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Computing Sharp Bounds of Metric Based Fractional Dimensions for the Sierpinski Networks

Arooba Fatima [1], Ahmed Alamer [2] and Muhammad Javaid [1,*]

[1] Department of Mathematics, School of Science, University of Management and Technology, Lahore 54770, Pakistan
[2] Department of Mathematics, University of Tabuk, Tabuk 71491, Saudi Arabia
* Correspondence: javaidmath@gmail.com or muhammad.javaid@umt.edu.pk

Abstract: The concept of metric dimension is widely applied to solve various problems in the different fields of computer science and chemistry, such as computer networking, integer programming, robot navigation, and the formation of chemical structuring. In this article, the local fractional metric dimension (LFMD) of the cycle-based Sierpinski networks is computed with the help of its local resolving neighborhoods of all the adjacent pairs of vertices. In addition, the boundedness of LFMD is also examined as the order of the Sierpinski networks approaches infinity.

Keywords: fractional metric dimension; Sierpinski networks; metric index; distance in networks

MSC: 05C12; 05C90; 05C15; 05C62

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1. Introduction

In connected networks, the distance between vertices (nodes) plays an important role in the study of the different structural properties such as connectivity, robustness, completeness, and complexity. Being a distance-based parameter, metric dimension is used to find the minimum number of nodes as object locations of the auto-machines in a network [1,2].

For a vertex $x \in V(\mathbb{N})$ and a set of vertices $P = \{p_1, p_2, p_3, \ldots, p_l\}$ of a network $\mathbb{N}$, the l-tuple metric form of P in terms of x is $r(x/P) = (d(x, p_1), d(x, p_2), d(x, p_3), \ldots, d(x, p_l))$. The set $P \subseteq \mathbb{N}$ becomes a resolving set if it has unique metric form with respect to each x in $\mathbb{N}$. Moreover, the metric dimension of $\mathbb{N}$ is defined as $dim(\mathbb{N}) = \min\{|P|: P$ is the resolving set of $\mathbb{N}\}$, where P of minimum cardinality is considered as a base set of $\mathbb{N}$ [3]. In 1975, Slater commenced the location number for the connected networks [4]. Subsequently, Melter and Harary worked on this concept in 1976, and they used the term metric dimension instead of location number [5]. Shahida and Sunitha found the metric dimension of the networks under different operations [6]. In 2000, Chartrand et al. developed the sharp bounds of unicylic networks by using metric dimension (MD) [7].

Currie and Ollermann used the concept of fractional metric dimension (FMD) as the optimal solution of non-integer programming problems (IPP) [8]. In 2012, Arumugam and Mathew gave the formal definition of FMD and also proved its different basic properties. They derived sufficient conditions for connected networks whose FMD is $\frac{|V|}{2}$, where $|V|$ is the order of the network under study [9]. In addition, they found the FMD of some basic networks such as Petersen, cycles, wheels, hypercubes, and friendship networks [10]. In addition, fractional metric dimensions of trees and uni-cyclic networks were discussed in [11]. Subsequently, many researchers worked on this concept and developed the FMDs for various networks under different operations of product such as comb, hierarchical, corona, and lexicographic [12–14]. The fractional metric dimension of generalized Jahangir networks, molecular networks, and metal-organic networks were discussed in [15–17]. The lower and upper bounds of FMD were discussed in [3]. The improved lower bound of FMD from unity can be found in [1]. The exact value of FMD of double connected networks, circular double ladder networks, and double path networks are computed in terms of their order [18]. For more results, we refer to [19,20].

Local fractional metric dimension (LFMD) was introduced in 2019 by Aisyah et al. They discussed the LFMD of corona products of networks [21]. The outcomes of the sharp bounds of LFMD on connected structures can be found in [1]. The sharp bounds of LFMD for cycle-related structures and prism-related structures are in [22]. In 2021, Javaid et al. discussed the boundedness of convex polytope networks using the concept of LFMD [23].

In this article, we computed the sharp lower and upper bounds of the local fractional metric dimensions for the cycle-based Sierpinski networks in terms of the order of the used cycles. Moreover, we checked the boundedness and unboundedness of the obtained results when the order of the used cycles approaches infinity. This article is organized as follows: Section 1 narrates the introduction, Section 2 describes preliminaries, Section 3 presents LRNs of the Sierpinski network of the cycle network, Section 4 deals with the LFMDs of the Sierpinski network of the cycle network, and Section 5 comprises the discussion and conclusion.

2. Preliminaries

A simple connected network $\mathbb{N}$ (without loops and multiple edges) can be expressed by a vertex set $V(\mathbb{N})$ and an edge set $E(\mathbb{N})$, which is a subset of $V(\mathbb{N})$ x $V(\mathbb{N})$, where $|V(\mathbb{N})|$ and $|E(\mathbb{N})|$ are the order and the size of network $\mathbb{N}$. A walk (sequence of vertices and edges) without repetition of vertices is called a path. A cycle is a nontrivial path in which only one vertex is repeated. The length of the path is the number of edges on it. The distance between u and v of $\mathbb{N}$ is the length of the shortest path between them and it is denoted by $d(u,v)$. For more information regarding network theory, we refer to [24–26].

For an edge $uv \in E(\mathbb{N})$, the vertex $z \in V(\mathbb{N})$ locally resolves uv if $d(u,z) \neq d(v,z)$. The set consisting of all such vertices is known as the local resolving neighborhood set (LRNs) of uv as $LR(uv) = \{z \in V(\mathbb{N}) : d(u,z) \neq d(v,z)\}$. Now, we discuss some basic definitions that help to compute the main results as follows:

Definition 1. *For a vertex set $V(\mathbb{N})$ of a connected network $\mathbb{N}$, a real valued function $f : V(\mathbb{N}) \to [0,1]$ is said to be a local resolving function (LRF) if $f(LR(uv)) \geq 1$ for every edge uv in network $\mathbb{N}$, where $f(LR(uv)) = \sum_{x \in LR(uv)} f(x)$ [21].*

Definition 2. *An LRF, f becomes minimal LRF if each $f^{'} : V(\mathbb{N}) \to [0,1]$ such that $f^{'}(x) \leq f(x)$ and $f^{'}(x) \neq f(x)$ for at least one $x \in V(\mathbb{N})$, which is not the LRF of network $\mathbb{N}$ [21].*

Definition 3. *The local fractional metric dimension ($dim_{lfm}(\mathbb{N})$) is defined as $dim_{lfm}(\mathbb{N})$ = min $\{|f|$: f is local minimal resolving function of $\mathbb{N}\}$, where $|f| = \sum_{u \in V(\mathbb{N})} f(u)$ [21].*

Lemma 1 ([1])**.** *Let $\mathbb{N} = (V(\mathbb{N}), E(\mathbb{N}))$ be a connected network and $LR(e)$ be the local resolving neighborhood set. Then,*

$$\frac{|V(\mathbb{N})|}{\beta} \leq dim_{lfm}(\mathbb{N}),$$

where $\beta = max\{|LR(e)| : e \in E(\mathbb{N})\}$ and $2 \leq \beta \leq |V(\mathbb{N})|$.

