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Fullness and Decidability in Continuous Propositional Logic

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Abstract: In this paper we consider general continuous propositional logics and prove some basic properties about them. First, we characterize full systems of continuous connectives of the form $\{\neg, \dot{-}, f\}$ where f is a unary connective. We also show that, in contrast to the classical propositional logic, a full system of continuous propositional logic cannot contain only one continuous connective. We then construct a closed full system of continuous connectives without any constants. Such a system does not have any tautologies. For the rest of the paper we consider the standard continuous propositional logic as defined by Yaacov, I.B and Usvyatsov, A. We show that Strong Compactness and Craig Interpolation fail for this logic, but approximated versions of Strong Compactness and Craig Interpolation hold true. In the last part of the paper, we introduce various notions of satisfiability, falsifiability, tautology, and fallacy, and show that they are either NP-complete or co-NP-complete.

Keywords: continuous propositional logic; full system of connectives; strong compactness; Craig interpolation; satisfiability; NP-complete

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1. Introduction

In [1] Ben Yaacov and Usvyatsov developed continuous first-order logic as a variant of the logic studied by Chang and Keisler [2]. This logic turns out to be very useful in the study of metric structures. For instance, Ben Yaacov [3] proved that the linear isometry group of the Gurarij space is universal among all Polish groups by viewing Banach spaces as continuous first-order structures. Another example is the metric Scott analysis developed in [4] where the infinitary continuous first-order logic is used.

In this paper we study *continuous propositional logic* in the framework of continuous first-order logic of [1], because some basic questions about continuous propositional logic have not been addressed in previous research. For example, in [1] the notion of full connective systems was defined, it was shown that the system $\{\wedge, \dot{-}, \frac{x}{2}\}$ is full, and this system had then been adopted as the standard connective system for the rest of the study. Here we give a more complete analysis of full connective systems. In particular, we give a characterization of the unary connectives f where $\{\wedge, \dot{-}, f\}$ forms a full system. Another curious issue is the existence of a full system of connectives with only one connective. In the case of classical propositional logic, such systems exist; an example is $\{|\}$ where $|$ denotes the Sheffer stroke (also known as the nand operation), which is a binary connective. We will show that the situation is quite different in continuous propositional logic, that no such singleton system can be full, regardless of the arity of the connective. We will also construct a closed full system of connectives which contains no constants. This is curious because in the corresponding continuous propositional logic there are no tautologies.

In [5] Ben Yaacov and Pedersen introduced a deductive system and showed that the Completeness theorem holds for continuous first-order logic. It follows that the Compactness theorem holds, which states that a set of formulas in continuous first-order logic is satisfiable iff every finite subset of it is satisfiable. The axioms they used for continuous propositional logic is a natural extension of the axioms of Łukasiewicz logic (cf. e.g., [6]). In fact Ben Yaacov in [7] gave a more explicit treatment of continuous propositional logic and verified directly its Completeness. Here we consider Strong Compactness, which is the possible equivalence between $\Sigma \models p$ and $\Sigma' \models p$ for a finite subset Σ' of Σ. We will see that this Strong Compactness fails, and instead we establish an Approximated Compactness theorem in a style similar to an Approximated Completeness theorem proved by Ben Yaacov [7]. Similarly, we give examples where the Craig Interpolation theorem fails, but prove an Approximated Interpolation theorem for continuous propositional logic.

In the last part of the paper we consider the complexity of decidability problems for continuous propositional logic. In analogy with the classical propositional logic, we show that the satisfiability problem for continuous propositional logic is NP-complete and the tautology problem for continuous propositional logic is co-NP-complete, except that the satisfiability problem and the tautology problem for continuous propositional logic take more than one form according to a set threshold. Formally, in Section 4 we define, for rationals $\alpha \in (0,1]$ and $\beta \in [0,1)$ the notions of $<\alpha$-satisfiability, $<\alpha$-tautologies, α-falsifiability, α-fallacies, β-satisfiability, β-tautologies, $>\beta$-falsifiability, and $>\beta$-fallacies. We define a particular full system of connectives $\mathcal{P}$, which is a natural extension of the standard system $\{\neg, \dot{-}, \frac{x}{2}\}$. We completely characterize the complexity for these notions as follows.

Theorem 1. *Let $\mathcal{F}$ be a finite subset of $\mathcal{P}$. Suppose $\overline{\mathcal{F}}$ contains $\neg, \dot{-}$. Then for all rational $\alpha \in (0,1]$ and $\beta \in [0,1)$:*

- *the following sets of formulas in $\overline{\mathcal{F}}$ are NP-complete:*
  1. *$<\alpha$-satisfiable formulas;*
  2. *α-falsifiable formulas;*
  3. *β-satisfiable formulas;*
  4. *$>\beta$-falsifiable formulas;*
- *the following sets of formulas in $\overline{\mathcal{F}}$ are co-NP-complete:*
  5. *$<\alpha$-tautologies;*
  6. *α-fallacies;*
  7. *β-tautologies;*
  8. *$>\beta$-fallacies.*

Mundici [8] had shown that for $\mathcal{F} = \{\neg, \dot{-}\}$, the set of all <1-satisfiable formulas in $\overline{\mathcal{F}}$ is NP-complete. Our result is a generalization.

The rest of the paper is organized as follows. In Section 2 we study fullness of connective systems. In Section 3 we discuss the Strong Compactness and Craig Interpolation theorems. In Sections 4 and 5 we investigate the decidability problems considered in Theorem 1 and determine their complexity.

2. Fullness of Connective Systems

Our presentation of the continuous propositional logic will (almost) follow [1], with only two exceptions, which we will explain soon below.

For $n \geqslant 0$, an n-ary *continuous connective* is a continuous function from $[0,1]^n$ to $[0,1]$.

The set of all 0-ary continuous connectives consists of all constant functions $C_a \equiv a$ for $a \in [0,1]$. They serve as *continuous truth values*. They generalize traditional discrete truth values $\top = T = 0$ and $\bot = F = 1$. Note that, following [1], truth corresponds to 0 and fallacy corresponds to 1, for good technical reasons.

The set of all unary continuous connectives consists of all continuous functions from $[0,1]$ to $[0,1]$. In classical propositional logic there is only one unary connective $\neg$ (*negation*). Here we define

$$\neg x = 1 - x$$

for $x \in [0,1]$. Obviously $\neg x = 0$ iff $x = 1$, and $\neg x = 1$ iff $x = 0$; thus the definition is consistent with the traditional definition of $\neg$.

In classical propositional logic, $\wedge$ (*conjunction*), $\vee$ (*disjunction*) are standard binary connectives. In continuous propositional logic, we define them as follows:

$$x \wedge y = \max\{x, y\},$$
$$x \vee y = \min\{x, y\}.$$

One can check that they generalize the traditional definitions, that is,

$$x \wedge y = 0 \iff x = 0 \text{ and } y = 0, \; x \wedge y = 1 \iff x = 1 \text{ or } y = 1$$
$$x \vee y = 0 \iff x = 0 \text{ or } y = 0, \; x \vee y = 1 \iff x = 1 \text{ and } y = 1$$

The reason we are presenting these obvious computations is that in [1] and all other recent literature their definitions were swapped. Of course, our system is isomorphic to theirs.

It is well known that in classical propositional logic, one can express any connective of arbitrary arity using either $\neg, \vee$ or $\neg, \wedge$. This is no longer true in continuous propositional logic. An important development is to consider the binary continuous connective $\dot{-}$, defined as

$$x \dot{-} y = \max\{x - y, 0\}.$$

In the presence of $\neg$, we do not lose expressive power by adopting $\dot{-}$ instead of $\vee$ or $\wedge$, because

$$x \vee y = x \dot{-} (x \dot{-} y),$$
$$x \wedge y = \neg(\neg x \vee \neg y).$$

Sometimes it is convenient to use the following redundant binary continuous connective as a dual of $\dot{-}$:

$$x \dot{+} y = \neg(\neg x \dot{-} y).$$

As for continuous connectives of higher arity, we only specify the *projections* as below. For $n \geqslant 1$ and $1 \leqslant i \leqslant n$,

$$P_i^n(x_1, \ldots, x_n) = x_i.$$

It is conventional to consider only connective systems of continuous propositional logic where all the P_i^n are present.

