mathematics

MDPI

Article

Fisher-like Metrics Associated with ϕ-Deformed (Naudts) Entropies

Cristina-Liliana Pripoae [1], Iulia-Elena Hirica [2], Gabriel-Teodor Pripoae [2,*] and Vasile Preda [2,3,4]

[1] Department of Applied Mathematics, The Bucharest University of Economic Studies, Piata Romana 6, RO-010374 Bucharest, Romania
[2] Faculty of Mathematics and Computer Science, University of Bucharest, Academiei 14, RO-010014 Bucharest, Romania
[3] "Gheorghe Mihoc-Caius Iacob" Institute of Mathematical Statistics and Applied Mathematics of Romanian Academy, 2. Calea 13 Septembrie, nr.13, Sect. 5, RO-050711 Bucharest, Romania
[4] "Costin C. Kiritescu" National Institute of Economic Research of Romanian Academy, 3. Calea 13 Septembrie, nr.13, Sect. 5, RO-050711 Bucharest, Romania
* Correspondence: gpripoae@yahoo.com or gpripoae@fmi.unibuc.ro

Abstract: The paper defines and studies new semi-Riemannian generalized Fisher metrics and Fisher-like metrics, associated with entropies and divergences. Examples of seven such families are provided, based on exponential PDFs. The particular case when the basic entropy is a ϕ-deformed one, in the sense of Naudts, is investigated in detail, with emphasis on the variation of the emergent scalar curvatures. Moreover, the paper highlights the impact on these geometries determined by the addition of some group logarithms.

Keywords: ϕ-deformed (Naudts) entropy; divergence; relative group entropy; generalized Fisher metric; Fisher-like metric; MaxEnt problem

MSC: 53B12; 22E70; 94A17; 53B20

Citation: Pripoae, C.-L.; Hirica, I.-E.; Pripoae, G.-T.; Preda, V. Fisher-like Metrics Associated with ϕ-Deformed (Naudts) Entropies. *Mathematics* **2022**, *10*, 4311. https://doi.org/10.3390/math10224311

Academic Editor: Sorin V. Sabau

Received: 12 October 2022
Accepted: 15 November 2022
Published: 17 November 2022

1. Introduction

1.1. History

Entropy is a a very versatile measure of order (or of chaos). In the last few several decades, the growing needs of modeling for stochastic phenomena contributed to the apparition of many new different families of entropy functionals, with increasing levels of generality, reliability and applicability [1–19]. One of the recent interesting new directions of study uses the relative group entropies, based on group logarithms (see [20,21] and references therein).

The geometrization method, a powerful tool in modelization, was applied in the investigation of some statistical relevant parameters sets, beginning with the work of the pioneers: Fisher, Rao, Efron and Amari [12,22,23]. This bridge allows the use of the differential geometric machinery to understand the local and the global behavior of statistical objects.

In particular, the Fisher (semi-Riemannian) metrics correspond to the Fisher Information matrices. Their invariants, especially those tensor fields expressing different kinds of curvature properties, are used in the parameters estimation theory as control tools. For example, the scalar curvature function measures the average statistical uncertainty of a density matrix [12,20,24].

Consider a statistical model, governed by a given entropy, and two or more fixed parameterized probability density functions (PDFs) within it. Various divergences ("distance-like functionals") can be defined in this framework, able to detect how these PDFs relate to each other. A kind of infinitesimal variation of such divergences, w.r.t. the parameters, may provide interpretations for some Fisher-like metrics. Several types of divergences are used, including the Kullback–Leibler and the Bregman ones. For recent viewpoints upon divergences, see [14,25,26].

In 2002, Naudts introduced ([27]) the "ϕ-deformed entropy", via a positive strictly increasing function ϕ, which plays the role of a "generalized logarithm". (We shall call it "ϕ-deformed (Naudts) entropy" and not simply "ϕ-deformed entropy", in order to avoid confusion and to distinguish it from other "deformed" entropies, all originating—sooner or later—from the Boltzman–Gibbs–Shannon (BGS) germ). This new entropy extends (with some technical precautions) the Tsallis and the Kaniadakis entropies, among other ones. Using it, new Fisher metrics were defined [27–29], ranging from simple ones to some more "baroque" constructions. Their applicability covers a wide area, from Physics (the starting point) to Information geometry [29–32].

Using a ϕ-deformed (Naudts) exponential family of PDFs, Matsuzoe et al. [33] investigated the geometry of statistical manifolds derived from a sequence of escort expectations.

Korbel et al. [30] studied properties of the Fisher metrics associated with the ϕ-deformed (Naudts) entropies, in the case of exponential-type PDFs. Particular choices of the function ϕ provided examples based on (c,d)-entropies. Dealing with the MaxEnt problem, they use the Fisher information of the ϕ-deformed (Naudts) exponential entropies, in order to reveal a duality between the cases with linear constraints and those based on escort constraints.

Inspired by these previous works, we believe that a systematic study of semi-Riemannian metrics, canonically associated with the ϕ-deformed (Naudts) entropy, is necessary and might provide useful statistical tools in the future. Our paper suggests a method of research, which combines the beaten path with some new speculative ideas.

1.2. The Content of the Paper

In Section 2, we recall (in a creative manner) the notations and fix the conventions concerning (the different variants of) entropy and divergence; we closely follow [34]. We make some comments about the place of the Naudts' ϕ-deformed entropy in the "Universe" of generalized entropies. We recall here some other examples of remarkable entropies (Tsallis, Kaniadakis, Sharma–Taneja–Mittal). Our main new idea is the distinction we made between the "quotient" divergence and the "difference" divergence, in the context of generalized logarithms; in the particular case of the Neperian logarithm, these two notions coincide, but in other cases (such that of the ϕ-deformed (Naudts) entropy) they are distinct.

In Section 3, we fix the needed notions concerning the generalized Fisher-like metrics associated with the entropies and to the relative (group) entropies, following (especially) [20]. Following the previous distinction we made in Section 2, between the two kinds of divergences, we introduce two generalized Fisher-like metrics (GFM1 and GFM2), which coincide in the classical setting with the Fisher metric. Three other Fisher-like metrics are defined, in a formal way, as auxiliary (but eventually useful) by-products of the former ones.

In Section 4, we determine the semi-Riemannian geometries of the generalized Fisher-like metrics, associated with group relative entropies based on ϕ-deformed (Naudts) entropies and divergences. Their coefficients are expressed in terms of both PDFs and of the ϕ-deformed logarithm and may depend on a group logarithm too.

In the next section, we give seven families of examples of such metrics, for the case when the involved PDFs are exponential. The scalar curvatures functions are computed, and their variation is studied.

In Section 6, we define and solve the MaxEnt problem based on the ϕ-deformed (Naudts) entropy, for univariate PDFs, and we generalize some thermodynamic relations.

1.3. Conventions

Implicitly, the integrals are supposed to be correctly defined and to commute with their derivatives. "Differentiable" means "smooth", even if, sometimes, a weaker assumption would be enough. When a symmetric matrix is called "a (semi-Riemannian) metric", we assume, implicitly that it is non-degenerate; the positive definiteness is not assumed, in general, unless otherwise stated.

2. Entropies and Divergences—A Breviary

We consider a real valued random variable x on a domain $X \subset \mathbb{R}^m$. We denote by $\rho = \rho(x)$ a fixed probability density function (PDF); then, $\rho(x) \geq 0$ and $\int_X \rho(x)dx = 1$. We fix a real valued differentiable function φ, as a "controlling tool". In this setting, the generalized (normalized) entropy is

$$H[\rho] = -\int_X \rho(x)\varphi(\rho(x))dx. \tag{1}$$

We shall use a similar notation for other entropy-like functionals too. In the literature, the avatars of the "generalized logarithm" φ are subject to additional restrictions, imposed through applications inspired axioms.

Let $F : [0, \infty) \times [0, \infty) \to \mathbb{R}$ a smooth function and σ an additional fixed PDF. We define

$$D(\rho, \sigma) := \int_X F(\rho(x), \sigma(x))dx. \tag{2}$$

We suppose that $D(\rho, \sigma) \geq 0$ and $D(\rho, \sigma) = 0$ if and only if $\rho = \sigma$. The number $D(\rho, \sigma)$ is called the (generalized) divergence between ρ and σ and measures to what extent σ influences ρ. Sometimes, additional properties of the divergence function are added, axiomatically.

Example 1. *With the previous notations, we recall some well-known examples of entropies ([35–37]).*

(i) In the particular case when $\varphi(y) := log(y)$, from Formula (1), we obtain the Boltzmann–Gibbs–Shannon (BGS) entropy.

(ii) Consider a fixed parameter $q \in \mathbb{R} \backslash \{1\}$. The Tsallis q-logarithm

$$\varphi^T_{\{q\}}(y) := \frac{y^{1-q} - 1}{1 - q} \tag{3}$$

provides a Tsallis entropy. Usually, for $\varphi^T_{\{q\}}$, we use the notation $log^T_{\{q\}}$. When $q \to 1$, the BGS entropy is recovered.

(iii) Let us fix $k \in [-1, 1] \backslash \{0\}$. The Kaniadakis k-logarithm

$$\varphi^K_{\{k\}}(y) := \frac{y^k - y^{-k}}{2k} \tag{4}$$

defines a Kaniadakis entropy (named also k-deformed entropy). Usually, $\varphi^K_{\{k\}}$ is denoted $log^K_{\{k\}}$. When $k \to 0$, we recover again the BGS entropy.

(iv) Fix two real parameters k and r. The Sharma–Taneja–Mittal (k, r)-logarithm

$$\varphi^{STM}_{\{(k,r)\}}(y) := y^r \cdot \frac{y^k - y^{-k}}{2k}$$

provides a Sharma–Taneja–Mittal (STM) entropy (also named (k, r)-deformed entropy). Instead of $\varphi^{STM}_{\{(k,r)\}}$, we shall denote $log^{STM}_{\{(k,r)\}}$. The Kaniadakis k- logarithm and the Tsallis q-logarithm are recovered as particular cases, for $r = 0$ and for $r = \pm \mid k \mid$, respectively. When $(k, r) \to (0, 0)$, we recover the BGS entropy. Sometimes, additional restrictions are imposed on the domain of the parameters, required by convergence conditions imposed on some integrals (see [38–40] for details).

(v) ([27]) Let $\phi : (0, \infty) \to \mathbb{R}$ a positive, differentiable, strictly-increasing function. (Sometimes, in the literature, "non-decreasing" is required, instead of the "strictly-increasing" condition). Define the ϕ-deformed (Naudts) logarithm

$$log_{\phi}^{N}(y) := \int_{1}^{y} \frac{1}{\phi(z)} dz. \tag{5}$$

The function $\varphi_{\phi}^{N} := log_{\phi}^{N}$ defines the ϕ-deformed (Naudts) entropy. The previous formula may also be read "backwards":

$$\phi(y) = (\frac{\partial}{\partial_x} \varphi_{\phi}^{N}(y))^{-1}. \tag{6}$$

Moreover, given an arbitrary "generalized logarithm" φ as in (1), Formula (6) always provides a differentiable function ϕ; if it is positive and strictly-increasing, we expressed φ like a ϕ-deformed (Naudts) logarithm. Sometimes, this procedure works for some restrictions of the involved parameters only. For example, the preceding four entropies are recovered as particular cases of ϕ-deformed (Naudts) entropies, as follows: BGS for $\phi := id$; Tsallis for $\phi(y) := y^q$ with the restrictions $q > 0$ and $y \in (0, \infty)$; Kaniadakis $\phi(y) := 2(y^{k-1} + y^{-k-1})^{-1}$ with the additional restriction

$$y^{2k} < \frac{k+1}{k-1},$$

for $y \in (0, \infty)$; STM for

$$\phi(y) := 2k[(k+r)y^{k+r-1} + (k-r)y^{r-k-1}]^{-1},$$

with the additional restriction

$$y^{2k} < \frac{(r-k)(r-k-1)}{(r+k)(r+k-1)},$$

for $y \in (0, \infty)$. These additional restrictions are imposed in order ϕ to be strictly-increasing.

(vi) Let $G = G(t)$ be a formal group logarithm, which is a differentiable real valued function with some special algebraic properties, inspired from the formal series linking Lie groups to Lie algebras. More precisely,

$$G(t) := \sum_{i=0}^{\infty} c_i \frac{t^{i+1}}{i+1},$$

where $c_0 = 1$ and $c_i \in \mathbb{Q}$. Its inverse is

$$F(s) := \sum_{i=0}^{\infty} \gamma_i \frac{s^{i+1}}{i+1},$$

where $\gamma_i \in \mathbb{Q}$, $\gamma_0 = 1$, $\gamma_1 = -c_1$, $\gamma_2 = \frac{3}{2}c_1^2 - c_2$ and so on. (We refer to [20,21,41] for details about these functions). The simplest example is $G(t) = t$.

We define the generalized group entropy functional (GGEF) associated with (1) by

$$S_G(\rho) := \int_X \rho(x) G(\varphi \circ \rho(x)) dx. \tag{7}$$

In particular, for $\varphi := -log$, we recover the well-known group entropy functional *([20,41]) associated with (1)*

$$S_G(\rho) := \int_X \rho(x) G(log \rho(x)^{-1}) dx. \tag{8}$$

Similar GGEFs can be provided by replacing the Neperian logarithm by other "generalized" logarithms (e.g., Tsallis, Kaniadakis, STM, etc). In Section 3, we shall introduce the geometries associated with the GGEF, based on ϕ-deformed (Naudts) entropies. Accordingly, we shall use the generalized logarithm $\log_{\phi}^{N}$ from (5).

