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On the Optimal Control Problem for Vibrations of the Rod/String Consisting of Two Non-Homogeneous Sections with the Condition at an Intermediate Time

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Abstract: We consider an optimal boundary control problem for a one-dimensional wave equation consisting of two non-homogenous segments with piecewise constant characteristics. The wave equation describes the longitudinal vibrations of a non-homogeneous rod or the transverse vibrations of a non-homogeneous string with given initial, intermediate, and final conditions. We assume that wave travel time for each of the sections is the same. The control is carried out by shifting one end with the other end fixed. The quality criterion is set on the entire time interval. A constructive approach to building an optimal boundary control is proposed. The results obtained are illustrated with an analytical example.

Keywords: optimal vibration control; longitudinal vibrations of a piecewise homogeneous rod; transverse vibrations of a piecewise homogeneous string; optimal boundary control; intermediate condition; separation of variables

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1. Introduction

Many researchers pay attention to the study of control problems and optimal control problems for vibration processes [1–15]. Modeling and control of dynamic systems with intermediate conditions is an actively developing direction in modern control theory. In particular, Refs. [2–15] address the study of such problems. This scientific direction has not yet been sufficiently studied, is in the process of formation, and there are only some results on it. The study of problems for such heterogeneous distributed systems is provided, in particular, in [7–20]. The conditions that determine the contact interactions of materials of heterogeneous bodies are of great importance. Therefore, in the course of mathematical modeling, taking into account these conditions of conjugation (joint, gluing) of two sections with different physical characteristics of materials should correspond to the conditions for the continuous outflow of excited wave processes. One of the first control problems for a distributed oscillatory system consisting of two piecewise homogeneous media was set by A.G. Butkovsky and studied in [8]. The problems of the optimization of the boundary control of vibrations of a rod consisting of heterogeneous sections were studied in [9,10] (and other works by the same author and his followers). For the study of these problems, the d'Alembert method was used. The authors of [13–20] studied boundary value problems for an equation describing the process of longitudinal vibrations of a rod with piecewise constant characteristics (consisting of at least two sections) with a free or fixed right end. Research was carried out in the class of generalized solutions.

This work aims to develop a constructive approach to building an optimal boundary control function for an inhomogeneous wave equation consisting of two heterogeneous sections with given initial, intermediate, and final conditions with a quality criterion given over the entire time interval.

2. Problem Statement

We consider longitudinal vibrations of a piecewise homogeneous rod located along the segment $-l_1 \le x \le l$ and comprising two subsegments. The segment $-l_1 \le x \le 0$ has a uniform density $\rho_1 = const$, Young's modulus $k_1 = const$ and a wave velocity $a_1 = \sqrt{\frac{k_1}{\rho_1}}$. The second segment $0 \le x \le l$ has a uniform density $\rho_2 = const$, Young's modulus $k_2 = const$ and a wave velocity $a_2 = \sqrt{\frac{k_2}{\rho_2}}$. As in [9], we assumed that the lengths l_1 and l of the rod segments are such that the wave velocity on $-l_1 \le x \le 0$ coincides with the wave velocity on $0 \le x \le l$, i.e.,

$$\frac{l_1}{a_1} = \frac{l}{a_2}. \tag{1}$$

Let the state (longitudinal vibrations) of the rod (or transverse vibrations of the string) be described by the function $Q\,(x,t)$, $-l_1 \le x \le l, 0 \le t \le T$, and the deviations from the equilibrium state satisfy the following wave equation,

$$\frac{\partial^2 Q(x,t)}{\partial t^2} = \begin{cases} a_1^2 \frac{\partial^2 Q(x,t)}{\partial x^2}, & -l_1 \le x \le 0, \quad 0 \le t \le T, \\ a_2^2 \frac{\partial^2 Q(x,t)}{\partial x^2}, & \quad 0 \le x \le l, \quad 0 \le t \le T, \end{cases} \tag{2}$$

with the boundary conditions

$$Q(-l_1,t) = \mu(t), \;\; Q(l,t) = 0, \;\; 0 \le t \le T, \tag{3}$$

and with the conjugation conditions at the connection point $x = 0$ of the segments,

$$Q(0-0,t) = Q(0+0,t), \;\; a_1^2\rho_1 \frac{\partial Q(x,t)}{\partial x}|_{x=0-0} = a_2^2\rho_2 \frac{\partial Q(x,t)}{\partial x}|_{x=0+0}. \tag{4}$$

The dynamics of a piecewise-homogeneous vibratory process represented by a homogeneous wave equation of variable structure (2) describes not only longitudinal vibrations of a piecewise-homogeneous rod (ρ is density, k is elasticity modulus), but also transverse vibrations of a piecewise-homogeneous string (ρ is density, k is string tension).

Let there be given initial (for $t = t_0 = 0$) and final (for $t = T$) conditions,

$$Q(x,0) = \varphi_0(x), \;\; \left.\frac{\partial Q(x,t)}{\partial t}\right|_{t=0} = \psi_0(x), \;\; -l_1 \le x \le l, \tag{5}$$

$$Q(x,T) = \varphi_T(x), \;\; \left.\frac{\partial Q(x,t)}{\partial t}\right|_{t=T} = \psi_T(x), \;\; -l_1 \le x \le l. \tag{6}$$

Additionally, let there be given at some intermediate moment of time t_1 $(0 = t_0 < t_1 < t_2 = T)$ an intermediate state in the form:

$$Q(x,t_1) = \varphi_1(x), \;\; -l_1 \le x \le l. \tag{7}$$

In conditions (3), functions $\mu(t)$ are control actions (boundary control).

