t,y_t) \le &- f_{10}A_1(t)|x| + \alpha_0(t)|x| - f_{20}A_2(t)|y| + \beta_0(t)|y| \\ &+ g_{10}|x| \int_t^{\infty} |C_1(u,t)| du + g_{20}|y| \int_t^{\infty} |C_2(u,t)| du \\ = &- \left[f_{10}A_1(t) - \alpha_0(t) - g_{10} \int_t^{\infty} |C_1(u,t)| du \right] |x| \\ &- \left[f_{20}A_2(t) - \beta_0(t) - g_{20} \int_t^{\infty} |C_2(u,t)| du \right] |y| \\ \le &- h_0|x| - h_1|y| < 0, (x \ne 0,\ y \ne 0). \end{aligned}$$

Hence, we arrive at the end of the proof of Theorem 2. □

We now present the following new integrability theorem of (6), which is equivalent to system (7).

Theorem 3. *If (A1)–(A3) hold and $E(t,x,y) \equiv 0$, then the square derivatives of solutions of (7) are integrable.*

Proof. According to (A1)–(A3) and $E(t,x,y) \equiv 0$, we obtain:

$$\frac{d}{dt}\mathrm{L}(t,x_t,y_t) \le -(\ell_0)y^2 \le 0.$$

Taking into account that the LKF $\mathrm{L}(t,x_t,y_t)$ is decreasing and then integrating the inequality above, we obtain:

$$\int_0^{\infty} y^2(\eta) d\eta < +\infty.$$

Thus, this result verifies the idea of Theorem 3. Here, the integrability concept is in the sense of Lebesgue. □

We now introduce the following new integrability result of (10).

Theorem 4. *If (C1)–(C3) hold, then the solutions of (10) are integrable.*

Proof. By virtue of (C1)–(C3), we have:

$$\frac{d}{dt}\mathrm{W}(t, x_t, y_t) \leq -h_0|x| - h_1|y| \leq 0.$$

This relation shows that the LKF $\mathrm{W}(t, x_t, y_t)$ is decreasing. According to this information, it follows that $\mathrm{W}(t, x_t, y_t) \leq \mathrm{W}(0, x_0, y_0) = \mathrm{W}_0$, $\mathrm{W}_0 > 0$, $\mathrm{W}_0 \in \mathbb{R}$. Integrating,

$$h_0 \int_0^t |x(s)|ds + h_1 \int_0^t |y(s)|ds \leq \mathrm{W}(0, x_0, y_0) - \mathrm{W}(t, x_t, y_t) \leq \mathrm{W}(0, x_0, y_0) = \mathrm{W}_0.$$

Consequently, we obtain:

$$\int_0^\infty |x(s)|ds \leq h_0^{-1}\mathrm{W}_0 < \infty \text{ and } \int_0^\infty |y(s)|ds \leq h_1^{-1}\mathrm{W}_0 < \infty,$$

where the integrability concept is in the sense of Lebesgue. This is the end of the proof. □

We now give an example as numerical applications of the global asymptotic stability and integrability theorems, Theorems 1 and 3.

Example 1. *For the case $E(t, x, y) \equiv 0$ of (7), we take into consideration the following nonlinear DIDE of second order with infinite delay:*

$$\begin{aligned} x'' &+ (2 - \exp(-t))\Big(25 + \exp(-t^2 - x^2 - (x')^2)\Big)x' + \left(1 + \frac{1}{1+t^6}\right)\Big(16 + x^4 + (x')^2\Big)x' \\ &+ (1 + \exp(-t))\Big(4 + (x')^4\Big)x' + \left(4 - \frac{3}{1 + 2\exp(t)}\right)x \\ &+ \int_{-\infty}^{t} \exp(-(t-s))\frac{2x'(s)}{\sqrt{1 + s^2 + (x'(s))^2}}ds = 0. \end{aligned} \tag{18}$$

