mathematics

Article

Semihypergroup-Based Graph for Modeling International Spread of COVID-*n* in Social Systems

Narjes Firouzkouhi [1], Reza Ameri [2], Abbas Amini [3,4] and Hashem Bordbar [5,*]

1 Department of Mathematics, Golestan University, Gorgan 15759-49138, Iran
2 School of Mathematics, Statistic, and Computer Science, University of Tehran, Tehran 79416-55665, Iran
3 Department of Mechanical Engineering, Australian University-Kuwait, Mishref, Safat 13015, Kuwait
4 Centre for Infrastructure Engineering, Western Sydney University, Penrith, NSW 2751, Australia
5 Center for Information Technologies and Applied Mathematics, University of Nova Gorica, 5000 Nova Gorica, Slovenia
* Correspondence: hashem.bordbar@ung.si

Abstract: Graph theoretic techniques have been widely applied to model many types of links in social systems. Also, algebraic hypercompositional structure theory has demonstrated its systematic application in some problems. Influenced by these mathematical notions, a novel semihypergroup-based graph (SBG) of $G = \langle H, E \rangle$ is constructed through the fundamental relation γ_n on H, where semihypergroup H is appointed as the set of vertices and E is addressed as the set of edges on SBG. Indeed, two arbitrary vertices x and y are adjacent if $x\gamma_n y$. The connectivity of graph G is characterized by $x\gamma^* y$, whereby the connected components SBG of G would be exactly the elements of the fundamental group H/γ^*. Based on SBG, some fundamental characteristics of the graph such as complete, regular, Eulerian, isomorphism, and Cartesian products are discussed along with illustrative examples to clarify the relevance between semihypergroup H and its corresponding graph. Furthermore, the notions of geometric space, block, polygonal, and connected components are introduced in terms of the developed SBG. To formulate the links among individuals/countries in the wake of the COVID (coronavirus disease) pandemic, a theoretical SBG methodology is presented to analyze and simplify such social systems. Finally, the developed SBG is used to model the trend diffusion of the viral disease COVID-n in social systems (i.e., countries and individuals).

Keywords: graph theory; hypergroup; fundamental relation; social systems; geometric space

MSC: 05C25; 20N20

check for updates

Citation: Firouzkouhi, N.; Ameri, R.; Amini, A.; Bordbar, H. Semihypergroup-Based Graph for Modeling International Spread of COVID-*n* in Social Systems. *Mathematics* **2022**, *10*, 4405. https://doi.org/10.3390/math10234405

Academic Editor: Patrick Solé

Received: 19 September 2022
Accepted: 17 November 2022
Published: 22 November 2022

1. Introduction

Graph theory with its systematic structure is applied to different complicated problems such as physical, biological, and social systems. By employing graph theory, social network structures can be modeled and analyzed to provide simplified knowledge of such systems, where nodes (vertices) are users and lines (edges) are the links among users. Graph theory was first proposed by Euler to solve Konigsberg's seven-bridge problem [1]. After that, he established a novel graph structure called an Eulerian graph [2]. The concepts of a complete graph [3] and a bipartite graph was defined along with tree structure and coloring problems [4]. With the integration of graph theory and fuzzy set theory, the notion of fuzzy graph theory was proposed by Kaufmann. Then, this theory was developed by Rosenfeld, where fuzzy relations on fuzzy sets were introduced to improve graph-theoretic concepts (e.g., bridges and trees) [5]. To eliminate new problems in science, especially combinatorics, hypergraph theory was initiated and formulated by Berge [6] as the generalization of graph theory, where the edges are arbitrary subsets of the vertices to effectively analyze and simplify complex relations in various spectra for real-world problems [7].

Algebraic hypercompositional structure theory, with its dynamic multi-valued systems, is enumerated as the extension of a classical algebraic structure. Marty introduced a hyperoperation (hypercomposition) on a nonvoid set H, which is a map from $H \times H$ to the power set $P(H)$ of H, such that with associative property and reproductivity, H would be hypergroup [8]. Then, the hypercompositional structure theory was improved in terms of theory and applications by Corsini et al. [9]. Freni determined a novel characterization of the derived hypergroup via strongly regular equivalence relation γ on a hypergroup H, and a binary operation on the quotient set H/γ^* so that H/γ^* is a group with relation γ^* as a fundamental relation (γ^* is the transitive closure of γ and H/γ^* is the fundamental group) [10,11]. Indeed, a fundamental relation is a powerful gadget for the derivation of universal algebra (group, ring, module, etc.) on algebraic hypercompositional structures as well as fuzzy algebraic hypercompositional structures. The present authors studied and formulated the fundamental relations on the fuzzy hypergroup, fuzzy hyperring, and fuzzy hypermodule, where their fundamental relations have the smallest equivalence relation resulting in their quotients being a group, ring, and module, respectively, [12–14]. In other studies, they appointed the fundamental functor between the category of fuzzy hyperrings (hypermodules) and the category of rings (modules) [15,16].

The relevance between graphs/hypergraphs and hypergroups has been investigated by many scholars such as Corsini [17] and Leoreanu [18]. Farshi et al. studied hypergroups associated with hypergraphs and established a ρ-hypergroup with a given hypergraph by describing a relation ρ which resulted in the fundamental relation of an ρ-hypergroup [19]. Kalampakas et al. surveyed path hypergroupoids, especially commutativity and graph connectivity, along with the directed graph isomorphism classes of C-hypergroupoids [20]. Nikkhah et al. developed hypergroups constructed from hypergraphs using a hyperoperation upon the set of vertice degrees of a hypergraph, where the established hypergroupoid is H_v-group [21]. Recently, the present authors proposed a Caley graph related to a semihypergroup (hypergroup) with some important features including the category of Cayley graphs and a functor with an application in social networks [22].

With dynamic and potential applications of graph theory in various fields of science, i.e., computer science, linguistics, physics, chemistry, social sciences, biology, mathematics, bioinformatics, etc., many studies have been conducted [23]. For example, Savinkov et al. analyzed and modeled human lymphatic systems via graph theory [24]. The systematic converter derivation/modeling and advanced control in an emerging/challenging power electronics converter was simulated by graph theory as a powerful mathematical structure [25]. Park et al. indicated important insights from complex travel mobility networks with graph-based spatiotemporal analytics [26]. In another work, an effective transductive learning technique was proposed by employing variational nonlocal graph theory for hyperspectral image classification [27]. Recently, the authors presented a soft hypergraph as the generalization of graph theory with the pragmatic application for modeling global interactions in social media networks [28].