It is assumed that $Q(x,t) \in C^2(\Omega_T)$, where

$$\Omega_T = \{(x,t) : \, x \in [-l_1, l], \, t \in [0,T]\},$$

and $\varphi_i(x) \in C^2[-l_1, l]$, $i = \overline{0,2}$, $\psi_0(x)$, $\psi_T(x) \in C^1[-l_1, l]$.

We also assume that for all functions the following consistency conditions are satisfied,

$$\mu(0) = \varphi_0(-l_1), \; \dot{\mu}(0) = \psi_0(-l_1), \; \varphi_0(l) = \psi_0(l) = 0,$$

$$\mu(t_1) = \varphi_1(-l_1),\ \varphi_1(l) = 0,\ \mu(T) = \varphi_T(-l_1), \tag{8}$$
$$\dot{\mu}(T) = \psi_T(-l_1),\ \varphi_T(l) = \psi_T(l) = 0.$$

Let us formulate the following problem of optimal boundary control of oscillations for system (2) with given values at intermediate times.

Among the possible controls $\mu(t)$, $0 \le t \le T$, condition (3) is required to find such an optimal control that provides transition of the oscillatory motion of system (2) from a given initial state (5) to the final state (6), at the same time ensuring the fulfillment of condition (7) and minimizing the functional

$$\int_0^T \mu^2(t)dt. \tag{9}$$

3. Reduction to the Problem with Zero Boundary Conditions

To solve the problem under study, introduce a new variable [20],

$$\xi = \begin{cases} \frac{a_2}{a_1}x, & -l_1 \le x \le 0, \\ x, & 0 \le x \le l, \end{cases} \tag{10}$$

which leads to stretching or compression of the segment $-l_1 \le x \le 0$ with respect to the point $x = 0$. Taking into account (1), the segment $-l_1 \le x \le 0$ turns into the segment $-l \le \xi \le 0$. For the function $Q(\xi, t)$, we obtain the same equations for the segments of the same length

$$\frac{\partial^2 Q(\xi,t)}{\partial t^2} = \begin{cases} a_2^2 \frac{\partial^2 Q(\xi,t)}{\partial \xi^2} & -l \le \xi \le 0, \quad 0 \le t \le T, \\ a_2^2 \frac{\partial^2 Q(\xi,t)}{\partial \xi^2}, & 0 \le \xi \le l, \quad 0 \le t \le T, \end{cases}$$

or

$$\frac{\partial^2 Q(\xi,t)}{\partial t^2} = a_2^2 \frac{\partial^2 Q(\xi,t)}{\partial \xi^2},\ \ -l \le \xi \le l,\ \ 0 \le t \le T, \tag{11}$$

with the corresponding initial conditions

$$Q(\xi, 0) = \varphi_0(\xi),\ \ \left.\frac{\partial Q(\xi,t)}{\partial t}\right|_{t=0} = \psi_0(\xi),\ \ -l \le x \le l, \tag{12}$$

boundary conditions

$$Q(-l, t) = \mu(t),\ \ Q(l, t) = 0,\ \ 0 \le t \le T, \tag{13}$$

intermediate conditions

$$Q(\xi, t_1) = \varphi_1(\xi),\ \ -l \le \xi \le l, \tag{14}$$

final conditions

$$Q(\xi, T) = \varphi_T(\xi),\ \ \left.\frac{\partial Q(\xi,t)}{\partial t}\right|_{t=T} = \psi_T(\xi),\ \ -l \le \xi \le l, \tag{15}$$

and conjugation conditions at the point $\xi = 0$ where the segments connect

$$Q(0-0, t) = Q(0+0, t),\ \ a_1\rho_1 \frac{\partial Q(\xi,t)}{\partial \xi}\big|_{\xi=0-0} = a_2\rho_2 \frac{\partial Q(\xi,t)}{\partial \xi}\big|_{\xi=0+0}. \tag{16}$$

Since conditions (13) are not homogeneous, the solution to (11) can be constructed as a sum,

$$Q(\xi, t) = V(\xi, t) + W(\xi, t), \tag{17}$$

where $V(\xi,t)$ is a function with boundary conditions,

$$V(-l,t) = V(l,t) = 0, \tag{18}$$

that require definitions, whereas $W(\xi,t)$ is a solution to (11) with non-homogeneous boundary conditions,

$$W(-l,t) = \mu(t), \;\; W(l,t) = 0. \tag{19}$$

The function $W(\xi,t)$ has the form

$$W(\xi,t) = \frac{1}{2l}(l-\xi)\mu(t). \tag{20}$$

Substitute (17) into (11). Taking into account (20), we obtain the following equations for defining the function $V(\xi,t)$:

$$\frac{\partial^2 V(\xi,t)}{\partial t^2} = a_2^2 \frac{\partial^2 V(\xi,t)}{\partial \xi^2} + F(\xi,t), \;\; -l \le \xi \le l, \;\; 0 \le t \le T, \tag{21}$$

where

$$F(\xi,t) = \frac{1}{2l}(\xi - l)\ddot{\mu}(t). \tag{22}$$

The function $V(\xi,t)$ satisfies the conjugation condition corresponding to (16) at the connection point $\xi = 0$ of the segments. Note that, according to (10), we have

$$\begin{gathered} \varphi_0(-l_1) = \varphi_0(-l), \;\; \varphi_i(-l_1) = \varphi_i(-l), \;\; \varphi_T(-l_1) = \varphi_T(-l), \\ \varphi_1(-l_1) = \varphi_1(-l), \;\; \psi_0(-l_1) = \psi_0(-l), \;\; \psi_T(-l_1) = \psi_T(-l). \end{gathered} \tag{23}$$