Construction of C_k-based Sierpinski Networks: Sierpinski networks are a family of fractal networks widely studied and applied in topology, computer sciences, and other fields. Many researchers worked on Sierpinski networks and they discussed their average eccentricity, variants of covering codes, and coloring [27–29]. Klavzar and Milutinovic proposed the construction method of the Sierpinski network of complete network K_k. In the first step, they considered the complete network of order k and denoted it as $S(1, K_k)$.

In the second step, they took k-copies of $S(1, K_k)$ and added an edge between each pair of $S(1, K_k)$ for the formation of $S(2, K_k)$. By the repetition of this process, they derive $S(n, K_k)$, where $n \leq k$ [30]. Gravier et al. investigated the generalized Sierpinski network $S(n, \mathbb{N})$ for any network $\mathbb{N}$ [31]. For more detail, see the survey [32].

Now we define a C_k-based Sierpinski network, where C_k is the cycle network of order k in which each vertex has degree 2. The vertex set and edge set of C_k-based Sierpinski network $S(2, C_k)$ are $V(S(2, C_k)) = \{x_i, v_i^j; 1 \leq i \leq k; 1 \leq j \leq k-1\}$ and $E(S(2, C_k)) = \{x_i v_{i+1}^{k-2t-2}, x_i v_{i+1}^{k-1}, v_{i+1}^j v_{i+1}^{j+1}, v_i^{k-2t-2} v_{i+1}^{k-1}; 1 \leq i \leq k; 1 \leq j \leq k-1\}$ respectively, where $t = \frac{k-3}{2}$. The order and size of $S(2, C_k)$ are k^2 and $k(k+1)$, respectively. The C_3-based Sierpinski network $S(2, C_3)$ is shown in Figure 1.

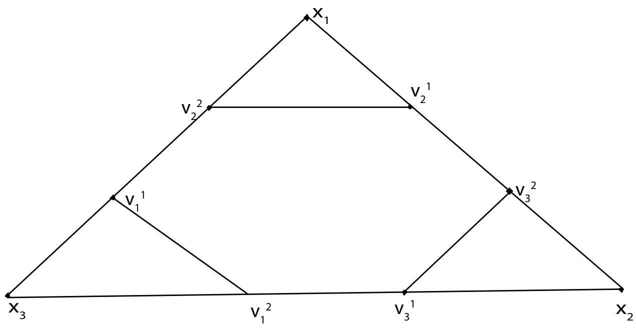

Figure 1. A C_3-based Sierpinski network with $n = 2$, $S(2, C_3)$.

The vertex set and edge set of C_k-based Sierpinski network $S(3, C_k)$ are $V(S(3, C_k)) = \{x_i^l, v_i^{j,l}; 1 \leq i \leq k; 1 \leq j \leq k-1; 1 \leq l \leq k\}$ and $E(S(3, C_k)) = \{x_i^l v_{i+1}^{k-2t-2,l}, x_i^l v_{i+1}^{k-1,l}, v_i^{k-2t-2,l} v_{i+1}^{k-1,l}, v_{i+1}^{j,l} v_{i+1}^{j+1,l}, x_i^{i+1} x_{i+1}^i; 1 \leq i \leq k; 1 \leq j \leq k-1; 1 \leq l \leq k\}$ respectively, where $t = \frac{k-3}{2}$. The order and size of $S(3, C_k)$ are k^3 and $k(k^2 + k + 1)$, respectively. The C_3-based Sierpinski network $S(3, C_3)$ is shown in Figure 2.

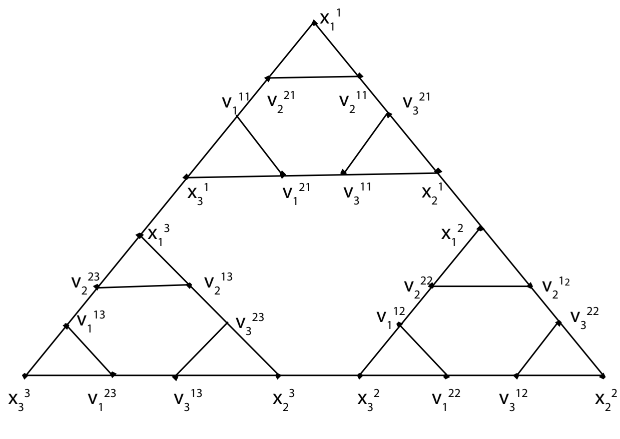

Figure 2. A C_3-based Sierpinski Network with $n = 3$, $S(3, C_3)$.

3. LRNs of C_k-Based Sierpinski Networks

In this section, the local resolving neighborhood sets of the C_k-based Sierpinski networks are classified.

Lemma 2. *Let $\mathbb{N} \cong S(2, C_k)$ be a C_k-based Sierpinski network with $k \geq 7$ and $k \cong 1(mod 2)$. Then we have*

1. $\frac{k^2+2k-5}{2} \leq |LR(e)| \; \forall \, e \in E(\mathbb{N})$,
2. $|LR(e) \cap \bigcup_{i=1}^{k} LR(e_i^w)| \geq |LR(e_i^w)|$, *where* $|LR(e_i^w)| = \frac{k^2+2k-5}{2}$, $w = 1, 2$ *and* $1 \leq i \leq k$.

Proof. 1. For $1 \leq i \leq k$, we have eight types of edges in a network $\mathbb{N}$. Three types of LRNs of these edges as follows:
$LR(e_i^1) = LR(v_{i+1}^{k-t-3} v_{i+1}^{k-t-2}) = V(\mathbb{N}) - \{x_{s+t+i+1}, v_{i+1}^{k-1}, v_{p+i+1}^{j}, v_{k-t+i}^{r}; k-t \leq p \leq k-1; 1 \leq s \leq k-t-2; 1 \leq r \leq k-t-1; 1 \leq j \leq k-1\}$,
$LR(a_i) = LR(x_i v_{i+1}^{k-2t-2}) = V(\mathbb{N}) - \{v_{i+1}^{k-t-1}, v_{k-t+i}^{p}; k-t \leq p \leq k-1\}$,
$LR(c_i) = LR(v_{i+1}^{m'} v_{i+1}^{m'+1}) = V(\mathbb{N}) - \{v_{i+1}^{k-t+m'-1}\}$, where $1 \leq m' \leq \frac{k-5}{2}$.

Similarly, we have obtained $LR(e_i^2) = LR(v_{i+1}^{k-t-1} v_{i+1}^{k-t})$, $LR(b_i) = LR(x_i v_{i+1}^{k-1})$, $LR(d_i) = LR(v_{i+1}^{k-t-2} v_{i+1}^{k-t-1}) = LR(e_i) = LR(v_{i+1}^{k-t-m'-1} v_{i+1}^{k-t+m'}) = LR(f_i) = LR(v_{i+1}^{k-2t-2} v_{i+2}^{k-1})$ and the cardinalities of all LRNs are in Table 1.