Then, the DIDE (18) is converted to the following system:

$$\begin{aligned} x' =& y, \\ y' =& -(2 - \exp(-t))\Big(25 + \exp(-t^2 - x^2 - y^2)\Big)y - \left(1 + \frac{1}{1+t^6}\right)\Big(16 + x^4 + y^2\Big)y \\ &- (1 + \exp(-t))\Big(4 + y^4\Big)y - \left(4 - \frac{3}{1 + 2\exp(t)}\right)x \\ &- \int_{-\infty}^{t} \exp(-(t-s))\frac{2y(s)}{\sqrt{1 + s^2 + y^2(s)}}ds. \end{aligned} \tag{19}$$

From the comparison of (19) and (7) and some elementary calculations, we have the following data:

$$a(t) = 2 - \exp(-t) \geq 1, t \geq 0,$$

$$b(t) = 1 + \frac{1}{1+t^6} \geq 1,$$

$$c(t) = 1 + \exp(-t) \geq 1, t \geq 0,$$

$$d(t) = 4 - \frac{3}{1 + 2\exp(t)} \geq 1,$$

$$d'(t) = \frac{6\exp(t)}{(1+2\exp(t))^2} \geq 0;$$

$$F(t,x,y) = \Big(25 + \exp(-t^2 - x^2 - y^2)\Big)y,$$

$$F(t,x,0) = 0,$$

$$yF(t,x,y) = y^2\Big(25 + \exp(-t^2 - x^2 - y^2)\Big) \geq 25y^2,\ F_0 = 25;$$

$$G(x,y) = \Big(16 + x^4 + y^2\Big)y, G(x,0) = 0,$$

$$yG(x,y) = y^2\Big(16 + x^4 + y^2\Big) \geq 16y^2,\ G_0 = 16;$$

$$H(y) = \Big(4 + y^4)\Big)y, H(0) = 0,$$

$$yH(y) = y^2\Big(4 + y^4)\Big) \geq 4y^2,\ H_0 = 4;$$

$$Q(x) = x, Q(0) = 0,$$

$$xQ(x) = x^2,\ Q_0 = 1;$$

$$U(t,y) = \frac{2y}{\sqrt{1+t^2+y^2}},$$

$$U(t,0) = 0, U^2(t,y) = \frac{4y^2}{1+t^2+y^2} \leq 4y^2,\ U_0^2 = 4;$$

$$F_0 a(t) + G_0 b(t) + H_0 c(t) - 2^{-1}U_0^2 - 2^{-1}$$

$$= 25 \times (2 - \exp(-t)) + 16 \times \left(1 + \frac{1}{1+t^6}\right)$$

$$+4 \times (1 + \exp(-t)) - \frac{5}{2} \geq \frac{85}{2} = \ell_0 > 0.$$

According to the data above, (A1)–(A3) of Theorems 1 and 3 hold. Thus, the trivial solution of DIDE (18) is globally asymptotically stable and the square of the derivatives of its solutions are integrable.

The next section studies the boundedness of solutions of (6) and (10).

3. Boundedness

In this section, we prove two new theorems, Theorems 5 and 6, on the bounded solutions of DIDE (6), which is equivalent to system (7), and system of DIDEs (10). In particular cases of (6) and (10), two examples are given as numerical applications of Theorems 5 and 6, respectively.

The following Theorem 5 investigates the boundedness of solutions of system (7).

Theorem 5. *If (A1), (A2), and (A4) hold, then the solutions of system (7) and their derivatives are bounded.*

Proof. Clearly, by virtue of (A1), (A2), and (A4), we obtain the inequality of (13) and the following result:

$$\begin{aligned}\frac{d}{dt}\mathrm{L}(t,x_t,y_t) \leq -& \left[F_0a(t)+G_0b(t)+H_0c(t)-2^{-1}U_0^2-2^{-1}\right]y^2+yE(t,x,y)\\ \leq -& \left[F_0a(t)+G_0b(t)+H_0c(t)-2^{-1}U_0^2-2^{-1}\right]y^2+|y|\,|E(t,x,y)|\\ \leq -& \left[F_0a(t)+G_0b(t)+H_0c(t)-|\lambda(t)|-2^{-1}U_0^2-2^{-1}\right]y^2\\ \leq -& (\hbar_0)y^2\leq 0. \end{aligned} \tag{20}$$