Example 2. *With the previous notations, we recall some well-known examples of divergences.*

(i) An important particular case is the generalized (quotient) relative entropy (a.k.a. generalized divergence) between ρ *and* σ *(see [34,42])*

$$\tilde{D}(\rho \parallel \sigma) := \int_X \rho(x)\varphi(\frac{\rho(x)}{\sigma(x)})dx. \quad (9)$$

The function $F(z,y) := z\varphi(\frac{z}{y})$*. We accept (formally) that* $0 \cdot \varphi(\frac{0}{\sigma}) = 0, \quad \rho \cdot \varphi(\frac{\rho}{0}) = 0$ *and* $\varphi(1) = 0$*. In particular, when* $\varphi := \log$*, we recover the Kullback–Leibler divergence ([20]).*

Another particular case considers $f : [0,\infty) \to (-\infty,\infty]$ *to be a convex function, with* $f(1) = 0$ *and* $f(0) = \lim_{t\to 0^+} f(t)$*. For* $\varphi(y) := \frac{1}{y}f(y)$*, we recover the* f*-divergence ([43] and references therein). The slightly more general notion of* (f,Γ)*-divergence (see [44]) may be recovered in a similar way.*

(ii) In a similar way, we define the generalized (difference) relative entropy between ρ *and* σ*, as*

$$D(\rho \parallel \sigma) := \int_X \rho(x)[\varphi(\rho(x)) - \varphi(\sigma(x))]dx. \quad (10)$$

The function $F(z,y) := z[\varphi(z) - \varphi(y)]$*. In particular, when* $\varphi := \log$*,* $\tilde{D}$ *coincides with* D *and we recover the Kullback–Leibler divergence, as in (i). When* $\varphi := \log_\phi^N$*, the divergence* D *was considered in [27]; we mention that, in this case,* $\tilde{D}$ *does not coincide with* D*.*

In general, a necessary and sufficient condition on φ*,* ρ *and* σ*, in order that* $D = \tilde{D}$*, is the vanishing of the mean function* $\varphi(\frac{\rho}{\sigma}) - \varphi(\rho) + \varphi(\sigma)$*. A sufficient (but quite strong) condition is provided by the functional equation* $\varphi(\frac{\rho}{\sigma}) = \varphi(\rho) - \varphi(\sigma)$*.*

(iii) In the hypothesis of Example 1 (vi), we can define generalized divergences as relative group entropies, which combine the formal group logarithm G*, the* φ*-likelihood function and the previous quotient or difference operation upon two PDFs. For example, the analogue of (10) is*

$$D_G(\rho \parallel \sigma) := \int_X \rho(x) \cdot G\Big(\varphi(\rho(x)) - \varphi(\sigma(x))\Big)dx.$$

(iv) Consider two fixed PDFs ρ_1 *and* ρ_2*. Denote* $\psi : \mathbb{R} \to \mathbb{R}$ *as a fixed convex differentiable function. In this setting, the Bregman divergence is*

$$D_\psi(\rho_1 \parallel \rho_2) := \int_X \{\psi(\rho_1(x)) - \psi(\rho_2(x)) - (\rho_1(x) - \rho_2(x))\psi'(\rho_2(x))\}dx. \quad (11)$$

We mention that the function $F(z,y) := \psi(z) - \psi(y) - (z-y) \cdot \psi'(y)$ *is convex too.*

Let $\rho = \rho(x,t)$ be a *time-dependent* PDF, where $x,t \in \mathbb{R}$. Then, the entropy in (1) will also depend on the parameter t, so $H[\rho] = H[\rho](t)$. We consider a *potential energy function* $V = V(x)$ and its associated *energy average function*

$$U[\rho](t) := \int_{\mathbb{R}} V(x)\rho(x,t)dx. \quad (12)$$

(If needed, restriction of these functions to open subsets is possible). This particular framework will be used in Section 6 only.

3. Fisher-like Metrics Associated with Generalized Entropies and Generalized Divergences

In this section, we recall the notion of Fisher metric associated with a family of (generalized) entropies or divergences, defined on the space of parameters of an arbitrary PDF, using mainly [20,34]. For a more general setting, see [34].

Consider the case when the PDF ρ in Section 2 depends, moreover, on n real parameters $\theta^1,\ldots,\theta^n$, with $\theta := (\theta^1,\ldots,\theta^n) \in \Theta$, where Θ is an open set of $\mathbb{R}^n$. Thus, $\rho : X \times \Theta \to \mathbb{R}$, $\rho = \rho(x,\theta)$. Let $\varphi : \mathbb{R} \to \mathbb{R}$ be a differentiable controlling function, $\varphi = \varphi(y)$. The dependence on θ leads to a generalized entropy function $H : \Theta \to \mathbb{R}$, canonically derived from Formula (1):

$$H(\theta) = -\int_X \rho(x,\theta) \cdot \varphi(\rho(x,\theta))dx. \quad (13)$$

In a similar natural way, we can define generalized divergence functions, by θ-parameterizing (2) and its avatars.

Define

$$g_{ij}(\theta) := -\int_X \rho(x,\theta) \frac{\partial^2 \varphi(\rho(x,\theta))}{\partial\theta^i \partial\theta^j} dx \quad , \quad i,j = \overline{1,n} \quad (14)$$

and

$$\tilde{g}_{ij}(\theta) := \int_X \rho(x,\theta) \frac{\partial \varphi(\rho(x,\theta))}{\partial\theta^i} \cdot \frac{\partial \varphi(\rho(x,\theta))}{\partial\theta^j} dx \quad , \quad i,j = \overline{1,n}. \quad (15)$$

We suppose that the matrices $(g_{ij})_{i,j=\overline{1,n}}$ and $(\tilde{g}_{ij})_{i,j=\overline{1,n}}$ are non-degenerated, and g has constant index on Θ. We call g and $\tilde{g}$ *generalized Fisher metrics of type 1 and type 2*, respectively, and denote GFM1 and GFM2. Both metrics are "means", w.r.t. ρ, of some φ-mediated "information matrices": the Hessian of $\varphi \circ \rho$ and the matrix of the gradient of $\varphi \circ \rho$ with its transpose, respectively. The diagonal coefficients $\tilde{g}_{ii}(\theta)$, $i = \overline{1,n}$, generalize the Fisher Information Numbers from [45], which can be recovered when φ is the Tsallis logarithm.

In general, the semi-Riemannian metric g and the Riemannian metric $\tilde{g}$ differ from each other and differ from the Hessian (semi-Riemannian metric if non-degenerated)

$$h_{ij}(\theta) := \frac{\partial^2 H(\theta)}{\partial\theta^i \partial\theta^j}. \quad (16)$$

We define, in a formal way, two auxiliary symmetric tensors of (0,2)-type α and β, given by

$$\alpha_{ij}(\theta) := \int_X \frac{\partial^2 \rho(x,\theta)}{\partial\theta^i \partial\theta^j} \cdot \varphi(\rho(x,\theta))dx \quad (17)$$

and

$$\beta_{ij}(\theta) := \int_X \left\{ \frac{\partial \rho(x,\theta)}{\partial\theta^i} \cdot \frac{\partial \varphi(\rho(x,\theta))}{\partial\theta^j} + \frac{\partial \rho(x,\theta)}{\partial\theta^j} \cdot \frac{\partial \varphi(\rho(x,\theta))}{\partial\theta^i} \right\} dx. \quad (18)$$

We remark that, if non-degenerated, α and β provide semi-Riemannian metrics. In this case, these metrics are also of Fisher type, as they express "means" w.r.t. the PDF ρ of two "derived information matrices", of coefficients $\rho^{-1} \cdot \rho_{ij} \cdot \varphi(\rho)$ and $\rho^{-1} \cdot (\rho_i \cdot \varphi_j(\rho) + \rho_j \cdot \varphi_i(\rho))$, respectively.

Example 3. *Consider the particular case of the BGS-entropy, with $\varphi := log$.*

(i) In this case, both previous GFM1 and GFM2 coincide with the classical (Riemannian) Fisher metric g^0 associated with H (or φ) [20].

In the general case, it would be interesting to find all the controlling functions φ, for which g coincides with $\tilde{g}$. Does this property necessarily imply that φ is proportional with log, modulo a non-null constant? A further step would be to look for appropriate functions φ, in order that g and $\tilde{g}$: be homothetic or conformal; have the same geodesics; have the same curvature, etc. To this differential geometric viewpoint, a statistical counterpart may eventually correspond.

(ii) Let $X \subset \mathbb{R}^m$ be an open set and let $C = C(x)$, $F_1 = F_1(x), \ldots, F_n = F_n(x)$, $\nu = \nu(\theta)$ be smooth functions on X. Consider $\rho : X \times \mathbb{R}^n \to \mathbb{R}$ the PDF of exponential type, given by

$$\rho(x,\theta) := exp\{C(x) - \nu(\theta) + \sum_{i=1}^{n} F_i(x)\theta^i\}.$$

The associated Fischer metric is $g = Hess_\nu$, which is a Hessian metric.

(iii) For this choice of the function φ, we obtain $\alpha_{ij} = \rho^{-1} \cdot \rho_{ij} \cdot log(\rho)$ and $\beta_{ij} = 2\rho^{-2} \cdot \rho_i \cdot \rho_j$, for $i,j = \overline{1,n}$. The "perturbed" Hessian matrix associated with α is similar to the one studied in some recent statistical applications (see, for example, [46]).

Remark 1. *(i) We give an interpretation and a motivation for the definition of the GFM1, in a slightly more general case than [20]. Consider φ a fixed controlling function. Let $\rho = \rho(x,\theta)$ and $\sigma := \rho(x,\theta_0)$ be two families of parameterized PDFs over $X \subset \mathbb{R}^m$, with $\theta, \theta_0 \in \mathbb{R}^n$, and let*

$$D(\rho \parallel \sigma)(\theta,\theta_0) := \int_X \rho(x,\theta) \cdot [\varphi(\rho(x,\theta)) - \varphi(\sigma(x,\theta_0))]dx$$

be the generalized (difference) relative entropy between them, as in (10). Denote $\Delta\theta := \theta - \theta_0$ and suppose its norm to be infinitesimally small. We know that $D(\rho \parallel \sigma)$ has a unique minimum for $\rho = \sigma$, i.e., for $\theta_0 = \theta$. The Taylor decomposition around $\theta = \theta_0$ gives

$$D(\rho \parallel \sigma)(\theta_0,\theta_0) = -\frac{1}{2}\int_X \rho(x,\theta_0) \cdot \Delta\theta^i \cdot \Delta\theta^j \cdot \left(Hess_{\varphi\circ\rho}\right)_{ij}(\theta_0)\, dx + \mathcal{O}((\Delta\theta)^3) =$$

$$= \frac{1}{2} \cdot \Delta\theta^i \cdot \Delta\theta^j \cdot g_{ij}(\theta_0) + \mathcal{O}((\Delta\theta)^3).$$

The second order approximation of this expression is precisely half of the GFM1 g, calculated in θ_0.

When $\varphi := log$, we recover the interpretation given in [20].

(ii) We do not know a similar interpretation for the GFM2 $\tilde{g}$.

(iii) The generalized group relative divergences from Example 2 (iii) provide analogous formulas. We shall study them in the next section, in the particular case of the ϕ-deformed (Naudts) entropy.

(iv) The definition of Fisher metrics described previously is closely related to the need for understanding a variation of a PDF w.r.t. another (reference) one; the output of this "variational calculus factory" are functions. We signal here the forthcoming book [47], containing new revolutionary ideas in Variational calculus, including invariants of tensorial type, motivated by differential geometric problems; this source provides new insights for the definition and the study of divergence-like tensor fields, as a path toward a new bundle spaces approach in Statistics.

(v) All the previous tensor fields g, $\tilde{g}$, h, α, and β have constant index, one each connected component of their definition domains.

An open problem is to find the more general hypothesis such that these tensor fields be non-degenerated (in order to define semi-Riemannian metrics). Locally, the answer is simple: let θ_0 be a point in the parameters space, such that the determinant of the corresponding matrix, calculated in θ_0, is not null. Then, the tensor field is non-degenerated in an open neighborhood of θ_0. For many families of examples (and in Section 5 we add several more ones), this property holds true. A common practice in the literature is to stop here, without investigating global conditions which are fulfilled in general cases. To our knowledge, global existence results for Fisher metrics, in the general setting, are not proven yet. Moreover, the eventual singular points have an interest in their own, as they may signal—in a suitable statistical model—a phase transition ([48]).

We consider it useful to point out here the paper [49], where a different but correlated problem is studied: namely, to what extent the Fisher metric is (globally) unique, modulo the action of a diffeomorphism group.

4. The Fisher Geometries Associated with GGEFs Based on ϕ-Deformed (Naudts) Entropies and Divergences

We particularize now the results from Section 3, for the case of the Naudts entropies. Let us fix the context more precisely.

