mathematics

MDPI

Article

Stability, Hopf Bifurcation and Optimal Control of Multilingual Rumor-Spreading Model with Isolation Mechanism

Shuzhen Yu [1,*], Zhiyong Yu [2] and Haijun Jiang [2]

[1] School of Mathematics Science, Xinjiang Normal University, Urumqi 830017, China
[2] College of Mathematics and System Sciences, Xinjiang University, Urumqi 830017, China
* Correspondence: yusz0505@163.com or yusz0505@xjnu.edu.cn

Abstract: The propagation of rumors on online social networks (OSNs) brings an awful lot of trouble to people's life and society. Aiming at combating rumors spreading on OSNs, two novel rumor-propagation models without and with time delays are proposed, which combine with the influence of the immune mechanism, isolation mechanism and network structure. Firstly, we analyze the existence of rumor equilibria and obtain some existence conditions of backward bifurcation. Secondly, the local stabilities of rumor-free and rumor equilibria are proved by using the Jacobian matrix method, and some critical conditions for the existence of Hopf bifurcation are acquired by selecting critical parameters and delays as bifurcation parameters. Furthermore, an optimal control method is proposed, which can prevent the spread of rumors within an expected time period and minimize the cost of control. Finally, some numerical simulations are provided to verify the effectiveness of the proposed theoretical results.

Keywords: stability; rumor spreading; online social networks; Hopf bifurcation; optimal control

MSC: 34D20; 37C75; 34H05; 49K15

check for updates

Citation: Yu, S.; Yu, Z.; Jiang, H. Stability, Hopf Bifurcation and Optimal Control of Multilingual Rumor-Spreading Model with Isolation Mechanism. *Mathematics* **2022**, *10*, 4556. https://doi.org/10.3390/math10234556

Academic Editor: Ioannis E. Livieris

Received: 7 November 2022
Accepted: 29 November 2022
Published: 1 December 2022

1. Introduction

The current information age has witnessed the emergence of various communication platforms and online social network software, which provide convenience for users to obtain all kinds of information [1]. As an attractive information statement, a rumor catches people's attention naturally. It spreads rapidly and widely on OSNs since the content is close to people's daily life [2–4]. For example, the rumors that 'eating garlic can cure cancer' and 'smearing ginger can cure hair loss' spread widely on OSNs. As a result, some users believe these rumors and miss the best treatment period. Another well-known example is about the treatment of COVID-19, such as the claims that drinking Banlangen, fumigation vinegar and Shuanghuanglian oral liquid can prevent COVID-19, and the thicker the mask, the better the antiviral effect. This rumor not only reduced the public's vigilance to the virus but also intensified the anxiety and panic to a certain extent. Hence, the spread of rumor has a negative impact on people's lives [5,6]. It is of great significance to explore spreading rules and propose some feasible control methods to prevent the spread of rumors on OSNs.

The research on the dynamics of rumor propagation is mainly inspired by the infectious disease model due to the similarity of transmission mechanism. The systematic study of rumor spreading model began in 1965. As a milestone of the rumor-spreading model, the DK [7,8] model and MT [9] model divide the population into three categories. Each person can only be in one of the following three states: ignorant (the user who does not know the rumor); spreader (the one who propagates the rumor); and removed (the one who knows the rumor but does not spread it). However, these models [7–9] are only suitable for describing the spread of rumor by word of mouth. With the rapid development of social science and technology, the spread of rumors on social media has gradually replaced oral communication and become a new mainstream way. Therefore, some new research methods have been proposed by scholars. In 2001, Zanette [10] first applied complex network theory to the study of rumor spreading. After that, an increasing number of scholars began to investigate the rumor propagation model on small world networks [11–13], uniform networks [14–16] and scale-free networks [17,18].

With the widespread use of new Internet media and the occurrence of frequent emergencies in the past few years, research about rumor spreading on OSNs has also been widely carried out [19–21]. Meanwhile, a growing number of attention has been paid to the influence of group psychological characteristics [22–26], government refutation [27–29], the immune mechanism [30] and the educational mechanism [31]. In addition, the authors studied a rumor spreading model with homogeneous network, and introduced expert intervention strategy into the model in [32]. In [33], an impulsive rumor-blocking strategy was proposed, in which the rumor users were isolated for a limited period and the optimal control method was applied to save cost. Different from the above articles, considering the difference in regulatory capabilities of network platforms, a kind of forced silence mechanism was introduced to suppress the spreading of rumors in [34,35]. In practical application, the forced silence mechanism can be realized by forbidding the rumor disseminator. Unfortunately, these useful models were all proposed in a single language environment, rarely involving multiple language environments.