According to (20), the LKF $\mathrm{L}(t,x_t,y_t)$ is decreasing, i.e.,

$$\mathrm{L}(t,x_t,y_t)\leq \mathrm{L}(t_0,x_{t_0},y_{t_0}),\forall t\geq t_0.$$

Hence, from (13) and (20), we obtain:

$$\frac{1}{2}Q_0x^2+\frac{1}{2}y^2\leq \mathrm{L}(t,x_t,y_t)\leq \mathrm{L}(t_0,x_{t_0},y_{t_0}),$$

where $\mathrm{L}(t_0,x_{t_0},y_{t_0})$ is a positive constant. This inequality verifies that the solutions of system (7) are bounded. □

The following Theorem 6 investigates the boundedness of solutions of system (10).

Theorem 6. *If (C1)–(C3) hold, then the solutions of system (10) are bounded.*

Proof. It is noted from Theorem 2 that

$$|x(t)|+|y(t)|\leq \mathrm{W}(t,x_t,y_t).$$

Next, by (C1)–(C3) of Theorem 6, we obtain that:

$$\frac{d}{dt}\mathrm{W}(t,x_t,y_t)\leq 0.$$

As a consequence of both of the results above, we can conclude that:

$$|x(t)|+|y(t)|\leq \mathrm{W}(t,x_t,y_t)\leq \mathrm{W}(t_0,x_{t_0},y_{t_0})\equiv \mathrm{W}_0>0,\mathrm{W}_0\in\mathbb{R},\ t\geq t_0.$$

Consequently,

$$|x(t)|\leq \mathrm{W}_0 \text{ and } |y(t)|\leq \mathrm{W}_0,\forall t\geq t_0.$$

Thus, clearly, the solutions of system (10) are bounded. □

We now give two examples as numerical applications of Theorems 2 and 5 and Theorems 4 and 6, respectively.

Example 2. *For the case $E(t,x,y)\neq 0$, we take into consideration the following nonlinear DIDE of second order with infinite delay:*

$$
\begin{aligned}
&x'' + (2-\exp(-t))\Big(25+\exp(-t^2-x^2-(x')^2)\Big)x' + \left(1+\frac{1}{1+t^6}\right)\Big(16+x^4+(x')^2\Big)x' \\
&\quad + (1+\exp(-t))\Big(4+(x')^4\Big)x' + \left(4-\frac{3}{1+2\exp(t)}\right)x \\
&\quad + \int_{-\infty}^{t} \exp(-(t-s))\frac{2x'(s)}{\sqrt{1+s^2+(x'(s))^2}}ds. \\
&= \frac{2x'}{1+t^2+x^2+(x')^2}.
\end{aligned} \tag{21}
$$

Then, the DIDE (21) is converted to the following system:

$$
\begin{aligned}
x' =& y, \\
y' =& -(2-\exp(-t))\Big(25+\exp(-t^2-x^2-y^2)\Big)y - \left(1+\frac{1}{1+t^6}\right)\Big(16+x^4+y^2\Big)y \\
&- (1+\exp(-t))\Big(4+y^4\Big)y - \left(4-\frac{3}{1+2\exp(t)}\right)x \\
&- \int_{-\infty}^{t} \exp(-(t-s))\frac{2y(s)}{\sqrt{1+s^2+y^2(s)}}ds + \frac{2y}{1+t^2+x^2+y^2}.
\end{aligned} \tag{22}
$$