2. Table 1 shows that $|LR(e_i^w)| \leq |LR(e)| \; \forall \, e \in E(\mathbb{N})$ and $|LR(e) \cap \bigcup_{i=1}^{k} LR(e_i^w)| > |LR(e_i^w)|$.

$\square$

Table 1. Local resolving neighborhood set (LRNs) of $S(2, C_k)$.

LRNs	Cardinalities
$LR(e_i^1)$, $LR(e_i^2)$	$\frac{k^2+2k-5}{2}$
$LR(a_i)$, $LR(b_i)$	$k^2 - (t+1)$
$LR(c_i)$, $LR(d_i)$, $LR(e_i)$, $LR(f_i)$	$k^2 - 1$

Lemma 3. *Let $\mathbb{N} \cong S(3, C_3)$ be a C_3-based Sierpinski network. Then we have*

1. $14 \leq LR(e) \; \forall \, e \in E(\mathbb{N})$,
2. $|LR(e) \cap \bigcup_{i=1}^{3} LR(e_i^y)| \geq |LR(e_i^y)|$, *where* $|LR(e_i^y)| = 14$, $y = 1, 2$ *and* $1 \leq i \leq 3$.

Proof. 1. For $1 \leq i \leq 3$, we have 13 types of edges in a network $\mathbb{N}$. Five types of LRNs of these edges are as follows:

$LR(e_i^1) = LR(x_i^i v_{i+1}^{1,i}) = V(\mathbb{N}) - \{v_{i+1}^{2,i}, x_{i+2}^{i}, v_i^{j,i}, x_l^{i+2}, v_l^{j,i+2}; 1 \leq l \leq 3; 1 \leq j \leq 2\}$,
$LR(u_i^1) = LR(x_{i+1}^{i} v_{i+2}^{1,i}) = V(\mathbb{N}) - \{v_{i+2}^{2,i}, x_i^i, v_{i+1}^{j,i}, v_{i+2}^{1,i+2}; 1 \leq j \leq 2\}$,
$LR(u_i^2) = LR(x_{i+1}^{i} v_{i+2}^{2,i}) = V(\mathbb{N}) - \{v_{i+2}^{1,i}, x_{i+2}^{i}, v_i^{j,i} x_i^{i+2}, x_{i+2}^{i+2}, v_i^{j,i+2}, v_{i+1}^{j,i+2}, v_{i+2}^{2,i+2}; 1 \leq j \leq 2\}$,
$LR(u_i^7) = LR(v_i^{1,i} v_{i+1}^{2,i}) = V(\mathbb{N}) - \{x_{i+1}^{i}, x_i^{i+1}, x_{i+1}^{i+1}, v_l^{j,i+1}; 1 \leq l \leq 3; 1 \leq j \leq 2\}$,
$LR(u_i^9) = LR(v_{i+1}^{1,i} v_{i+1}^{2,i}) = V(\mathbb{N}) - \{x_i^i\}$.

Similarly, we obtained $LR(e_i^2) = LR(x_i^i v_{i+1}^{2,i})$, $LR(u_i^5) = LR(x_{i+2}^{i} v_i^{2,i})$, $LR(u_i^3) = LR(v_{i+2}^{1,i} v_{i+2}^{2,i})$, $LR(u_i^4) = LR(x_{i+2}^{i} v_i^{1,i})$, $LR(u_i^6) = LR(v_i^{1,i} v_i^{2,i})$, $LR(u_i^8) = LR(v_{i+1}^{1,i} v_{i+2}^{2,i})$, $LR(u_i^{10}) = LR(v_{i+2}^{1,i} v_i^{2,i})$ and $LR(u_i^{11}) = LR(x_{i+1}^{i} x_i^{i+1})$ and the cardinalities of all LRNs are in Table 2.

2. Table 2 shows that $|LR(e_i^y)| \leq |LR(u_i^z)|$ and $|LR(u_i^z) \cap \bigcup_{i=1}^{3} LR(e_i^y)| > |LR(e_i^y)|$.

$\square$

Table 2. Local resolving neighborhood set (LRNs) $LR(e_i^y)$ for $1 \le y \le 2$ and $LR(u_i^z)$ for $1 \le z \le 11$.

LRNs	Cardinalities
$LR(e_i^1)$, $LR(e_i^2)$	14
$LR(u_i^1)$, $LR(u_i^5)$	22
$LR(u_i^2)$, $LR(u_i^3)$, $LR(u_i^4)$, $LR(u_i^6)$	16
$LR(u_i^7)$, $LR(u_i^8)$	18
$LR(u_i^9)$, $LR(u_i^{10})$, $LR(u_i^{11})$	26

Lemma 4. *Let $\mathbb{N} \cong S(3, C_5)$ be a C_5-based Sierpinski network. Then we have*

1. $63 \le LR(e)\ \forall\ e \in E(\mathbb{N})$,
2. $|LR(e) \cap \bigcup_{i=1}^{5} LR(e_i^y)| \ge |LR(e_i^y)|$, *where* $|LR(e_i^y)| = 63$, $y = 1, 2$ *and* $1 \le i \le 5$.