The COVID-19 (coronavirus disease 2019) pandemic is considered the most fatal global health catastrophe to date with its serious negative and destructive impact on human life, i.e., social, economical, and environmental challenges. After its detection, the virus extended globally and caused innumerable death. At present, there is no definitive treatment of clinical antiviral drugs or vaccines against the virus [29]. Almost whole nations attempted to decline the transition of the disease via examination and treating patients, quarantining suspected persons through contact tracing, limiting large gatherings, maintaining complete or partial lockdowns, etc. The impact of COVID-19 on various societies and useful ways for controlling viral disease were investigated in [30].

The principal objective of this study is to establish a novel framework of a graph called SBG using a specific relation of algebraic hypercompositional structures in the context of social systems, i.e., the spread trend of the coronavirus disease among societies and individuals. After the Introduction and the Preliminary sections, in Section 3, we appoint a neoteric graph $G = \langle H, E \rangle$ by applying a fundamental relation γ^* on a semihypergroup H.

The elements of H are vertices and two vertices x and y are adjacent if $x\gamma_n y$, that is, they are considered edges. The connectivity SBG of G is defined as $x\gamma^* y$, where the connected components of G are precisely the elements of the fundamental group H/γ^*. Certain fundamental properties of graph theory such as complete, regular, Eulerian, isomorphism, and Cartesian products are proposed. In addition, elucidatory examples are applied to demonstrate the relationship between semihypergroup (hypergroup) H and its associated graph. The mathematical notions of geometric space, block, polygonal, and connected components are discussed. In the end, in Section 4, the developed SBG is utilized to model the global outbreak of COVID-n in social systems (i.e., individuals as well as countries) (Figure 1).

$$x\gamma_n y \Longleftrightarrow \exists a_i \in H,\ \exists \sigma \in S_n,\ 1 \le i \le n;\ x \in \prod_{i=1}^{n} a_i,\ y \in \prod_{i=1}^{n} a_{\sigma(i)}$$

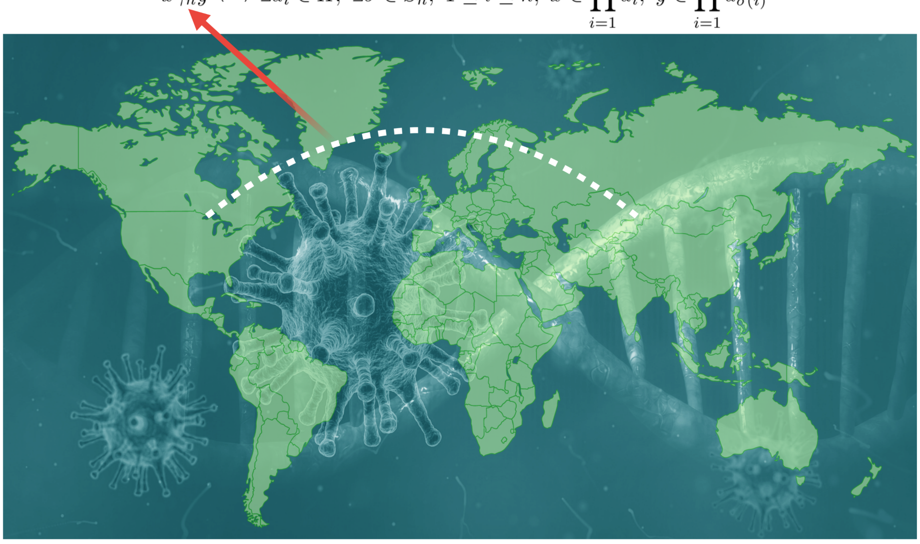

Figure 1. SBG for modeling global spread of COVID-n.

2. Preliminaries

Definition 1. *A hypergroupoid $(L, \diamond)$ is a nonvoid set L with a hyperoperation $\diamond$, which is a map $\diamond : L \times L \to P^*(L)$, where $P^*(L)$ implies the family of all nonvoid subsets of L [9]. Denote $c \diamond d$ as the hyperproduct of c and d for every $c, d \in L$. A hypergroupoid $(L, \diamond)$ is described as a semihypergroup if L has associative property, i.e., $(c \diamond d) \diamond e = c \diamond (d \diamond e)$ for all $c, d, e \in L$. A hypergroup is a semihypergroup along with reproductivity axiom, that is $e \diamond L = L \diamond e = L$ for all $e \in L$. A hypergroupoid $(L, \diamond)$ is called quasihypergroup if the reproductivity property holds. The hypergroup is commutative if $e \diamond f = f \diamond e$ for all $e, f \in L$. A nonvoid subset M of a hypergroup L is a subhypergroup of L if $z \diamond M = M \diamond z = M$ for every $z \in M$.*

Assume E and F are nonvoid subsets of L, hence $E \diamond F = \bigcup_{e \in E, f \in F} e \diamond f$. Moreover, $l \in L$ and $E \subseteq L$, we have $l \diamond E = \bigcup_{e \in E} l \diamond e$. If associativity holds, then we denote the hyperproduct of elements $x_1, \ldots, x_n$ of L by $\prod_{i=1}^{n} x_i := x_1 \diamond x_2 \diamond \ldots \diamond x_n$.

Suppose that $(L, \diamond)$ and $(L', \diamond')$ are two hypergroups. A map $\psi : L \longrightarrow L'$ is determined as a homomorphism *if $\psi(k \diamond l) = \psi(k) \diamond' \psi(l)$ for all $k, l \in L$. Furthermore, ψ is named an* isomorphism *if it is one to one and onto homomorphism written by $L \cong L'$.*

The following Definition 2, Proposition 1, Theorem 1, Proposition 2, and Theorem 2 are taken from [31].

Definition 2. *Assume that L is a nonvoid set and σ is a binary relation on L. Consider the following hypercomposition "$\circ$" on L as:*

$$x \circ y = \{z \in L : (x, z) \in \sigma,\ (z, y) \in \sigma\} \quad (1)$$

$(L, \circ)$ is a hypergroupoid provided there exists $z \in L$ so that $(x, z) \in \sigma$ and $(z, y) \in \sigma$ for every couple of elements $x, y \in L$.

Denote the hypercompositional structure in Equation (1) by L_σ. The reproductivity property in L_σ is satisfied if and only if $(x, y) \in \sigma$ for all $x, y \in L_\sigma$.

Proposition 1.

- *L_σ is a quasihypergroup if and only if $(x, y) \in \sigma$ for all $x, y \in L_\sigma$.*
- *L_σ is a semihypergroup if and only if $(x, y) \in \sigma$ for all $x, y \in L_\sigma$.*

Theorem 1. *Let σ be a binary relation on the nonvoid set L. Then, the hypercomposition $x \circ y$ satisfies the reproductivity or associativity only when L_σ is total (i.e., $x \circ y = L_\sigma$).*

Each relation σ on finite set $L = \{a_1, a_2, \ldots, a_n\}$ can be represented through a Boolean matrix M_σ with $n \times n$ elements. The Boolean matrix $M_\sigma = (m_{ij})$ is defined as follows:

$$m_{ij} = \begin{cases} 1, & if\ (a_i, a_j) \in \sigma \\ 0, & otherwise \end{cases}$$

In Boolean algebra, we have

$$0 + 1 = 1 + 0 = 1 + 1 = 1,\ 0 + 0 = 0$$
$$0.0 = 0.1 = 1.0 = 0,\ 1.1 = 1$$

L_σ is hypergroupoid if and only if $M_\sigma^2 = S$, where $S = (s_{ij})$ with $s_{ij} = 1$ for all i, j.