By virtue of conditions (12), (14), and (15), the function $V(\xi,t)$ should satisfy the following set of conditions: initial

$$V(\xi,0) = \varphi_0(\xi) - \frac{1}{2l}(l-\xi)\mu(0), \;\; \left.\frac{\partial V(\xi,t)}{\partial t}\right|_{t=0} = \psi_0(\xi) - \frac{1}{2l}(l-\xi)\dot{\mu}(0), \tag{24}$$

intermediate

$$V(\xi,t_1) = \varphi_1(\xi) - \frac{1}{2l}(l-\xi)\mu(t_1), \tag{25}$$

and final

$$V(\xi,T) = \varphi_T(\xi) - \frac{1}{2l}(l-\xi)\mu(T), \;\; \left.\frac{\partial V(\xi,t)}{\partial t}\right|_{t=T} = \psi_T(\xi) - \frac{1}{2l}(l-\xi)\dot{\mu}(T). \tag{26}$$

Taking into account conditions (7) and (23), conditions (24)–(26) can be written as follows, respectively:

$$V(\xi,0) = \varphi_0(\xi) - \frac{1}{2l}(l-\xi)\varphi_0(-l), \;\; \left.\frac{\partial V(\xi,t)}{\partial t}\right|_{t=0} = \psi_0(\xi) - \frac{1}{2l}(l-\xi)\psi_0(-l), \tag{27}$$

$$V(\xi,t_1) = \varphi_1(\xi) - \frac{1}{2l}(l-\xi)\varphi_1(-l), \tag{28}$$

$$V(\xi,T) = \varphi_T(\xi) - \frac{1}{2l}(l-\xi)\varphi_T(-l), \;\; \left.\frac{\partial V(\xi,t)}{\partial t}\right|_{t=T} = \psi_T(\xi) - \frac{1}{2l}(l-\xi)\psi_T(-l). \tag{29}$$

Thus, the original problem has been reduced to the problem of motion optimal control described by Equation (21) with homogeneous boundary conditions (18), which is formulated as follows: it is required to find such optimal boundary control $\mu(t)$, $0 \le t \le T$, that provides a transition of the oscillation described by Equation (21) with boundary conditions (18) from the given initial state (27) to the final state (29) through the intermediate states (28).

4. Reduction of the Problem with Zero Boundary Conditions to the Problem of Moments

Considering that the boundary conditions (18) are homogeneous, the consistency conditions are satisfied and the functions used belong to the indicated corresponding spaces, we seek the solution to (21) in the form:

$$V(\xi,t)=\sum_{k=1}^{\infty}V_k(t)\sin\frac{\pi k\xi}{l},\quad V_k(t)=\frac{1}{l}\int_{-l}^{l}V(\xi,t)\sin\frac{\pi k\xi}{l}d\xi. \tag{30}$$

Use the Fourier series with the basis $\left\{\sin\frac{\pi k\xi}{l}\right\}$ $(k=1,2,\ldots)$ to write down functions $F(\xi,t)$, $\varphi_i(\xi)$ $(i=\overline{0,m+1})$, $\psi_0(\xi)$ and $\psi_T(\xi)$. Substitute their values together with $V(\xi,t)$ into Equations (21), (22) and conditions (27)–(29). We obtain

$$\ddot{V}_k(t)+\lambda_k^2V_k(t)=F_k(t),\quad \lambda_k^2=\left(\frac{a_2\pi k}{l}\right)^2, \tag{31}$$

$$F_k(t)=-\frac{a_2}{\lambda_k l}\ddot{\mu}(t), \tag{32}$$

$$V_k(0)=\varphi_k^{(0)}-\frac{a_2}{\lambda_k l}\varphi_0(-l),\quad \dot{V}_k(0)=\psi_k^{(0)}-\frac{a_2}{\lambda_k l}\psi_0(-l), \tag{33}$$

$$V_k(t_1)=\varphi_k^{(i)}-\frac{a_2}{\lambda_k l}\varphi_1(-l), \tag{34}$$

$$V_k(T)=\varphi_k^{(T)}-\frac{a_2}{\lambda_k l}\varphi_T(-l),\quad \dot{V}_k(T)=\psi_k^{(T)}-\frac{a_2}{\lambda_k l}\psi_T(-l). \tag{35}$$

Here, the Fourier coefficients of the functions $F(\xi,t)$, $\varphi_i(\xi)$ $(i=\overline{0,2})$, $\psi_0(\xi)$, and $\psi_T(\xi)$ are denoted by $F_k(t)$, $\varphi_k^{(i)}$ $(i=0,\,1,\,2)$, $\psi_k^{(0)}$, and $\psi_k^{(T)}$, respectively.