Proof. 1. For $1 \le i \le 5$, we have 31 types of edges in a network $\mathbb{N}$. Eleven types of LRNs of these edges are as follows:
$LR(e_i^1) = LR(v_{i+3}^{1,i} v_{i+3}^{2,i}) = V(\mathbb{N}) - \{v_{i+3}^{4,i}, x_{i+1}^i, v_{i+2}^{j,i}, x_i^i, v_{i+1}^{1,i}, v_{i+1}^{2,i}, v_{i+1}^{3,i}, x_l^{i+1}, v_l^{j,i+1}, x_l^{i+2}, v_l^{j,i+2}, x_{i+2}^{i+3}, v_{i+3}^{4,i+3}; 1 \le l \le 5; 1 \le j \le 4\}$,
$LR(u_i^1) = LR(x_i^i v_{i+1}^{1,i}) = V(\mathbb{N}) - \{v_{i+1}^{3,i}, v_{i+4}^{4,i}, v_{i+3}^{4,i+3}, x_{i+2}^{i+3}\}$,
$LR(u_i^{11}) = LR(x_{i+2}^i v_{i+3}^{4,i}) = V(\mathbb{N}) - \{v_{i+3}^{2,i}, v_i^{1,i}, x_{i+1}^{i+3}\}$,
$LR(u_i^{13}) = LR(x_{i+1}^i v_{i+2}^{1,i}) = V(\mathbb{N}) - \{v_{i+2}^{3,i}, v_i^{4,i}\}$,
$LR(u_i^{15}) = LR(v_{i+2}^{3,i} v_{i+2}^{4,i}) = V(\mathbb{N}) - \{v_{i+2}^{1,i}, x_{i+2}^i, v_{i+3}^{j,i}, x_{i+3}^i, v_{i+4}^{2,i}, v_{i+4}^{3,i}, v_{i+4}^{4,i}; 1 \le j \le 4\}$,
$LR(u_i^{17}) = LR(v_{i+2}^{2,i} v_{i+2}^{3,i}) = V(\mathbb{N}) - \{x_{i+1}^i, x_l^{i+1}, v_l^{j,i+1}, x_l^{i+2}, v_l^{j,i+2}, x_{i+2}^{i+3}, v_{i+3}^{j,i+3}, x_{i+1}^{i+3}, v_{i+2}^{1,i+3}, v_{i+2}^{2,i+3}; 1 \le l \le 5; 1 \le j \le 4\}$,
$LR(u_i^{19}) = LR(v_{i+1}^{1,i} v_{i+1}^{2,i}) = V(\mathbb{N}) - \{v_{i+1}^{4,i}, x_{i+4}^i, x_{i+3}^i, v_i^{j,i+3}, v_{i+4}^{1,i}, v_{i+4}^{2,i}, v_{i+4}^{3,i}, x_l^{i+4}, v_l^{j,i+4}, x_i^{i+3}, x_{i+1}^{i+3}, x_{i+3}^{i+3}, x_{i+4}^{i+3}, v_i^{j,i}, v_{i+1}^{j,i+3}, v_{i+2}^{j,i+3}, v_{i+4}^{j,i+3}, v_{i+3}^{1,i+3}, v_{i+3}^{2,i+3}, v_{i+3}^{3,i+3}; 1 \le l \le 5; 1 \le j \le 4\}$,
$LR(u_i^{21}) = LR(v_{i+2}^{1,i} v_{i+2}^{2,i}) = V(\mathbb{N}) - \{v_{i+2}^{4,i}, x_i^i, v_{i+1}^{j,i}, x_{i+4}^i, x_l^{i+4}, v_l^{j,i+4}, x_i^{i+3}, x_{i+3}^{i+3}, x_{i+4}^{i+3}, v_{i+1}^{j,i+3}, v_{i+2}^{4,i+3}, v_i^{j,i+3}, v_i^{1,i}, v_i^{2,i}, v_i^{3,i}, v_{i+4}^{1,i+3}, v_{i+4}^{2,i+3}, v_{i+4}^{3,i+3}; 1 \le l \le 5; 1 \le j \le 4\}$,
$LR(u_i^{23}) = LR(v_{i+3}^{3,i} v_{i+3}^{4,i}) = V(\mathbb{N}) - \{v_{i+3}^{1,i}, x_{i+3}^i, v_{i+4}^{j,i}, x_{i+4}^i, x_l^{i+4}, v_l^{j,i+4}, x_{i+3}^{i+3}, x_{i+4}^{i+3}, v_i^{j,i+3}, v_{i+1}^{j,i+3}, v_{i+4}^{j,i+3}, v_i^{2,i}, v_i^{3,i}, v_i^{4,i}, v_{i+2}^{3,i+3}, v_{i+2}^{4,i+3}, x_i^{i+3}; 1 \le l \le 5; 1 \le j \le 4\}$,
$LR(u_i^{25}) = LR(v_{i+2}^{1,i} v_{i+3}^{4,i}) = V(\mathbb{N}) - \{x_{i+4}^i, x_l^{i+4}, v_l^{j,i+4}, v_{i+2}^{4,i+3}, x_{i+3}^{i+3}, x_{i+4}^{i+3}, v_i^{j,i+3}, v_{i+1}^{j,i+3}, v_{i+4}^{1,i+3}, v_{i+4}^{2,i+3}, v_{i+4}^{3,i+3}, x_i^{i+3}; 1 \le l \le 5; 1 \le j \le 4\}$,
$LR(u_i^{27}) = LR(v_{i+1}^{2,i} v_{i+1}^{3,i}) = V(\mathbb{N}) - \{x_i^i\}$.

Similarly, we obtained $LR(e_i^2) = LR(v_{i+4}^{3,i} v_{i+4}^{4,i})$, $LR(u_i^2) = LR(x_i^i v_{i+1}^{4,i})$, $LR(u_i^3) = LR(x_{i+1}^i v_{i+2}^{4,i})$, $LR(u_i^4) = LR(x_{i+2}^i v_{i+3}^{1,i})$, $LR(u_i^5) = LR(v_{i+3}^{2,i} v_{i+3}^{3,i})$, $LR(u_i^6) = LR(v_{i+4}^{2,i} v_{i+4}^{3,i})$, $LR(u_i^7) = LR(x_{i+3}^i v_{i+4}^{4,i})$, $LR(u_i^8) = LR(x_{i+4}^i v_i^{1,i})$, $LR(u_i^9) = LR(v_i^{1,i} v_{i+1}^{4,i})$, $LR(u_i^{10}) = LR(v_{i+1}^{1,i} v_{i+2}^{4,i})$, $LR(u_i^{12}) = LR(x_{i+3}^i v_{i+4}^{1,i})$, $LR(u_i^{14}) = LR(x_{i+4}^i v_i^{4,i})$, $LR(u_i^{16}) = LR(v_i^{1,i} v_i^{2,i})$, $LR(u_i^{18}) = LR(v_i^{2,i} v_i^{3,i})$, $LR(u_i^{20}) = LR(v_{i+1}^{3,i} v_{i+1}^{4,i})$, $LR(u_i^{22}) = LR(v_i^{3,i} v_i^{4,i})$, $LR(u_i^{24}) = LR(v_{i+4}^{1,i} v_{i+4}^{2,i})$, $LR(u_i^{26}) = LR(v_{i+4}^{1,i} v_i^{4,i})$, $LR(u_i^{28}) = LR(v_{i+3}^{1,i} v_{i+4}^{4,i})$ and $LR(u_i^{29}) = LR(x_{i+1}^i x_i^{i+1})$ and the cardinalities of the sets of all LRNs are in Table 3.

2. Table 3 shows that $|LR(e_i^y)| \le |LR(u_i^z)|$ and $|LR(u_i^z) \cap \bigcup_{i=1}^{5} LR(e_i^y)| > |LR(e_i^y)|$.

□

Table 3. Local resolving neighborhood set (LRNs) $LR(e_i^y)$ for $1 \leq y \leq 2$ and $LR(u_i^z)$ for $1 \leq z \leq 29$.

LRNs	Cardinalities
$LR(e_i^1)$, $LR(e_i^2)$	63
$LR(u_i^j)$; $1 \leq j \leq 10$	121
$LR(u_i^{11})$, $LR(u_i^{12})$	122
$LR(u_i^{13})$, $LR(u_i^{14})$	123
$LR(u_i^{15})$, $LR(u_i^{16})$	115
$LR(u_i^{17})$, $LR(u_i^{18})$	66
$LR(u_i^{19})$, $LR(u_i^{20})$	67
$LR(u_i^{21})$, $LR(u_i^{22})$	75
$LR(u_i^{23})$, $LR(u_i^{24})$	73
$LR(u_i^{25})$, $LR(u_i^{26})$	84
$LR(u_i^{27})$, $LR(u_i^{28})$, $LR(u_i^{29})$	124

Lemma 5. *Let $\mathbb{N} \cong S(3, C_k)$ be a C_k-based Sierpinski network with $k \geq 7$ and $k \cong 1(mod2)$. Then we have*

1. $\frac{k^2+2k-5}{2} \leq LR(e)\ \forall\ e \in E(\mathbb{N})$,
2. $|LR(e) \cap \bigcup_{i=1}^{k} LR(e_i^y)| \geq |LR(e_i^y)|$, *where* $|LR(e_i^y)| = \frac{k^2+2k-5}{2}$, $y = 1, 2$ *and* $1 \leq i \leq k$.