Proposition 2.

- *L_σ is a quasihypergroup if and only if $M_\sigma = S$.*
- *L_σ is a semihypergroup if and only if $M_\sigma = S$.*

Theorem 2. *The only relation σ which results in a quasihypergroup or semihypergroup is the one with $M_\sigma = S$. Additionally, L_σ is the total hypergroup.*

It was revealed that with a few lines of the Mathematica program, the results were constructed for the enumeration of the hypergroupoid associated with binary relations of orders 2, 3, 4, and 5 by a significantly simpler procedure [31].

Definition 3. *A graph G is a pair $G = (V, E)$, where V is a set of elements described as vertices and E is a set of edges [32]. The two vertices associated with an edge are called endpoints. If $x = y$, then the edge is considered as a loop. A vertex is isolated if it is incident with no edges. The graph G is simple if it has no loops and no two distinct edges have the same pair of ends. The graph G is called null graph when its edges set is empty. Graph H is named a subgraph of graph G if $V(H) \subseteq V(G)$, $E(H) \subseteq E(G)$, and the ends of an edge $e \in E(H)$ are the same as its ends in G. Denote $d(x)$ as the degree of vertex x as well as the number of edges incident with x.*

A path in graph G consists of a sequence $x_1, e_1, x_2, e_2, \ldots, e_k, x_k$ that the edges e_i are distinct. Furthermore, if $x_1 = x_k$ then, we call the path a cycle. Consider that $d(x, y)$ is the length of the shortest path between two vertices x and y. Note that $diam(G) = sup\{d(a, b)\}$ for all a and b that are vertices of G, which is called the diameter of graph G. The graph G is connected if there exists a path from vertex x to vertex y, or graph G includes several connected components. A tree is a connected graph that includes no simple cyclic path. Denote k_n as a complete graph, where every pair of vertices is adjacent. An Eulerian circuit is a closed path through a graph applying each edge once and an Eulerian graph is a graph that has this property. Furthermore, graph G is called a Hamiltonian graph if it has a cycle that passes each vertex exactly once. If every vertex has the same degree, the graph is regular, or k-regular if $\forall x \in V, d(x) = k$.

Theorem 3. *A finite graph G without isolated vertices is Eulerian if and only if G is connected and each vertex has an even degree [32].*

Definition 4. *The Cartesian product of two graphs $G_1 = \langle V_1, E_1 \rangle$ and $G_2 = \langle V_2, E_2 \rangle$ is denoted by $G_1 \Box G_2$, that is a graph with vertices set $V_1 \times V_2$, where vertices $(t_1, t_2), (w_1, w_2)$ are adjacent if and only if $t_1 = w_1, (t_2, w_2) \in E_2$ or $t_2 = w_2, (t_1, w_1) \in E_1$ for $t_1, w_1 \in V_1,\ t_2, w_2 \in V_2$ [33].*

3. Semihypergroup-Based Graph (SBG) Based on Relation γ

Consider an SBG of $G = \langle H; E = (\gamma_n)_{n \in \mathbb{N}} \rangle$, where $(H, \circ)$ is a semihypergroup and γ_n is the relation on H. The order of G is $o(G) = |H|$. The elements of H are represented as vertices and the relations γ_n are appointed as edges. We assign x and y to be adjacent, if $x\gamma_n y$. Clearly, for $n = 1$ and $x\gamma_1 x$, the edge is a loop.

Indeed, γ_n was determined in [10] as follows:

$$x\gamma_n y \Longleftrightarrow \exists (a_1, \ldots, a_n) \in H^n,\ \exists \sigma \in S_n :\ x \in \prod_{i=1}^{n} a_i,\ y \in \prod_{i=1}^{n} a_{\sigma(i)} \tag{2}$$

Consider $\gamma_1 = \{(a, a)\ |a \in H\}$. Clearly, the relations γ_n have symmetric property and relation γ has a reflexive and symmetric property for every $n \in \mathbb{N}$, where $\gamma = \bigcup_{n \geq 1} \gamma_n$. Let γ^* be the transitive closure of γ. The class of H/γ^* was addressed as $\gamma^*(z) = \{w\ |z\gamma^* w\}$, for $z, w \in H$. It was proven that for hypergroup H, the relation γ is transitive and γ^* has the smallest strongly regular equivalence property that results H/γ^* is an Abelian group (fundamental group).

Theorem 4. *Assume that H is a hypergroup. Then, for an SBG of $G = \langle H; E = (\gamma_n)_{n \in \mathbb{N}} \rangle$, the following statements hold:*

(i) A path exists between two vertices x and y of G if and only if $x\gamma^* y$.

(ii) The SBG of G is connected if and only if the fundamental group H/γ^* is a singleton, that is $|H/\gamma^*| = 1$.

Proof. Proof of (i): Consider a path from vertex x to vertex y. Then, there exists a sequence $(a_1, \ldots, a_k) \in H^k$ so that $x = a_1\gamma_1 a_2 \ldots \gamma_k a_k = y$, that is equal to $x\gamma^* y$. Conversely, if $x\gamma^* y$, then $\exists (a_1, \ldots, a_k) \in H^k$ such that $x = a_1\gamma_1 a_2 \ldots \gamma_k a_k = y$. Therefore, there exists a path from vertex x to vertex y.

Proof of (ii): By applying (i), for $x, y \in H$, a path exists from vertex x to vertex y if and only if $x\gamma^* y$. Therefore, the SBG of G is a connected graph if and only if $\gamma^* = H \times H$ (i.e., clearly, $\gamma^* \subseteq H \times H$. Furthermore, for all $x, y \in H$, since $(x, y) \in \gamma^*$, then $H \times H \subseteq \gamma^*$). Since $x\gamma^* y$, we have $\gamma^*(x) = \gamma^*(y)$ which means that the fundamental group $H/\gamma^* = \{\gamma^*(x)|x \in H\}$ is a singleton, i.e., $|H/\gamma^*| = 1$. □

Theorem 5. *The connected components SBG of G are precisely the elements of the fundamental group H/γ^*.*

Proof. Let x, y be two vertices SBG of G. By employing Theorem 4, vertex x is connected to vertex y if and only if $x\gamma^* y$. Then, for all $a \in H$, every element of $\gamma^*(a)$ is connected. With the equivalence relation of γ^*, the elements of H/γ^* would be the connected components SBG of G. □