Multilingual OSNs refer that there are multiple languages on social platforms. The emergence of a large number of social network platforms has greatly facilitated people's communication. At the same time, online translation and other software also provide convenience for information exchange between multiple languages. Therefore, it is of great practical significance to study the spread of rumors on multilingual OSNs. There are some results [36–39] about rumor spreading in multilingual environment. In [36], an I2S2R rumor-spreading model was proposed in homogeneous complex networks, and the global stability of the rumor-free and rumor equilibria was proved. In [37], a multilingual SIR rumor-spreading model with a cross-transmitted mechanism was established. In [38,39], two rumor-propagation models with heterogeneous networks were proposed. In the process of rumor spreading, it is an effective way to isolate the rumor disseminator or forbid the users. Therefore, it is meaningful to study the rumor-transmission model with the isolation mechanism. However, as far as we know, there is no research on the modeling and control of rumor propagation with the isolation mechanism in a multilingual environment.

Based on the above-mentioned factors, in this paper, we propose two new rumor-propagation models with an isolation mechanism on multilingual OSNs. In the model, all users are divided into six categories, including two types of ignorants, two types of spreaders, quarantined users and recovered users. We abbreviate this type of model as the 2I2SQR model. Moreover, we study the dynamic behaviors of the model, and propose an optimal control by using the Pontriagin maximum principle to minimize the control cost. The main contributions of this paper are as follows. Firstly, based on some basic assumptions, a rumor-spreading model with quarantine control is designed on multilingual OSNs. The existence of a rumor equilibrium is analyzed, and some conditions for the existence of backward bifurcation are obtained. Secondly, we prove the local stability of the rumor-free and rumor equilibria without and with time delays. Concurrently, by selecting the critical parameters and the time delays, some conditions for the existence of Hopf bifurcation are given. In addition, using the optimal control theory, an optimal control is obtained, which can minimize the objective function and suppress the spread of rumor.

The rest of this paper is arranged as follows. In Section 2, a novel rumor-propagation model without time delay is proposed, and the dynamic analysis of the model is given. An optimal control is proposed to optimize the isolation mechanism in Section 3. In Section 4, a rumor-propagation model with time delays is given, and the dynamics of the model is studied. The validity of the theoretical results is verified by several numerical simulations in Section 5. Section 6 concludes this paper.

2. 2I2SQR Rumor Model without Time-Delay

In this section, a rumor spreading model without time-delay is proposed on OSNs with multilingual environment, and the dynamics of the model is carefully analyzed.

2.1. Model Formulation

Users may speak multiple languages on an OSN when they come from different regions or countries. According to the language ability, we classify users into two groups. The first group is the ones who can speak both the official language and unofficial language, but they are more likely to browse the information published in the unofficial language when they are skimming through the web, and they also prefer to edit it in the unofficial language. The second group is the ones who can only speak the official language. In our model, we divide users into six categories. Ignorants-1($I_1(t)$) and Ignorants-2($I_2(t)$) represent the ratios of users who are not aware of the rumor in the first and second groups. Spreaders-1($S_1(t)$) and Spreaders-2($S_2(t)$) denote the ratios of users who know the rumor and spread it in the first and second groups, respectively. Quarantined ($Q(t)$) denotes the ratio of the user whose communicator is temporarily banned. Recovered ($R(t)$) represents the ratio of the individual who knows the rumor and no longer spreads it. Moreover, the densities satisfy $I_1(t) + I_2(t) + S_1(t) + S_2(t) + Q(t) + R(t) = 1$.

Based on the mention above, the following detailed assumptions are presented before we establish the rumor-spreading model on OSNs with a multilingual environment:

(1) Assume that the immigration rates are denoted by $B_i (i = 1, 2)$ for the first and second groups, respectively. The removed rate is represented by d.