Proof. 1. For $1 \leq i \leq k$ and $t = \frac{k-3}{2}$, the LRNs of 57 types of edges in a network $\mathbb{N}$. Thirteen types of LRNs of these edges are as follows:

$LR(e_i^1) = LR(v_{m'+i+2}^{k-t-3,i} v_{m'+i+2}^{k-t-2,i}) = V(\mathbb{N}) - \{v_{m'+i+2}^{k-t,i}, x_{k-t+s+m'+i-1}^{i}, v_{k-t+m+m'+i-1}^{j,i}, x_{k-t+m'+i-1}^{r,i}, x_l^{l'}, v_l^{j,l'}; 3 \leq m \leq k-t-1; 2 \leq l' \leq k; 1 \leq s \leq k-t-2; 1 \leq r \leq k-t-1; 1 \leq j \leq k-1; 1 \leq l \leq k\}$, where $1 \leq m' \leq \frac{k-5}{2}$,

$LR(u_i^1) = LR(v_{k-t+i-1}^{k-t-3,i} v_{k-t+i-1}^{k-t-2,i}) = V(\mathbb{N}) - \{v_{k-t+i-1}^{k-1,i}, x_{m+i-1}^{k-t+i-1}, v_{k-t+i-1}^{p,k-t+i-1}, v_{k-m'-t+i-1}^{j,k-t+i-1}, v_{i+2}^{r,i}, v_l^{j,s+i}, x_l^{s+i}, x_{s+i-1}^{i}, v_{m+i-1}^{j,i}; k-t \leq p \leq k-1; 1 \leq s \leq k-t-2; 1 \leq m' \leq \frac{k-5}{2}; 1 \leq r \leq k-t-1; 1 \leq j \leq k-1; 1 \leq l \leq k\}$,

$LR(u_i^3) = LR(v_{i+2}^{k-t-2,i} v_{i+2}^{k-t-1,i}) = V(\mathbb{N}) - \{x_{i+1}^{i}, x_l^{s+i}, v_l^{j,s+i}, x_{s+i}^{k-t+i-1}, v_{m+i}^{j,k-t+i+1}, v_{i+2}^{s,k-t+i-1}; 1 \leq j \leq k-1; 1 \leq l \leq k; 3 \leq m \leq k-t-1; 1 \leq s \leq k-t-2\}$,

$LR(u_i^5) = LR(v_{i+1}^{k-t-3,i} v_{i+1}^{k-t-2,i}) = V(\mathbb{N}) - \{v_{i+1}^{k-1,i}, v_{p+i+1}^{j,i}, v_{r'+t+i+2}^{j,k-t+i-1}, v_{k-t+i}^{r,i}, v_{k-t+i-1}^{r,k-t+i-1}, x_{r'+t+i+1}^{k-t+i-1}, v_l^{j,p+i}, x_l^{p+i}, x_{s+t+i+1}^{i}; k-t \leq p \leq k-1; 1 \leq s \leq k-t-2; 1 \leq r \leq k-t-1; 1 \leq j \leq k-1; 1 \leq l \leq k; 1 \leq r' \leq k-t\}$,

$LR(u_i^7) = LR(v_{k-t+i-1}^{k-t-1,i} v_{k-t+i-1}^{k-t,i}) = V(\mathbb{N}) - \{v_{k-t+i-1}^{1,i}, x_{k-s+i}^{i}, v_{p+i}^{j,i}, v_i^{r+t,i}, x_l^{p+i}, x_{r+t+i+1}^{k-t+i-1}, v_{r+t+i+2}^{j,k-t+i-1}, v_l^{j,p+i}, v_{i+2}^{s+t+1,k-t+i-1}; 1 \leq s \leq k-t-2; 1 \leq r \leq k-t-1; 1 \leq l \leq k; 1 \leq j \leq k-1; k-t \leq p \leq k-1\}$,

$LR(u_i^9) = LR(v_{i+2}^{k-t-3,i} v_{i+2}^{k-t-2,i}) = V(\mathbb{N}) - \{v_{i+2}^{k-1,i}, x_{m+t+i+1}^{i}, v_{m+t+i+2}^{j,i}, x_{k-t+i}^{i}, v_{k-t+i+1}^{r,i}, x_l^{p+i}, x_{r+t+i+1}^{k-t+i-1}, v_l^{j,p+i}, v_{k-t+s+i}^{j,k-t+i-1}, v_{i+2}^{p,k-t+i-1}, v_{k-t+i}^{r,k-t+i-1}; 3 \leq m \leq k-t-1; 1 \leq l \leq k; k-t \leq p \leq k-1; 1 \leq s \leq k-t-2; 1 \leq r \leq k-t-1; 1 \leq j \leq k-1\}$,

$LR(u_i^{11}) = LR(v_{k-t+i-2}^{k-2t-2,i} v_{k-t+i-1}^{k-1,i}) = V(\mathbb{N}) - \{x_{k+i-1}^{i}, x_{k-t+r+i-2}^{k-t+i-1}, v_{k-t+s+i}^{j,k-t+i-1}, v_{i+2}^{p,k-t+i-1}, v_{k-t+i}^{r,k-t+i-1}, x_l^{p+i}, v_l^{j,p+i}; 1 \leq j \leq k-1; 1 \leq l \leq k; 1 \leq s \leq k-t-2; 1 \leq r \leq k-t-1; k-t \leq p \leq k-1\}$,

$LR(u_i^{17}) = LR(x_i^i v_{i+1}^{k-2t-2,i}) = V(\mathbb{N}) - \{v_{i+1}^{k-t-1,i}, v_{k-t+i}^{p,i}, v_{k-t+i-1}^{p,k-t+i-1}, x_{m+i-1}^{k-t+i-1}, v_{k-t-m'+i}^{j,k-t+i-1}; k-t \leq p \leq k-1; 3 \leq m \leq k-t-1; 1 \leq m' \leq \frac{k-5}{2}; 1 \leq j \leq k-1\}$,

$LR(u_i^{19}) = LR(x_{i+1}^{i} v_{i+2}^{k-1,i}) = V(\mathbb{N}) - \{v_{i+2}^{k-t-2,i}, v_{k-t+i}^{k-m-t,i}, v_{k-t+i}^{p,k-t+i-1}, v_{i+2}^{k-t-1,k-t+i-1}; 3 \leq m \leq k-t-1; k-t \leq p \leq k-1\}$,
$LR(u_i^{23}) = LR(v_{k-t+i-1}^{k-t-2,i} v_{k-t+i-1}^{k-t-1,i}) = V(\mathbb{N}) - \{x_{k-t+i-2}^{i}, v_{k-t+i-1}^{s,k-t+i-1}, v_{i+2}^{k-t-2,k-t+i-1}; 1 \leq s \leq k-t-2\}$,
$LR(u_i^{27}) = LR(v_{i+1}^{k-2t-2,i} v_{i+2}^{k-1,i}) = V(\mathbb{N}) - \{x_{k-t+i-1}^{i}, v_{k-t+i-1}^{s,k-t+i-1}; 1 \leq s \leq k-t-2\}$,
$LR(u_i^{29}) = LR(x_{i+1}^{i} v_{i+2}^{k-2t-2,i}) = V(\mathbb{N}) - \{v_{i+2}^{k-t-1,i}, v_{k-t+i+1}^{p,i}; k-t \leq p \leq k-1;\}$,
$LR(u_i^{35}) = LR(v_{i+1}^{m',i} v_{i+1}^{m'+1,i}) = V(\mathbb{N}) - \{v_{i+1}^{k-t+m'-1,i}\}$, where $1 \leq m' \leq \frac{k-5}{2}$.