Theorem 6. *Let H be a semihypergroup. If the SBG of $G = \langle H, E \rangle$ is complete, then the relation γ is transitive.*

Proof. Let $x\gamma y$ and $y\gamma z$. For some $n_1, n_2 \in \mathbb{N}$, we have $x\gamma_{n_1} y$ and $y\gamma_{n_2} z$. Since the SBG of G is complete, therefore, for some $n \in \mathbb{N}$, we have $x\gamma_n z$ that yields $x\gamma z$. □

Remark 1. *Note that a loop is not considered an edge. If $x\gamma x$, then for every $a_i \in H$, $\exists\sigma \in S_n$ we have $x \in \prod_{i=1}^{n} a_i$ and $x \in \prod_{i=1}^{n} a_{\sigma(i)}$. Hence, $\prod_{i=1}^{n} a_i = \prod_{i=1}^{n} a_{\sigma(i)}$.*

Definition 5. *Let H be a nonvoid set and let γ^* be the defined relation in Equation (2). Consider the hypercomposition "$\odot$" on H as follows:*

$$x \odot y = \{w \in H : (x, w) \in \gamma^*,\ (w, y) \in \gamma^*\} \tag{3}$$

We denote the hypercompositional structure $(H, \odot)$ by H_{γ^*}. The H_{γ^*} is a hypergroupoid if $\exists w \in H$ so that $(x, w) \in \gamma^*$ and $(w, y) \in \gamma^*$ for every $x, y \in H$. Since γ^* is transitive, we have $(x, y) \in \gamma^*$ for all $x, y \in H_{\gamma^*}$, then the reproductivity property holds. In fact, for the arbitrary element $x \in H_{\gamma^*}$, the reproductivity axiom $y \in x \odot H_{\gamma^*}$ holds for all $y \in H_{\gamma^*}$, as per the transitive property of γ^*.

Proposition 3.
(i) H_{γ^*} is a semihypergroup if and only if $(x, y) \in \gamma^*$ for all $x, y \in H_{\gamma^*}$.
(ii) H_{γ^*} is a quasihypergroup if and only if $(x, y) \in \gamma^*$ for all $x, y \in H_{\gamma^*}$.
(iii) The SBG of $G = \langle H_{\gamma^*}, E\rangle$ is a connected graph if and only if $(x, y) \in \gamma^*$ for all $x, y \in H_{\gamma^*}$.
(iv) The SBG of $G = \langle H_{\gamma^*}, E\rangle$ is a complete graph if and only if H_{γ^*} is total, i.e., $x \odot y = H_{\gamma^*}$ for all $x, y \in H_{\gamma^*}$.

Proof. Proof of (i): It is derived by applying Proposition 1.

Proof of (ii): With the validity of the reproductivity property, the statement is proven.

Proof of (iii): Since H_{γ^*} is a quasihypergroup and considering part (i), we have H_{γ^*} as a hypergroup. By Theorem 4, we have $x\gamma^* y$ for all $x, y \in H_{\gamma^*}$ if and only if the SBG of G is connected.

Proof of (iv): The statement is attained from Equation (3). □

Example 1. *Consider $(H, \circ)$ as a semihypergroup that is given in Table 1.*

Table 1. Semihypergroup $(H, \circ)$

$\circ$	**0**	**1**	**2**
0	0	1	2
1	1	{0,2}	1
2	2	1	{0,2}

It is seen that $1 \in 1 \circ 2$, $1 \in 2 \circ 1$, then $1\gamma 1$. Furthermore, we have $0\gamma 0$, $2\gamma 2$ and $0\gamma 2$. The corresponding SBG of G is depicted in Figure 2. Moreover, $H/\gamma^* = \{\{0, 2\}, 1\}$ and $|H/\gamma^*| \neq 1$.

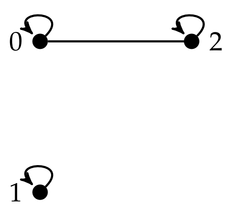

Figure 2. SBG of G.

γ is transitive and the SBG of G is not connected, because vertices 0 and 1 are not adjacent. The SBG of G is not complete, which results in the invalidity of the reverse Theorem 6.

Corollary 1. *Let $G = \langle H, E\rangle$ be an SBG, and let H be a semihypergroup. If the SBG of G is complete, then H/γ^* is a singleton, and $diam(G) = 1$.*

Proof. By applying Theorems 4 and 6, the relation γ is transitive and H/γ^* is a singleton. Since the SBG of G is complete, then every path from vertex x to vertex y has a maximum length of 1, which means $diam(G) = 1$. □

Proposition 4. *Suppose that H is a hypergroup on the SBG of $G = \langle H, E \rangle$. Then, the degree of vertex x in SBG of G is equal to $|\gamma^*(x)|$.*

Proof. Let H be a hypergroup. By employing Theorem 4 and $\gamma^*(x)$ as an equivalence class of x, the results show that the number of edges incident with vertex x is equal to $|\gamma^*(x)|$. □

Corollary 2. *Let $G = \langle H, E \rangle$ be an SBG, and let H be a hypergroup. Assume that $|\gamma^*(x)| = k$ for all $x \in H$. Then, the SBG of G is a k-regular graph.*

Theorem 7. *Let H_1 be a hypergroup on SBG of $G_1 = \langle H_1, E_1 \rangle$. Let H_2 be a subhypergroup of H_1 on SBG of $G_2 = \langle H_2, E_2 \rangle$. Then, the SBG of G_2 is a sub-SBG of G_1.*

Proof. Assume that H_2 is a subhypergroup of H_1, then $H_2 \subset H_1$. Therefore, the vertices SBG of G_2 is contained in the vertices SBG of G_1 and the edges G_2 is included in the edges of G_1. Then, the SBG of G_2 is a sub-SBG of G_1. □

Theorem 8. *Let H be a hypergroup. The SBG of $G = \langle H, E \rangle$ is Eulerian if and only if $|\gamma^*(z)| = 2k$ for all $z \in H, k \in \mathbb{N}$.*

Proof. Let H be a hypergroup. Then, the relation γ is transitive [9]. By applying Theorem 4, the SBG of G is a connected graph. Additionally, with Proposition 4, $d(z) = |\gamma^*(z)|$, for all $z \in H$ and by Theorem 3, the proof is completed. □

Example 2. *Let $(H, \circ)$ be a hypergroup in [34] (Example 28 (3)).*

The corresponding SBG of G is shown in Figure 3, which is a connected and complete graph. Moreover, $H/\gamma^* = \{\{a, b, c\}\}$ and $|H/\gamma^*| = 1$. Additionally, $|\gamma^*(a)| = |\gamma^*(b)| = |\gamma^*(c)| = 2$, that means $d(a) = d(b) = d(c) = 2$. Furthermore, $diam(G) = 1$ and the SBG of G is a 2-regular and Eulerian graph.