(2) In the process of rumor spreading, $I_1(t)$(or $I_2(t)$) will turn into $S_1(t)$ (or $S_2(t)$) with conversion rate α_1 (or α_2). In addition, S_1 will change to S_2 with a certain probability ρ to expand the spread range of the rumor.

(3) When $I_1(t)$ (or $I_2(t)$) glance over a rumor, they will choose to disbelieve the rumor and transform into $R(t)$ with probability μ_1(or μ_2) because of the level of their education (identifying the rumor) or not being interested in the rumor. After spreading the rumor for a period of time, $S_1(t)$ (or $S_2(t)$) will change into $R(t)$ with probability β_1 (or β_2) on account of losing interest or other factors. Similarly, the $Q(t)$ will turn into $R(t)$ with probability c after passing the compulsory prohibition.

(4) When a rumor spreads on OSNs, the network regulatory authorities will prohibit the propagation of the rumor according to the number of communicators with a certain proportion. We describe it by a quarantine function $h_i(S_i) = \frac{r_i S_i(t)}{\sigma_i + S_i(t)}$, $(i = 1, 2)$, where $r_i > 0$ is the probability of partition, and $\sigma_i > 0$ is a half-saturation constant that measures the isolation delay index. If σ_i is small, then the efficiency is high.

On the basis of these facts, B_1, B_2, α_1, α_2, μ_1, μ_2, β_1, β_2, d are all positive constants. The process of rumor spreading is shown in Figure 1. The dynamics of the rumor-propagation model in a multilingual environment is established as follows:

$$\begin{cases} \dfrac{\mathrm{d}I_1(t)}{\mathrm{d}t} = B_1 - \langle k\rangle\alpha_1 I_1(t)S_1(t) - \mu_1 I_1(t) - dI_1(t), \\ \dfrac{\mathrm{d}I_2(t)}{\mathrm{d}t} = B_2 - \langle k\rangle\alpha_2 I_2(t)S_2(t) - \mu_2 I_2(t) - dI_2(t), \\ \dfrac{\mathrm{d}S_1(t)}{\mathrm{d}t} = \langle k\rangle\alpha_1 I_1(t)S_1(t) - \dfrac{r_1 S_1(t)}{\sigma_1 + S_1(t)} - \beta_1 S_1(t) - \rho S_1(t) - dS_1(t), \\ \dfrac{\mathrm{d}S_2(t)}{\mathrm{d}t} = \langle k\rangle\alpha_2 I_2(t)S_2(t) + \rho S_1(t) - \dfrac{r_2 S_2(t)}{\sigma_2 + S_2(t)} - \beta_2 S_2(t) - dS_2(t), \\ \dfrac{\mathrm{d}Q(t)}{\mathrm{d}t} = \dfrac{r_1 S_1(t)}{\sigma_1 + S_1(t)} + \dfrac{r_2 S_2(t)}{\sigma_2 + S_2(t)} - cQ(t) - dQ(t), \\ \dfrac{\mathrm{d}R(t)}{\mathrm{d}t} = \beta_1 S_1(t) + \beta_2 S_2(t) + \mu_1 I_1(t) + cQ(t) + \mu_2 I_2(t) - dR(t), \end{cases} \tag{1}$$

where $\langle k \rangle = \sum_{k=1}^{n} kP(k)$ denotes the average degree of the individual, $P(k)$ stands for the probability of an individual with degree k and satisfies $\sum_{k=1}^{n} P(k) = 1$. The initial conditions satisfy

$$I_1(0) \geq 0, I_2(0) \geq 0, S_1(0) > 0, S_2(0) > 0, Q(0) \geq 0, R(0) \geq 0.$$

Thus, according to the above conditions, the positive invariant set of system (1) is

$$\Omega = \{(I_1, I_2, S_1, S_2, Q, R) \in R_6^+ : I_1 + I_2 + S_1 + S_2 + Q + R = 1\}.$$