The general solution to the non-homogeneous Equation (31) with conditions (33) having the form

$$V_k(t)=V_k(0)\cos\lambda_k t+\frac{1}{\lambda_k}\dot{V}_k(0)\sin\lambda_k t+\frac{1}{\lambda_k}\int_0^t F_k(\tau)\sin\lambda_k(t-\tau)d\tau. \tag{36}$$

Further, taking into account the intermediate (34) and final (35) conditions, we apply the approaches given in [2–4] to (36). Then, the control functions $\mu(t)$ for each k should satisfy the following integral relations:

$$\int_0^T\mu(\tau)\sin\lambda_k(T-\tau)d\tau=C_{1k},\ \int_0^T\mu(\tau)\cos\lambda_k(T-\tau)d\tau=C_{2k},$$

$$\int_0^T\mu(\tau)h_k^{(1)}(\tau)d\tau=C_{1k}(t_1), \tag{37}$$

where

$$h_k^{(1)}(\tau)=\begin{cases}\sin\lambda_k(t_1-\tau), & 0\le\tau\le t_1,\\ 0, & t_1<\tau\le T,\end{cases}$$

$$C_{1k}=\frac{1}{\lambda_k^2}\left[\frac{\lambda_k l}{a_2}\tilde{C}_{1k}+X_{1k}\right],\quad C_{2k}=\frac{1}{\lambda_k^2}\left[\frac{\lambda_k l}{a_2}\tilde{C}_{2k}+X_{2k}\right],$$

$$C_{1k}(t_1) = \frac{1}{\lambda_k^2}\left[\frac{\lambda_k l}{a_2}\tilde{C}_{1k}(t_1) + X_{1k}^{(1)}\right],$$
$$\tilde{C}_{1k} = \lambda_k V_k(T) - \lambda_k V_k(0)\cos\lambda_k T - \dot{V}_k(0)\sin\lambda_k T,$$
$$\tilde{C}_{2k} = \dot{V}_k(T) + \lambda_k V_k(0)\sin\lambda_k T - \dot{V}_k(0)\cos\lambda_k T,$$
$$\tilde{C}_{1k}(t_1) = \lambda_k V_k(t_1) - \lambda_k V_k(0)\cos\lambda_k t_1 - \dot{V}_k(0)\sin\lambda_k t_1,$$
$$X_{1k} = \lambda_k\varphi_T(-l) - \psi_0(-l)\sin\lambda_k T - \lambda_k\varphi_0(-l)\cos\lambda_k T, \tag{38}$$
$$X_{2k} = \psi_T(-l) - \psi_0(-l)\cos\lambda_k T + \lambda_k\varphi_0(-l)\sin\lambda_k T,$$
$$X_{1k}^{(1)} = \lambda_k\varphi_1(-l) - \psi_0(-l)\sin\lambda_k t_1 - \lambda_k\varphi_0(-l)\cos\lambda_k t_1.$$

Relation (37) entails the validity of the following

Proposition 1. *For each n, the process described by (31) with conditions (33)–(35) is completely controllable if and only if for any values of $C_{1k}(T)$, $C_{2k}(T)$ and $C_{1k}(t_1)$ determined by (38) one can find a control $\mu(t)$, $t \in [0, T]$, satisfying (37).*

Thus, the solution of the optimal control problem under study is reduced to finding such boundary controls $\mu(t)$, $0 \le t \le T$, that for each $k = 1, 2, \ldots$ satisfy the integral relations (37) and provide a minimum to the functional (9). The optimal control problem for the functional (9) with integral conditions (38) can be considered as a conditional extremum problem from the calculus of variations.

5. Problem Solution

Since the functional (9) is the square of the norm of a linear normed space and the integral relations (37) generated by the functions $\mu(t)$ are linear, the problem of determining the optimal control for each $k = 1, 2, \ldots$ can be considered as a problem of moments [1,21]. Therefore, the solution can be constructed using the algorithm for solving the problem of moments.

In practice, it is common to select the first few n harmonics of elastic oscillations and solve the problem of control synthesis using methods of control theory for finite-dimensional systems. Therefore, we construct a solution to problems (9) and (37) for $k = \overline{1, n}$ using the algorithm for solving the problem of moments. Following [21], to solve the finite-dimensional (for $k = \overline{1, n}$) problem of moments (9) and (37), it is necessary to find p_k, q_k, γ_k, $k = \overline{1, n}$, linked by the condition

$$\sum_{k=1}^{n}\left[p_k C_{1k}(T) + q_k C_{2k}(T) + \gamma_k C_{1k}(t_1)\right] = 1, \tag{39}$$

for which

$$(\rho_n^0)^2 = \min_{(39)} \int_0^T h_{1n}^2(\tau)d\tau, \tag{40}$$

where

$$h_{1n}(\tau) = \sum_{k=1}^{n}\left[p_k \sin\lambda_k(T-\tau) + q_k\cos\lambda_k(T-\tau) + \gamma_k h_k^{(1)}(\tau)\right]. \tag{41}$$

The notation in Equation (40) means that the minimum of the functional is calculated by condition (39).

Without giving further constructions of the solution (since they are similar to the constructions given in [4,5]), note that the optimal boundary control $\mu_n^0(\tau)$ for any $n = 1, 2, \ldots$ can be represented as:

$$\mu_n^0(\tau) = \begin{cases} \frac{1}{(\rho_n^0)^2} \sum\limits_{k=1}^{n} \left[G_k\left(p_k^0, q_k^0, \lambda_k, T, \tau\right) + \gamma_k^0 \sin \lambda_k(t_1 - \tau)\right], & 0 \le \tau \le t_1, \\ \frac{1}{(\rho_n^0)^2} \sum\limits_{k=1}^{n} G_k\left(p_k^0, q_k^0, \lambda_k, T, \tau\right), & t_1 < \tau \le t_2 = T, \end{cases} \tag{42}$$

where

$$G_k\left(p_k^0, q_k^0, \lambda_k, T, \tau\right) = p_k^0 \sin \lambda_k(T - \tau) + q_k^0 \cos \lambda_k(T - \tau).$$