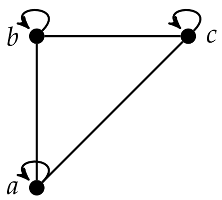

Figure 3. SBG of G.

Definition 6. *The SBG of G is isomorphic to the SBG of G', if there exists a bijection ϕ from the set vertices of G to the set vertices of G', such that $x\gamma_G y \Longleftrightarrow \phi(x)\gamma_{G'}\phi(y)$, written by $G \cong G'$.*

Theorem 9. *Let $(H_1, \circ_1)$ and $(H_2, \circ_2)$ be two isomorphic hypergroups and let G_1 and G_2 be two SBGs associated with H_1 and H_2, respectively. Then, the SBG of G_1 and the SBG of G_2 are isomorphisms.*

Proof. Assume H_1 and H_2 are isomorphisms. Then, $|H_1| = |H_2|$ and we have $|G_1| = |G_2|$. Furthermore, if vertex x is connected to vertex y, then $x\gamma^* y$ and we have $\exists (a_1, \ldots, a_n) \in H^n$, $\exists \sigma \in S_n$; $x \in \prod_{i=1}^n a_i$, $y \in \prod_{i=1}^n a_{\sigma(i)}$. Let $\phi : H_1 \longrightarrow H_2$ be an isomorphism and let $\phi(x) = x'$, $\phi(y) = y'$ and $\phi(a_i) = a'_i$. Furthermore, $\phi(\prod_{i=1}^n a_i) = \prod_{i=1}^n \phi(a_i) = \prod_{i=1}^n a'_i$, which yields x' that is connected to y'. Hence, $G_1 \cong G_2$. □

Example 3. *To show that the reverse of Theorem 9 is not satisfied, consider two hypergroups $(H, \circ)$ and $(H', \circ')$ in [34] (Example 16 (3)).*

Let $f : H \longrightarrow H'$ with $f(a) = 1,\ f(b) = 1,\ f(c) = 2$. Since $f(a \circ b) = f(H) = \{1, 2\}$ and $f(a) \circ' f(b) = 1 \circ 1 = 1$, means that f is not an isomorphism (i.e., $f(a \circ b) \neq f(a) \circ' f(b)$). The two SBGs are isomorphisms, as depicted in Figure 4.

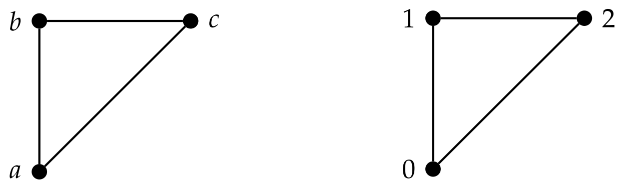

Figure 4. SBGs of G and G' associated with H and H'.

Definition 7. *Let $G = \langle H, E\rangle$ and $G' = \langle H', E'\rangle$ be two SBGs, where H and H' are two hypergroups and $E = \{E_1, \ldots, E_m\}$ and $E' = \{E'_1, \ldots, E'_n\}$. Define the Cartesian product $G \divideontimes G'$ with the vertices set $H \times H'$ and edges set $E_l \times E'_k$ for $1 \leq l \leq m,\ 1 \leq k \leq n$.*

Example 4. *Consider two SBGs in Example 3. By considering $G = \langle H, E\rangle$ and $G' = \langle H', E'\rangle$, the Cartesian product of two SBGs G and G' is depicted in Figure 5. The vertices of $G \divideontimes G'$ are $H \times H' = \{(a, 0), (a, 1), (a, 2), (b, 0), (b, 1), (b, 2), (c, 0), (c, 1), (c, 2)\}$ and the corresponding edges are $E \times E' = \{[(a, 0), (b, 0)], [(a, 0), (a, 1)], \ldots, [(a, 2), (c, 2)]\}$.*

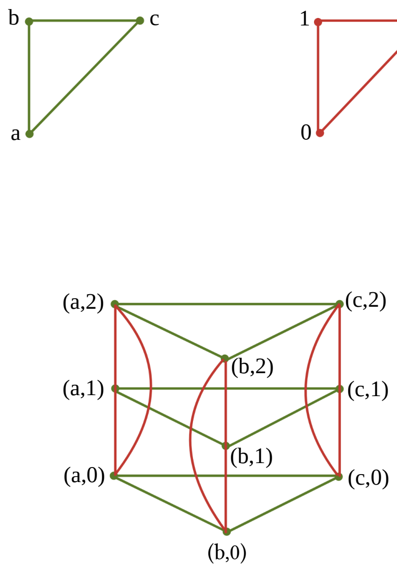

Figure 5. Cartesian product of SBGs G and G'.

Proposition 5. *Let $G = \langle H, E\rangle$ and $G' = \langle H', E'\rangle$ be two SBGs and let $(a_1, b_1), (a_2, b_2) \in H \times H'$. Then,*

$$(a_1, b_1)\gamma_{G \times G'}(a_2, b_2) \Longleftrightarrow a_1 \gamma_G a_2,\ b_1 \gamma_{G'} b_2.$$

Geometric Concept of SBG

A *geometric space* is a couple (S, V) where S is a nonvoid set and V is the family of a nonvoid subset of S. The elements of S are considered points and the elements of V are represented as blocks. If V covers S, then a *polygonal* of (S, V) is an n-tuple of blocks $(V_1, V_2, \ldots, V_n)$ so that $V_i \cap V_{i+1} \neq \emptyset$, for every $i \in \{1, 2, \ldots, n-1\}$. Introduce the relation $\approx$ on S as follows:

$$x \approx y \Longleftrightarrow \exists (V_1, V_2, \ldots, V_n);\ x \in V_1,\ y \in V_n.$$

If V covers S, then the relation is an equivalence relation. The equivalence class $[x]$ is determined as a *connected component* of x in S [10,11].