Definition 1. *A* system of continuous connectives *is a sequence* $\mathcal{F} = \{F_n : n < \omega\}$ *where each* F_n *is a collection of continuous functions from* $[0,1]^n$ *to* $[0,1]$. *The* closure *of* $\mathcal{F}$, *denoted* $\overline{\mathcal{F}}$, *is the smallest system* $\mathcal{G} = \{G_n : n < \omega\}$ *of continuous connectives satisfying:*

- *for all* $n < \omega$, $F_n \subseteq G_n$;
- *for all* $n \geqslant 1$ *and* $1 \leqslant i \leqslant n$, $P_i^n \in G_n$; *and*
- *if* $f \in G_n$ *and* $g_1, \ldots, g_n \in G_m$, *then* $f \circ (g_1, \ldots, g_n) \in G_m$.

$\mathcal{F}$ *is* closed *if* $\overline{\mathcal{F}} = \mathcal{F}$.

Although it is a slight abuse of notation, we usually present a system $\mathcal{F}$ as a single set of connectives if the arities of the connectives are clear from the context. The above definition is slightly different from that in [1]; this definition formalizes the above convention that all projections are considered part of any connective system.

Definition 2 ([1])**.** *A system of continuous connectives* $\mathcal{F}$ *is* full *if, letting* $\overline{\mathcal{F}} = \{F_n : n < \omega\}$, *for every* $0 < n < \omega$, *the set* F_n *is dense in the space of all continuous functions from* $[0,1]^n$ *to* $[0,1]$, *equipped with the compact-open topology (equivalently, uniform convergence topology).*

Note that fullness does not require that the closure of the system has constants. Later in this section we will give an example of a closed full system without any constants.

The basic tool to study full connective systems in continuous propositional logic is the following lattice version of the Stone–Weierstrass theorem.

Theorem 2 (Stone–Weierstrass Theorem, lattice version [1]). *Let X be a compact Hausdorff space containing at least two points. Equip $C(X,[0,1])$ with the uniform convergence topology. Let $\mathcal{S} \subseteq C(X,[0,1])$ be a sub-lattice (i.e., for $g,h \in \mathcal{S}$ we have $\min\{g,h\}, \max\{g,h\} \in \mathcal{S}$) such that for every distinct $x,y \in X$, $a,b \in [0,1]$, and $\epsilon > 0$, there is $f \in \mathcal{S}$ such that $|f(x)-a|, |f(y)-b| < \epsilon$. Then $\mathcal{S}$ is dense in $C(X,[0,1])$.*

The following theorem was proved in [1].

Theorem 3 ([1]). *Let X be a compact Hausdorff space. Assume that $\mathcal{S} \subseteq C(X,[0,1])$ is closed under $\neg$ and $\dot{-}$, separates points in X (i.e., for any two distinct $x,y \in X$, there is $f \in \mathcal{S}$ such that $f(x) \neq f(y)$), and satisfies either of the following two additional properties:*

(i) *The set $C = \{a \in [0,1] : C_a \in \mathcal{S}\}$ is dense in $[0,1]$.*
(ii) *$\mathcal{S}$ is closed under the function $x \mapsto \frac{x}{2}$.*
Then $\mathcal{S}$ is dense in $C(X,[0,1])$.

The proof of Theorem 3 gives that the following system of continuous connectives is full:

$$\{(C_a)_{a \in A},\ C_1,\ \dot{-}\}$$

where $A \subseteq [0,1]$ is dense.

Theorem 3 also implies that

$$\left\{\neg,\ \dot{-}, \frac{x}{2}\right\}$$

is a full set of continuous connectives. This system has been adopted as the standard connective system for continuous propositional logic. However, before we focus on this system, we will prove some basic results about general connective systems.

What follows is a characterization of unary continuous functions f where $\{\neg, \dot{-}, f\}$ is full.

Definition 3. *A set $A \subseteq [0,1]$ is a $\{\neg, \dot{-}\}$*-algebra *if A contains 0 and is closed under $\neg$ and $\dot{-}$. Let $S \subseteq [0,1]$. The $\{\neg, \dot{-}\}$-algebra* generated by *S, denoted A_S, is the smallest $\{\neg, \dot{-}\}$-algebra containing all elements of S. A is* finitely generated *if $A = A_S$ for some finite set $S \subseteq [0,1]$.*

Lemma 1. *For any nonempty finitely generated $\{\neg, \dot{-}\}$-algebra A, exactly one of the following holds:*

(i) *There is an irrational $0 < \alpha < 1$ such that*

$$\{a \in [0,1]\ :\ a \equiv k\alpha \ \mathit{mod}\ 1, k \geqslant 1\} \subseteq A.$$

In particular, A is dense in $[0,1]$.

(ii) *A is finite, and there is an $N \geqslant 1$ such that $A = \{k/N : 0 \leqslant k \leqslant N\}$.*

Proof. Note that any $\{\neg, \dot{-}\}$-algebra contains 0 and 1 and is closed under $\dot{-}$ and $\dot{+}$. If A contains an irrational $\alpha \in [0,1]$, then $\{a \in [0,1]\ :\ a \equiv k\alpha \bmod 1, k \geqslant 1\} \subseteq A$ and A is dense. Otherwise, A contains only rational numbers. Since A is closed under $\dot{-}$ and $\dot{+}$, for any $a,b \in A$, $\gcd(a,b) \in A$, where $\gcd(a,b)$ is the great common divisor of a and b, i.e., the largest rational c such that a/c and b/c are both integers. It follows that A is finite if A has only finitely many rational generators. Letting $a \in A$ be the smallest nonzero number in A, then $\gcd(a,1) = a$. This means that $a = 1/N$ for some $N \geqslant 1$. We have $A = \{k/N : 0 \leqslant k \leqslant N\}$. □

Definition 4. *Let $f : [0,1] \to [0,1]$ be a continuous function. Define*

$$A_0 = \textit{the } \{\neg, \dot{-}\}\textit{-algebra generated by } \{f(0), f(1)\}$$
$$A_{n+1} = \textit{the } \{\neg, \dot{-}\}\textit{-algebra generated by } A_n \cup f(A_n)$$

Let $A_f = \bigcup_n A_n$. A_f *is called the* $\{\neg, \dot{-}\}$*-algebra* generated by f.

Proposition 1. *Let* $f : [0,1] \to [0,1]$ *be a continuous function. Then the following are equivalent:*

1. $\{\neg, \dot{-}, f\}$ *is a full system of continuous connectives.*
2. A_f *is infinite.*

Proof. Since $+$ can be expressed from $\neg$ and $\dot{-}$, when A_f is infinite the set of all constants in the closure of $\{\neg, \dot{-}, f\}$ is dense in $[0,1]$. By Theorem 3, $\{\neg, \dot{-}, f\}$ is full. Conversely, suppose A_f is finite. It is easy to see by induction that for any n-ary function g of the closure of $\{\neg, \dot{-}, f\}$, $n \geqslant 1$, if $x \in A_f$ then $g(x, \ldots, x) \in A_f$. Thus the set of n-ary functions of the closure of $\{\neg, \dot{-}, f\}$ is not dense in $C([0,1]^n, [0,1])$. □

Some examples of f that give fullness include

1. any continuous f that is strictly increasing and satisfies $f(0) > 0$ or $f(1) < 1$;
2. any continuous f that is strictly decreasing and satisfies $f(0) < 1$ or $f(1) > 0$;
3. any continuous f with $f(1)$ irrational.