Here the values p_k^0, q_k^0, γ_k^0, $k = \overline{1, n}$, are the solution to Equation (40) by condition (39), whereas

$$(\rho_n^0)^2 = \int\limits_0^T \left\{ \sum_{k=1}^{n} \left[p_k^0 \sin \lambda_k(T - \tau) + q_k^0 \cos \lambda_k(T - \tau) + \gamma_k^0 h_k^{(1)}(\tau) \right] \right\}^2 d\tau.$$

It should be highlighted that the values of the optimal control $\mu_n^0(\tau)$ at the end of the interval $[0, t_1]$ coincide with the values at the beginning of the interval $(t_1, T]$, and this value has the following form:

$$\mu_n^0(t_1) = \frac{1}{\left(\rho_n^0\right)^2} \sum_{k=1}^{n} \left[p_k^0 \sin\lambda_k(T - t_1) + q_k^0 \cos\lambda_k(T - t_1) \right].$$

Therefore, the obtained optimal boundary controls $\mu_n^0(\tau)$ are continuous on $[0, T]$ as functions with respect to time.

Substituting the resulting expression for the optimal function $\mu_n^0(\tau)$ into (32) and the expression that we found for $F_k^0(t)$—into (36), we obtain the function $V_k^0(t)$, $t \in [0, T]$. Further, (30) entails that

$$V_n^0(\xi, t) = \sum_{k=1}^{n} V_k^0(t) \sin \frac{\pi k}{l} \xi, \tag{43}$$

and, using (17) and (20), the optimal vibration function $Q_n^0(\xi, t)$ for the first n harmonics will have the form

$$Q_n^0(\xi, t) = V_n^0(\xi, t) + W_n^0(\xi, t), \tag{44}$$

where

$$W_n^0(\xi, t) = \frac{1}{2l}(l - \xi)\mu_n^0(t). \tag{45}$$

Taking into account notations from (10), and following (43)–(45), the optimal function $Q_n^0(x, t)$ for $-l_1 \le x \le l$ can be represented as

$$Q_n^0(x, t) = \begin{cases} \sum\limits_{k=1}^{n} V_k^0(t) \sin \frac{\pi k}{l_1} x + \frac{1}{2}(1 - \frac{x}{l_1})\mu_n^0(t), & -l_1 \le x \le 0, \quad 0 \le t \le T, \\ \sum\limits_{k=1}^{n} V_k^0(t) \sin \frac{\pi k}{l} x + \frac{1}{2}(1 - \frac{x}{l})\mu_n^0(t), & 0 \le x \le l, \quad 0 \le t \le T. \end{cases} \tag{46}$$

The function $Q_n^0(x, t)$ is continuous and it can be verified that it satisfies the conjugation condition at the point $x = 0$ where the segments (5) meet.

6. Constructing a Solution for $n = 1$

Applying the approach proposed above, construct an optimal boundary control for $n = 1$ ($k = 1$) and the corresponding string deflection function.

For simplicity, we construct the boundary control function $\mu_n(t)$ for $n=1$ (hence, $k=1$). In this case, to find p_1^0, q_1^0, γ_1^0, we will have the following system of algebraic equations:

$$\begin{gathered} a_{11}p_1+b_{11}q_1+c_{11}\gamma_1=-\frac{\beta_1}{2}C_{11}(T),\ \ d_{11}p_1+e_{11}q_1+f_{11}\gamma_1=-\frac{\beta_1}{2}C_{21}(T), \\ a_{11}^{(1)}p_1+b_{11}^{(1)}q_1+g_{11}\gamma_1=-\frac{\beta_1}{2}C_{11}(t_1),\ \ p_1C_{11}(T)+q_1C_{21}(T)+\gamma_1C_{11}(t_1)=1, \end{gathered} \tag{47}$$

where

$$a_{11}=\int_0^T(\sin\lambda_1(T-\tau))^2d\tau,\ \ b_{11}=\int_0^T\cos\lambda_1(T-\tau)\sin\lambda_1(T-\tau)d\tau,$$

$$c_{11}=\int_0^T h_1^{(1)}(\tau)\sin\lambda_1(T-\tau)d\tau,\ \ d_{11}=\int_0^T\sin\lambda_1(T-\tau)\cos\lambda_1(T-\tau)d\tau,$$

$$e_{11}=\int_0^T(\cos\lambda_1(T-\tau))^2d\tau,\ \ f_{11}=\int_0^T h_1^{(1)}(\tau)\cos\lambda_1(T-\tau)d\tau,$$

$$a_{11}^{(1)}=\int_0^T\sin\lambda_1(T-\tau)h_1^{(1)}(\tau)d\tau,\ \ b_{11}^{(1)}=\int_0^T\cos\lambda_1(T-\tau)h_1^{(1)}(\tau)d\tau,$$

$$g_{11}=\int_0^T h_1^{(1)}(\tau)h_1^{(1)}(\tau)d\tau.$$

$$a_{11}=\frac{T}{2}-\frac{1}{2\lambda_1}\ \sin\lambda_1T\cos\lambda_1T,\ \ b_{11}=d_{11}=\frac{1}{2\lambda_1}\ \sin^2\lambda_1T,$$

$$c_{11}=a_{11}^{(1)}=\frac{t_1}{2}\cos\lambda_1(T-t_1)-\frac{1}{2\lambda_1}\ \sin\lambda_1t_1\cos\lambda_1T,$$

$$f_{11}=b_{11}^{(1)}=\frac{1}{2\lambda_1}\ \sin\lambda_1t_1\ \sin\lambda_1T-\frac{t_1}{2}\sin\lambda_1(T-t_1),$$

$$e_{11}=\frac{T}{2}+\frac{1}{2\lambda_1}\ \sin\lambda_1T\cos\lambda_1T,\ \ g_{11}=\frac{t_1}{2}-\frac{1}{2\lambda_1}\ \sin\lambda_1t_1\cos\lambda_1t_1,$$