According to the SBG of G, we consider a pair $\langle H, E\rangle$ as a *geometric space of SBG*, where H is a semihypergroup (set of vertices) and E is the set of relations γ_n (set of edges) for $n \in \mathbb{N}$ on H. For every $x, y \in H$, we have $xE_iy \Longleftrightarrow x\gamma_n y$ with the given relation γ_n as follows:

$$x\gamma_n y \Longleftrightarrow \exists(a_1, \ldots, a_n) \in H^n,\ \exists\sigma \in S_n :\ x \in \prod_{i=1}^{n} a_i,\ y \in \prod_{i=1}^{n} a_{\sigma(i)}$$

Take a polygonal SBG of $G = \langle H, E\rangle$ as $(E_1, E_2, \ldots, E_n)$, so that $E_i \cap E_{i+1} \neq \emptyset$ (i.e., $(x, x') \in E_i, (x', x'') \in E_{i+1}$) for $1 \leq i \leq n-1$. By applying the polygonal concept of SBG, the relation $\approx$ is defined as follows:

$$x \approx y \Longleftrightarrow \exists E_i, 1 \leq i \leq n;\ (x, z) \in E_1,\ (z, y) \in E_n$$

The relation $\approx$ is an equivalence relation. The SBG of G is connected and the equivalence class $[x] = \{y\ |x\gamma^* y\} = |\gamma^*(x)\ |$, where $[x]$ is a connected component by Theorem 4. Indeed, the connected components SBG of $G = \langle H, E\rangle$ are equivalence classes modulo γ^*. The geometric space $G = \langle H, E\rangle$ is connected if it includes only one connected component, i.e., $H = [x]$, for $x \in H$. Clearly, the relation $\approx$ is the transitive closure of the relation $\gamma = \bigcup_{n \in N} \gamma_n$. The blocks of the geometric space SBG of $G = \langle H, E\rangle$ using relation γ_n are the constructed sets with permuting finite hyperproducts of distinct finite points (vertices).

4. SBG for Modeling the Spread Trend of COVID-*n*

SBG can be utilized to model the spread trend of COVID-*n* by travelers in different countries and on a large scale, involved countries. In this pattern, the vertices represent individuals/countries and edges appoint the relationship among individuals/countries which are based on a fundamental relation.

4.1. Application 1

Let H be the number of individuals. Consider $H = \{\text{Michael}, \text{Robert}, \text{Emma}, \text{Olivia}\}$. Then, the SBG of $G = \langle H, E\rangle$ is determined in the following way:

- Each vertex addresses an individual
- An edge addresses the relationship between two vertices

Define a binary relation "∘" on H as follows:

$$a \circ b = \{x\ |x \text{ get infected to COVID} - n \text{ by person } a \text{ or person } b\}$$

In Table 2, the pair $(H, \circ)$ is a hypergroup.

Table 2. Hypergroup $(H, \circ)$.

∘	**Michael = 1**	**Robert = 2**	**Emma = 3**	**Olivia = 4**
Michael = 1	1	2	3	4
Robert = 2	2	{1,2}	{3,4}	3
Emma = 3	3	{3,4}	H	{2,3}
Olivia = 4	4	3	{2,3}	{1,4}

The following statements are attained from Table 2:

- Either Robert, Michael, or Emma infected Olivia with COVID.
- Emma is the most infectious the person for the transmission of the coronavirus disease and all members get infected by Emma ($3 \circ 3 = H$).

Consider the relation γ_n as edges for two arbitrary vertices x and y as:

$$x\gamma_n y \Longleftrightarrow \exists(a_1,\ldots,a_n) \in H^n,\ \exists\sigma \in S_n:\ x \in \prod_{i=1}^{n} a_i,\ y \in \prod_{i=1}^{n} a_{\sigma(i)}$$

Note that $\prod_{i=1}^{n} a_i$ is regarded as a hyperproduct of distinct elements a_i for $i \in \{1,2,\ldots,n\}$, that is $a_1 \circ a_2 \circ \ldots a_n$. We follow the procedure for all components, i.e.,

$$\begin{aligned}
1\gamma_2 2 &\Longleftrightarrow 1 \in 2\circ 2,\ 2 \in 2\circ 2\\
3\gamma_2 4 &\Longleftrightarrow 3 \in 2\circ 3,\ 4 \in 3\circ 2\\
2\gamma_2 3 &\Longleftrightarrow 2 \in 3\circ 4,\ 3 \in 4\circ 3\\
1\gamma_2 4 &\Longleftrightarrow 1 \in 4\circ 4,\ 4 \in 4\circ 4\\
2\gamma_2 4 &\Longleftrightarrow 2 \in 3\circ 3,\ 4 \in 3\circ 3\\
1\gamma_2 3 &\Longleftrightarrow 1 \in 3\circ 3,\ 3 \in 3\circ 3
\end{aligned}$$

This means that $(1,2) \in e_1, (3,4) \in e_2, (2,3) \in e_3, (1,4) \in e_4, (2,4) \in e_5, (1,3) \in e_6$ where, $E = \{e_1, e_2, e_3, e_4, e_5, e_6\}$ are the edges of SBG. The corresponding SBG of G is depicted in Figure 6a and Table 3.

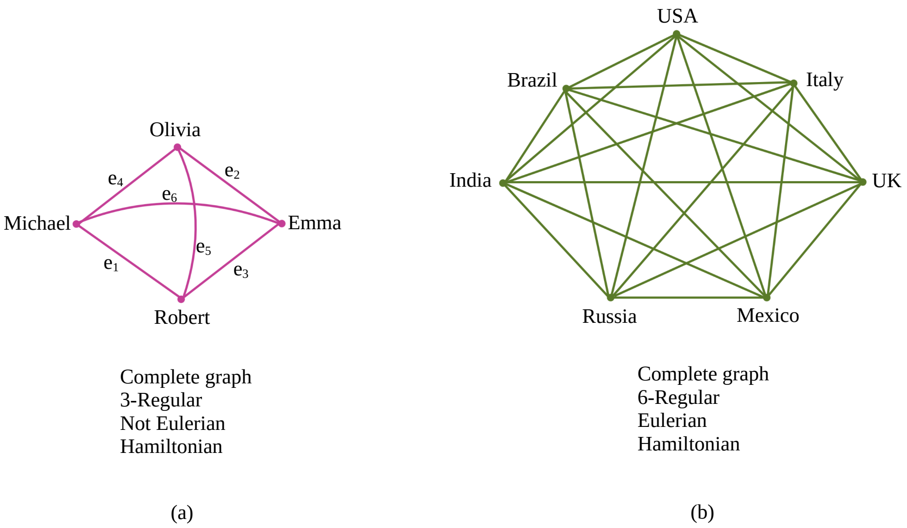

Figure 6. SBGs of G corresponding to (**a**) Application 1 and (**b**) Application 2.

Table 3. SBGs of G.

	Michael	Robert	Emma	Olivia
e_1	1	1	0	0
e_2	0	0	1	1
e_3	0	1	1	0
e_4	1	0	0	1
e_5	0	1	0	1
e_6	1	0	1	0

Furthermore, the equivalence class of $[x]$ is considered as the individuals who transmit viral disease COVID to specific person x, that is $[x] = \{y \mid x\gamma^* y\}$, where γ^* is the transitive closure of γ and $\gamma = \bigcup_{n\geq 1} \gamma_n$. Therefore, the class $[\text{Michael}] = \{\text{Robert}, \text{Emma}, \text{Olivia}\}$, and so on. By applying Proposition 4, the degree of Michael is $|\gamma^*(\text{Michael})\ | = 3$ and by Corollary 2, the SBG is 3-regular.