Since $Q(t)$ and $R(t)$ do not affect other equations of system (1), we simplify system (1) as follows:

$$\begin{cases} \dfrac{\mathrm{d}I_1(t)}{\mathrm{d}t} = B_1 - \langle k \rangle \alpha_1 I_1(t) S_1(t) - \mu_1 I_1(t) - dI_1(t), \\ \dfrac{\mathrm{d}I_2(t)}{\mathrm{d}t} = B_2 - \langle k \rangle \alpha_2 I_2(t) S_2(t) - \mu_2 I_2(t) - dI_2(t), \\ \dfrac{\mathrm{d}S_1(t)}{\mathrm{d}t} = \langle k \rangle \alpha_1 I_1(t) S_1(t) - \dfrac{r_1 S_1(t)}{\sigma_1 + S_1(t)} - \beta_1 S_1(t) - dS_1(t) - \rho S_1(t), \\ \dfrac{\mathrm{d}S_2(t)}{\mathrm{d}t} = \langle k \rangle \alpha_2 I_2(t) S_2(t) + \rho S_1(t) - \dfrac{r_2 S_2(t)}{\sigma_2 + S_2(t)} - \beta_2 S_2(t) - dS_2(t), \end{cases} \tag{2}$$

with the initial conditions $I_1(0) \geq 0$, $I_2(0) \geq 0$, $S_1(0) > 0$, and $S_2(0) > 0$.

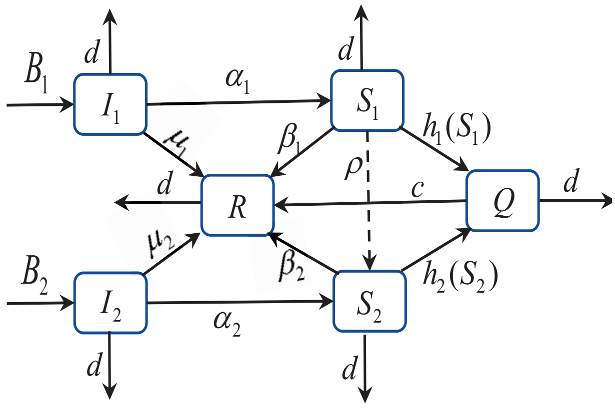


Figure 1. The state transition diagram of 2I2SQR model.

Remark 1. *Although some rumor-spreading models on OSNs have been proposed in [19–21,34,35], they are only used to describe the spread of rumors in a single language environment. In this paper, an new multilingual rumor propagation model is established. In particular, when $I_1(t) = S_1(t) = 0$ in system (1), it will be simplified into a language rumor propagation model. Therefore, the model proposed in this paper has a wider application.*

2.2. Existence of Equilibria

It is easily to verify that system (2) always has a rumor-free equilibrium, which is given by

$$E_0 = \left\{ \frac{B_1}{d + \mu_1}, \frac{B_2}{d + \mu_2}, 0, 0 \right\}.$$

By using the next generation matrix method [37], we calculate the basic reproduction number $\Re_0$, which is defined as

$$\Re_0 = \max\{\Re_{01}, \Re_{02}\}, \tag{3}$$

where $\Re_{01} = \frac{\langle k\rangle\alpha_1 B_1\sigma_1}{(d+\mu_1)[r_1+(\beta_1+\rho+d)\sigma_1]}$, $\Re_{02} = \frac{\langle k\rangle\alpha_2 B_2\sigma_2}{(d+\mu_2)[r_2+(\beta_2+d)\sigma_2]}$.

Next, we will determine the existence of a rumor equilibrium. Supposing that $E^* = (I_1^*, I_2^*, S_1^*, S_2^*)$ is a rumor equilibrium, then it satisfies

$$\begin{cases} B_1 - \langle k\rangle\alpha_1 I_1^* S_1^* - \mu_1 I_1^* - dI_1^* = 0, \\ B_2 - \langle k\rangle\alpha_2 I_2^* S_2^* - \mu_2 I_2^* - dI_2^* = 0, \\ \langle k\rangle\alpha_1 I_1^* S_1^* - \frac{r_1 S_1^*}{\sigma_1 + S_1^*} - \beta_1 S_1^* - dS_1^* - \rho S_1^* = 0, \\ \langle k\rangle\alpha_2 I_2^* S_2^* + \rho S_1^* - \frac{r_2 S_2^*}{\sigma_2 + S_2^*} - \beta_2 S_2^* - dS_2^* = 0. \end{cases}$$