Similarly, we obtained $LR(e_i^2) = LR(v_{k-t+m'+i}^{k-t-1,i} v_{k-t+m'+i}^{k-t,i})$, $LR(u_i^2) = LR(v_{k-t+i}^{k-t-1,i} v_{k-t+i}^{k-t,i})$, $LR(u_i^4) = LR(v_i^{k-t-2,i} v_i^{k-t-1,i})$, $LR(u_i^6) = LR(v_{i+1}^{k-t-1,i} v_{i+1}^{k-t,i})$, $LR(u_i^8) = LR(v_{k-t+i}^{k-t-3,i} v_{k-t+i}^{k-t-2,i})$, $LR(u_i^{10}) = LR(v_i^{k-t-1,i} v_i^{k-t,i})$, $LR(u_i^{12}) = LR(v_{k-t+i}^{k-2t-2,i} v_{k-t+i+1}^{k-1,i})$, $LR(u_i^{13}) = LR(v_{i+2}^{k-t-1,i} v_{i+2}^{k-t,i})$, $LR(u_i^{14}) = LR(v_{m'+i+2}^{k-t-1,i} v_{m'+i+2}^{k-t,i})$, $LR(u_i^{15}) = LR(v_{k-t+m'+i}^{k-t-3,i} v_{k-t+m'+i}^{k-t-2,i})$, $LR(u_i^{16}) = LR(v_i^{k-t-3,i} v_i^{k-t-2,i})$, $LR(u_i^{18}) = LR(x_i^{i} v_{i+1}^{k-1,i})$, $LR(u_i^{20}) = LR(x_{k-t+i-2}^{i} v_{k-t+i-1}^{k-2t-2,i})$, $LR(u_i^{21}) = LR(x_{k-t+i-1}^{i} v_{k-t+i}^{k-1,i})$, $LR(u_i^{22}) = LR(x_{k+i-1}^{i} v_i^{k-2t-2,i})$, $LR(u_i^{24}) = LR(x_{k-t+i-2}^{i} v_{k-t+i-1}^{k-1,i})$, $LR(u_i^{25}) = LR(v_{k-t+i}^{k-t-2,i} v_{k-t+i}^{k-t-1,i})$, $LR(u_i^{26}) = LR(v_i^{k-2t-2,i} v_{i+1}^{k-1,i})$, $LR(u_i^{28}) = LR(x_{k-t+i-1}^{i} v_{k-t+i}^{k-2t-2,i})$, $LR(u_i^{30}) = LR(x_{m'+i+1}^{i} v_{m'+i+2}^{k-2t-2,i})$, $LR(u_i^{31}) = LR(x_{m'+i+1}^{i} v_{m'+i+2}^{k-1,i})$, $LR(u_i^{32}) = LR(x_{k-t+m'+i-1}^{i} v_{k-t+m'+i}^{k-2t-2,i})$, $LR(u_i^{33}) = LR(x_{k-t+m'+i-1}^{i} v_{k-t+m'+i}^{k-1,i})$, $LR(u_i^{34}) = LR(x_{k+i-1}^{i} v_i^{k-1,i})$, $LR(u_i^{36}) = LR(v_{i+1}^{k-t-2,i} v_{i+1}^{k-t-1,i})$, $LR(u_i^{37}) = LR(v_{i+1}^{k-t+m'-1,i} v_{i+1}^{k-t+m',i})$, $LR(u_i^{38}) = LR(v_{i+2}^{m',i} v_{i+2}^{m'+1,i})$, $LR(u_i^{39}) = LR(v_{i+2}^{k-t+m'-1,i} v_{i+2}^{k-t+m',i})$, $LR(u_i^{40}) = LR(v_{m'+i+2}^{q,i} v_{m'+i+2}^{q+1,i})$, where $1 \leq q \leq \frac{k-5}{2}$, $LR(u_i^{41}) = LR(v_{m'+i+2}^{k-t-2,i} v_{m'+i+2}^{k-t-1,i})$, $LR(u_i^{42}) = LR(v_{m'+i+2}^{k-t+q-1,i} v_{i+2}^{k-t+q,i})$, where $1 \leq q \leq \frac{k-5}{2}$, $LR(u_i^{43}) = LR(v_{k-t+i-1}^{m',i} v_{k-t+i-1}^{m'+1,i})$, $LR(u_i^{44}) = LR(v_{k-t+i-1}^{k-t+m'-1,i} v_{k-t+i-1}^{k-t+m',i})$, $LR(u_i^{45}) = LR(v_{k-t+i}^{m',i} v_{k-t+i}^{m'+1,i})$, $LR(u_i^{46}) = LR(v_{k-t+i}^{k-t+m'-1,i} v_{k-t+i}^{k-t+m',i})$, $LR(u_i^{47}) = LR(v_{k-t+m'+i}^{q,i} v_{k-t+m'+i}^{q+1,i})$, where $1 \leq q \leq \frac{k-5}{2}$, $LR(u_i^{48}) = LR(v_{k-t+m'+i}^{k-t-2,i} v_{k-t+m'+i}^{k-t-1,i})$, $LR(u_i^{49}) = LR(v_{k-t+m'+i}^{k-t+q-1,i} v_{k-t+m'+i}^{k-t+q,i})$, where $1 \leq q \leq \frac{k-5}{2}$, $LR(u_i^{50}) = LR(v_i^{m',i} v_i^{m'+1,i})$, $LR(u_i^{51}) = LR(v_i^{k-t+m'-1,i} v_i^{k-t+m',i})$, $LR(u_i^{52}) = LR(v_{m'+i+1}^{k-2t-2,i} v_{m'+i+2}^{k-1,i})$, $LR(u_i^{53}) = LR(v_{k-t+i-1}^{1,i} v_{k-t+i}^{k-1,i})$, $LR(u_i^{54}) = LR(v_{k-t+m'+i}^{k-2t-2,i} v_{k-t+m'+i+1}^{k-1,i})$, $LR(u_i^{55}) = LR(x_{i+1}^{i} x_i^{i+1})$ and the cardinalities of the sets of all LRNs are in Table 4.