Next we turn to some examples of non-full systems.

It follows from Proposition 1 that $\{\neg, \dot{-}\}$ is not full. Since the closure of $\{\neg, \dot{-}\}$ contains $0, 1, \neg, \wedge, \vee$ it follows that $\{0, 1, \neg, \wedge, \vee\}$ is also not full.

Lemma 2. *Suppose* $\overline{\mathcal{F}}$ *contains only functions that are* 1*-Lipschitz in each variable. Then* $\mathcal{F}$ *is not full.*

Proof. The set of 1-Lipschitz unary functions is not dense in $C([0,1],[0,1])$. □

Example 1. *The system* $\{0, 1, \neg, \wedge, \vee, \frac{x}{2}\}$ *is not full. By induction one can verify that every function in the closure of the system is* 1*-Lipschitz in every variable.*

In classical propositional logic the system $\{\,|\,\}$ is full, where $|$ is the Sheffer stroke (or nand), defined as

$$x \mid y = \neg(x \wedge y).$$

The following result shows that in continuous propositional logic there is not a single function f such that $\{f\}$ is full.

Proposition 2. *For every continuous function* $f : [0,1]^n \to [0,1]$, $\{f\}$ *is not a full system of continuous connectives.*

Proof. Define $g(x) = f(x, \ldots, x)$. Then $g : [0,1] \to [0,1]$ and has a fixed point x_0. Now we claim that for all h in the closure of $\{f\}$, $h(x_0, \ldots, x_0) = x_0$. This is proved by induction. When $h = f$ or h is a projection, this is obvious. For compositions, it is true by induction. Now $\{f\}$ is not full since the unary functions in the closure of $\{f\}$ is not dense in $C([0,1],[0,1])$. □

We do not know if there is a unary function f such that $\{0, \neg, \wedge, f\}$ is full.

Definition 5. *A* tautology *in continuous propositional logic is a formula* $f(x_1, \ldots, x_n)$ *such that* $f(x_1, \ldots, x_n) \equiv 0$ *for all* $x_1, \ldots, x_n \in [0,1]$. *A formula* $f(x_1, \ldots, x_n)$ *is* satisfiable *if there are* $x_1, \ldots, x_n \in [0,1]$ *such that* $f(x_1, \ldots, x_n) = 0$.

Note that the projection functions are always satisfiable, so the set of satisfiable formulas is always nonempty.

Proposition 3. *There exists a closed full system of continuous connectives without any constants. In particular, there are no tautologies in this system.*

Proof. Let F_1 consist of all piecewise linear functions f on $[0,1]$ (i.e., there is a finite sequence $x_0 = 0 < x_1 < \cdots < x_m = 1$ such that $\forall 0 \leqslant i \leqslant m-1, f$ is linear on $[x_i, x_{i+1}]$) such that f is not constant on any interval. Consider the system $\mathcal{F} = \{F_n : n < \omega\}$ where F_1 is defined as above, $F_2 = \{\wedge, \vee\}$, and $F_n = \varnothing$ for $n \neq 1, 2$. Then $\overline{\mathcal{F}}$ is a closed system, $\overline{\mathcal{F}}$ separates points, and the set of all n-ary functions of $\overline{\mathcal{F}}$, for any $n \geqslant 1$, is a sub-lattice in $C([0,1]^n, [0,1])$. Theorem 2 implies that $\overline{\mathcal{F}}$ is full.

We verify that $\overline{\mathcal{F}}$ does not contain constant functions. First, note that F_1 is closed under composition, $\wedge$, and $\vee$, and it does not contain constant functions. Next, we claim that for any n-ary g of $\overline{\mathcal{F}}$, $n \geqslant 1$, $f(x) = g(x, \ldots, x) \in F_1$. This can be seen by noting that it is true for P_i^n for all $n \geqslant 1$ and $1 \leqslant i \leqslant n$, and is preserved under composition, $\wedge$, and $\vee$. Thus g is not a constant function. □

We will use the following observation in the next section.

Lemma 3. *The constant formulas in the closure of the system* $\{\neg, \dot{-}, \frac{x}{2}\}$ *takes values only in dyadic rationals.*

Proof. By induction one can verify that the linear functions used to express any formula in the closure of $\{\neg, \dot{-}, \frac{x}{2}\}$ has dyadic rationals as coefficients. Now if $f(x_1, \ldots, x_n)$ is a constant function in the closure of $\{\neg, \dot{-}, \frac{x}{2}\}$ then the constant value is $f(0, \ldots, 0)$, which is a dyadic rational. □

3. Strong Compactness and Craig Interpolation

Throughout this section we fix the full system $\{\neg, \dot{-}, \frac{x}{2}\}$ for continuous propositional logic.

The deduction system for continuous propositional logic is an adaptation of the Łukasiewicz axioms studied in many-valued logics, particularly fuzzy logic. The reader can refer to [9,10] for more background information about fuzzy logic, many-valued logic, and their basic model theory.

The *Łukasiewicz axioms* are:

(A1) $(p \dot{-} q) \dot{-} p$
(A2) $((r \dot{-} p) \dot{-} (r \dot{-} q)) \dot{-} (q \dot{-} p)$
(A3) $(p \dot{-} (p \dot{-} q)) \dot{-} (q \dot{-} (q \dot{-} p))$
(A4) $(p \dot{-} q) \dot{-} (\neg q \dot{-} \neg p)$

The Modus Ponens rule specifies the procedure to make deductions:

$$\frac{p,\ q \dot{-} p}{q}$$

This deduction system consistitutes the *Łukasiewicz logic*, denoted as Ł, which is a many-valued logic originally proposed by Łukasiewicz.

Following [5,7] we consider two more axioms in continuous propositional logic:

(A5) $\frac{1}{2}p \dot{-} (p \dot{-} \frac{1}{2}p)$

(A6) $(p \dot{-} \frac{1}{2}p) \dot{-} \frac{1}{2}p$

Denote the deduction system as CŁ. We write $\Sigma \vdash^{\text{CŁ}} p$ if p is deducible from the formulas in Σ, together with axioms (A1)–(A6), by repeatedly applying the Modus Ponens rule in CŁ. Similarly, we also write $\Sigma \vDash^{\text{CŁ}} p$ if all truth value assignments that evaluate all formulas in Σ to be 0 also evaluate p to be 0. If the context is clear we omit the superscripts for notational simplicity.

Satisfiability and consistency of a set of formulas are defined in the most natural way. The soundness of the continuous propositional logic is obvious. The following Completeness theorem was proved in [7].

Theorem 4 ([7])**.** *Let Σ be a set of formulas in continuous propositional logic. Then Σ is consistent iff Σ is satisfiable.*

The following weak Compactness theorem is a corollary of the above theorem.

Theorem 5. *Let Σ be a set of formulas in continuous propositional logic. Then Σ is satisfiable iff any finite subset Σ' of Σ is satisfiable.*

The following Approximated Strong Completeness for continuous propositional logic is also a corollary of the Completeness theorem.

Theorem 6 ([7])**.** *Let Σ be set of formulas and p be a formula in continuous propositional logic. Then $\Sigma \models p$ iff $\Sigma \vdash p \dot{-} 2^{-n}$ for all $n < \omega$.*

This is the best one can do; the Strong Completeness for continuous propositional logic fails. To see this, consider $\Sigma = \{p \dot{-} 2^{-n} \,:\, n \geqslant 1\}$, where p is an atomic formula. Then $\Sigma \models p$ but $\Sigma \nvdash p$.

This same example also shows that the Strong Compactness for continuous propositional logic fails, since we have $\Sigma \models p$ but there is no finite $\Sigma' \subseteq \Sigma$ with $\Sigma' \models p$.