Consider ϕ a positive, differentiable and strictly-increasing function as in Example 1 (v) and the ϕ-deformed (Naudts) logarithm log_ϕ^N defined in Formula (5). Let $\rho : X \times \Theta \to \mathbb{R}$, $\rho = \rho(x,\theta)$ be a family of parameterized PDFs, as in Section 3. The associated GFM1 g and the GFM2 $\tilde{g}$ are obtained as particular cases from (14) and (15):

$$g_{ij}(\theta) := -\int_X \rho(x,\theta)\frac{\partial^2 log_\phi^N(\rho(x,\theta))}{\partial\theta^i\partial\theta^j}dx \quad , \quad i,j = \overline{1,n} \tag{19}$$

and

$$\tilde{g}_{ij}(\theta) := \int_X \rho(x,\theta) \frac{\partial log_\phi^N(\rho(x,\theta))}{\partial\theta^i} \cdot \frac{\partial log_\phi^N(\rho(x,\theta))}{\partial\theta^j} dx \quad , \quad i,j = \overline{1,n}. \tag{20}$$

We suppose, as usual, that g and $\tilde{g}$ are non-degenerated and that $\tilde{g}$ has a constant index on X.

We also consider, via (16), the associated Hessian metric $h = h(\theta)$

$$h_{ij}(\theta) = -\frac{\partial^2}{\partial\theta^i\partial\theta^j}\Big\{ \int_X \rho(x,\theta) \cdot log_\phi^N(\rho(x,\theta))dx \Big\} \quad , \quad i,j = \overline{1,n}. \tag{21}$$

Proposition 1. *With the previous notations, for every $i,j = \overline{1,n}$, we have*

$$g_{ij}(\theta) = \int_X \rho(x,\theta)\Big\{ \frac{\partial\rho(x,\theta)}{\partial\theta^i} \cdot \frac{\partial\rho(x,\theta)}{\partial\theta^j} \cdot \phi^{-2}(\rho(x,\theta)) \cdot \phi'(\rho(x,\theta)) - \tag{22}$$

$$-\frac{\partial^2\rho(x,\theta)}{\partial\theta^i\partial\theta^j} \cdot \phi^{-1}(\rho(x,\theta))\Big\} dx,$$

$$\tilde{g}_{ij}(\theta) := \int_X \rho(x,\theta) \cdot \frac{\partial\rho(x,\theta)}{\partial\theta^i} \cdot \frac{\partial\rho(x,\theta)}{\partial\theta^j} \cdot \phi^{-2}(\rho(x,\theta))dx, \tag{23}$$

and

$$h_{ij}(\theta) = \int_X \Big\{ \rho(x,\theta) \frac{\partial\rho(x,\theta)}{\partial\theta^i} \cdot \frac{\partial\rho(x,\theta)}{\partial\theta^j} \cdot \phi^{-2}(\rho(x,\theta)) \cdot \phi'(\rho(x,\theta)) - \tag{24}$$

$$-\frac{\partial^2\rho(x,\theta)}{\partial\theta^i\partial\theta^j} \cdot log_\phi^N(\rho(x,\theta)) - 2\frac{\partial\rho(x,\theta)}{\partial\theta^i} \cdot \frac{\partial\rho(x,\theta)}{\partial\theta^j} \cdot \phi^{-1}(\rho(x,\theta)) -$$

$$-\rho(x,\theta) \cdot \frac{\partial^2\rho(x,\theta)}{\partial\theta^i\partial\theta^j} \cdot \phi^{-1}(\rho(x,\theta))\Big\} dx.$$

In this case, α and β are given by

$$\alpha_{ij}(\theta) := \int_X \frac{\partial^2\rho(x,\theta)}{\partial\theta^i\partial\theta^j} \cdot log_\phi^N(\rho(x,\theta))dx$$

and

$$\beta_{ij}(\theta) := \int_X \Big\{ \frac{\partial\rho(x,\theta)}{\partial\theta^i} \cdot \frac{\partial log_\phi^N(\rho(x,\theta))}{\partial\theta^j} + \frac{\partial\rho(x,\theta)}{\partial\theta^j} \cdot \frac{\partial log_\phi^N(\rho(x,\theta))}{\partial\theta^i} \Big\} dx.$$

Corollary 1. *In a condensed form, we have the following relation*

$$h = g - \alpha - \beta.$$

We consider now, in addition, a fixed formal group logarithm G, as in Example 1 (vi). Let $\sigma := \rho(x,\theta_0)$ be the associated parameterized PDFs and $D_{G,\phi} = D_{G,\phi}(\rho \parallel \sigma)(\theta,\theta_0)$ be the generalized (difference) group relative entropy (a.k.a. the generalized (difference) group divergence), as particularization from (10) and Remark 1 (i), (iii), written as

$$D_{G,\phi}(\rho \parallel \sigma)(\theta,\theta_0) = \int_X \rho(x,\theta) \cdot G\Big(log_\phi^N(\rho(x,\theta)) - log_\phi^N(\rho(x,\theta_0))\Big) dx.$$

Denote the generalized group Fisher metric associated with $D_{G,\phi}$ by

$$\hat{g}_{jk}(\theta_0) := \frac{\partial^2 D_{G,\phi}(\rho \parallel \sigma)(\theta,\theta_0)}{\partial\theta^j\partial\theta^k} \,|_{\theta=\theta_0} . \tag{25}$$

This Hessian-type metric will be calculated in the next result.

Proposition 2. *With the previous notations, we have the relation*

$$\hat{g}_{jk}(\theta_0) = G'(0) \cdot \Big\{ \int_X \frac{\partial^2 \rho(x,\theta_0)}{\partial\theta^j \partial\theta^k} \cdot \frac{\rho(x,\theta_0)}{\phi(\rho(x,\theta_0))} dx + \tag{26}$$

$$+2 \int_X \phi(\rho(x,\theta_0)) \cdot \frac{\partial}{\partial\theta^j} log_\phi^N(\rho(x,\theta_0)) \cdot \frac{\partial}{\partial\theta^k} log_\phi^N(\rho(x,\theta_0)) dx -$$

$$- \int_X \rho(x,\theta_0) \cdot \phi'(\rho(x,\theta_0)) \cdot \frac{\partial}{\partial\theta^j} log_\phi^N(\rho(x,\theta_0)) \cdot \frac{\partial}{\partial\theta^k} log_\phi^N(\rho(x,\theta_0)) dx \Big\} +$$

$$+G''(0) \cdot \int_X \rho(x,\theta_0) \cdot \frac{\partial}{\partial\theta^j} log_\phi^N(\rho(x,\theta_0)) \cdot \frac{\partial}{\partial\theta^k} log_\phi^N(\rho(x,\theta_0)) dx,$$

which may be re-written as depending only on ϕ *and* ρ*, in*

$$\hat{g}_{jk}(\theta_0) = G'(0) \cdot \Big\{ \int_X \frac{\partial^2 \rho(x,\theta_0)}{\partial\theta^j \partial\theta^k} \cdot \frac{\rho(x,\theta_0)}{\phi(\rho(x,\theta_0))} dx + \tag{27}$$

$$+2 \int_X \phi^{-1}(\rho(x,\theta_0)) \cdot \frac{\partial}{\partial\theta^j} \rho(x,\theta_0) \cdot \frac{\partial}{\partial\theta^k} \rho(x,\theta_0) dx -$$

$$- \int_X \rho(x,\theta_0) \cdot \phi'(\rho(x,\theta_0)) \cdot \phi^{-2}(\rho(x,\theta_0)) \cdot \frac{\partial}{\partial\theta^j} \rho(x,\theta_0) \cdot \frac{\partial}{\partial\theta^k} \rho(x,\theta_0) dx \Big\} +$$

$$+G''(0) \cdot \int_X \rho(x,\theta_0) \cdot \phi^{-2}(\rho(x,\theta_0)) \cdot \frac{\partial}{\partial\theta^j} \rho(x,\theta_0) \cdot \frac{\partial}{\partial\theta^k} \rho(x,\theta_0) dx.$$

Proof. We follow the line of reasoning from [20]. As

$$log_\phi^N(\rho(x,\theta)) = \int_1^{\rho(x,\theta)} \frac{1}{\phi(y)} dy,$$

we calculate

$$\frac{\partial}{\partial\theta^k} log_\phi^N(\rho(x,\theta)) = \frac{\partial\rho(x,\theta)}{\partial\theta^k} \cdot \frac{1}{\phi(\rho(x,\theta))}.$$

Suppose, for the moment, that θ_0 is constant. Denote

$$A(\theta) := D_{G,\phi}(\rho(x,\theta) \parallel \rho(x,\theta_0)).$$

We calculate successively

$$\frac{\partial A}{\partial\theta^k}(\theta) = \int_X \Big\{ \frac{\partial\rho(x,\theta)}{\partial\theta^k} \cdot G\Big(log_\phi^N(\rho(x,\theta)) - log_\phi^N(\rho(x,\theta_0))\Big) +$$

$$+\rho(x,\theta) \cdot G'\Big(log_\phi^N(\rho(x,\theta)) - log_\phi^N(\rho(x,\theta_0))\Big) \cdot \frac{\partial}{\partial\theta^k} log_\phi^N(\rho(x,\theta)) \Big\} dx =$$

$$= \int_X \Big\{ \frac{\partial\rho(x,\theta)}{\partial\theta^k} \cdot G\Big(log_\phi^N(\rho(x,\theta)) - log_\phi^N(\rho(x,\theta_0))\Big) +$$

$$+\rho(x,\theta) \cdot G'\Big(log_\phi^N(\rho(x,\theta)) - log_\phi^N(\rho(x,\theta_0))\Big) \cdot \frac{1}{\phi(\rho(x,\theta))} \frac{\partial\rho(x,\theta)}{\partial\theta^k} \Big\} dx =$$

$$= \int_X \frac{\partial\rho(x,\theta)}{\partial\theta^k} \cdot \Big\{ G\Big(log_\phi^N(\rho(x,\theta)) - log_\phi^N(\rho(x,\theta_0))\Big) +$$

$$+ \frac{\rho(x,\theta)}{\phi(\rho(x,\theta))} \cdot G'\Big(log_\phi^N(\rho(x,\theta)) - log_\phi^N(\rho(x,\theta_0))\Big) \Big\} dx$$

and

$$\frac{\partial^2 A}{\partial\theta^j \partial\theta^k}(\theta) = \int_X \frac{\partial^2 \rho(x,\theta)}{\partial\theta^j \partial\theta^k} \cdot \Big\{ G\Big(log_\phi^N(\rho(x,\theta)) - log_\phi^N(\rho(x,\theta_0))\Big) +$$

$$+\frac{\rho(x,\theta)}{\phi(\rho(x,\theta))}\cdot G'\Big(log_\phi^N(\rho(x,\theta))-log_\phi^N(\rho(x,\theta_0))\Big)\Big\}+$$

$$+\frac{\partial\rho(x,\theta)}{\partial\theta^k}\cdot\Big\{\frac{\partial}{\partial\theta^j}log_\phi^N(\rho(x,\theta))\cdot G'\Big(log_\phi^N(\rho(x,\theta))-log_\phi^N(\rho(x,\theta_0))\Big)+$$

$$+G'\Big(log_\phi^N(\rho(x,\theta))-log_\phi^N(\rho(x,\theta_0))\Big)\cdot\frac{\frac{\partial\rho(x,\theta)}{\partial\theta^j}\cdot[\phi(\rho(x,\theta))-\rho(x,\theta)\cdot\phi'(\rho(x,\theta)]}{\phi^2(\rho(x,\theta))}+$$

$$+\frac{\rho(x,\theta)}{\phi(\rho(x,\theta))}\cdot G''\Big(log_\phi^N(\rho(x,\theta))-log_\phi^N(\rho(x,\theta_0))\Big)\cdot\frac{\partial}{\partial\theta^j}log_\phi^N(\rho(x,\theta))\Big\}dx=$$

$$=\int_X\frac{\partial^2\rho(x,\theta)}{\partial\theta^j\partial\theta^k}\cdot\Big\{G\Big(log_\phi^N(\rho(x,\theta))-log_\phi^N(\rho(x,\theta_0))\Big)+$$

$$+\frac{\rho(x,\theta)}{\phi(\rho(x,\theta))}\cdot G'\Big(log_\phi^N(\rho(x,\theta))-log_\phi^N(\rho(x,\theta_0))\Big)\Big\}+$$

$$+\frac{\partial\rho(x,\theta)}{\partial\theta^j}\cdot\frac{\partial\rho(x,\theta)}{\partial\theta^k}\cdot\frac{1}{\phi(\rho(x,\theta))}\cdot\Big\{G'\Big(log_\phi^N(\rho(x,\theta))-log_\phi^N(\rho(x,\theta_0))\Big)+$$

$$+G'\Big(log_\phi^N(\rho(x,\theta))-log_\phi^N(\rho(x,\theta_0))\Big)\cdot\frac{\phi(\rho(x,\theta))-\rho(x,\theta)\cdot\phi'(\rho(x,\theta)}{\phi(\rho(x,\theta))}+$$

$$+\frac{\rho(x,\theta)}{\phi(\rho(x,\theta))}\cdot G''\Big(log_\phi^N(\rho(x,\theta))-log_\phi^N(\rho(x,\theta_0))\Big)\Big\}dx.$$