2. Table 4 shows that $|LR(e_i^y)| \leq |LR(u_i^z)|$ and $|LR(u_i^z) \cap \bigcup_{i=1}^{k} LR(e_i^y)| > |LR(e_i^y)|$.

$\square$

Table 4. Local resolving neighborhood set (LRNs) $LR(e_i^y)$ for $1 \le y \le 2$ and $LR(u_i^z)$ for $1 \le z \le 55$.

LRNs	Cardinalities
$LR(e_i^1)$, $LR(e_i^2)$	$\frac{k^2+2k-5}{2}$
$LR(u_i^1)$, $LR(u_i^2)$	$k^3-(t+2)k^2+2(k+t)+1$
$LR(u_i^3)$, $LR(u_i^4)$	$k^3-(t+1)k^2+k(1+t)+1$
$LR(u_i^5)$, $LR(u_i^6)$	$k^3-(t+1)k^2-k-3$
$LR(u_i^7)$, $LR(u_i^8)$	$k^3-(t+1)k^2-2$
$LR(u_i^9)$, $LR(u_i^{10})$	$k^3-(t+1)k^2+2$
$LR(u_i^{11})$, $LR(u_i^{12})$, $LR(u_i^{13})$, $LR(u_i^{14})$, $LR(u_i^{15})$, $LR(u_i^{16})$	$k^3-tk^2-(t+2)k-1$
$LR(u_i^{17})$, $LR(u_i^{18})$	k^3-tk+1
$LR(u_i^{19})$, $LR(u_i^{20})$, $LR(u_i^{21})$, $LR(u_i^{22})$	k^3-k+1
$LR(u_i^{23})$, $LR(u_i^{24})$, $LR(u_i^{25})$, $LR(u_i^{26})$	$k^3-(t+3)$
$LR(u_i^{27})$, $LR(u_i^{28})$	$k^3-(t+2)$
$LR(u_i^{29})$, $LR(u_i^{30})$, $LR(u_i^{31})$, $LR(u_i^{32})$, $LR(u_i^{33})$, $LR(u_i^{34})$	$k^3-(t+1)$
$LR(u_i^j)$; $35 \le j \le 55$	k^3-1

4. LFMD of C_k-Based Sierpinski Networks

The LFMD of the C_k-based Sierpinski networks are computed as follows.

Theorem 1. *Let $\mathbb{N} \cong S(2, C_k)$ be a C_k—based Sierpinski network with $k \ge 7$, and $k = 1(mod2)$. Then*

$$\frac{k^2}{k^2-1} \le dim_{lfm}(\mathbb{N}) \le \frac{2k^2}{k^2+2k-5}.$$

Proof. From *Lemma 2*, we have $|LR(e_i^w)| = \frac{k^2+2k-5}{2}$, where $w = 1, 2$ and $|LR(e_i^w)| \le |LR(e)|$ $\forall\, e \in E(\mathbb{N})$. In addition, $|LR(e) \cap \bigcup_{i=1}^{k} LR(e_i^w)| \ge |LR(e_i^w)| = \frac{k^2+2k-5}{2}$. Define a local resolving function $\eta : V(\mathbb{N}) \longrightarrow [0,1]$ as

$$\eta(x) = \begin{cases} \frac{2}{k^2+2k-5} & \text{if } x \in \bigcup_{w=1,2} LR(e_i^w) \\ 0 & \text{otherwise.} \end{cases}$$

Since for each $e \in E(\mathbb{N})$, $\eta(LR(e)) = \sum_{x \in LR(e)} \eta(x) = \sum_{x \in LR(e) \cap \bigcup_{i=1}^{k} LR(e_i^w)} \frac{2}{k^2+2k-5} = |LR(e) \cap \bigcup_{i=1}^{k} LR(e_i^w)| \frac{2}{k^2+2k-5} \ge |LR(e_i^w)| \frac{2}{k^2+2k-5} \ge 1$.

We note that $\bigcap_{i=1}^{k} LR(e_i^w) = \emptyset$ and the pairwise intersection of $LR(e_i^w)$'s is non-empty. Therefore $\exists$ a LRF, $\vartheta : V(\mathbb{N}) \longrightarrow [0,1]$ such that $\vartheta > \eta$ then $dim_{lfm}(\mathbb{N}) \le \sum_{i=1}^{X} \frac{2}{k^2+2k-5}$, where $X = |\bigcup_{i=1}^{w} LR(e_i^w)| = |V(\mathbb{N})|$. Therefore, $dim_{lfm}(\mathbb{N}) \le \frac{2k^2}{k^2+2k-5}$.

The cardinalities of $LR(c_i)$, $LR(d_i)$, $LR(e_i)$ and $LR(f_i)$ are greater than other remaining LRNs. Using *Lemma 1*, $dim_{lfm}(\mathbb{N}) \ge \frac{k^2}{k^2-1}$. Using *Lemma 1*, $dim_{lfm}(\mathbb{N}) \ge \frac{k^2}{k^2-1}$. Hence,

$$\frac{k^2}{k^2-1} \le dim_{lfm}(\mathbb{N}) \le \frac{2k^2}{k^2+2k-5}.$$

□

Theorem 2. *Let $\mathbb{N} \cong S(3, C_k)$ be a C_k-based Sierpinski network with $k \geq 3$, and $k = 1(mod2)$. Then*

1. $\frac{27}{26} \leq dim_{lfm}(\mathbb{N}) \leq \frac{27}{14}$, *if* $k = 3$.
2. $\frac{125}{124} \leq dim_{lfm}(\mathbb{N}) \leq \frac{125}{63}$, *if* $k = 5$.
3. $\frac{k^3}{k^3-1} \leq dim_{lfm}(\mathbb{N}) \leq \frac{2k^3}{k^2+2k-5}$, *if* $k \geq 7$.

Proof. 1. From *Lemma 3*, we have $|LR(e_i^y)| = 14$, where $y = 1, 2$ and $|LR(e_i^y)| \leq |LR(e)|$ $\forall\ e \in E(\mathbb{N})$. In addition, $|LR(e) \bigcap \bigcup_{i=1}^{k} LR(e_i^y)| \geq |LR(e_i^y)|$= 1. Define a local resolving function $\eta : V(\mathbb{N}) \longrightarrow [0, 1]$ as

$$\eta(x) = \begin{cases} \frac{1}{14} & \text{if } x \in \bigcup_{i=1}^{3} LR(e_i^y) \\ 0 & \text{otherwise.} \end{cases}$$

Since for each $e \in E(\mathbb{N})$, $\eta(LR(e))$ = $\sum_{x \in LR(e)} \eta(x)$ = $\sum_{x \in LR(e) \bigcap \bigcup_{i=1}^{3} LR(e_i^y)} \frac{1}{14}$ = $|LR(e) \bigcap \bigcup_{i=1}^{3} LR(e_i^y)| \frac{1}{14} \geq |LR(e_i^y)| \frac{1}{14} \geq 1$.

We note that $\bigcap_{i=1}^{3} LR(e_i^y) = \emptyset$ and the pairwise intersection of $LR(e_i^y)$'s is non-empty. Therefore $\exists$ a LRF, $\vartheta : V(\mathbb{N}) \longrightarrow [0, 1]$ such that $\vartheta > \eta$ then $dim_{lfm}(\mathbb{N}) \leq \sum_{i=1}^{X} \frac{1}{14}$, where $X = |\bigcup_{i=1}^{3} LR(e_i^y)| = |V(\mathbb{N})|$. Therefore, $dim_{lfm}(\mathbb{N}) \leq \frac{27}{14}$.