As for the next step, the discussions and the estimates relation to the functions $a(t)$, $b(t)$, $c(t)$, $d(t)$, F, G, H, Q, and U of Example 1 hold for Example 2, too. Hence, the former previous discussions will not be given here for these functions once again. As for the next step for the function E of Example 2, it is given by:

$$
E(t,x,y) = \frac{2y}{1+t^2+x^2+y^2}.
$$

Hence, we derive:

$$
|E(t,x,y)| = \frac{2|y|}{1+t^2+x^2+y^2} \le \frac{2|y|}{1+t^2} = \lambda(t)|y|,
$$

where

$$
\begin{aligned}
\lambda(t) =& \frac{2}{1+t^2}, t \ge 0. \\
F_0a(t) &+ G_0b(t) + H_0c(t) - |\lambda(t)| - 2^{-1}U_0^2 - 2^{-1} \\
&= 25\times(2-\exp(-t)) + 16\times\left(1+\frac{1}{1+t^6}\right) \\
&\quad + 4\times(1+\exp(-t)) - \frac{2}{1+t^2} - \frac{5}{2} \ge \frac{81}{2} = \hbar_0 > 0.
\end{aligned}
$$

Then, the conditions of Theorem 5 hold. Thus, the solutions of system (22) and their derivatives are bounded.

Remark 1. *It is seen from Theorems 1–3 that we do need the differentiability of the functions $a(t)$, F, $b(t)$, G, $c(t)$, H, Q, U, and E. This case is an advantage for the results of this paper, Theorems 1–3, and leads to a weaker condition for these results.*

Example 3. *We consider the following nonlinear system of DIDEs with infinite delay, which is included by (10):*

$$\begin{cases} x' = -(1+\exp(t^2))\left(24\pi x + \frac{x}{1+\exp(x^2)}\right) + \int\limits_{-\infty}^{t} \frac{1}{1+t^2+s^2}\frac{x(s)}{1+x^4(s)}ds \\ \qquad + \frac{x}{(1+t^6)(1+\exp(x^2))}, \\ y' = -(1+\exp(t^4))\left(24\pi y + \frac{y}{1+\exp(y^2)}\right) + \int\limits_{-\infty}^{t} \frac{1}{1+4t^2+4s^2}\frac{3y(s)}{1+y^4(s)}ds \\ \qquad + \frac{y}{(1+t^6)(1+\exp(y^2))}. \end{cases} \tag{23}$$

From the comparison of systems (23) and (10) and some elementary calculations, we have the following data:

$$A_1(t) = 1+\exp(t^2),$$

$$A_2(t) = 1+\exp(t^4),$$

$$f_1(x) = 24\pi x + \frac{x}{1+\exp(x^2)},$$

$$f_1(0) = 0, x f_1(x) = 24\pi x^2 + \frac{x^2}{1+\exp(x^2)} \geq 24\pi x^2,\ f_{10} = 24\pi;$$

$$f_2(y) = 24\pi y + \frac{y}{1+\exp(y^2)},$$

$$f_2(0) = 0, y f_2(y) = 24\pi y^2 + \frac{y^2}{1+\exp(y^2)} \geq 24\pi y^2,\ f_{20} = 24\pi;$$

$$C_1(t,s) = \frac{1}{1+t^2+s^2},$$

$$\int\limits_t^\infty C_1(u,t)du = \int\limits_t^\infty \frac{1}{1+u^2+t^2}du \leq \int\limits_t^\infty \frac{1}{1+u^2}du$$

$$\leq \int\limits_0^\infty \frac{1}{1+u^2}du = \frac{\pi}{2} < \infty;$$

$$C_2(t,s) = \frac{1}{1+4t^2+4s^2},$$

$$\int\limits_t^\infty C_2(u,t)du = \int\limits_t^\infty \frac{1}{1+4u^2+4t^2}du \leq \int\limits_t^\infty \frac{1}{1+4u^2}du$$