Therefore, it has

$$I_1^* = \frac{B_1}{\langle k\rangle\alpha_1 S_1^* + \mu_1 + d}, \qquad I_2^* = \frac{B_2}{\langle k\rangle\alpha_2 S_2^* + \mu_2 + d}.$$

Moreover, S_1^* and S_2^* are determined by the following equations:

$$a_2(S_1^*)^2 + a_1 S_1^* + a_0 = 0, \tag{4}$$

$$b_3(S_2^*)^3 + b_2(S_2^*)^2 + b_1 S_2^* + b_0 = 0, \tag{5}$$

where

$$\begin{aligned} a_0 &= (\mu_1 + d)[r_1 + \sigma_1(\beta_1 + d + \rho)](1 - \Re_{01}), \\ a_1 &= r_1\alpha_1\langle k\rangle + \alpha_1\sigma_1(\beta_1 + d + \rho)\langle k\rangle + (\mu_1 + d)(\beta_1 + d + \rho) - B_1\alpha_1\langle k\rangle, \\ a_2 &= \alpha_1(\beta_1 + d + \rho)\langle k\rangle, \\ b_0 &= \rho(\mu_2 + d)\sigma_2 S_1^*, \\ b_1 &= [r_2 + (\beta_2 + d)\sigma_2](\mu_2 + d)(\Re_{02} - 1) + [\rho\alpha_2\sigma_2\langle k\rangle + \rho(\mu + d)]S_1^*, \\ b_2 &= \alpha_2 B_2\langle k\rangle + \alpha_2\rho S_1^*\langle k\rangle - \alpha_2\langle k\rangle[r_2 + (\beta_2 + d)\sigma_2] - (\beta_2 + d)(\mu_2 + d), \\ b_3 &= -\langle k\rangle\alpha_2(\beta_2 + d). \end{aligned}$$

Firstly, we will solve the positive solutions of Equation (4) with respect to S_1^*. For convenience, we denote

$$\hat{\Re}_{01} = 1 - \frac{[(r_1 - B_1)\alpha_1\langle k\rangle + (\alpha_1\sigma_1\langle k\rangle + \mu_1 + d)(\beta_1 + d + \rho)]^2}{4\alpha_1\langle k\rangle(\beta_1 + d + \rho)(\mu_1 + d)[r_1 + \sigma_1(\beta_1 + d + \rho)]}.$$

By simple calculation, the following results can be obtained.

(i). If $\Re_{01} > 1$, Equation (4) has a positive solution.

(ii). If $\Re_{01} = 1$, Equation (4) is transformed into

$$a_2(S_1^*)^2 + a_1 S_1^* = 0. \tag{6}$$

Obviously, $S_1^* = 0$ or $S_1^* = -\frac{a_1}{a_2} > 0$ if $a_1 < 0$.

(iii). If $\Re_{01} < 1$ and $a_1 < 0$, the following results can be easily verified.
(1). If $\Re_{01} > \hat{\Re}_{01}$, Equation (4) has two positive roots.
(2). If $\Re_{01} = \hat{\Re}_{01}$, Equation (4) has two equal positive roots $-\frac{a_1}{2a_2}$.
(3). If $\Re_{01} < \hat{\Re}_{01}$, Equation (4) has no positive root.

Next, we will consider the positive solutions of Equation (5) with respect to S_2^*.

Case (1). It is clear that $b_0 = 0$ when $S_1^* = 0$, then Equation (5) is transformed into

$$b_3(S_2^*)^3 + \tilde{b}_2(S_2^*)^2 + \tilde{b}_1 S_2^* = 0, \tag{7}$$

where $\tilde{b}_1 = [r_2 + (\beta_2 + d)\sigma_2](\mu_2 + d)(\Re_{02} - 1)$, $\tilde{b}_2 = \alpha_2 B_2\langle k\rangle - \alpha_2\langle k\rangle[r_2 + (\beta_2 + d)\sigma_2] - (\beta_2 + d)(\mu_2 + d)$, $b_3 = -\langle k\rangle\alpha_2(\beta_2 + d)$.