$$\begin{aligned} C_{11}(T)= &\ \tfrac{l}{2a\lambda_1}\left[\lambda_1V_1(T)-\lambda_1V_1(0)\cos\lambda_1T-\dot V_1(0)\sin\lambda_1T\right]+ \\ &+\tfrac{1}{\lambda_1^2}[\lambda_1\varphi_T(0)-\psi_0(0)\sin\lambda_1T-\lambda_1\varphi_0(0)\cos\lambda_1T], \end{aligned}$$

$$\begin{aligned} C_{21}(T)= &\ \tfrac{l}{2a\lambda_1}\left[\dot V_1(T)+\lambda_1V_1(0)\sin\lambda_1T-\dot V_1(0)\cos\lambda_1T\right]+ \\ &+\tfrac{1}{\lambda_1^2}[\psi_T(0)-\psi_0(0)\cos\lambda_1T+\lambda_1\varphi_0(0)\sin\lambda_1T], \end{aligned}$$

$$\begin{aligned} C_{11}(t_1)= &\ \tfrac{l}{2a\lambda_1}\left[\lambda_1V_1(t_1)-\lambda_1V_1(0)\cos\lambda_1t_1-\dot V_1(0)\sin\lambda_1t_1\right]+ \\ &+\tfrac{1}{\lambda_1^2}[\lambda_1\varphi_1(0)-\psi_0(0)\sin\lambda_1t_1-\lambda_1\varphi_0(0)\cos\lambda_1t_1]. \end{aligned}$$

Next, we find a solution to system (47), i.e., the values of p_1^0, q_1^0, γ_1^0 and β_1^0:

$$p_1^0=\frac{1}{2\Delta}\Big(\Big(f_{11}^2-e_{11}g_{11}\Big)C_{11}(T)+(b_{11}g_{11}-c_{11}f_{11})C_{21}(T)+(e_{11}c_{11}-b_{11}f_{11})C_{11}(t_1)\Big),$$

$$q_1^0=\frac{1}{2\Delta}\Big((b_{11}g_{11}-c_{11}f_{11})C_{11}(T)+\Big(c_{11}^2-a_{11}g_{11}\Big)C_{21}(T)+(a_{11}f_{11}-b_{11}c_{11})C_{11}(t_1)\Big),$$

$$\gamma_1^0=\frac{1}{2\Delta}\Big((e_{11}c_{11}-b_{11}f_{11})C_{11}(T)+(a_{11}f_{11}-b_{11}c_{11})C_{21}(T)+\Big(b_{11}^2-a_{11}e_{11}\Big)C_{11}(t_1)\Big),$$

$$\beta_1^0=\frac{1}{\Delta}\Big(a_{11}g_{11}e_{11}+2c_{11}b_{11}f_{11}-a_{11}f_{11}^2-e_{11}c_{11}^2-g_{11}b_{11}^2\Big),$$

where

$$\begin{aligned}\Delta = \; & \tfrac{1}{2}\left[(f_{11}^2 - e_{11}g_{11})C_{11}^2(T) + (c_{11}^2 - a_{11}g_{11})C_{21}^2(T) + (b_{11}^2 - a_{11}e_{11})C_{11}^2(t_1)\right] + \\ & + (e_{11}c_{11} - b_{11}f_{11})C_{11}(t_1)C_{11}(T) + (b_{11}g_{11} - c_{11}f_{11})C_{21}(T)C_{11}(T) + \\ & + (a_{11}f_{11} - b_{11}c_{11})C_{11}(t_1)C_{21}(T).\end{aligned}$$

According to (36), we obtain:

$$\mu_1^0(\tau) = \begin{cases} \mu_1^{(1)0}(\tau) = \frac{p_1^0 \sin\lambda_1(T-\tau) + q_1^0 \cos\lambda_1(T-\tau) + \gamma_1^0 \sin\lambda_1(t_1-\tau)}{(\rho_1^0)^2}, & 0 \le \tau \le t_1, \\ \mu_1^{(2)0}(\tau) = \frac{p_1^0 \sin\lambda_1(T-\tau) + q_1^0 \cos\lambda_1(T-\tau)}{(\rho_1^0)^2}, & t_1 < \tau \le t_2 = T, \end{cases}$$

where

$$\begin{aligned}(\rho_1^0)^2 = \; & \tfrac{T}{2}\left((q_1^0)^2 + (p_1^0)^2\right) + \tfrac{t_1}{2}\left((\gamma_1^0)^2 + 2\gamma_1^0\left(p_1^0 \cos\lambda_1(T-t_1) - q_1^0 \sin\lambda_1(T-t_1)\right)\right) + \\ & + \tfrac{1}{\lambda_1}\left(p_1^0 q_1^0 \sin^2\lambda_1 T - \frac{(\gamma_1^0)^2}{2} \sin\lambda_1 t_1 \cos\lambda_1 t_1 + \gamma_1^0\left(q_1^0 \sin\lambda_1 T - p_1^0 \cos\lambda_1 T\right)\sin\lambda_1 t_1 + \right. \\ & \left. + \frac{(q_1^0)^2 - (p_1^0)^2}{2} \sin\lambda_1 T \cos\lambda_1 T\right).\end{aligned}$$

Following (43)–(45), the optimal state function $Q_n^0(\xi, t)$ will have the form:

$$Q_1^0(\xi, t) = V_1^0(\xi, t) + W_1^0(\xi, t) = V_1^0(t) \sin\frac{\pi}{l}\xi + \left(1 - \frac{\xi}{l}\right)\mu_1^0(t).$$