4.2. Application 2

Let H be a set of countries with the most reported cases and death in the world. Consider $H = \{\text{USA}, \text{Brazil}, \text{India}, \text{Russia}, \text{Mexico}, \text{UK}, \text{Italy}\}$. Thus, the SBG of $G = \langle H, E \rangle$ is defined as follows

- Every vertex is appointed to a country.
- An edge is appointed to the relationship between two vertices.

Introduce the hyperoperation "$\oplus$" for all $x, y \in H$, as follows:

$x \oplus y \quad = \quad$ The country or set of countries that causes disease outbreak from country x to country y

The couple $(H, \oplus)$ is a hypergroupoid, as given in Table 4.

Table 4. Hypergroupoid $(H, \oplus)$.

$\oplus$	**USA = 1**	**Brazil = 2**	**India = 3**	**Russia = 4**	**Mexico = 5**	**UK = 6**	**Italy = 7**
USA = 1	{1,2}	2	{1,2,3}	4	{1,2,5}	{1,6,2}	{1,7,2}
Brazil = 2	2	2	2	{2,4}	2	{2,6}	{1,2}
India = 3	{1,2}	{2,3}	3	{2,3,4}	{1,2,3,5}	{1,2,3,6}	{1,2,3,7}
Russia = 4	{1,4}	2	{3,4}	4	{2,4,5}	{2,4,6}	{1,2,4,7}
Mexico = 5	{1,2,5}	2	{2,3,5}	{1,2,4,5}	5	{1,2,5,6}	H
UK = 6	{1,6}	2	{2,3}	{2,4}	{1,2,5,6}	6	{1,2,4,6,7}
Italy = 7	{1,7}	H	{2,3}	{2,4}	{1,2,5,7}	{2,6,7}	7

Consider the relation γ given below:

$$x \gamma_n y \Longleftrightarrow \exists (a_1, \ldots, a_n) \in H^n,\ \exists \sigma \in S_n :\ x \in \prod_{i=1}^{n} a_i,\ y \in \prod_{i=1}^{n} a_{\sigma(i)}$$

we continue the procedure for all elements of H, according to Table 4, that is

$$1\gamma_2 2 \Longleftrightarrow 1 \in 3 \oplus 1,\ 2 \in 1 \oplus 3$$
$$1\gamma_2 3 \Longleftrightarrow 1 \in 3 \oplus 5,\ 3 \in 5 \oplus 3$$
$$1\gamma_2 4 \Longleftrightarrow 1 \in 4 \oplus 1,\ 4 \in 1 \oplus 4$$
$$1\gamma_2 5 \Longleftrightarrow 1 \in 3 \oplus 5,\ 5 \in 5 \oplus 3$$
$$1\gamma_2 6 \Longleftrightarrow 1 \in 1 \oplus 6,\ 6 \in 6 \oplus 1$$
$$1\gamma_2 7 \Longleftrightarrow 1 \in 1 \oplus 7,\ 7 \in 7 \oplus 1$$
$$2\gamma_2 3 \Longleftrightarrow 2 \in 3 \oplus 4,\ 3 \in 4 \oplus 3$$
$$2\gamma_2 4 \Longleftrightarrow 2 \in 3 \oplus 4,\ 4 \in 4 \oplus 3$$
$$2\gamma_2 5 \Longleftrightarrow 2 \in 4 \oplus 5,\ 5 \in 5 \oplus 4$$
$$2\gamma_2 6 \Longleftrightarrow 2 \in 6 \oplus 2,\ 6 \in 2 \oplus 6$$
$$2\gamma_2 7 \Longleftrightarrow 2 \in 1 \oplus 7,\ 7 \in 7 \oplus 1$$
$$3\gamma_2 4 \Longleftrightarrow 3 \in 3 \oplus 4,\ 4 \in 4 \oplus 3$$
$$3\gamma_2 5 \Longleftrightarrow 3 \in 3 \oplus 5,\ 5 \in 5 \oplus 3$$
$$3\gamma_2 6 \Longleftrightarrow 3 \in 6 \oplus 3,\ 6 \in 3 \oplus 6$$
$$3\gamma_2 7 \Longleftrightarrow 3 \in 7 \oplus 3,\ 7 \in 3 \oplus 7$$
$$4\gamma_2 5 \Longleftrightarrow 4 \in 4 \oplus 5,\ 5 \in 5 \oplus 4$$
$$4\gamma_2 6 \Longleftrightarrow 4 \in 6 \oplus 7,\ 6 \in 7 \oplus 6$$
$$4\gamma_2 7 \Longleftrightarrow 4 \in 6 \oplus 7,\ 7 \in 7 \oplus 6$$
$$5\gamma_2 6 \Longleftrightarrow 5 \in 5 \oplus 6,\ 6 \in 6 \oplus 5$$
$$5\gamma_2 7 \Longleftrightarrow 5 \in 5 \oplus 7,\ 7 \in 7 \oplus 5$$
$$6\gamma_2 7 \Longleftrightarrow 6 \in 6 \oplus 7,\ 7 \in 7 \oplus 6$$

Therefore, $E = \{e_1, \ldots, e_{21}\}$ and the corresponding SBG of G is demonstrated in Figure 6b. By applying Proposition 4, the degree of each vertex is $|\gamma^*(z)| = 6$, and G is complete, and 6-regular. It also has an Eulerian circuit because of connectivity and has an even degree of each vertex; therefore, graph G is Eulerian. The SBG of G is connected and Hamiltonian and the relation γ is transitive.

5. Conclusions

The neoteric structure of a semihypergroup-based graph (SBG) is established using a fundamental relation to advance the mathematical concept of an algebraic hypercompositional structure, namely the hypergroup, in the form of graph theory. Additionally, to model and analyze the links in social systems, the developed SBG approach is recommended to intuitively simplify the complicated procedure. Some significant characteristics of SBG are proposed, including connected, complete, regular, Eulerian, isomorphism, and Cartesian products along with illustrative examples and graphical attitude. As per the engagement of all nations and individuals after the global COVID-n pandemic, the resulting SBG is applied to address the trend of transmission of the coronavirus disease in social systems, particularly countries and individuals. The next phase can be the development of fuzzy SBG and intuitionistic fuzzy SBG with further applicable platforms.

Author Contributions: Conceptualization, N.F. and R.A.; methodology, N.F. and R.A.; software, N.F. and H.B.; validation, A.A. and H.B.; formal analysis, N.F.; investigation, N.F. and R.A.; resources, N.F. and R.A.; data curation, N.F.; writing—original draft preparation, N.F.; writing—review and editing, N.F., R.A., A.A. and H.B.; visualization, N.F. and R.A.; supervision, R.A. and A.A.; project administration, N.F.; funding acquisition, H.B. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Acknowledgments: We gratefully acknowledge BioRender software, Canada, for assisting in the figure drawing.