We replace $\theta:=\theta_0$, and we use the property $G(0)=0$. It follows that

$$\hat{g}_{jk}(\theta_0):=\frac{\partial^2 A}{\partial\theta^j\partial\theta^k}\mid_{\theta=\theta_0}=\int_X\frac{\partial^2\rho(x,\theta_0)}{\partial\theta^j\partial\theta^k}\cdot\Big\{G(0)+\frac{\rho(x,\theta_0)}{\phi(\rho(x,\theta_0))}\cdot G'(0)\Big\}+$$

$$+\frac{\partial\rho(x,\theta_0)}{\partial\theta^j}\cdot\frac{\partial\rho(x,\theta_0)}{\partial\theta^k}\cdot\frac{1}{\phi(\rho(x,\theta_0))}\cdot\Big\{G'(0)+\frac{\rho(x,\theta_0)}{\phi(\rho(x,\theta_0))}\cdot G''(0)+$$

$$+G'(0)\cdot\frac{\phi(\rho(x,\theta_0))-\rho(x,\theta_0)\cdot\phi'(\rho(x,\theta_0)}{\phi(\rho(x,\theta_0))}\Big\}dx=$$

$$=G'(0)\cdot\int_X\frac{\partial^2\rho(x,\theta_0)}{\partial\theta^j\partial\theta^k}\cdot\frac{\rho(x,\theta_0)}{\phi(\rho(x,\theta_0))}dx+$$

$$+\int_X\phi(\rho(x,\theta_0))\cdot\frac{\partial}{\partial\theta^j}log_\phi^N(\rho(x,\theta_0))\cdot\frac{\partial}{\partial\theta^k}log_\phi^N(\rho(x,\theta_0))\cdot$$

$$\cdot G'(0)\cdot\Big\{2-\frac{\rho(x,\theta_0)\cdot\phi'(\rho(x,\theta_0))}{\phi(\rho(x,\theta_0))}\Big\}dx+$$

$$+G''(0)\cdot\int_X\rho(x,\theta_0)\cdot\frac{\partial}{\partial\theta^j}log_\phi^N(\rho(x,\theta_0))\cdot\frac{\partial}{\partial\theta^k}log_\phi^N(\rho(x,\theta_0))dx.$$

From the last suite of formulas, we obtain both (26) and (27). □

Suppose, moreover, that $G(t)=t$. Then, we have

$$\hat{g}_{jk}(\theta_0)=\int_X\Big\{\frac{\partial^2\rho(x,\theta_0)}{\partial\theta^j\partial\theta^k}\frac{\rho(x,\theta_0)}{\phi(\rho(x,\theta_0))}+2\frac{\partial\rho(x,\theta_0)}{\partial\theta^j}\cdot\frac{\partial\rho(x,\theta_0)}{\partial\theta^k}\cdot\frac{1}{\phi(\rho(x,\theta_0))}- \tag{28}$$

$$-\frac{\partial\rho(x,\theta_0)}{\partial\theta^j}\cdot\frac{\partial\rho(x,\theta_0)}{\partial\theta^k}\cdot\frac{\rho(x,\theta_0)\cdot\phi'(\rho(x,\theta_0))}{\phi^2(\rho(x,\theta_0))}\Big\}dx.$$

We re-write this formula in a condensed form, and we obtain the following result, which completes Corollary 1.

Corollary 2. *With the previous notations, for $G(t) = t$, we obtain*

$$\hat{g} = -h - \alpha.$$

By analogy, starting with a generalized (quotient) group relative entropy (a.k.a. the generalized (quotient) group divergence) $\tilde{D}_{G,\phi} = \tilde{D}_{G,\phi}(\rho \parallel \sigma)(\theta, \theta_0)$, as particularization from (9), we shall obtain, in the sequel, other Fisher-like metrics, similar to the ones in Proposition 2 and Corollary 2.

Denote the generalized group Fisher metric associated with $\tilde{D}_{G,\phi}$ by

$$\overline{g}_{jk}(\theta_0) := \frac{\partial^2 \tilde{D}_{G,\phi}(\rho \parallel \sigma)(\theta, \theta_0)}{\partial \theta^j \partial \theta^k} \mid_{\theta = \theta_0} . \tag{29}$$

Proposition 3. *With the previous notations, we have the relation*

$$\overline{g} = \left\{ G'(0) \cdot \left[\frac{2}{\phi(1)} - \frac{\phi'(1)}{\phi^2(1)} \right] + G''(0) \cdot \frac{1}{\phi^2(1)} \right\} \cdot g^0, \tag{30}$$

where g^0 denotes the classical Fisher metric.

Proof. We adapt the proof of Proposition 2, from the divergence $D_{G,\phi}$ to the divergence $\tilde{D}_{G,\phi}$. Suppose that θ_0 is constant. Denote

$$\tilde{A}(\theta) := \tilde{D}_{G,\phi}(\rho(x,\theta) \parallel \rho(x,\theta_0)).$$

It follows that

$$\frac{\partial \tilde{A}}{\partial \theta^k}(\theta) = \int_X \left\{ \frac{\partial \rho(x,\theta)}{\partial \theta^k} \cdot G\left(log_\phi^N \left[\frac{\rho(x,\theta)}{\rho(x,\theta_0)} \right] \right) + \right.$$

$$\left. + \rho(x,\theta) \cdot G'\left(log_\phi^N \left[\frac{\rho(x,\theta)}{\rho(x,\theta_0)} \right] \right) \cdot \frac{\partial}{\partial \theta^k} log_\phi^N \left[\frac{\rho(x,\theta)}{\rho(x,\theta_0)} \right] \right\} dx =$$

$$= \int_X \left\{ \frac{\partial \rho(x,\theta)}{\partial \theta^k} \cdot G\left(log_\phi^N \left[\frac{\rho(x,\theta)}{\rho(x,\theta_0)} \right] \right) + \right.$$

$$\left. + \rho(x,\theta) \cdot G'\left(log_\phi^N \left[\frac{\rho(x,\theta)}{\rho(x,\theta_0)} \right] \right) \cdot \phi^{-1}\left(\frac{\rho(x,\theta)}{\rho(x,\theta_0)} \right) \cdot \rho^{-1}(x,\theta_0) \cdot \frac{\partial \rho(x,\theta)}{\partial \theta^k} \right\} dx =$$

$$= \int_X \frac{\partial \rho(x,\theta)}{\partial \theta^k} \cdot \left\{ G\left(log_\phi^N \left[\frac{\rho(x,\theta)}{\rho(x,\theta_0)} \right] \right) + \right.$$

$$\left. + \rho(x,\theta) \cdot G'\left(log_\phi^N \left[\frac{\rho(x,\theta)}{\rho(x,\theta_0)} \right] \right) \cdot \phi^{-1}\left(\frac{\rho(x,\theta)}{\rho(x,\theta_0)} \right) \cdot \rho^{-1}(x,\theta_0) \right\} dx$$

and

$$\frac{\partial^2 A}{\partial \theta^j \partial \theta^k}(\theta) = \int_X \frac{\partial^2 \rho(x,\theta)}{\partial \theta^j \partial \theta^k} \cdot \left\{ G\left(log_\phi^N \left[\frac{\rho(x,\theta)}{\rho(x,\theta_0)} \right] \right) + \right.$$

$$\left. + \rho(x,\theta) \cdot \phi^{-1}\left(\frac{\rho(x,\theta)}{\rho(x,\theta_0)} \right) \cdot \rho^{-1}(x,\theta_0) \cdot G'\left(log_\phi^N \left[\frac{\rho(x,\theta)}{\rho(x,\theta_0)} \right] \right) \right\} +$$

$$+ \frac{\partial \rho(x,\theta)}{\partial \theta^k} \cdot \left\{ \phi^{-1}\left(\frac{\rho(x,\theta)}{\rho(x,\theta_0)} \right) \cdot \rho^{-1}(x,\theta_0) \cdot \frac{\partial \rho(x,\theta)}{\partial \theta^j} \cdot G'\left(log_\phi^N \left[\frac{\rho(x,\theta)}{\rho(x,\theta_0)} \right] \right) + \right.$$

$$+ G'\left(log_\phi^N \left[\frac{\rho(x,\theta)}{\rho(x,\theta_0)} \right] \right) \cdot \phi^{-2}\left(\frac{\rho(x,\theta)}{\rho(x,\theta_0)} \right) \cdot \frac{\partial \rho(x,\theta)}{\partial \theta^j} \cdot$$

$$\cdot\Big[\rho^{-1}(x,\theta_0)\cdot\phi(\frac{\rho(x,\theta)}{\rho(x,\theta_0)})-\rho(x,\theta)\cdot\rho^{-2}(x,\theta_0)\cdot\phi'(\frac{\rho(x,\theta)}{\rho(x,\theta_0)}\Big]+$$

$$+\rho(x,\theta)\cdot\phi^{-2}(\frac{\rho(x,\theta)}{\rho(x,\theta_0)})\cdot\rho^{-2}(x,\theta_0)\cdot G''\Big(log_\phi^N\Big[\frac{\rho(x,\theta)}{\rho(x,\theta_0)}\Big]\Big)\cdot\frac{\partial\rho(x,\theta)}{\partial\theta^j}\Big\}dx \quad .$$

We assign $\theta := \theta_0$, and we use the property $G(0) = 0$. We obtain

$$\overline{g}_{jk}(\theta_0) := \frac{\partial^2 A}{\partial\theta^j\partial\theta^k}\,|_{\theta=\theta_0} = \int_X \frac{\partial^2\rho(x,\theta_0)}{\partial\theta^j\partial\theta^k}\cdot\Big\{G(0)+\frac{1}{\phi(1)}\cdot G'(0)\Big\}+$$

$$+\frac{\partial\rho(x,\theta_0)}{\partial\theta^j}\cdot\frac{\partial\rho(x,\theta_0)}{\partial\theta^k}\cdot\frac{1}{\rho(x,\theta_0)}\cdot\Big\{\frac{1}{\phi(1)}\cdot G'(0)+\frac{1}{\phi^2(1)}\cdot G''(0)+$$

$$+G'(0)\cdot\frac{\phi(1))-\phi'(1)}{\phi^2(1)}\Big\}dx = \Big[G'(0\cdot\Big(\frac{2}{\phi(1)}-\frac{\phi'(1)}{\phi^2(1)}\Big)+G''(0)\cdot\frac{1}{\phi^2(1)}\Big]\cdot$$

$$\cdot\int_X \rho^{-1}(x,\theta_0))\cdot\frac{\partial\rho(x,\theta_0)}{\partial\theta^j}\cdot\frac{\partial\rho(x,\theta_0)}{\partial\theta^k}+G'(0)\cdot\frac{1}{\phi(1)}\cdot\int_X\frac{\partial^2\rho(x,\theta_0)}{\partial\theta^j\partial\theta^k}dx.$$

The first integral equals $g^0_{jk}(\theta_0)$. The second integral is null because $\int_X \rho(x,\theta)dx = 1$. We obtained Formula (30). □

Remark 2. *(i) In Proposition 1, we establish the basic formulas for the future development of associated Riemannian geometries determined by g, $\tilde{g}$, h, α, β, in terms of the function ϕ-deformed (Naudts) entropy (curvature, geodesics, Riemannian distance in the positive definite case). Examples of scalar curvature functions derived from these formulas will be shown in the next section. The coefficients of GFM1 g extend known ones from [29], derived for PDFs of exponential type and for particular functions ϕ. The other Fisher metrics are new.*

An interesting consequence of Proposition 1 is the fact that g and $\tilde{g}$ do not coincide, as in the case of the Neperian logarithm. This can be seen directly, by comparing their ϕ-dependent coefficients.

(ii) In Proposition 2, we derive the Fisher-like metric $\hat{g}$ associated with the divergence $D_{G,\phi}$, as a generalization of a construction in [30] for the case of a Kullback–Leibler divergence, of a trivial group logarithm $G = id$ and for PDFs of exponential type.

(iii) In Proposition 3, the Fisher-like metric $\hat{g}$ associated with the divergence $\tilde{D}_{G,\phi}$ is—to our knowledge—completely new.

The metrics in Formula (30) are homothetic, via a constant $k_{G,\phi}$ supposed—implicitly—to be not null. It is interesting that $k_{G,\phi}$ depends only on the behavior of the deformation function ϕ, for or around 1 and on G, around 0. Its independence on the PDFs gives $k_{G,\phi}$ an "universality" feature, which corresponds—probably—to some special uncovered property of the statistical model.

Suppose, moreover, that $G(t) = t$. We replace in (30) the values $G'(0) = 1$ and $G''(0) = 0$, and we obtain

$$\overline{g} = \Big[\frac{2}{\phi(1)}-\frac{\phi'(1)}{\phi^2(1)}\Big]\cdot g^0. \tag{31}$$

5. Examples

We particularize now the results from Section 4, for the case when ρ is an exponential PDF and $m = 1$, $n = 2$. The deforming function ϕ will be chosen conveniently, in order to be able to compute the integrals.

Let $X := \mathbb{R}$ and $\rho : \mathbb{R}\times\mathbb{R}\times(0,\infty)\to\mathbb{R}$ be the exponential (normal) PDF given by

$$\rho(x;\theta^1,\theta^2) = \frac{1}{\sqrt{2\pi}\theta^2}\cdot e^{-\frac{(x-\theta^1)^2}{2(\theta^2)^2}}. \tag{32}$$

We denote the partial derivatives of ρ, with respect to the variables θ^1 and θ^2, by ρ_1, ρ_2, ρ_{11}, ρ_{12}, ρ_{22}. A short calculation ([34]) leads to the formulas

$$\rho_1 = \frac{x-\theta^1}{(\theta^2)^2}\cdot\rho \quad , \quad \rho_2 = \{\frac{(x-\theta^1)^2}{(\theta^2)^3} - \frac{1}{\theta^2}\}\cdot\rho,$$

$$\rho_{11} = \{\frac{(x-\theta^1)^2}{(\theta^2)^4} - \frac{1}{(\theta^2)^2}\}\cdot\rho \quad , \quad \rho_{12} = \{\frac{(x-\theta^1)^3}{(\theta^2)^5} - \frac{3(x-\theta^1)}{(\theta^2)^3}\}\cdot\rho,$$

$$\rho_{22} = \{\frac{(x-\theta^1)^4}{(\theta^2)^6} - \frac{5(x-\theta^1)^2}{(\theta^2)^4} + \frac{2}{(\theta^2)^2}\}\cdot\rho.$$

The classical Fisher metric g^0 has the coefficients $g^0_{11} = (\theta^2)^{-2}$, $g^0_{12} = g^0_{21} = 0$ and $g^0_{22} = 2(\theta^2)^{-2}$ (see, for example, [2,34]).