We do, however, have an approximated version of the Strong Compactness theorem as a corollary of the Approximated Strong Completeness theorem.

Theorem 7 (Approximated Strong Compactness)**.** *Let Σ be set of formulas and p be a formula in continuous propositional logic. If $\Sigma \models p$ and $n < \omega$ then there is a finite $\Sigma' \subseteq \Sigma$ such that $\Sigma' \models p \dot{-} 2^{-n}$.*

Proof. Suppose $\Sigma \models p$ and $n < \omega$. By Theorem 6 $\Sigma \vdash p \dot{-} 2^{-n}$. Thus there is a finite $\Sigma' \subseteq \Sigma$ such that $\Sigma' \vdash p \dot{-} 2^{-n}$. By Theorem 4 we have $\Sigma' \models p \dot{-} 2^{-n}$. □

Next we note that the Craig Interpolation theorem for continuous propositional logic fails. Let x, y, z be atomic propositions and consider

$$\begin{aligned}
&p(x,y) = (1 \dot{-} (x \dot{-} y)) \dot{-} (x \dot{-} y)\\
&q(x,y) = p(x,y) \vee p(y,x)\\
&r(x,y) = |x-y| = (x \dot{-} y) \dot{+} (y \dot{-} x) = (x \dot{-} y) \vee (y \dot{-} x)\\
&\varphi(x,y) = q(x,y) \wedge r(x,y)\\
&\psi(y,z) = (1 \dot{-} \varphi(y,z)) \dot{-} \varphi(y,z)
\end{aligned}$$

Observe that for all $x, y, z \in [0,1]$ we have

$$\varphi(x,y) \geqslant \frac{1}{3} \geqslant \psi(y,z).$$

Thus $\models \psi \dot{-} \varphi$. Assume $\theta(y)$ is any formula satisfying $\models \psi \dot{-} \theta$ and $\models \theta \dot{-} \varphi$. Observe further that for any $y \in [0,1]$,

$$\inf_x \varphi(x,y) = \frac{1}{3} = \sup_z \psi(y,z).$$

We conclude that $\theta(y) \equiv \frac{1}{3}$, that is, it is a constant unary function that takes value $\frac{1}{3}$, contradicting Lemma 3.

The following is an Approximated Craig Interpolation theorem for continuous propositional logic.

Theorem 8 (Approximated Craig Interpolation). *Let $\varphi(\vec{x},\vec{y})$ and $\psi(\vec{y},\vec{z})$ be formulas in continuous propositional logic. Suppose $\models \varphi \dot{-} \psi$. Then for any $n < \omega$ there is a formula $\theta(\vec{y})$ such that $\models \varphi \dot{-} \theta$ and $\models (\theta \dot{-} \psi) \dot{-} 2^{-n}$. Similarly, for any $n < \omega$ there is a formula $\theta'(\vec{y})$ such that $\models (\varphi \dot{-} \theta') \dot{-} 2^{-n}$ and $\models \theta' \dot{-} \psi$.*

Proof. From $\models \varphi \dot{-} \psi$ we conclude that for any $\vec{y}$,

$$\sup_{\vec{x}} \varphi(\vec{x},\vec{y}) \leqslant \inf_{\vec{z}} \psi(\vec{y},\vec{z}).$$

Let $f(\vec{y}) = \inf_{\vec{z}} \psi(\vec{y},\vec{z})$. Then f is continuous. By the fullness of continuous propositional logic, there is a formula $\theta_0(\vec{y})$ such that

$$\sup_{\vec{y}} |\theta_0(\vec{y}) - f(\vec{y})/2 - 2^{-n-2}| \leqslant 2^{-n-2}.$$

Let $\theta = \theta_0 \dot{+} \theta_0$. Then for any $\vec{y}$,

$$\theta(\vec{y}) = \theta_0(\vec{y}) \dot{+} \theta_0(\vec{y}) \geqslant f(\vec{y})/2 \dot{+} f(\vec{y})/2 = f(\vec{y}) \geqslant \sup_{\vec{x}} \varphi(\vec{x},\vec{y}) \geqslant \varphi(\vec{x},\vec{y})$$

and thus $\models \varphi \dot{-} \theta$. On the other hand,

$$\theta(\vec{y}) - \psi(\vec{x},\vec{y}) - 2^{-n} \leqslant 2(\theta_0(\vec{y}) - f(\vec{y})/2 - 2^{-n-2}) - 2^{-n-1} \leqslant 0.$$

Thus $\models (\theta \dot{-} \psi) \dot{-} 2^{-n}$. □

4. Complexity of Decidability Problems, Part I

In this and the next sections we prove Theorem 1. First we define the relevant concepts. Recall that we have defined the notion of tautology and satisfiability. Here we expand to some other notions.

Definition 6. *A formula $f(x_1,\ldots,x_n)$ of continuous propositional logic is*

- *a* fallacy *if $\neg f$ is a tautology, i.e., for all $x_1,\ldots,x_n \in [0,1]$,*

$$f(x_1,\ldots,x_n) = 1;$$

- falsifiable *if there are $x_1,\ldots,x_n \in [0,1]$ such that $f(x_1,\ldots,x_n) = 1$.*

Definition 7. *Fix $\alpha \in [0,1]$. A formula $f(x_1,\ldots,x_n)$ of continuous propositional logic is*

- *an α-*tautology *if for all $x_1,\ldots,x_n \in [0,1]$, $f(x_1,\ldots,x_n) \leqslant \alpha$;*
- *an α-*fallacy *if for all $x_1,\ldots,x_n \in [0,1]$, $f(x_1,\ldots,x_n) \geqslant \alpha$;*
- α-satisfiable *if there are $x_1,\ldots,x_n \in [0,1]$ such that $f(x_1,\ldots,x_n) \leqslant \alpha$;*
- α-falsifiable *if there are $x_1,\ldots,x_n \in [0,1]$ such that $f(x_1,\ldots,x_n) \geqslant \alpha$.*

One can similarly define the notion of $<\alpha$-tautology and $<\alpha$-satisfiability for $\alpha \in (0,1]$ and that of $>\alpha$-fallacy and $>\alpha$-falsifiability for $\alpha \in [0,1)$.

Note that

tautology = 0-tautology
fallacy = 1-fallacy
satisfiable = 0-satisfiable
falsifiable = 1-falsifiable

In this section we investigate the computational complexity of these sets of formulas in continuous propositional logic. The following lemma is easy to prove; it provides P-time reductions between various sets.

Lemma 4. *Suppose $\overline{\mathcal{F}}$ contains $\neg$. Let $p \in \overline{\mathcal{F}}$. Then for any $\alpha \in (0,1]$ and $\beta \in [0,1)$, the following hold:*

- *p is $<\alpha$-satisfiable iff p is not an α-fallacy iff $\neg p$ is $>(1-\alpha)$-falsifiable iff $\neg p$ is not a $(1-\alpha)$-tautology;*
- *p is β-satisfiable iff p is not an $>\beta$-fallacy iff $\neg p$ is $(1-\beta)$-falsifiable iff $\neg p$ is not a $<(1-\beta)$-tautology.*

Our objective is to show that satisfiability in continuous propositional logic is in NP. To do this we first construct a particular full system $\mathcal{P}$ of continuous connectives as follows.