Conflicts of Interest: The authors declare no conflict of interests.

References

1. Chen, W.K. *Applied Graph Theory*; Elsevier: Amsterdam, The Netherlands, 2012; Volume 13.
2. Dankelmann, P. Proof of a conjecture on the Wiener index of Eulerian graphs. *Discret. Appl. Math.* **2021**, *301*, 99–108. [CrossRef]
3. Li, Y.; Liu, X.; Zhang, S.; Zhou, S. Perfect state transfer in NEPS of complete graphs. *Discret. Appl. Math.* **2021**, *289*, 98–114. [CrossRef]
4. Dokeroglu, T.; Sevinc, E. Memetic Teaching-Learning-Based Optimization algorithms for large graph coloring problems. *Eng. Appl. Artif. Intell.* **2021**, *102*, 104282. [CrossRef]
5. Rosenfeld, A. Fuzzy graphs. In *Fuzzy Sets and Their Applications to Cognitive and Decision Processes*; Academic Press: Cambridge, MA, USA, 1975; pp. 77–95.
6. Bretto, A. *Hypergraph Theory. An introduction*; Mathematical Engineering; Springer: Cham, Switzerland, 2013.
7. Gerbner, D. A note on the uniformity threshold for Berge hypergraphs. *Eur. J. Comb.* **2022**, *105*, 103561. [CrossRef]
8. Massouros, C.; Massouros, G. An overview of the foundations of the hypergroup theory. *Mathematics* **2021**, *9*, 1014. [CrossRef]
9. Corsini, P.; Leoreanu, V. *Applications of Hyperstructure Theory*; Springer Science and Business Media: Berlin/Heidelberg, Germany, 2013; Volume 5.
10. Freni, D. A new characterization of the derived hypergroup via strongly regular equivalences. *Commun. Algebra* **2002**, *30*, 3977–3989. [CrossRef]
11. Freni, D. *Strongly Transitive Geometric Spaces: Applications to Hypergroups and Semigroups Theory*; Taylor & Francis: Abingdon, UK, 2004.
12. Firouzkouhi, N.; Ameri, R. New Fundamental Relation on Fuzzy Hypersemigroups. *New Math. Nat. Comput.* **2021**, *18*, 209–217. [CrossRef]
13. Davvaz, B.; Firouzkouhi, N. Commutative Rings Derived from Fuzzy Hyperrings. *Honam Math. J.* **2020**, *42*, 219–234.
14. Davvaz, B.; Firouzkouhi, N. Fundamental relation on fuzzy hypermodules. *Soft Comput.* **2019**, *23*, 13025–13033. [CrossRef]
15. Firouzkouhi, N.; Davvaz, B. New fundamental relation and complete part of fuzzy hypermodules. *J. Discreet. Math. Sci. Cryptogr.* **2020**, *25*, 1225–1237. [CrossRef]
16. Firouzkouhi, N.; Davvaz, B. Γ^*-Relation on Fuzzy Hyperrings and Fundamental Ring. *New Math. Nat. Comput.* **2021**, *17*, 691–701. [CrossRef]
17. Corsini, P.; Leoreanu-Fotea, V.; Iranmanesh, A. On the Sequence of Hypergroups and Membership Functions Determined by a Hypergraph. *J. Mult.-Valued Log. Soft Comput.* **2008**, *14*, 565–577.
18. Leoreanu-Fotea, V.; Corsini, P. Isomorphisms of hypergroups and of n-hypergroups with applications. *Soft Comput.* **2009**, *13*, 985–994. [CrossRef]
19. Farshi, M.; Davvaz, B.; Mirvakili, S. Hypergraphs and hypergroups based on a special relation. *Commun. Algebra* **2014**, *42*, 3395–3406. [CrossRef]
20. Kalampakas, A.; Spartalis, S. Path hypergroupoids: Commutativity and graph connectivity. *Eur. J. Comb.* **2015**, *44*, 257–264. [CrossRef]
21. Nikkhah, A.; Davvaz, B.; Mirvakili, S. Hypergroups constructed from hypergraphs. *Filomat* **2018**, *32*, 3487–3494. [CrossRef]
22. Shamsi, K.; Ameri, R.; Mirvakili, S. Cayley graph associated to a semihypergroup. *Algebr. Struct. Their Appl.* **2020**, *7*, 29–49.
23. Gross, J.L.; Yellen, J. *Graph Theory and Its Applications*; CRC Press: Boca Raton, FL, USA, 2005.
24. Savinkov, R.; Grebennikov, D.; Puchkova, D.; Chereshnev, V.; Sazonov, I.; Bocharov, G. Graph Theory for Modeling and Analysis of the Human Lymphatic System. *Mathematics* **2020**, *8*, 2236. [CrossRef]
25. Li, Y.; Kuprat, J.; Li, Y.; Liserre, M. Graph-Theory-Based Derivation, Modeling and Control of Power Converter Systems. *IEEE J. Emerg. Sel. Top. Power Electron.* **2022**. [CrossRef]
26. Park, S.; Yuan, Y.; Choe, Y. Application of graph theory to mining the similarity of travel trajectories. *Tour. Manag.* **2021**, *87*, 104391. [CrossRef]
27. Huang, B.; Ge, L.; Chen, G.; Radenkovic, M.; Wang, X.; Duan, J.; Pan, Z. Nonlocal graph theory based transductive learning for hyperspectral image classification. *Pattern Recognit.* **2021**, *116*, 107967. [CrossRef]
28. Amini, A.; Firouzkouhi, N.; Gholami, A.; Gupta, A.R.; Cheng, C.; Davvaz, B. Soft hypergraph for modeling global interactions via social media networks. *Expert Syst. Appl.* **2022**, *203*, 117466. [CrossRef]
29. Mercer, T.R.; Salit, M. Testing at scale during the COVID-19 pandemic. *Nat. Rev. Genet.* **2021**, *22*, 415–426. [CrossRef]
30. Chakraborty, I.; Maity, P. COVID-19 outbreak: Migration, effects on society, global environment, and prevention. *Sci. Total Environ.* **2020**, *728*, 138882. [CrossRef] [PubMed]

31. Massouros, C.G.; Tsitouras, C. Enumeration of hypercompositional structures defined by binary relations. *Ital. J. Pure Appl. Math.* **2011**, *28*, 43–54.
32. Rahman, M.S. *Basic Graph Theory*; Springer International Publishing: Cham, Switzerland, 2017.
33. Imrich, W.; Klavzar, S.; Rall, D.F. *Topics in Graph Theory: Graphs and Their Cartesian Product*; CRC Press: Boca Raton, FL, USA, 2008.
34. Corsini, P. *Prolegomena of Hypergroup Theory*; Aviani Editore: Udine, Italy, 1993.