For future calculations, we shall use the following simple result.

Lemma 1. *Let c, k_1, k_2 be fixed real constants, with $k_1 \neq 0$, $k_2 \neq 0$. Then, the semi-Riemannian metric*

$$y^{-c}\cdot\begin{bmatrix} k_1 & 0 \\ 0 & k_2 \end{bmatrix}$$

on the set $y \neq 0$ in $\mathbb{R}^2$ has the scalar curvature

$$-\frac{c}{2k_2}\cdot y^{c-2}.$$

In the sequel, we give examples of the semi-Riemannian metrics from Propositions 1–3, under various particular assumptions.

I—The case of g. Suppose $\phi(t) := t^c$, with $c \in (0,2)$ an arbitrary fixed parameter. From Formula (22), we calculate the coefficients

$$g_{11} = K_1(c)\cdot(\theta^2)^{c-3} \quad , \quad g_{12} = g_{21} = 0 \quad , \quad g_{22} = K_2(c)\cdot(\theta^2)^{c-3},$$

where

$$K_1(c) = (2-c)^{-\frac{3}{2}}\cdot(\sqrt{2\pi})^{c-1} \quad , \quad K_2(c) = (c^3 - 4c^2 + 6c - 1)\cdot(2-c)^{-\frac{5}{2}}\cdot(\sqrt{2\pi})^{c-1}.$$

There exists a unique $c_0 \in (0.18, 0.19)$ such that $K_2(c_0) = 0$. For this value, g is degenerated. The metric g is Lorentzian, when $c \in (0, c_0)$ and is Riemannian, when $c \in (c_0, 2)$.

The scalar curvature $S^{\{c\}} = S^{\{c\}}(\theta)$ of g is

$$S^{\{c\}}(\theta) = \frac{1}{2K_2(c)}\cdot(c-3)\cdot(\theta^2)^{1-c}.$$

The scalar curvature $S^{\{c\}}$ does not vanish anywhere, and its sign is the opposite sign of $K_2(c)$. Moreover, $S^{\{c\}}$ is constant if and only if $c = 1$, i.e., only in the case when g is the classical Fisher metric g_0. If we decide to use the scalar curvature as a control, this may lead to a quick criterion to distinguish the BGS entropy case from the ϕ-deformed (Naudts) entropy case. (The statistical interpretation of the scalar curvature of the Fisher metrics may be found in [20]).

We depicted in Figure 1 (and magnified in Figure 2 around $c = 1$ and in Figure 3 around $c = 0.19$) how $S^{\{c\}}$ varies w.r.t. c and θ^2 (denoted t).

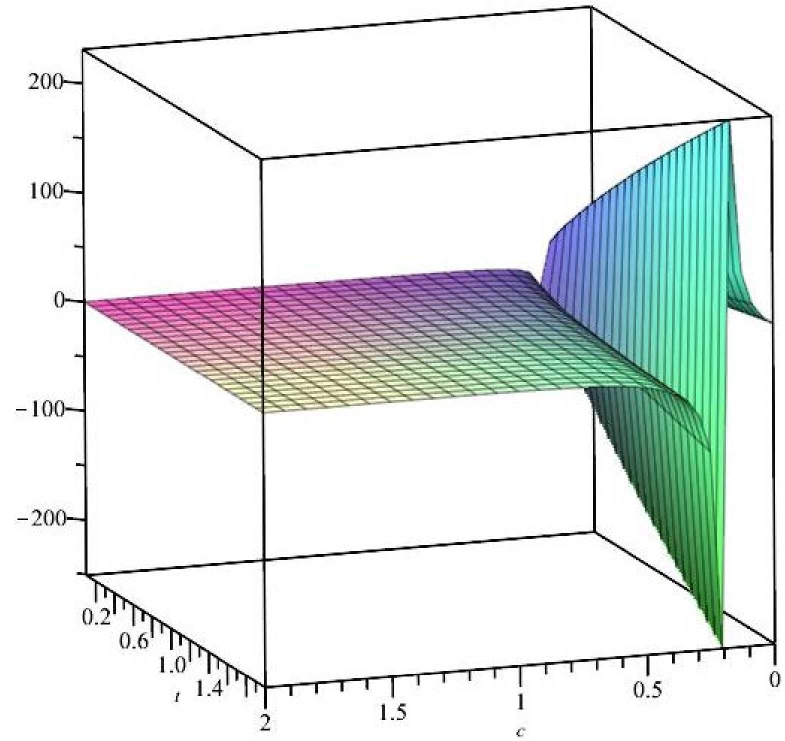

Figure 1. The variation of $S^{\{c\}}$ w.r.t. $c \in (0, c_0) \bigcup (c_0, 2)$ and $\theta^2 := t$.

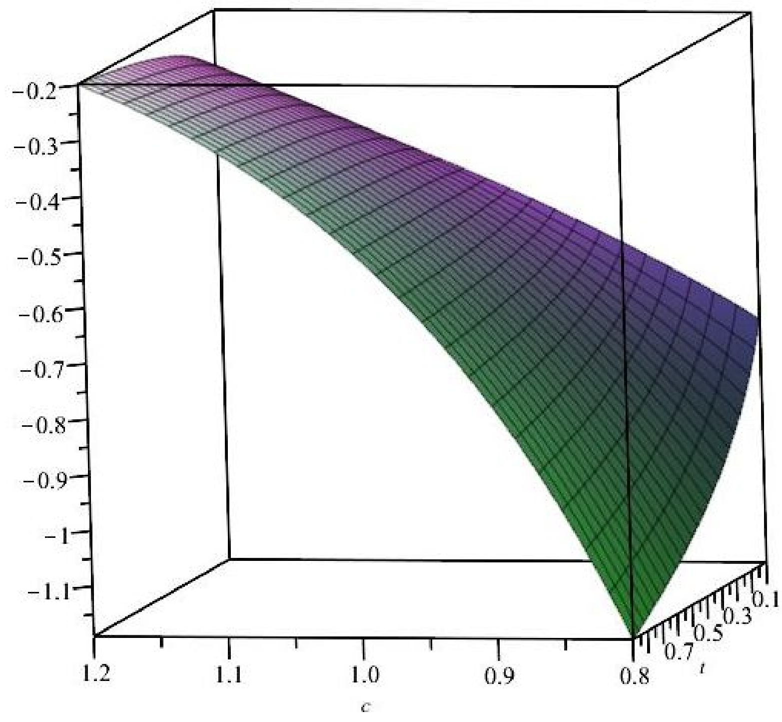

Figure 2. The variation of $S^{\{c\}}$ w.r.t. $c \in (0.8, 1.2)$ and $\theta^2 := t$.

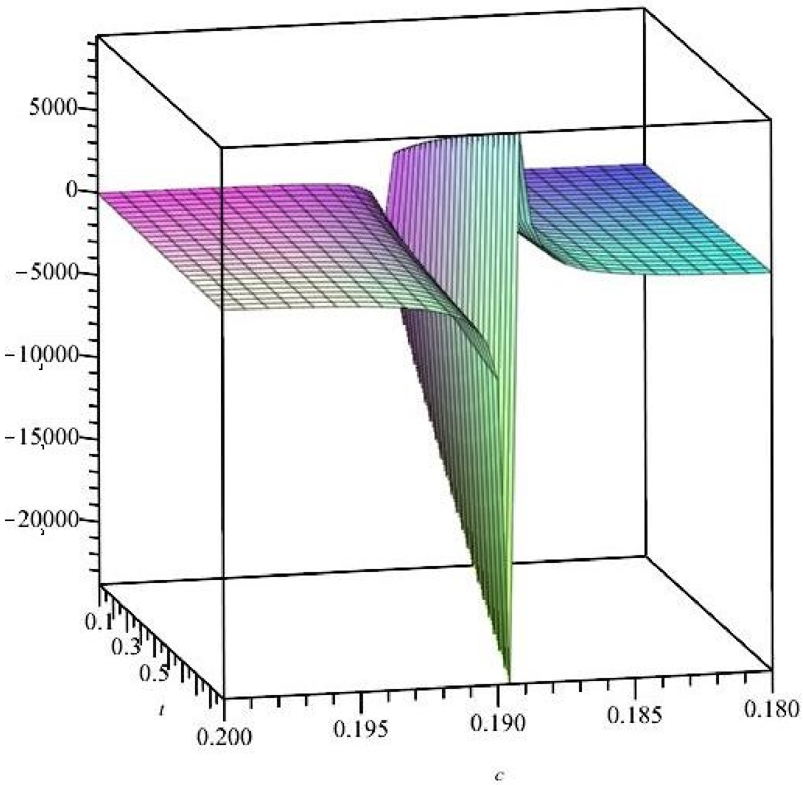

Figure 3. The variation of $S^{\{c\}}$ w.r.t. $c \in (0.18, 0.20)$ and $\theta^2 := t$.

II—The case of $\tilde{g}$. Suppose $\phi(t) := t^c$, with $c \in (0, \frac{3}{2})$ an arbitrary fixed parameter. From Formula (23), we calculate the coefficients

$$\tilde{g}_{11} = \tilde{K}_1(c) \cdot (\theta^2)^{2c-4} \quad , \quad \tilde{g}_{12} = \tilde{g}_{21} = 0 \quad , \quad \tilde{g}_{22} = \tilde{K}_2(c) \cdot (\theta^2)^{2c-4},$$

where

$$\tilde{K}_1(c) = (3-2c)^{-\frac{3}{2}} \cdot (\sqrt{2\pi})^{2c-2} \quad , \quad \tilde{K}_2(c) = (4c^2 - 8c + 6) \cdot (3-2c)^{-\frac{5}{2}} \cdot (\sqrt{2\pi})^{2c-2}.$$

The scalar curvature $\tilde{S}^{\{c\}} = \tilde{S}^{\{c\}}(\theta)$ of the Riemannian metric $\tilde{g}$ is

$$\tilde{S}^{\{c\}}(\theta) = \frac{1}{\tilde{K}_2(c)} \cdot (c-2) \cdot (\theta^2)^{2-2c}.$$

We mention that: the scalar curvature is negative; it decreases indefinitely as the variable θ^2 grows and the parameter c goes to 0; it tends to 0 as c goes to $\frac{3}{2}$. We depicted in Figure 4 how $\tilde{S}^{\{c\}}$ varies w.r.t. c and θ^2 (denoted t).

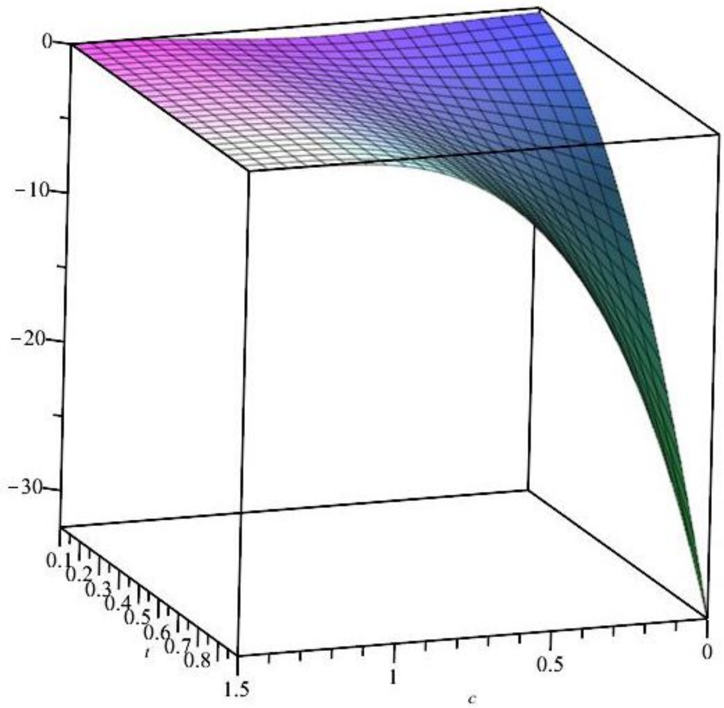

Figure 4. The variation of $\tilde{S}^{\{c\}}$ w.r.t. c and $\theta^2 := t$.

III—The case of h**.** Suppose $\phi(t) := t^c$, with $c \in (0,2)$ an arbitrary fixed parameter. From Formula (24), we calculate the coefficients

$$h_{11} = h_{12} = h_{21} = 0 \quad , \quad h_{22} = K_4(c) \cdot (\theta^2)^{c-3},$$

where

$$K_4(c) = -(2-c)^{\frac{1}{2}} \cdot (\sqrt{2\pi})^{c-1}.$$

As the (0,2)-type tensor field h is degenerated, it does not define a semi-Riemannian metric. In this case, there is no scalar curvature to compute.