For $n \geqslant 1$, an element $x = (a_1, \dots, a_n) \in [0,1]^n$ is a *rational point* if $a_1, \dots, a_n$ are all rational numbers. Let V_n be the set of all vertices of $[0,1]^n$, i.e., $x = (a_1, \dots, a_n) \in V_n$ iff each $a_i \in \{0,1\}$ for $1 \leqslant i \leqslant n$. Given finitely many points $x_1, \dots, x_k \in [0,1]^n$, a *polyhedronization* of $[0,1]^n$ with extreme points $x_1, \dots, x_k$ is a decomposition of $[0,1]^n$ as a complex that consists of convex polyhedra with extreme points that are among $x_1, \dots, x_k$ and the elements of V_n. For each $n \geqslant 1$, let F_n be the set of all continuous functions f from $[0,1]^n$ to $[0,1]$ such that there is a polyhedronization of $[0,1]^n$ with finitely many rational extreme points such that f is a linear function with rational coefficients on each of the polyhedron in the polyhedronization. Let $\mathcal{P} = \{F_n : n < \omega\}$.

Lemma 5. *$\mathcal{P}$ is full.*

Proof. When $n = 1$ F_1 is a dense subset of the set of unary functions considered in the previous proof. Obviously F_1 separates points. Now it is easy to check that $\wedge, \vee \in F_2$. Thus the n-ary functions of $\overline{\mathcal{P}}$, for any $n \geqslant 1$, form a sub-lattice of $C([0,1]^n, [0,1])$. By Theorem 2, $\mathcal{P}$ is full. □

Lemma 6. *$\bigcup \mathcal{P}$ is closed under composition.*

Proof. Suppose $g_1, \dots, g_m$ are n-ary functions of $\mathcal{P}$, f is an m-ary function of $\mathcal{P}$, and $h = f \circ (g_1, \dots, g_m)$. Each g_i, $i = 1, \dots, m$, is piecewise linear with a polyhedronization P_i of $[0,1]^n$ (i.e., f is linear on every convex polyhedron and agrees on their boundaries) . Let P be the largest common refinement of all $P_1, \dots, P_m$. Then the extreme points of P are among

- the extreme points of $P_1, \dots, P_m$,
- elements of V_n, and
- extreme points of the intersections of polyhedra in $P_1, \dots, P_m$.

For the last kind, note that the intersection of any number of polyhedra is still convex, and therefore is itself a polyhedron. Their extreme points are solutions of linear equations with rational coefficients, and therefore are also rational points. Let P' be a polyhedronization of $[0,1]^m$ such that f is linear on each polyhedron in P'. Consider a particular polyhedron in P, and denote it as S. Note that each $g_1, \dots, g_m$ is a linear function on S. Let $\gamma_1, \dots, \gamma_m$ be these linear functions corresponding respectively to $g_1, \dots, g_m$. They have rational coefficients. Let S' be a polyhedron in P'. Now S' is given by a number of linear inequalities in m variables. Denote these inequalities as, for instance, $\varphi_i(y_1, \dots, y_m) \leqslant c_i$, for $i = 1, \dots, k$. The coefficients of these inequalities are all rational. Consider the subset of $[0,1]^n$ that satisfies

$$\varphi_i(\gamma_1(x_1, \dots, x_n), \dots, \gamma_m(x_1, \dots, x_n)) \leqslant c_i$$

for all $i = 1, \dots, k$. This is again a system of linear inequalities with rational coefficients. Note that the solution set is convex, and thus its intersection with S, if nonempty, is also convex, and therefore is a polyhedron. Traversing all S' in T' would give a complete polyhedronization of S into polyhedra with rational extreme points, and on each polyhedron the function $h = f \circ (g_1, \dots, g_m)$ is linear. This shows that h is an n-ary function of $\mathcal{P}$. □

Thus $\mathcal{P}$ is a full, closed system of continuous connectives. It is easy to check that the standard connective system $\{\neg, \dot{-}, \frac{x}{2}\}$ is a subsystem of $\mathcal{P}$. Hence the satisfiability problem for the standard continuous propositional logic is a subproblem of that for $\mathcal{P}$. Also, since each function of $\mathcal{P}$ is determined by a polyhedronization with finitely many rational extreme points and linear functions with rational coefficients, we can code all functions of $\mathcal{P}$ by natural numbers.

Now consider an arbitrary finite subset $\mathcal{F}$ of the system $\mathcal{P}$. For any rational $\alpha \in (0,1]$ and $\beta \in [0,1)$, we show that the sets of $<\alpha$-satisfiable formulas and β-satisfiable formulas in $\overline{\mathcal{F}}$ are in NP.

Recall that each n-ary function $f \in \mathcal{F} \subseteq \mathcal{P}$ is determined by a polyhedronization Δ of $[0,1]^n$ with rational extreme points and a linear function with rational coefficients on each polyhedron in Δ. Furthermore, each polyhedron in Δ is given by a number of linear inequalities of the form

$$c_1x_1 + \cdots + c_nx_n + d \geqslant 0$$

or

$$c_1x_1 + \cdots + c_nx_n + d \leqslant 0$$

where the coefficients $c_1, \ldots, c_n, d$ are rational numbers. We refer to the linear functions appearing on the left hand sides of the inequalities used in the polyhedronization of $[0,1]^n$ as *type I linear forms* and the linear functions in the definition of f as *type II linear forms*.

For each linear form λ of either type, we define its *standard form* to be the form

$$\frac{a_1}{d}x_1 + \cdots + \frac{a_n}{d}x_n + \frac{b}{d}$$

where $a_1, \ldots, a_n, b$ and $d > 0$ are integers such that

$$\gcd(a_1, \ldots, a_n, b, d) = 1.$$

Also define

$$M_\lambda = \max\left\{d, \frac{|a_1|}{d}, \ldots, \frac{|a_n|}{d}, \frac{|b|}{d}\right\}.$$

Similarly, for a tuple of rational numbers $r = (r_1, \ldots, r_n)$ we also define its *standard form* to be the form

$$\left(\frac{a_1}{d}, \ldots, \frac{a_n}{d}\right)$$

where $a_1, \ldots, a_n$ and $d > 0$ are integers and $\gcd(a_1, \ldots, a_n, d) = 1$, and let

$$M_r = \max\left\{d, \frac{|a_1|}{d}, \ldots, \frac{|a_n|}{d}\right\}$$

when $r \neq (0, \ldots, 0)$, and $M_r = 0$ when $r = (0, \ldots, 0)$.

Lemma 7. *Let $\lambda_1, \ldots, \lambda_n$ be linear forms with variables $x_1, \ldots, x_n$ such that the system*

$$\lambda_1 = 0, \ldots, \lambda_n = 0$$

has a unique solution $r = (r_1, \ldots, r_n)$. Then $M_r \leqslant n! M_{\lambda_1}^2 \cdots M_{\lambda_n}^2$.

Proof. This is a direct consequence of Cramer's rule. Let A be the matrix consisting of the coefficients of $x_1, \ldots, x_n$ in $\lambda_1, \ldots, \lambda_n$, and for each $j = 1, \ldots, n$ let A_j be the matrix obtained from A by replacing its j-th column by the constant terms of $\lambda_1, \ldots, \lambda_n$. Then

$$r_j = \det(A_j)/\det(A).$$

Let $d_1, \ldots, d_n$ be the common denominators appearing in the forms $\lambda_1, \ldots, \lambda_n$, respectively. We can write

$$r_j = \frac{\det(A_j)d_1 \ldots d_n}{\det(A)d_1 \ldots d_n},$$

noting that both the numerator and the denominator are integers. We have

$$|\det(A)d_1\cdots d_n| \leqslant n!M_{\lambda_1}^2\cdots M_{\lambda_n}^2.$$

Similarly

$$|\det(A_j)d_1\cdots d_n| \leqslant n!M_{\lambda_1}^2\cdots M_{\lambda_n}^2.$$

The conclusion of the lemma follows. □

We will not need the following lemma but it is a converse as well as a consequence of Lemma 7.