Apparently, $S_2^* = 0$ or $b_3(S_2^*)^2 + \tilde{b}_2(S_2^*) + \tilde{b}_1 = 0$. Let us discuss the solutions of the following equation:

$$b_3(S_2^*)^2 + \tilde{b}_2(S_2^*) + \tilde{b}_1 = 0. \tag{8}$$

By calculation, it has

$$\tilde{b}_1 > 0 \Leftrightarrow \Re_{02} > 1, \;\; \tilde{b}_1 = 0 \Leftrightarrow \Re_{02} = 1, \;\; \tilde{b}_1 < 0 \Leftrightarrow \Re_{02} < 1.$$

We denote

$$\hat{\Re}_{02} = 1 - \frac{[\alpha_2 B_2\langle k\rangle - \alpha_2\langle k\rangle[r_2 + (\beta_2 + d)\sigma_2] - (\beta_2 + d)(\mu_2 + d)]^2}{4\langle k\rangle\alpha_2(\beta_2 + d)[[r_2 + (\beta_2 + d)\sigma_2](\mu_2 + d)]}.$$

Then, the following conclusions are obtained.

(i). If $\Re_{02} > 1$, Equation (8) has a positive solution.

(ii). If $\Re_{02} = 1$, Equation (8) has a solution $S_2^* = 0$ or $S_2^* = -\frac{\tilde{b}_2}{b_3} > 0$ if and only if $\tilde{b}_2 > 0$.

(iii). If $\Re_{02} < 1$ and $\tilde{b}_2 > 0$, one of the following three cases holds
(1). If $\Re_{02} > \hat{\Re}_{02}$, Equation (8) has two positive roots.
(2). If $\Re_{02} = \hat{\Re}_{02}$, Equation (8) has two identical positive roots $S_2^* = -\frac{\tilde{b}_2}{2b_3}$.
(3). If $\Re_{02} < \hat{\Re}_{02}$, Equation (8) does not have a positive root.

Case (2). When $S_1^* > 0$, it has $b_0 > 0$. We denote

$$G(S_2^*) = b_3(S_2^*)^3 + b_2(S_2^*)^2 + b_1 S_2^* + b_0 = 0. \tag{9}$$

Then, the discriminant of the cubic polynomial $G(S_2^*)$ is given by

$$D = b_1^2 b_2^2 + 18 b_0 b_1 b_2 b_3 - 4 b_0 b_2^3 - 4 b_1^2 b_3 - 27 b_0^2 b_3^2.$$

By simple calculation, $G(S_2^*)$ has one real root and two complex roots if $D < 0$. $G(S_2^*)$ has three real roots (at least two of which are equal) if $D = 0$, and G has three distinct real roots if $D > 0$. In order to analyze the roots of Equation (5), the following lemmas are given:

Lemma 1. *Suppose that $\alpha_2\sigma_2\langle k\rangle > \mu_2 + d$, then $b_2 < 0$ if $b_1 < 0$.*

Proof. According to $\alpha_2\sigma_2\langle k\rangle > \mu_2 + d$, we have

$$\begin{aligned}
b_2 &= \alpha_2 B_2\langle k\rangle + \alpha_2\rho S_1^*\langle k\rangle - \alpha_2\langle k\rangle[r_2 + (\beta_2 + d)\sigma_2] - (\beta_2 + d)(\mu_2 + d) \\
&= \alpha_2 B_2\langle k\rangle + \alpha_2\rho S_1^*\langle k\rangle - \alpha_2 r_2\langle k\rangle - (\beta_2 + d)(\sigma_2\alpha_2\langle k\rangle + \mu_2 + d), \\
b_1 &= \alpha_2\sigma_2 B_2\langle k\rangle - [r_2 + (\beta_2 + d)\sigma_2](\mu_2 + d) + [\rho\alpha_2\sigma_2\langle k\rangle + \rho(\mu + d)]S_1^* \\
&= \alpha_2\sigma_2 B_2\langle k\rangle - r_2(\mu_2 + d) - \sigma_2(\beta_2 + d)(\mu_2 + d) + \rho\sigma_2\alpha_2\langle k\rangle S_1^* + \rho(\mu + d)S_1^* \\
&\geq \sigma_2[\alpha_2 B_2\langle k\rangle - r_2\alpha_2\langle k\rangle - (\beta_2 + d)(\mu_2 + d) + \rho\alpha_2 S_1^*\langle k\rangle] + \rho(\mu + d)S_1^* \\
&\geq \sigma_2[\alpha_2 B_2\langle k\rangle - r_2\alpha_2\langle k\rangle - (\beta_2 + d)(\alpha_2\sigma_2\langle k\rangle + \mu_2 + d) + \rho\alpha_2 S_1^*\langle k\rangle] + \rho(\mu + d)S_1^* \\
&\geq \sigma_2 b_2.
\end{aligned}$$