IV—The case of α**.** Suppose $\phi(t) := t^c$, with $c \in (0,2)$ an arbitrary fixed parameter. From Formula (17) or from Proposition 1, we calculate the coefficients

$$\alpha_{11} = K_5(c) \cdot (\theta^2)^{c-3} \quad , \quad \alpha_{12} = \alpha_{21} = 0 \quad , \quad \alpha_{22} = K_6(c) \cdot (\theta^2)^{c-3},$$

where

$$K_5(c) = -(\sqrt{2\pi})^{c-1} \cdot (2-c)^{-\frac{3}{2}} \quad , \quad K_6(c) = (1-2c) \cdot (2-c)^{-\frac{5}{2}} \cdot (\sqrt{2\pi})^{c-1}.$$

The (0,2)-type tensor field α is degenerated for $c = \frac{1}{2}$. If $c \in (0, \frac{1}{2})$, then α is a Lorentzian metric. If $c \in (\frac{1}{2}, 2)$, then $(-\alpha)$ is a Riemannian metric.

The scalar curvature $U^{\{c\}} = U^{\{c\}}(\theta)$ of $(-\alpha)$ is

$$U^{\{c\}}(\theta) = \frac{1}{2K_6(c)} \cdot (3-c) \cdot (\theta^2)^{1-c}.$$

and has the sign of K_6. We depicted in Figure 5 how $U^{\{c\}}$ varies w.r.t. c and θ^2 (denoted t).

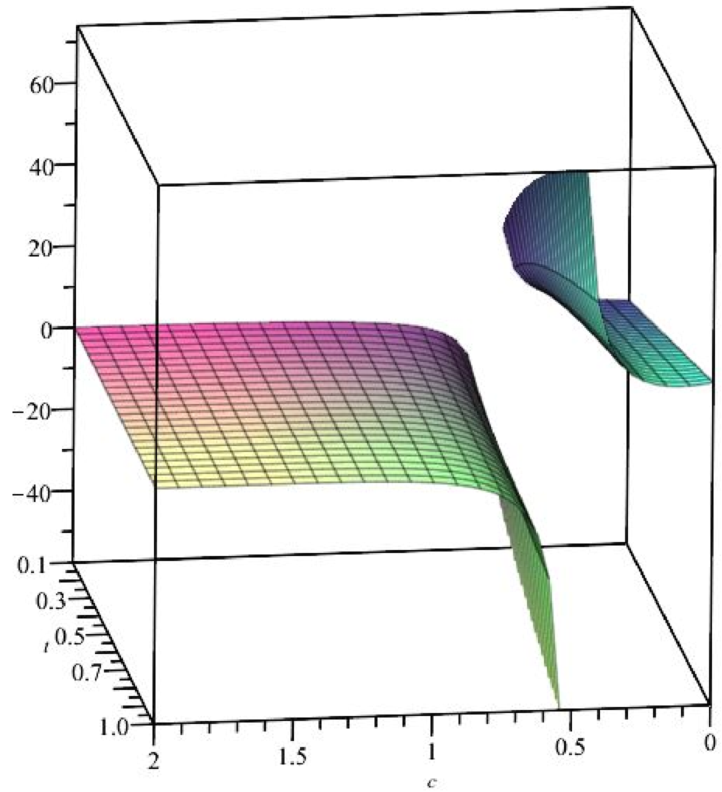

Figure 5. The variation of $U^{\{c\}}$ w.r.t. c and $\theta^2 := t$.

V—The case of β. Suppose $\phi(t) := t^c$, with $c \in (0,2)$ an arbitrary fixed parameter. From Formula (18) or from Proposition 1, we calculate the coefficients

$$\beta_{11} = K_7(c) \cdot (\theta^2)^{c-3} \quad , \quad \beta_{12} = \beta_{21} = 0 \quad , \quad \beta_{22} = K_8(c) \cdot (\theta^2)^{c-3},$$

where

$$K_7(c) = 2(\sqrt{2\pi})^{c-1}(2-c)^{-\frac{3}{2}} \quad , \quad K_8(c) = 2(c^2 - 2c + 3) \cdot (2-c)^{-\frac{5}{2}} \cdot (\sqrt{2\pi})^{c-1}.$$

The scalar curvature $V^{\{c\}} = V^{\{c\}}(\theta)$ of β is

$$V^{\{c\}}(\theta) = \frac{1}{2K_8(c)} \cdot (c-3) \cdot (\theta^2)^{1-c}.$$

and takes negative values. We depicted in Figure 6 how $V^{\{c\}}$ varies w.r.t. c and θ^2 (denoted t).

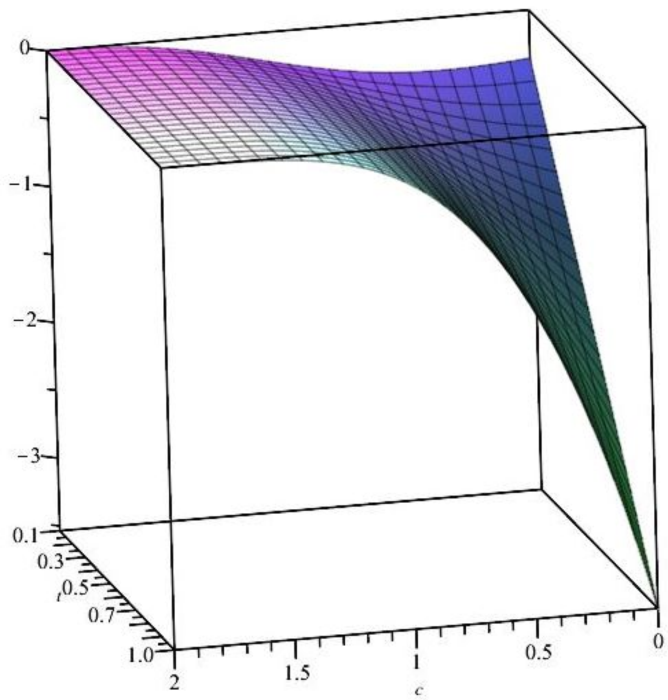

Figure 6. The variation of $V^{\{c\}}$ w.r.t. c and $\theta^2 := t$.

VI—The case of $\hat{g}$. Suppose $\phi(t) := t^c$, with $c \in (0, \frac{3}{2})$ an arbitrary fixed parameter. From Formula (27), we calculate the coefficients

$$\hat{g}_{11} = K_9(c) \cdot (\theta^2)^{c-3} + K_{10}(c) \cdot (\theta^2)^{2c-3},$$

$$\hat{g}_{12} = \hat{g}_{21} = 0,$$

$$\hat{g}_{22} = K_{11}(c) \cdot (\theta^2)^{c-3} + K_{12}(c) \cdot (\theta^2)^{2c-3},$$

where

$$K_9(c) = G'(0) \cdot (\sqrt{2\pi})^{c-1} \cdot (2-c)^{-\frac{3}{2}},$$

$$K_{10}(c) = G''(0) \cdot (\sqrt{2\pi})^{2c-2} \cdot (3-2c)^{-\frac{3}{2}},$$

$$K_{11}(c) = -G'(0) \cdot (\sqrt{2\pi})^{c-1} \cdot (2-c)^{-\frac{5}{2}} \cdot (c^3 - 6c^2 + 10c - 7),$$

$$K_{12}(c) = G''(0) \cdot (\sqrt{2\pi})^{2c-2} \cdot (3-2c)^{-\frac{5}{2}} \cdot (4c^2 - 8c + 6).$$

We suppose that the group logarithm G is chosen such that $\hat{g}$ be non-degenerated. The scalar curvature $\hat{S}^{\{c\}} = \hat{S}^{\{c\}}(\theta)$ of $\hat{g}$ is calculated using MAPLE:

$$\hat{S}^{\{c\}}(\theta) = \frac{1}{4} \cdot (\theta^2)^{-2} \cdot (K_{12}(\theta^2)^c + K_{11})^{-2} (K_{10}(\theta^2)^c + K_9)^{-3} \cdot \Big\{ (\theta^2)^3 \cdot \Big[K_9^3 K_{12} c^2 -$$

$$-K_9^3 K_{12} c - 18 K_9^2 K_{10} K_{11} - 3 K_9^2 K_{10} K_{11} c^2 + 11 K_9 K_{10} K_{11} c - 6 K_9^3 K_{12} \Big] +$$

$$+(\theta^2)^{3+c} \cdot \Big[-18 K_9^2 K_{10} K_{12} - 18 K_9 K_{10}^2 K_{11} + 2 K_9^2 K_{10} K_{12} c - 5 K_9 K_{10}^2 K_{11} c^2 +$$

$$+16 K_9 K_{10}^2 K_{11} c + K_9^2 K_{10} K_{12} c^2 \Big] + (\theta^2)^{3+2c} \cdot \Big[-18 K_9 K_{10}^2 K_{12} - 2 K_{10}^3 K_{11} c^2 +$$

$$+7 K_{10}^3 K_{11} c + 7 K_9 K_{10}^2 K_{12} c - 6 K_{10}^3 K_{11} \Big] + (\theta^2)^{3-c} \cdot \Big[2 K_9^3 K_{11} c - 6 K_9^3 K_{11} \Big] +$$

$$+(\theta^2)^{3+3c} \cdot \Big[4 K_{10}^3 K_{12} c - 6 K_{10}^3 K_{12} \Big] \Big\}.$$

Interestingly, the scalar curvature $\hat{S}^{\{c\}}$ is a rational function of θ^2 and $(\theta^2)^c$.

We particularize now the setting for the BGS group logarithm $G(t) := t$ and replace $G'(0) = 1$ and $G''(0) = 0$ in the previous formulas. Then,

$$\hat{g}_{11} = K_9(c) \cdot (\theta^2)^{c-3} \quad , \quad \hat{g}_{12} = \hat{g}_{21} = 0 \quad , \quad \hat{g}_{22} = K_{11}(c) \cdot (\theta^2)^{c-3},$$

where

$$K_9(c) = (\sqrt{2\pi})^{c-1} \cdot (2-c)^{-\frac{3}{2}},$$
$$K_{11}(c) = -(\sqrt{2\pi})^{c-1} \cdot (2-c)^{-\frac{5}{2}} \cdot (c^3 - 6c^2 + 10c - 7).$$

In this particular case, the scalar curvature $\hat{S}^{\{c\}} = \hat{S}^{\{c\}}(\theta)$ of the Riemannian metric $\hat{g}$ has the form:

$$\hat{S}^{\{c\}}(\theta) = \frac{1}{2K_{11}(c)} \cdot (c-3) \cdot (\theta^2)^{1-c}.$$

(The same formula may be recovered, directly, by using Lemma 1.) We mention that $\hat{S}^{\{c\}}$ takes negative values, for every $c \in (0,2)$. In Figure 7, we depicted how this particular $\hat{S}^{\{c\}}$ varies w.r.t. c and θ^2 (denoted t).

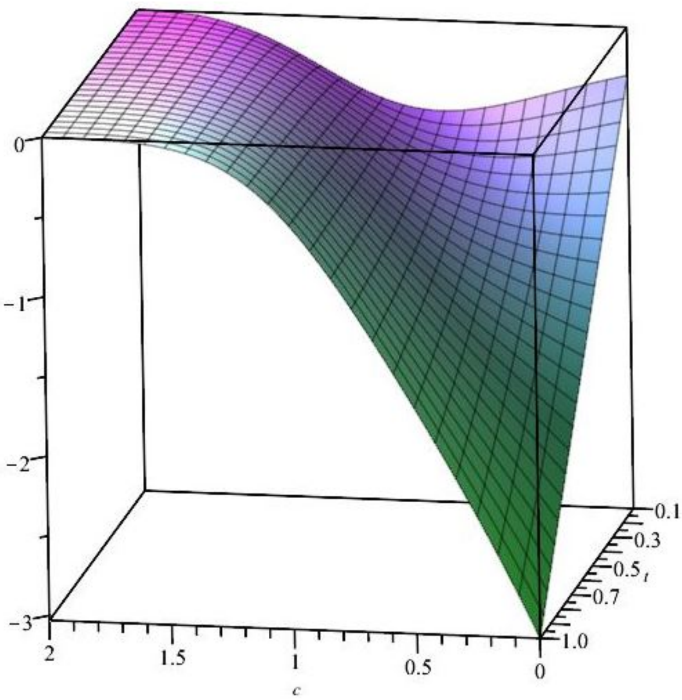

Figure 7. The variation of $\hat{S}^{\{c\}}$ w.r.t. c and $\theta^2 := t$.

VII—The case of $\overline{g}$. From (30), we have the coefficients of $\overline{g}$:

$$\overline{g}_{11} = k_{G,\phi} \cdot (\theta^2)^{-2} \quad , \quad \overline{g}_{12} = \overline{g}_{21} = 0 \quad , \quad \overline{g}^0_{22} = 2k_{G,\phi} \cdot (\theta^2)^{-2},$$

where

$$k_{G,\phi} = G'(0) \cdot \left[\frac{2}{\phi(1)} - \frac{\phi'(1)}{\phi^2(1)}\right] + G''(0) \cdot \frac{1}{\phi^2(1)}.$$

For the moment, we suppose that G and ϕ are suitable chosen, such that $k_{G,\phi} > 0$. It follows that $\overline{g}$ is a Riemannian metric. As the scalar curvature of g^0 is a negative constant $S^0 = -\frac{1}{2}$, we deduce the scalar curvature of $\overline{g}$ is a negative constant $\overline{S} = -\frac{1}{2} \cdot k_{G,\phi}$ w.r.t. θ too. In what follows, we study the variance of $\overline{S}$ in two particular cases.