Lemma 8. *Let λ be a linear form in variables $x_1,\dots,x_n$ with 1 as its constant term and let $s_1,\dots,s_n$ be rational points in $[0,1]^n$ which determine the hyperplane $\lambda = 0$. Then $M_\lambda \leqslant n!M_{s_1}^2\cdots M_{s_n}^2$.*

Proof. Suppose

$$\lambda = \frac{a_1}{d}x_1 + \cdots + \frac{a_n}{d}x_n + 1$$

such that $a_1,\dots,a_n,d$ are integers, $d > 0$ and $\gcd(a_1,\dots,a_n,d) = 1$. For each $i = 1,\dots,n$, say $s_i = (s_{i,1},\dots,s_{i,n})$, also consider the form

$$\mu_i = s_{i,1}y_1 + \cdots + s_{i,n}y_n + 1.$$

Then

$$\left(\frac{a_1}{d},\dots,\frac{a_n}{d}\right)$$

is the unique solution of the system

$$\mu_1 = 0,\dots,\mu_n = 0.$$

By Lemma 7,

$$d, \frac{|a_1|}{d},\dots,\frac{|a_n|}{d} \leqslant n!M_{\mu_1}^2\cdots M_{\mu_n}^2,$$

which implies that

$$M_\lambda = \max\{d,|a_1|,\cdots,|a_n|,1\} \leqslant n!M_{s_1}^2\cdots M_{s_n}^2.$$

□

Lemma 9. *Let μ be a linear form in m variables and let $\lambda_1,\dots,\lambda_m$ be linear forms in variables $x_1,\dots,x_n$. Let*

$$\nu = \mu(\lambda_1,\dots,\lambda_m).$$

Then $M_\nu \leqslant \max\{M_\mu M_{\lambda_1}\cdots M_{\lambda_m}, M_\mu(M_{\lambda_1} + \cdots + M_{\lambda_m} + 1)\}$.

Proof. Write $\mu,\lambda_1,\dots,\lambda_m$ in standard linear forms. The conclusion of the lemma is by straightforward computations. In fact, the common denominator of the form ν is bounded by the product of the common denominators of forms $\mu,\lambda_1,\dots,\lambda_m$, hence bounded by $M_\mu M_{\lambda_1}\cdots M_{\lambda_m}$. The coefficients of the form ν for each variable x_i is bounded by $M_\mu(M_{\lambda_1} + \cdots + M_{\lambda_m})$. The constant term of the form ν is bounded by

$$M_\mu(M_{\lambda_1} + \cdots + M_{\lambda_m} + 1).$$

□

Let $M \geqslant 3$ be a positive integer that is larger than all of the following:

- the number of functions in $\mathcal{F}$,
- the arities of the functions in $\mathcal{F}$,

- for each n-ary $f \in F$, the number of polyhedra, as well as the number of faces in all such polyhedra, in the polyhedronization Δ in the definition of f,
- M_λ, for each $f \in F$ and a standard linear form λ of either type in the definition of f.

Definition 8. *Given any system $\mathcal{F}$, a* connective tree *is a finite labeled, rooted, ordered tree (T, λ) such that*

- *if t is a terminal node of T, then $\lambda(t)$ is a variable or a constant;*
- *if t is a non-terminal node and t has n-many children for $n \geqslant 1$, then $\lambda(t)$ is an n-ary function of $\mathcal{F}$.*

Here ordered *means that for every node t there is a linear ordering of the children of t.*

Every connective tree T gives rise to a formula f. Let $x_1, \ldots, x_n$ be all the variables appearing as a label of a terminal node of T. Inductively, we can define a formula for each non-terminal node as follows. If a non-terminal node t has label f, which is an m-ary function of $\mathcal{F}$, and suppose inductively the children of f has been associated with formulas $g_1, \ldots, g_m$, listed in the order of the children, then the formula associated with t is $f \circ (g_1, \ldots, g_m)$. Let f_T denote the formula given by the tree T. It is easy to see that every formula f in $\overline{\mathcal{F}}$ admits a connective tree T with $f_T = f$.

Now we come back to the consideration of formulas in $\overline{\mathcal{F}}$. As mentioned above each $p \in \overline{\mathcal{F}}$ is associated with a connective tree T_p with either variables (atomic propositions) and constants (0-ary functions) as labels for terminal nodes and elements of $\mathcal{F}$ as labels for non-terminal nodes. We let $|p|$ denote the size (cardinality) of T_p. Thus $|p|$ represents the size of the formula p.

By Lemma 6 each n-ary $p \in \overline{\mathcal{F}}$ is also associated with a polyhedronization Δ_p such that on each polyhedron in Δ_p, p is a linear function with rational coefficients. We similarly refer to the linear forms appearing in this description of p as *type I* and *type II*, respectively.

Lemma 10. *Let $p \in \overline{\mathcal{F}}$ and let λ be a standard type II linear form for p. Then $M_\lambda \leqslant M^{2|p|}$.*

Proof. We prove this by induction on $|p|$. When $|p| = 1$ this is obvious. Consider a general p where $|p| > 1$. Suppose

$$p(x_1, \ldots, x_n) = f(q_1(x_1, \ldots, x_n), \ldots, q_m(x_1, \ldots, x_n))$$

where $f \in \mathcal{F}$ is m-ary, and $|p| = |q_1| + \cdots + |q_m| + 1$. Let λ be a type II linear form for p. Then

$$\lambda(x_1, \ldots, x_n) = \mu(\nu_1(x_1, \ldots, x_n), \nu_m(x_1, \ldots, x_n))$$

where μ is an m-ary linear form of type II for f and $\nu_1, \ldots, \nu_m$ are linear forms of type II for $q_1, \ldots, q_m$, respectively. By the inductive hypothesis, $M_{\nu_i} \leqslant M^{2|q_i|}$ for all $i = 1, \ldots, m$. Then by Lemma 9,

$$\begin{aligned} M_\lambda &\leqslant \max\{M_\mu M_{\lambda_1} \cdots M_{\lambda_m}, M_\mu(M_{\lambda_1} + \cdots + M_{\lambda_m} + 1)\} \\ &\leqslant M_\mu M^{2|q_1|} \cdots M^{2|q_m|} + M_\mu \\ &\leqslant M \cdot M^{2(|p|-1)} + M \leqslant M^{2|p|-1} + M \leqslant M^{2|p|}. \end{aligned}$$

□

Lemma 11. *Let $p \in \overline{\mathcal{F}}$ and let λ be a standard type I linear form for p. Then $M_\lambda \leqslant M^{|p|^2}$.*

Proof. We prove this by induction on $|p|$. When $|p| = 1$ this is obvius. Consider a general p where $|p| > 1$. Suppose

$$p(x_1, \dots, x_n) = f(q_1(x_1, \dots, x_n), \dots, q_m(x_1, \dots, x_n))$$

where $f \in \mathcal{F}$ is m-ary, and $|p| = |q_1| + \cdots + |q_n| + 1$. Let λ be a type I linear form for p. Then either λ is a type I linear form of some q_i, or there is a type I linear form μ for f and type II linear forms $\nu_1, \dots, \nu_m$ for $q_1, \dots, q_m$ respectively, such that

$$\lambda = \mu(\nu_1, \dots, \nu_m).$$

In the former case, by the inductive hypothesis $M_\lambda \leqslant M^{|q_i|^2}$ for some $i = 1, \dots, m$, and hence $M_\lambda \leqslant M^{|p|^2}$. In the latter case, by Lemma 10, $M_{\nu_i} \leqslant M^{2|q_i|}$ for $i = 1, \dots, m$. Then by Lemma 9 we have

$$\begin{aligned} M_\lambda &\leqslant \max\{M_\mu M_{\nu_1} \cdots M_{\nu_m}, M_\mu(M_{\nu_1} + \cdots + M_{\nu_m} + 1)\} \\ &\leqslant M \cdot M^{2|q_1|} \cdots M^{2|q_m|} + M \leqslant M^{|p|^2}. \end{aligned}$$