According to (46), the explicit expression of the optimal function $Q_1^0(x, t)$ for $-l_1 \le x \le l$ can be written as

$$Q_1^0(x,t) = \begin{cases} V_1^0(t) \sin\frac{\pi}{l_1}x + \frac{1}{2}(1 - \frac{x}{l_1})\mu_1^0(t), & -l_1 \le x \le 0, \quad 0 \le t \le T, \\ V_1^0(t) \sin\frac{\pi}{l}x + \frac{1}{2}(1 - \frac{x}{l})\mu_1^0(t), & 0 \le x \le l, \quad 0 \le t \le T, \end{cases}$$

or for $0 \le t \le t_1$

$$Q_1^0(x,t) = \begin{cases} V_1^0(t) \sin\frac{\pi}{l_1}x + \frac{1}{2}(1 - \frac{x}{l_1})\mu_1^{(1)0}(t), & -l_1 \le x \le 0, \\ V_1^0(t) \sin\frac{\pi}{l}x + \frac{1}{2}(1 - \frac{x}{l})\mu_1^{(1)0}(t), & 0 \le x \le l, \end{cases}$$

for $t_1 < t \le t_2 = T$

$$Q_1^0(x,t) = \begin{cases} V_1^0(t) \sin\frac{\pi}{l_1}x + \frac{1}{2}(1 - \frac{x}{l_1})\mu_1^{(2)0}(t), & -l_1 \le x \le 0, \\ V_1^0(t) \sin\frac{\pi}{l}x + \frac{1}{2}(1 - \frac{x}{l})\mu_1^{(2)0}(t), & 0 \le x \le l. \end{cases}$$

7. The Example with Numerical Experiment

Let $n = 1$, $a_1 = \frac{1}{4}$, $a_2 = \frac{1}{3}$, $l = 1$, $l_1 = \frac{3}{4}$, $t_0 = 0$, $t_1 = 3\frac{l}{a_2} = 9$, $T = 6\frac{l}{a_2} = 18$, $\lambda_1 = \frac{\pi}{3}$. Let the following initiate state be set for $t = 0$

$$\varphi_0(x) = \begin{cases} x^3 + l_1 x^2, & -l_1 \le x \le 0, \\ x^3 - l x^2, & 0 \le x \le l, \end{cases} \qquad \psi_0(x) = 0, \quad -l_1 \le x \le l,$$

the intermediate state for $t = t_1$ be given as

$$\varphi_1(x) = \begin{cases} -x^3 - l_1 x^2, & -l_1 \le x \le 0, \\ -x^3 + l x^2, & 0 \le x \le l, \end{cases}$$

and the trivial final states be defined for $t = T$ as

$$\varphi_T(x) = 0, \quad \psi_T(x) = 0.$$

From formula (47) we will have the following system of algebraic equations

$$9p_1 - \frac{9}{2}\gamma_1 + \frac{C_{11}(T)}{2}\beta_1 = 0, \quad -\frac{9}{2}p_1 + \frac{9}{2}\gamma_1 = 0, \quad 9q_1 = 0, \quad C_{11}(T)p_1 = 1,$$

where $C_{11}(T) = \frac{273}{32\pi^3}$. The solution is

$$p_1^0 = \gamma_1^0 = \frac{32}{273}\pi^3, \quad q_1^0 = 0, \quad \beta_1^0 = -\frac{1024}{8281}\pi^6,$$

so that

$$\left(\rho_k^0\right)^2 = \frac{512}{8281}\pi^6.$$

The function $\mu_1^0(t)$ has the form

$$\mu_1^0(t) = \begin{cases} 0, & 0 \le t \le t_1, \\ -\frac{91}{48\,\pi^3}\sin\frac{1}{3}\pi t, & t_1 \le t \le T. \end{cases}$$

For the function $V_1^0(t)$, we have

$$V_1^0(t) = \begin{cases} -\frac{91}{32\,\pi^3}\cos\frac{1}{3}\pi t, & 0 \le t \le t_1, \\ \left(-\frac{91}{16\pi^3} + \frac{91t}{288\,\pi^3}\right)\cos\frac{1}{3}\pi t - \frac{91}{96\pi^4}\sin\frac{1}{3}\pi t, & t_1 \le t \le T. \end{cases}$$

The graphical view of the function $V_1^0(t)$ is illustrated in Figure 1.

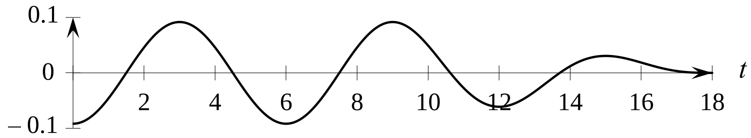

Figure 1. Graphic $V_1^0(t)$.