VII_1. Let $G(t) := t$ be the BGS grup logarithm function and consider $\phi(t) := t^{a^2} + e^{t^{b^3}}$, where the real parameters a and b satisfy $a^2 + eb^3 < 2(1+e)$. Denote the respective metrics by $\overline{g}^{\{a,b\}}$ and their scalar curvatures by $\overline{S}^{\{a,b\}}$. Then,

$$\overline{S}^{\{a,b\}} = -\frac{1}{2} \cdot \Big\{ \frac{2}{1+e} - \frac{a^2 + eb^3}{(1+e)^2} \Big\}.$$

We mention that $k_{G,\phi} > 0$ (and hence $\overline{S}^{\{a,b\}} < 0$). The dependency of $\overline{S}^{\{a,b\}}$ w.r.t. a and b may be seen in Figure 8.

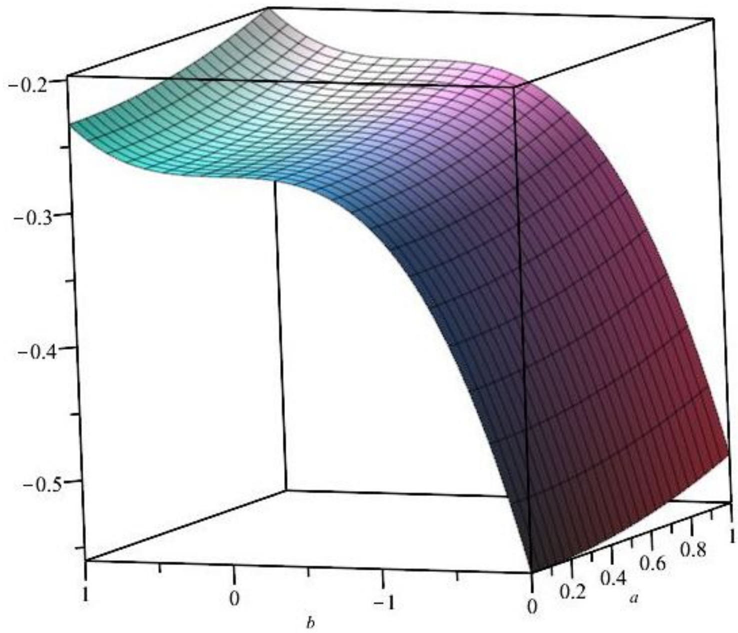

Figure 8. The variation of $\overline{S}^{\{a,b\}}$ w.r.t. a and b.

The family of Fisher-like Riemannian metrics $\overline{g}^{\{a,b\}}$ may be considered as evolving from the classical Fisher metric g^0. Their evolution may be controlled through their scalar curvature.

VII_2. Let $G(t) := \frac{e^{(1-q)t}-1}{1-q}$ be the Tsallis grup logarithm function, where $q \neq 1$. Let us define $\phi(t) := t^{a^2} + e^{t^{b^3}}$, with real parameters a and b satisfying $a^2 + e \cdot b^3 + q - 1 < 2(1+e)$. We denote the associated metric by $\overline{g}^{\{a,b;q\}}$ and its scalar curvature by $\overline{S}^{\{a,b;q\}}$. Then,

$$\overline{S}^{\{a,b;q\}} = -\frac{1}{2} \cdot \Big\{ \frac{2}{1+e} - \frac{a^2 + eb^3 + q - 1}{(1+e)^2} \Big\}.$$

We mention that $k_{G,\phi} > 0$ (and hence $\overline{S}^{\{a,b;q\}} < 0$). The dependency of $\overline{S}^{\{a,b;q\}}$ w.r.t. a and b may be seen in Figure 9, for q taking successively the values 1,11,21,31 (from bottom to top). The value $q = 1$ is no longer a forbidden (singular) one!

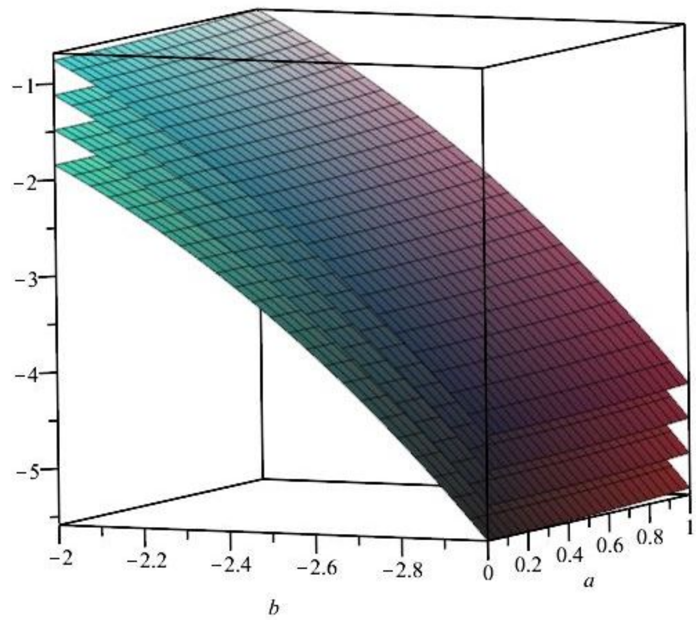

Figure 9. The variation of $\overline{S}^{\{a,b;q\}}$ w.r.t. a and b, when $q \in \{1, 11, 21, 31\}$.

The family of Fisher-like Riemannian metrics $\overline{g}^{\{a,b;q\}}$ may be considered as evolving from the classical Fisher metric g^0, and also as "expanding" from the BGS group logarithm to the q-dependent Tsallis group logarithm. The evolution of these metrics may be controlled through their scalar curvature, which, in addition to the previous case VII_1, "foliates" following the values of q.

Remark 3. *(i) The parameters' domains are subsets of $\mathbb{R} \times (0, \infty)$, which is two-dimensional. Therefore, for all the metrics in this section, the scalar curvature coincides with the Gaussian curvature. The coefficients of the metrics depend on the variable θ^2 only, which has the signification of standard deviation. It follows that the scalar curvature functions are also independent on the mean of the PDF modeled by θ^1. This dependence of the geometric invariants only on the standard deviation suggests applications where a similar property appears: see, for example, [50–54].*

(ii) Using general differential geometric arguments, we knew a priori that the metrics must be (locally) conformal with the Euclidean (or Minkowskian) metric of the plane. However, we obtained more: the conformal factors are explicitly derived, they are global and, as expected, they are also independent of the mean θ^1. Moreover, the metric $\overline{g}$ in example VII is even homothetic with the Euclidean metric.

If we consider a curve in the parameters space, its length (w.r.t any of the respective metrics) depends only on the standard deviation; instead, the angle of two such curves does not depend on either the mean or the standard deviation.

(iii) The statistical significance of the sectional curvature of Fisher-like metrics g, $\tilde{g}$, h, β, $\hat{g}$, $\overline{g}$ can be obtained by analogy with Ruppeiner's geometric modelization of the Gaussian thermodynamic fluctuations [55]. His "thermodynamic curvature" (R) corresponds to the sectional curvature and measures the inter-particles interaction: when $R = 0$, there is no interaction, and the cases $R > 0$ or $R < 0$ correspond to repulsive or attractive interactions, respectively ([55], apud [48,56]). This approach was developed and generalized by the Geometrothermodynamics theory [57].

Another viewpoint interprets the scalar curvature as a measure of the stability of the statistical model, in a direct proportionality relation ([58], apud [59]).

(iv) It may be worth noting the following special property, apparently collateral to the main path of the discourse. Let us fix a value of the Tsallis parameter q_0 and a value of the scalar curvature $\overline{S}^{\{a,b;q_0\}}$ in example VII_2, denoted by s_0. Then, the solution of the equation

$$s_0 = -\frac{1}{2} \cdot \left\{ \frac{2}{1+e} - \frac{a^2 + eb^3 + q_0 - 1}{(1+e)^2} \right\}$$

is an elliptic curve in the plane of coordinates (a, b), written in Weierstrass form. In Figure 10, we drew these elliptic curves, corresponding to $s_0 = -1$ and to $q_0 \in \{1, -51, -101, -1001\}$ (from left to right).

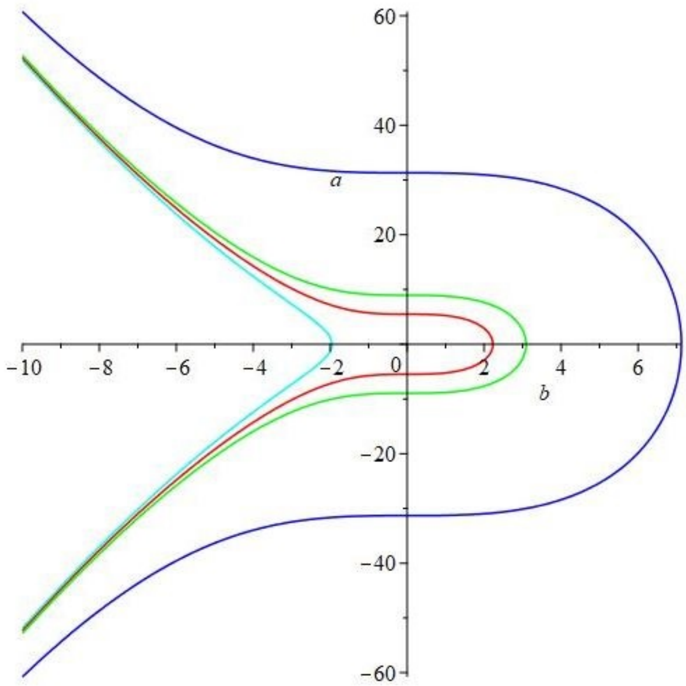

Figure 10. The elliptic curves associated with $s_0 = -1$ and $q_0 \in \{1, -51, -101, -1001\}$.

6. The MaxEnt Problem for the ϕ-Deformed (Naudts) Entropy

Let $V = V(x)$ be a fixed potential energy function, ϕ be a fixed positive strictly-increasing function and $U_0 > 0$ be a fixed real number. Consider $\rho = \rho(x)$ a univariate PDF, satisfying

$$\int_{\mathbb{R}} V(x)\rho(x)dx = U_0$$

and let $H_\phi^N[\rho]$ be its associated ϕ-deformed (Naudts) entropy, based on (5).

Theorem 1. *The optimization problem*

$$max \ H_\phi^N[\rho]$$

has the solution

$$\rho_\phi^{ME}(x) = exp_{\{\phi\}}^N \Big[\gamma + \beta V(x) \Big], \tag{33}$$

where $exp_{\{\phi\}}^N$ is the inverse function of $log_{\{\phi\}}^N$; β and γ are the Lagrange multipliers determined by the constraints, and satisfy the inequality $\gamma + \beta V(x) > 0$.

Proof. The proof is a standard one; see, for example, [60], §12.1. □

Remark 4. *Under the previous hypothesis, we denote: the (maximal) ϕ-deformed (Naudts) entropy $H := H^N_\phi[\rho^{ME}_\phi]$; the mean force with respect to ρ^{ME}_ϕ*

$$U := \int_{\mathbb{R}} V(x) \cdot \rho^{ME}_\phi(x)dx;$$

the ϕ-deformed generalized free energy

$$F := -\frac{\gamma}{\beta}.$$

We obtain ϕ-deformed generalizations of the thermodynamic relations:

$$F = U + \frac{1}{\beta}H \quad , \quad \frac{d}{d\beta}(\beta F) = U.$$

In the previous relations, all the notions depend on ϕ; we skipped it, in order to keep the formalism simpler. For some physical interpretations, we recommend [29,61,62]. In the particular cases when the ϕ-deformed (Naudts) entropy is of Tsallis or of Kaniadakis type, we recover the formulas from [38,39].

7. Conclusions

(**i**) In this paper, we refined the search of relevant semi-Riemannian metrics associated in a canonical manner to manifolds of parameterized PDFs, via remarkable entropies and divergences. We stress the main general ideas:

- We made the distinction between quotient divergence and difference divergence, leading to different metrics g and $\tilde{g}$ (see Example 2 (i), (ii) and Formulas (14) and (15));
- We defined the (0,2)-type tensor fields α and β, as possible candidates for Fisher-like metrics (see (17) and (18));
- We gave an interpretation of the GFM1, whose coefficients may be derived from a variation of a generalized (difference) divergence (Remark 1 (i)).

(**ii**) In particular, based on the ϕ-deformed (Naudts) entropy, we focused on the following topics:

- We calculated the coefficients of the metrics g, $\tilde{g}$, h, α, β, $\hat{g}$, $\overline{g}$ in terms of log^N and of the PDF ρ (Propositions 1–3);
- When the PDFs are normal, univariate and depending on two parameters, we provided seven families of examples of the previous metrics; we determined formulas for their scalar curvature and we discussed its variation w.r.t. parameters;
- We proved a MaxEnt result (Theorem 1) for univariate PDFs and some extensions of the thermodynamic relations (Remark 4).

(**iii**) Future work will be directed toward:

- The search of the statistical relevance of α and β and a statistical interpretation for quotient divergences, similar to that for difference divergences (in the Remark 1 (i));
- The characterization of the case when the quotient divergence coincides with the difference divergence; this kind of result might bring into light unexpected—and eventually important—families of entropies;
- Refining the known families of deformation functions ϕ and finding new ones, relevant for applications. The interplay between the choice of ϕ and of the group logarithm G offers many modeling opportunities.

(**iv**) There exist two different but connected approaches to entropy: in Thermodynamics and in Statistical mechanics. Its geometrization by means of Fisher metrics follows two apparently different paths. The procedures to construct Fisher-like metrics from entropy are analogous, as they originate from the same general differential geometric methods. Instead, the basic manifold these metrics act upon (i.e., the space of the parameters) is essentially different. Moreover, entropy in Thermodynamics is "more deterministic" and one does not use a log-likelihood function which "produces" it.