The cardinalities of $LR(u_i^9)$, $LR(u_i^{10})$ and $LR(u_i^{11})$ are greater than other remaining LRNs. Using *Lemma 1*, $dim_{lfm}(\mathbb{N}) \geq \frac{27}{26}$. Hence,

$$\frac{27}{26} \leq dim_{lfm}(\mathbb{N}) \leq \frac{27}{14}.$$

2. From *Lemma 4*, we have $|LR(e_i^y)| = 63$, where $y = 1, 2$ and $|LR(e_i^y)| \leq |LR(e)|$ $\forall$ $e \in E(\mathbb{N})$. In addition, $|LR(e) \bigcap \bigcup_{i=1}^{5} LR(e_i^y)| \geq |LR(e_i^y)|$ = 63. Define a local resolving function $\eta : V(\mathbb{N}) \longrightarrow [0, 1]$ as

$$\eta(x) = \begin{cases} \frac{1}{63} & \text{if } x \in \bigcup_{i=1}^{5} LR(e_i^y) \\ 0 & \text{otherwise.} \end{cases}$$

Since for each $e \in E(\mathbb{N})$, $\eta(LR(e))$ = $\sum_{x \in LR(e)} \eta(x)$ = $\sum_{x \in LR(e) \bigcap \bigcup_{i=1}^{5} LR(e_i^y)} \frac{1}{63}$ = $|LR(e) \bigcap \bigcup_{i=1}^{5} LR(e_i^y)| \frac{1}{63} \geq |LR(e_i^y)| \frac{1}{63} \geq 1$.

We note that $\bigcap_{i=1}^{5} LR(e_i^y) = \emptyset$ and the pairwise intersection of $LR(e_i^y)$'s is non-empty. Therefore $\exists$ a LRF, $\vartheta : V(\mathbb{N}) \longrightarrow [0, 1]$ such that $\vartheta > \eta$ then $dim_{lfm}(\mathbb{N}) \leq \sum_{i=1}^{X} \frac{1}{63}$, where $X = |\bigcup_{i=1}^{5} LR(e_i^y)| = |V(\mathbb{N})|$. Therefore, $dim_{lfm}(\mathbb{N}) \leq \frac{125}{63}$.

The cardinalities of $LR(u_i^{27})$, $LR(u_i^{28})$ and $LR(u_i^{29})$ are greater than other remaining LRNs. Using *Lemma 1*, $dim_{lfm}(\mathbb{N}) \geq \frac{125}{124}$. Hence,

$$\frac{125}{124} \leq dim_{lfm}(\mathbb{N}) \leq \frac{125}{63}.$$

3. From *Lemma 5*, we have $|LR(e_i^y)| = \frac{k^2+2k-5}{2}$, where $y = 1, 2$ and $|LR(e_i^y)| \leq |LR(e)|$ $\forall\ e \in E(\mathbb{N})$. In addition, $|LR(e) \cap \bigcup_{i=1}^{k} LR(e_i^y)| \geq |LR(e_i^y)| = \frac{k^2+2k-5}{2}$. Define a local resolving function $\eta : V(\mathbb{N}) \longrightarrow [0,1]$ as

$$\eta(x) = \begin{cases} \frac{1}{\frac{k^2+2k-5}{2}} & \text{if } x \in \bigcup_{i=1}^{k} LR(e_i^y) \\ 0 & \text{otherwise.} \end{cases}$$

Since for each $e \in E(\mathbb{N})$, $\eta(LR(e)) = \sum_{x \in LR(e)} \eta(x) = \sum_{x \in LR(e) \cap \bigcup_{i=1}^{k} LR(e_i^y)} \frac{2}{k^2+2k-5} = |LR(e) \cap \bigcup_{i=1}^{k} LR(e_i^y)| \frac{2}{k^2+2k-5} \geq |LR(e_i^y)| \frac{2}{k^2+2k-5} \geq 1$.

We note that $\bigcap_{i=1}^{k} LR(e_i^y) = \emptyset$ and the pairwise intersection of $LR(e_i^y)$'s is non-empty. Therefore $\exists$ a LRF, $\vartheta : V(\mathbb{N}) \longrightarrow [0,1]$ such that $\vartheta > \eta$ then $dim_{lfm}(\mathbb{N}) \leq \sum_{i=1}^{X} \frac{2}{k^2+2k-5}$, where $X = |\bigcup_{i=1}^{k} LR(e_i^y)| = |V(\mathbb{N})|$. Therefore, $dim_{lfm}(\mathbb{N}) \leq \frac{2k^3}{k^2+2k-5}$.

The cardinality of $LR(u_i^h)$, where $35 \leq h \leq 55$ is greater than other remaining LRNs. Using *Lemma 1*, $dim_{lfm}(\mathbb{N}) \geq \frac{k^3}{k^3-1}$. Hence,

$$\frac{k^3}{k^3-1} \leq dim_{lfm}(\mathbb{N}) \leq \frac{2k^3}{k^2+2k-5}.$$

□

5. Conclusions

This paper covered the newly developed version of metric dimension named local fractional metric dimension (LFMD). We computed the bounds for $k \geq 3$ of cycle based Sierpinski networks $S(n, C_k)$ with the help of local resolving neighborhood sets. The obtained results are presented in Table 5. We note that, as the order of the networks increased, the LFMD also increased. Thus, the limiting values of the LFMD for the Sierpinski networks $S(2, C_k), k \geq 3$ and upper bound is unbounded for $S(3, C_k), k \geq 7$. Consequently, we found that LFMD of the Sierpinski networks remains bounded as k approaches infinity. It is important to mention that if we consider the Sierpinski network as a communication network consisting of mobile users, then the bounded computed value of LFMD shows that the network does not dump even that the number of users approaches to infinity.

Table 5. Boundness of LFMD of Sierpinski network $S(2, C_k)$.

Networks	Bounds of LFMD	Limiting Values of LFMD as $k \to \infty$	Remarks
$S(2, C_k), k \geq 3$	$\frac{k^2}{k^2-1} \leq dim_{lfm}(S(2, C_k)) \leq \frac{2k^2}{k^2+2k-5}$	$1 \leq dim_{lfm}(S(2, C_k)) \leq 2$	bounded
$S(3, C_3)$	$\frac{27}{26} \leq dim_{lfm}(S(3, C_3)) \leq \frac{27}{14}$	-	bounded
$S(3, C_5)$	$\frac{125}{124} \leq dim_{lfm}(S(3, C_5)) \leq \frac{125}{63}$	-	bounded
$S(3, C_k), k \geq 7$	$\frac{k^3}{k^3-1} \leq dim_{lfm}(S(3, C_k)) \leq \frac{2k^3}{k^2+2k-5}$	$1 \leq dim_{lfm}(S(3, C_k)) \leq \infty$	unbounded

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