□

Lemma 12. *Let $p \in \overline{\mathcal{F}}$ and r be an extreme point of the polyhedronization for p. Then $M_r \leqslant M^{3|p|^3}$.*

Proof. Suppose p is n-ary. Note that $|p| \geqslant n$, and that r is the unique solution of a system of linear equations $\lambda_1 = 0, \dots, \lambda_n = 0$ where each λ_i is a type I linear form for p. By Lemmas 7 and 11, we have

$$\begin{aligned} M_r &\leqslant n! M_{\lambda_1}^2 \cdots M_{\lambda_n}^2 \\ &\leqslant 2^{n^2} \cdot M^{2n|p|^2} \leqslant M^{|p|^2} \cdot M^{2|p|^3} \leqslant M^{3|p|^3}. \end{aligned}$$

□

Theorem 9. *Let $\mathcal{F}$ be any finite subset of $\mathcal{P}$. For any rational $\alpha \in (0, 1]$ and $\beta \in [0, 1)$, the set of $<\alpha$-satisfiable formulas and the set of β-satisfiable formulas in $\overline{\mathcal{F}}$ are in NP.*

Proof. Let $p \in \overline{\mathcal{F}}$. We prove the statement for $<\alpha$-satisfiability. The statement for β-satisfiability is similar. Note that p is $<\alpha$-satisfiable iff there is an extreme point r of the polyhedronization Δ_p for p such that $p(r) < \alpha$. By Lemma 12,

$$p \text{ is } {<}\alpha\text{-satisfiable} \iff \exists r\ (M_r \leqslant M^{3|p|^3} \wedge p(r) < \alpha).$$

Suppose p is n-ary. Then $n \leqslant |p|$. Write $r = (r_1, \dots, r_n)$ and let $\|r\|$ be the size of r as an input to a Turing machine. Then $\|r\| \leqslant 3n|p|^3 \log M \leqslant 3|p|^4 \log M$. To complete the proof it suffices to show that the statement $p(r) < \alpha$ can be decided in P-time in terms of $|p|$.

We prove that it takes P-time to compute $p(r)$. We need the following notation for our discussion. Let $(T = T_p, \lambda)$ be the connective tree for p. For each node t, let T_t be the subtree of T below t, and let $p_t = f_{T_t}$. Our algorithm to compute $p(r)$ is by induction on $t \in T$ to compute the value $p_t(r)$, and finally $p(r) = p_{t_0}(r)$ where t_0 is the root of T. Note that for each terminal node $t \in T$, $p_t(r) = \lambda(t)(r)$, and for each non-terminal node $t \in T$ with children $t_1, \dots, t_n$,

$$p_t(r) = \lambda(t)(p_{t_1}(r), \dots, p_{t_n}(r)).$$

The entire algorithm will take exactly $|p|$ steps, corresponding to $|p|$ many nodes of T.

Before counting the computation time at each step, we claim that $M_{p_t(r)} \leqslant \max\{M, M_r\}^{|p_t|^2}$. We prove the claim by induction on $t \in T$. If t is a terminal node then $M_{p_t(r)} \leqslant \max\{M, M_r\} = \max\{M, M_r\}^{|p_t|^2}$. Suppose t is a non-terminal node with children $t_1, \ldots, t_n$. For $1 \leqslant i \leqslant n$ let $x_i = p_{t_i}(r)$. By the inductive hypothesis $M_{x_i} \leqslant \max\{M, M_r\}^{|p_{t_i}|^2}$ for $1 \leqslant i \leqslant n$. Also note that $|p_t| = |p_{t_1}| + \cdots + |p_{t_n}| + 1$. Since $M_{\lambda(t)} \leqslant M$, we have that

$$\begin{aligned} M_{p_t(r)} &\leqslant M^{n+1} M_r{}^{|p_{t_1}|^2 + \cdots + |p_{t_n}|^2} \\ &\leqslant \max\{M, M_r\}^{n+1+|p_{t_1}|^2 + \cdots + |p_{t_n}|^2} \leqslant \max\{M, M_r\}^{|p_t|^2}. \end{aligned}$$

Now assume $M_r \leqslant M^{3|p|^3}$. We count the time needed to compute $p_t(r)$ from $p_{t_1}(r), \ldots, p_{t_n}(r)$, which consists of n many binary multiplications of rational numbers and an $(n+1)$-ary addition of rational numbers. Each of the binary multiplication takes time no more than

$$c_1 \log(M^2 \max\{M, M_r\}^{2|p_t|^2}) \leqslant c \log(M^{2+6|p|^5}) = c(2 + 6|p|^5) \log M$$

for some constant c_1. The $(n+1)$-ary addition takes time no more than

$$c_2 n^2 \log(M^2 \max\{M, M_r\}^{|p_t|^2}) \leqslant c|p|^2(2 + 3|p|^5) \log M$$

for some constant c_2. Therefore, the computation of $p_t(r)$ takes time no more than

$$c(2 + 2|p|^2 + 6|p|^5 + 3|p|^7) \log M.$$

In summary, the computation of $p(r) = P_{t_0}(r)$ takes time no more than

$$c|p|(2 + 2|p|^2 + 6|p|^5 + 3|p|^7) \log M,$$

which is a polynomial in $|p|$. □

Corollary 1. *Let $\mathcal{F}$ be a finite subset of $\mathcal{P}$. Suppose $\overline{\mathcal{F}}$ contains $\neg$. Then for all rational $\alpha \in (0, 1]$ and $\beta \in [0, 1)$:*

- *the following sets of formulas in $\overline{\mathcal{F}}$ are in NP:*
  1. *$<\alpha$-satisfiable formulas;*
  2. *α-falsifiable formulas;*
  3. *β-satisfiable formulas;*
  4. *$>\beta$-falsifiable formulas;*
- *the following sets of formulas in $\overline{\mathcal{F}}$ are in co-NP:*
  5. *$<\alpha$-tautologies;*
  6. *α-fallacies;*
  7. *β-tautologies;*
  8. *$>\beta$-fallacies.*

Proof. This follows immediately from Theorem 9 and Lemma 4. □

5. Complexity of Decidability Problems, Part II

In this section we prove the other direction of Theorem 1, namely that satisfiability for continuous propositional logic is NP-complete.

Lemma 13. *Let $\mathcal{F} = \{\neg, \wedge\}$. Then for every $n \geqslant 1$ and n-ary function p of $\overline{\mathcal{F}}$, for all $r = (r_1, \ldots, r_n), s = (s_1, \ldots, s_n) \in [0, 1]^n$,*

$$|p(r) - p(s)| \leqslant \max_i |r_i - s_i|.$$

Proof. By induction on p. If $p = x_i$ for some variable x_i the statement is obvious. The case $p = \neg q$ is straightforward. Consider $p = q \wedge u$. Let $a = \max_i |r_i - s_i|$. Then by the inductive hypothesis, $|q(r) - q(s)|, |u(r) - u(s)| \leqslant a$. Without loss of generality we may assume $q(r) \geqslant u(r)$. If $q(s) \geqslant u(s)$ then $|p(r) - p(s)| = |q(r) - q(s)| \leqslant a$ and we are done. Suppose $q(s) \leqslant u(s)$. Then consider two cases.

Case 1: $q(r) \geqslant u(s)$. In this case, $q(s) \leqslant u(s) \leqslant q(r)$. Since $q(r) - q(s) \leqslant a$, we have $0 \leqslant q(r) - u(s) \leqslant a$.