$$\leq \int\limits_0^\infty \frac{1}{1+4u^2}du = \frac{\pi}{4} < \infty;$$

$$g_1(x) = \frac{2x}{1+x^4}, g_1(0) = 0, |g_1(x)| \leq 2|x|,\ g_{10} = 2;$$

$$g_2(y) = \frac{3y}{1+y^4}, g_2(0) = 0, |g_2(y)| \leq 3|y|,\ g_{20} = 3;$$

$$\ell_1(t,x) = \frac{x}{(1+t^6)(1+\exp(x^2))}, \ell_1(t,0) = 0,$$

$$|\ell_1(t,x)| \leq \frac{|x|}{1+t^6},\ \alpha_0(t) = \frac{1}{1+t^6};$$

$$\ell_2(t,y) = \frac{y}{(1+t^6)(1+\exp(y^2))}, \ell_2(t,0) = 0,$$

$$|\ell_2(t,y)| \le \frac{|y|}{1+t^6}, \beta_0(t) = \frac{1}{1+t^6};$$

$$f_{10}A_1(t) - \alpha_0(t) - g_{10}\int_t^\infty |C_1(u,t)|du = 24\pi\Big(1+\exp(t^2)\Big) - \frac{1}{1+t^6} - \pi$$

$$> 23\pi = h_0;$$

$$f_{20}A_2(t) - \beta_0(t) - g_{20}\int_t^\infty |C_2(u,t)|du = 24\pi\Big(1+\exp(t^4)\Big) - \frac{1}{1+t^6} - \frac{3\pi}{4}$$

$$> 23\pi = h_1.$$

According the data above, (C1)–(C3) of Theorems 2, 4, and 6 hold. Thus, the trivial solution of the non-linear system of DIDEs (23) is globally asymptotically stable, and the solutions of (23) are also integrable and bounded.

4. Conclusions and Discussion

In this part, we compare Theorems 1–6 of this paper with some articles in the references of this paper.

(1) DIDEs (1) and (3) are particular cases of our DIDE (6). Next, our LKF (11) is different from the LKFs (2), (4), and (5). This is our first contribution.

(2) The system of IDEs (8) with infinite delay is linear. Our system of IDEs (10) with infinite delay is nonlinear. The system of IDEs (10) with infinite delay generalizes and improves the linear system (8). Next, our LKF (12) is different from the LKF (9). This is our second contribution.

(3) The uniform stability of solutions of DIDE (1) and the uniform and equi-asymptotic stability of the zero solution of DIDE (3) are investigated using the LKF method. In our paper, we investigate the global asymptotic stability, boundedness, and integrability of solutions of DIDE (6) using the LKF method. As it is seen our results, we established the different qualitative concepts of our solutions. Next, in the past literature, some stability concepts are discussed. In our paper, in addition to the global asymptotic stability concept, we also study boundedness and integrability of solutions, which are different from the uniform and equi-asymptotic stability concepts. These are our third contributions.

(4) h-uniformly stability, h-uniformly asymptotically stability, and h-bounded solutions of the system of IDEs (8) with infinite delay are discussed by using a phase space and the LKF method. In this paper, the global asymptotic stability of zero solution, boundedness, and integrability of solutions of (10) are discussed by the LKF method. These qualitative concepts are a bit different from the h-uniformly stability, h-uniformly asymptotically stability, and h-bounded solutions because of the defined norm. These are our next contributions.

(5) As numerical applications of the results of this paper, we provide three examples, Examples 1–3, to illustrate the applications of Theorems 1–6 of this paper. Examples 1–3 are also new contributions of this paper.

(6) To the best of our knowledge, the scalar nonlinear DIDE (6) of second order and the non-linear system of IDEs (10) with infinite delays are new mathematical models. Qualitative behaviors of solutions of these mathematical models have not been investigated in the relevant literature as of yete. Hence, the new results of this paper, Theorems 1–6, and the illustrative Examples 1–3 are complementary outcomes of this paper to the theory of FDEs.

As some open problems for future researches, we would like to suggest that qualitative properties of fractional forms of the scalar nonlinear DIDE (6) of second order with infinite delay and the non-linear system of DIDEs (10) with infinite delay can be investigated.

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