From $Q_1^0(x,t)$ for $-l_1 \le x \le l$, we obtain
at $0 \le t \le t_1$:

$$Q_1^0(x,t) = \begin{cases} -\frac{91}{32\pi^3}\cos\frac{1}{3}\pi t \sin\frac{4}{3}\pi x, & -l_1 \le x \le 0, \\ -\frac{91}{32\pi^3}\cos\frac{1}{3}\pi t \sin\pi x, & 0 \le x \le l, \end{cases} \tag{48}$$

at $t_1 \le t \le T$:

$$Q_1^0(x,t) = \begin{cases} \left(\left(-\frac{91}{16\pi^3} + \frac{91t}{288\pi^3}\right)\cos\frac{1}{3}\pi t - \frac{91}{96\pi^4}\sin\frac{1}{3}\pi t\right)\sin\frac{4}{3}\pi x - \\ \qquad -\frac{91}{96\pi^3}\left(1 - \frac{4}{3}x\right)\sin\frac{1}{3}\pi t, & -l_1 \le x \le 0, \\ \left(\left(-\frac{91}{16\pi^3} + \frac{91t}{288\pi^3}\right)\cos\frac{1}{3}\pi t - \frac{91}{96\pi^4}\sin\frac{1}{3}\pi t\right)\sin\pi x - \\ \qquad -\frac{91}{96\pi^3}(1 - x)\sin\frac{1}{3}\pi t, & 0 \le x \le l. \end{cases} \tag{49}$$

Substituting $t = 0$, 9, 18 into (48), (49), we obtain the following relationship:

$$Q_1^0(x,0) = \begin{cases} -\frac{91}{32\pi^3}\sin\frac{4}{3}\pi x, & -l_1 \le x \le 0, \\ -\frac{91}{32\pi^3}\sin\pi x, & 0 \le x \le l, \end{cases} \quad \left.\frac{\partial Q_1^0(x,t)}{\partial t}\right|_{t=0} = 0,$$

$$Q_1^0(x,9) = \begin{cases} \frac{91}{32\pi^3}\sin\frac{4}{3}\pi x, & -l_1 \le x \le 0, \\ \frac{91}{32\pi^3}\sin\pi x, & 0 \le x \le l, \end{cases}$$

$$Q_1^0(x,18) = 0, \quad \left.\frac{\partial Q_1^0(x,t)}{\partial t}\right|_{t=18} = \begin{cases} -\frac{91}{288\pi^2}\left(1-\frac{4}{3}x\right), & -l_1 \le x \le 0, \\ -\frac{91}{288\pi^2}(1-x), & 0 \le x \le l. \end{cases}$$

Let us present the results of a comparative analysis based on residual

$$\varepsilon_1(x,\ t_j) = \left|Q_1^0(x,t_j) - \varphi_j(x)\right|,\ \ j = 0,\ 1;\ \ \widehat{\varepsilon}_1\ (x,\ t_k) = \left|\dot{Q}_1^0(x,t_k) - \psi_k(x)\right|,\ \ k = 2.$$

We obtained:

$$\max_{-\frac{3}{4}\le x\le 1} \varepsilon_1(x,0) = \max_{-\frac{3}{4}\le x\le 1} \varepsilon_1(x,9) \approx 0.0833, \quad \max_{-\frac{3}{4}\le x\le 1} \widehat{\varepsilon}_1\ (x,18) \approx 0.0566,$$

$$\int_{-\frac{3}{4}}^{1} \varepsilon_1(x,0)dx = \int_{-\frac{3}{4}}^{1} \varepsilon_1(x,9)dx \approx 0.0903,\ \int_{-\frac{3}{4}}^{1} \widehat{\varepsilon}_1\ (x,18)dx \approx 0.0459.$$

Graphical representations of the functions $Q_1^0(x,0)$, $Q_1^0(x,9)$, $\left.\frac{\partial Q_1^0(x,t)}{\partial t}\right|_{t=18}$ are given in Figures 2–4.

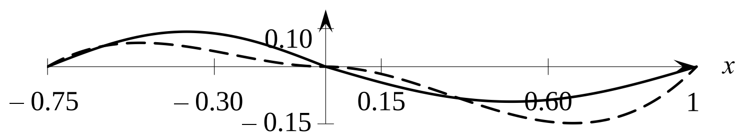

Figure 2. The solid line denotes $Q_1^0(x,0)$; the dotted line denotes $\varphi_0(x)$.

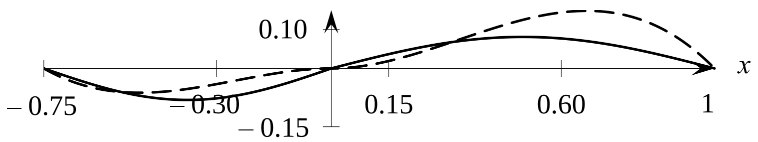

Figure 3. The solid line denotes $Q_1^0(x,9)$; the dotted line denotes $\varphi_1(x)$.

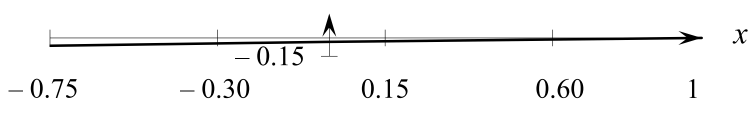

Figure 4. Graphical representation of the function $\dot{Q}_1^0(x,18)$.

Explicit expressions for the optimal function of boundary control $\mu_1^0(t)$ and the corresponding function of deflection of an inhomogeneous string $Q_1^0(x,t)$ are constructed for $n = 1$. The performed calculations and comparisons of the results showed that the behavior of the functions of deflection of an inhomogeneous string is quite close to the given initial functions.

8. Conclusions

In this paper, we considered the problem of optimal boundary control of a one-dimensional wave equation describing transverse vibrations of a piecewise homogeneous string or longitudinal vibrations of a piecewise homogeneous rod. A constructive approach was proposed for building an optimal boundary control function for one-dimensional non-homogeneous oscillatory processes. In this case, the explicit expression of the optimal boundary control function is represented through the given initial and final functions of the deflection and velocities of the points of the distributed system. The results can be used when designing the optimal boundary control of non-homogeneous oscillation processes in physical and technological systems.

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