Case 2: $q(r) \leqslant u(s)$. In this case, $u(r) \leqslant q(r) \leqslant u(s)$. Since $u(s) - u(r) \leqslant a$, we have $0 \leqslant u(s) - q(r) \leqslant a$. □

Proposition 4. *Let $\mathcal{F}$ be a finite subset of $\mathcal{P}$. Suppose $\overline{\mathcal{F}}$ contains $\neg, \wedge$. Then for all rational $\alpha \in (0, \frac{1}{2}]$ and $\beta \in [0, \frac{1}{2})$, the set of all $<\alpha$-satisfiable formulas and the set of all β-satisfiable formulas of $\overline{\mathcal{F}}$ are NP-hard.*

Proof. Let L be classical propositional logic with only connectives $\neg, \wedge$. By our assumption on $\overline{\mathcal{F}}$, for each p a formula of L we can associate a $\tilde{p} \in \overline{\mathcal{F}}$ by replacing all occurrences of $\neg$ and $\wedge$ by appropriate formulas in $\overline{\mathcal{F}}$. The mapping $p \mapsto \tilde{p}$ is P-time computable.

Let SAT be the set of all satisfiable formulas in L. Then SAT is NP-complete by Cook's Theorem. We show that for all rational $\alpha \in (0, \frac{1}{2}]$ and $\beta \in [0, \frac{1}{2})$ and $p \in$ L, $p \in$ SAT iff $\tilde{p}$ is $<\alpha$-satisfiable iff $\tilde{p}$ is β-satisfiable.

Suppose first $p \in$ SAT. Then there is a truth value assignment r such that $\tilde{p}(r) = p(r) = 0$. Thus $\tilde{p}$ is $<\alpha$-satisfiable for all rational $\alpha \in (0, \frac{1}{2}]$ and β-satisfiable for all rational $\beta \in [0, \frac{1}{2})$.

For the converse, suppose $p \notin$ SAT. Assume p is n-ary. Then for every truth value assignment $r \in \{0,1\}^n$, $\tilde{p}(r) = p(r) = 1$. Let $s = (s_1, \dots, s_n) \in [0,1]^n$. For each $i = 1, \dots, n$, let $t_i = 1$ if $s_i \in (\frac{1}{2}, 1]$ and $t_i = 0$ if $s_i \in [0, \frac{1}{2}]$. Let $t = (t_1, \dots, t_n)$. Then $\max_i |s_i - t_i| \leqslant \frac{1}{2}$. By Lemma 13,

$$|\tilde{p}(s) - \tilde{p}(t)| \leqslant \max_i |s_i - t_i| \leqslant \frac{1}{2}.$$

Since $t \in \{0,1\}^n$, $\tilde{p}(t) = 1$ and $\tilde{p}(s) \geqslant \frac{1}{2}$. This shows that $\tilde{p}$ is not $<\alpha$-satisfiable for all rational $\alpha \in (0, \frac{1}{2}]$ and is not β-satisfiable for all rational $\beta \in [0, \frac{1}{2})$. □

Proposition 5. *Let $\mathcal{F} = \{\neg, \wedge\}$. Then every formula of $\overline{\mathcal{F}}$ is $\frac{1}{2}$-satisfiable.*

Proof. By induction one can verify that p takes value $\frac{1}{2}$ at $(\frac{1}{2}, \dots, \frac{1}{2})$ for all $p \in \overline{\mathcal{F}}$. □

Proposition 6. *Let $\mathcal{F}$ be a finite subset of $\mathcal{P}$. Suppose $\overline{\mathcal{F}}$ contains $\neg, \wedge, \frac{x}{2}$. Then for all rational $\alpha \in (0, 1)$ and $\beta \in [0, 1)$, the set of all $<\alpha$-satisfiable formulas and the set of all β-satisfiable formulas of $\overline{\mathcal{F}}$ are NP-hard.*

Proof. By Proposition 4, for all rational $\alpha \in (0, \frac{1}{2}]$ and $\beta \in [0, \frac{1}{2})$, the set of all $<\alpha$-satisfiable formulas and the set of all β-satisfiable formulas of $\overline{\mathcal{F}}$ are NP-hard. Now consider $\alpha \in (\frac{1}{2}, 1)$. We show that the set of $<\alpha$-satisfiable formulas is NP-hard. Since $1 - \alpha \in (0, \frac{1}{2})$, there is a unique $k \geqslant 1$ such that $1 - \alpha \in [2^{-k-1}, 2^{-k})$. For any $p \in \overline{\mathcal{F}}$ let $q = \neg((\neg p)/2^k)$. Then $p \mapsto q$ is P-time computable, and, letting $\eta = 1 - 2^k(1 - \alpha) \in (0, \frac{1}{2}]$,

$$p \text{ is } <\eta\text{-satisfiable} \Leftrightarrow q \text{ is } <\alpha\text{-satisfiable}.$$

Since the set of all $<\eta$-satisfiable formulas is NP-hard, so is the set of all $<\alpha$-satisfiable formulas.

Next consider $\beta \in [\frac{1}{2}, 1)$. We show that the set of β-satisfiable formulas is NP-hard. Since $1 - \beta \in (0, \frac{1}{2}]$, there is a unique $k \geqslant 1$ such that $1 - \beta \in (2^{-k-1}, 2^{-k}]$. Define the map $p \mapsto q$ similarly as above, and let $\gamma = 1 - 2^k(1 - \beta) \in [0, \frac{1}{2})$, then

$$p \text{ is } \gamma\text{-satisfiable } \Leftrightarrow q \text{ is } \beta\text{-satisfiable.}$$

Since the set of all γ-satisfiable formulas is NP-hard, so is the set of all β-satisfiable formulas. □

Note that in the above proof the case of <1-satisfiability is not addressed. In fact, for $\mathcal{F} = \{\neg, \wedge, x/2\}$ a subsystem of that defined in Proposition 3, every $p \in \overline{\mathcal{F}}$ is <1-satisfiable. Hence the set of all <1-satisfiable formulas is not NP-hard

Theorem 10. *Let $\mathcal{F}$ be a finite subset of $\mathcal{P}$. Suppose $\overline{\mathcal{F}}$ contains $\neg, \dot{-}$. Then for all rational $\alpha \in (0, 1]$ and $\beta \in [0, 1)$, the set of all $<\alpha$-satisfiable formulas and the set of all β-satisfiable formulas are NP-hard.*

Proof. For all $p \in \overline{\mathcal{F}}$ let $q = p \dot{+} p$. Then $p \mapsto q$ is P-time computable. Moreover, for all rational $\alpha \in [0, 1)$, we have

$$p \text{ is } <\tfrac{\alpha}{2}\text{-satisfiable} \iff q \text{ is } <\alpha\text{-satisfiable.}$$

Since $\frac{\alpha}{2} \in (0, \frac{1}{2}]$, by Proposition 4 the set of all $<\frac{\alpha}{2}$-satisfiable formulas is NP-hard. Hence the set of all $<\alpha$-satisfiable formulas is NP-hard. Similarly, for all $\beta \in [0, 1)$, the set of all β-satisfiable formulas is NP-hard. □

Corollary 2 (Mundici [8])**.** *Let $\mathcal{F} = \{\neg, \dot{-}\}$. The set of all <1-satisfiable formulas in $\overline{\mathcal{F}}$ is NP-complete.*

Proof. This immediately follows from Theorems 9 and 10. □

Now Theorem 1 immediately follows from Corollary 1 and Theorem 10. In particular the conclusions hold for the continuous propositional logic with connectives $\neg, \dot{-}, \frac{x}{2}$.

6. Conclusions

Through the characterization we gave in Section 1, we conclude that there are many unary functions f such that the system $\{\neg, \dot{-}, f\}$ is full. On the other hand, no single connectives can make a full system.

While Strong Compactness and Craig Interpolation fail for continuous propositional logic, we showed that some versions of Approximated Strong Compactness and Approximated Craig Interpolation hold.

We also defined and studied different notions of satisfiability, falsifiablity, tautology, and fallacy, and established the NP-completeness and co-NP-completeness for these notions.

For future research we plan to extend these results to continuous predicate logic.

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