The first formalism is dominated by the ideas of Weinhold, Ruppeiner and Quevedo [55,57,63], and is extensively used in models for the entropy of black holes (see [64] and references therein).

Our paper engaged in the second path and is dependent of log-likelihood functions, especially of the ϕ-deformed (Naudts) one. However, we are aware that more connections between the two theories are needed, with refined comparisons of the Riemannian models they both rely on.

Author Contributions: Conceptualization, C.-L.P., I.-E.H., G.-T.P. and V.P. ; validation, C.-L.P., I.-E.H., G.-T.P. and V.P.; writing, C.-L.P., I.-E.H., G.-T.P. and V.P.; visualization, G.-T.P.; supervision, C.-L.P., I.-E.H., G.-T.P. and V.P. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Acknowledgments: We are grateful to the reviewers for their valuable enlightening remarks.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Brechtl, J.; Liaw, P.K. *High-Entropy Materials: Theory, Experiments, and Applications*; Springer: Cham, Switzerland, 2021.
2. Calin, O.; Udriste, C. *Geometric Modeling in Probability and Statistics*; Springer: New York, NY, USA, 2014.
3. Di Crescenzo, A.; Longobardi, M. On cumulative entropies and lifetime estimation. In *Methods and Models in Artificial and Natural Computation*; Mira, J.M., Ferrández, J.M., Alvarez-Sachez, J.-P., Paz, F., Toledo, J., Eds.; IWINAC Part I, LNCS 5601; Springer: Berlin, Germany, 2009; pp. 132–141.
4. Furuichi, S. On the maximum entropy principle and the minimization of the Fisher information in Tsallis statistics. *J. Math. Phys.* **2009**, *50*, 013303. [CrossRef]
5. Gell-Mann, M.; Tsallis, C. *Non-Extensive Entropy- Interdisciplinary Applications*; Oxford University Press: Oxford, UK, 2004.
6. Gray, R.M. *Entropy and Information Theory*; Springer: New York, NY, USA, 2011.
7. Guiasu, S. *Information Theory with Applications*; McGraw-Hill: New York, NY, USA, 1977.
8. Kelbert, M.; Stuhlb, I.; Suhov, Y. Weighted entropy: Basic inequalities. *Mod. Stochastics Theory Appl.* **2017**, *4*, 233–252. [CrossRef]
9. Klein, I.; Mangold, B.; Doll, M. Cumulative Paired ϕ-Entropy. *Entropy* **2016**, *18*, 248. [CrossRef]
10. Klein, I.; Doll, M. (Generalized) Maximum Cumulative Direct, Residual, and Paired ϕ Entropy Approach. *Entropy* **2020**, *22*, 91. [CrossRef] [PubMed]
11. Martin, N.F.G.; England, J.W.; Brooks, J.K. *Mathematical Theory of Entropy*; Addison-Wesley: Reading, PA, USA, 1981.
12. Nielsen, F. An Elementary Introduction to Information Geometry. *Entropy* **2020**, *22*, 1100. [CrossRef] [PubMed]
13. Papadimitriou, F. *Spatial Entropy and Landscape Analysis*; Springer: Wiesbaden, Germany, 2022.
14. Sagawa, T. *Entropy, Divergence, and Majorization in Classical and Quantum Thermodynamics*; Springer Nature: Singapore, 2022.
15. Sfetcu, R.-C.; Sfetcu, S.-C.; Preda, V. Ordering Awad–Varma Entropy and Applications to Some Stochastic Models. *Mathematics* **2021**, *9*, 280. [CrossRef]
16. Sherman, T.F. *Energy, Entropy, and the Flow of Nature*; Oxford University Press: Oxford, UK, 2018.
17. Tame, J.R.H. *Approaches to Entropy*; Springer: Singapore, 2019.
18. Popkov, Y.S.; Popkov, A.Y.; Dubnov, Y.A. *Entropy Randomization in Machine Learning*; CRC Press: Boca Raton, FL, USA, 2023.
19. Wei, I.; Ting, K.; Tebourbi, I.; Lu, W.-M.; Kweh, Q.L. The effects of managerial ability on firm performance and the mediating role of capital structure: Evidence from Taiwan. *Financ. Innov.* **2021**, *7*, 89.
20. Gomez, I.; Portesi, M.; Borges, E.P. Universality classes for the Fisher metric derived from relative group entropy. *Phys. A Stat. Mech. Its Appl.* **2020**, *547*, 123827. [CrossRef]
21. Tempesta, P. Multivariate group entropies, super-exponentially growing complex systems and functional equations. *Chaos* **2020**, *30*, 123119. [CrossRef]
22. Amari, S. *Information Geometry and Its Applications*; Springer: Tokyo, Japan, 2016.
23. Ay, N.; Jost, J.; Lê, H.V.; Schwachhöfer, L. *Information Geometry*; Springer: Cham, Switzerland, 2017.
24. Udriste, C.; Tevy, I. Information Geometry in Roegenian Economics. *Entropy* **2022**, *24*, 932. [CrossRef]
25. Eguchi, S.; Komori, O. *Minimum Divergence Methods in Statistical Machine Learning from an Information Geometric Viewpoint*; Springer: Tokyo, Japan, 2022.

26. Sason, I. Divergence Measures: Mathematical Foundations and Applications in Information-Theoretic and Statistical Problems. *Entropy* **2022**, *24*, 712. [CrossRef] [PubMed]
27. Naudts, J. Deformed exponentials and logarithms in generalized thermostatistics. *Phys. A* **2002**, *316*, 323–334. [CrossRef]
28. Naudts, J. Continuity of a class of entropies and relative entropies. *Rev. Math. Phys.* **2004**, *16*, 809–822. [CrossRef]
29. Naudts, J. *Generalized Thermostatistics*; Springer: London, UK, 2011.
30. Korbel, J.; Hanel, R.; Thurner, S. Information Geometric Duality of ϕ-Deformed Exponential Families. *Entropy* **2019**, *21*, 112. [CrossRef]
31. Naudts, J. Update of Prior Probabilities by Minimal Divergence. *Entropy* **2021**, *23*, 1668. [CrossRef]
32. Trivellato, B. Deformed Exponentials and Applications to Finance. *Entropy* **2013**, *15*, 3471–3489. [CrossRef]
33. Matsuzoe, H.; Scarfone, A.M.; Wada, T. A sequential structure of statistical manifolds on deformed exponential family. In *Geometric Science of Information*; GSI 2017, Lecture Notes in Computer Science; Nielsen, F., Barbaresco, F., Eds.; Springer: Berlin, Germany, 2017; Volume 10589, pp. 223–230.
34. Hirica, I.E.; Pripoae, C.-L.; Pripoae, G.-T.; Preda, V. Weighted Relative Group Entropies and Associated Fisher Metrics. *Entropy* **2022**, *24*, 120. [CrossRef]
35. Preda, V.; Balcau, C. Convex quadratic programming with weighted entropic perturbation. *Bull. Math. Soc. Sci. Math. Roum.* **2009**, *52*, 57–64.
36. Preda, V.; Balcau, C. *Entropy Optimization with Applications*; Academiei Romane: Bucharest, Romania, 2010.
37. Sfetcu, R.C.; Sfetcu, S.C.; Preda, V. On Tsallis and Kaniadakis Divergences. *Math. Phys. Anal. Geom.* **2022**, *25*, 7. [CrossRef]
38. Hirica, I.E.; Pripoae, C.-L.; Pripoae, G.-T.; Preda, V. Lie Symmetries of the Nonlinear Fokker–Planck Equation Based on Weighted Kaniadakis Entropy. *Mathematics* **2022**, *10*, 2776. [CrossRef]
39. Pripoae, C.-L.; Hirica, I.E.; Pripoae, G.-T.; Preda, V. Lie symmetries of the nonlinear Fokker–Planck equation based on weighted Tsallis entropy. *Carpathian J. Math.* **2022**, *38*, 597–617. [CrossRef]
40. Scarfone, A.M.; Wada, T. Lie symmetries and related group-invariant solutions of a nonlinear Fokker–Planck equation based on the Sharma–Taneja–Mittal entropy. *Braz. J. Phys.* **2009**, *39*, 475–482. [CrossRef]
41. Tempesta, P. Group entropies, correlation laws, and zeta functions. *Phys. Rev. E* **2011**, *84*, 021121. [CrossRef] [PubMed]
42. Csiszar, I. Why least squares and maximum entropy? An axiomatic approach to inference for linear inverse problems. *Ann. Stat.* **1991**, *19*, 2032–2066. [CrossRef]
43. Sason, I. On f-Divergences: Integral Representations, Local Behavior, and Inequalities. *Entropy* **2018**, *20*, 383. [CrossRef]
44. Birrell, J.; Dupuis, P.; Katsoulakis, M.A.; Pantazis, Y.; Rey-Bellet, L. $(f;\Gamma)$-Divergences: Interpolating between f-Divergences and Integral Probability Metrics. *J. Mach. Learn. Res.* **2022**, *23*, 1–70.
45. Suter, F.; Cernat, I.; Dragan, M. Some Information Measures Properties of the GOS-Concomitants from the FGM Family. *Entropy* **2022**, *24*, 1361. [CrossRef]
46. Futami, F.; Iwata, T.; Ueda, N.; Sato, I. Accelerated Diffusion- Based Sampling by the Non-Reversible Dynamics with Skew-Symmetric Matrices. *Entropy* **2021**, *23*, 993. [CrossRef]
47. Udriste, C.; Tevy, I. *Variational Calculus with Engineering Applications*; John Wiley & Sons: Hoboken, NJ, USA, 2023.
48. Dimov, H.; Mladenov, S.; Rashkov, R.C.; Vetsov, T. Entanglement entropy and Fisher information metric for closed bosonic strings in homogeneous plane wave background. *Phys. Rev. D* **2017**, *96*, 126004. [CrossRef]
49. Bauer, M.; Bruveris, M.; Michor, P.W. Uniqueness of the Fisher-Rao metric on the space of smooth densities. *Bull. Lond. Math. Soc.* **2016**, *48*, 499–506. [CrossRef]
50. Javaudin, B.; Gilblas, R.; Sentenac,T.; Le Maoult, Y. Experimental validation of the diffusion function model for accuracy-enhanced thermoreflectometry. *Quant. InfraRed Thermogr. J.* **2021**, *18*, 18–33. [CrossRef]
51. Lederer, A.; Zhang, M.; Tesfazgi, S.; Hirche, S. Networked Online Learning for Control of Safety-Critical Resource-Constrained Systems based on Gaussian Processes. *arXiv* **2022**, arXiv:2202.11491v1.
52. Rajaram, S.; Heinrich, L.E.; Gordan, J.D.; Avva, J.; Bonness, K.M.; Witkiewicz, A.K.; Malter, J.S.; Atreya, C.E.; Warren, R.S.; Wu, L.F.; et al. Sampling to capture single-cell heterogeneity. *Nat. Methods* **2017**, *14*, 967–970. [CrossRef]
53. Sharp, J.A.; Browning, A.P.; Burrage, K.; Simpson, M.J. Parameter estimation and uncertainty quantification using information geometry. *J. R. Soc. Interface* **2022**, *19*, 20210940. [CrossRef]
54. Zhao, T.; Pan, B.; Song, X.; Sui, D.; Xiao, H.; Zhou, J. Heuristic Approaches Based on Modified Three-Parameter Model for Inverse Acoustic Characterisation of Sintered Metal Fibre Materials. *Mathematics* **2022**, *10*, 3264. [CrossRef]
55. Ruppeiner, G. Riemannian geometry in thermodynamic fluctuation theory. *Rev. Mod. Phys.* **1995**, *67*, 605–659. [CrossRef]
56. Dehyadegari, A.; Sheykhi, A.; Wei, S.W. Microstructure of charged AdS black hole via $P-V$ criticality. *Phys. Rev. D* **2020**, *102*, 104013. [CrossRef]
57. Quevedo, H. Geometrothermodynamics. *J. Math. Phys.* **2007**, *48*, 013506. [CrossRef]
58. Janyszek, H.; Mrugala, R. Riemannian geometry and the thermodynamics of model magnetic systems. *Phys. Rev. A* **1989**, *39*, 6515. [CrossRef]
59. Felice, D.; Cafaro, C.; Mancini, S. Information geometric methods for complexity. *Chaos* **2018**, *28*, 032101. [CrossRef]
60. Cover, T.M.; Thomas J.A. *Elements of Information Theory*, 2nd ed.; Wiley-Interscience: Hoboken, NJ, USA, 2006.
61. Wada, T.; Scarfone, A.M. On the nonlinear Fokker–Planck equation associated with k-entropy. *AIP Conf. Proc.* **2007**, *965*, 177–180.

62. Wada, T.; Scarfone, A.M. Asymptotic solutions of a nonlinear diffusive equation in the framework of a k-generalized statistical mechanics. *Eur. Phys. J. B* **2009**, *70*, 65–71. [CrossRef]
63. Brody, D.; Rivier, N. Geometrical aspects of statistical mechanics. *Phys. Rev. E* **1995**, *51*, 1006–1011. [CrossRef] [PubMed]
64. Dimov, H.; Iliev, I.N.; Radomirov, M.; Rashkov, R.C.; Vetsov, T. Holographic Fisher information metric in Schrödinger spacetime. *Eur. Phys. J. Plus* **2021**, *136*, 1128. [CrossRef]