Since $\sigma_2 > 0$, it is easy to obtain that $b_2 < 0$ if $b_1 < 0$. □

Lemma 2. *If $\alpha_2\sigma_2\langle k\rangle > \mu_2 + d$ holds, Equation (9) has one positive root.*

Proof. By deriving Equation (5), we can obtain

$$G(S_2^*)' = 3b_3(S_2^*)^2 + 2b_2S_2^* + b_1.$$

Denote the roots of $G(S_2^*)' = 0$ by $x^{\pm}$, then one of the following three cases holds:

(i). $D < 0$. Using the fact that $G(0) = b_0 > 0$ and $\lim_{S_2^* \to \infty} G(S_2^*) = -\infty$, it follows that the real root is positive.
(ii). $D = 0$. Solving the roots x^+ and x^-, we find that Equation (5) has one positive root.
(iii). $D > 0$. The analysis method is similar to (ii), and Equation (5) has one positive root.

Note that there is only one positive root in the previous three cases. Therefore, Equation (5) only has one positive solution S_2^* if $\alpha_2\sigma_2\langle k\rangle > \mu_2 + d$ and $b_0 > 0$.

Based on the above analysis, the following theorem is given to ensure the existence of the rumor equilibria. □

Theorem 1. *For system (2), $\Re_{01}$ and $\Re_{02}$ are defined as Equation (3). The following conclusions about the existence of the rumor equilibria are presented.*

(1). For the case of $b_0 > 0$ and $\alpha_2\sigma_2\langle k\rangle > \mu_2 + d$:

If $\Re_{01} > 1$, there is a unique rumor equilibrium point E_1^*.

If $\Re_{01} = 1$ and $a_1 < 0$, system (2) admits a unique rumor equilibrium point E_1^*.

If $\hat{\Re}_{01} < \Re_{01} < 1$ and $a_1 < 0$, there are two rumor equilibrium points E_1^* and E_2^* in system (2), and no rumor equilibrium point when $a_1 \geq 0$.

If $\hat{\Re}_{01} = \Re_{01} < 1$ and $a_1 < 0$, system (2) admits a unique rumor equilibrium point E_1^*, and no rumor equilibrium point when $a_1 \geq 0$.

If $\Re_{01} < \hat{\Re}_{01} < 1$, system (2) has no rumor equilibrium point. (2). For the case of $S_1^* = 0(b_0 = 0)$:

If $\Re_{02} > 1$, there is a unique rumor equilibrium point E_1^* in system (2).

If $\Re_{02} = 1$ and $\tilde{b}_2 > 0$, system (2) admits a unique rumor equilibrium points E_1^*.

If $\hat{\Re}_{02} < \Re_{02} < 1$ and $\tilde{b}_2 > 0$, there are two rumor equilibria E_1^* and E_2^* in system (2), and no rumor equilibrium point when $\tilde{b}_2 \leq 0$.

If $\hat{\Re}_{02} = \Re_{02} < 1$ and $\tilde{b}_2 > 0$, system (2) admits a unique rumor equilibrium point E_1^*, and no rumor equilibrium point when $\tilde{b}_2 \leq 0$.

If $\Re_{02} < \hat{\Re}_{02} < 1$, system (2) has no rumor equilibrium.

We find that the conditions for the existence of a rumor equilibrium are very complicated. Under different conditions, system (2) may have different equilibria. Based on the above theorem, it can be seen that there exist rumor equilibria in system (2) when $\Re_{01} < 1$ and $\Re_{02} < 1$. For instance, when $\hat{\Re}_{01} = 0.89$ and $\hat{\Re}_{02} = 0.86$, the backward bifurcation graph of the system (2) is shown in Figure 2.

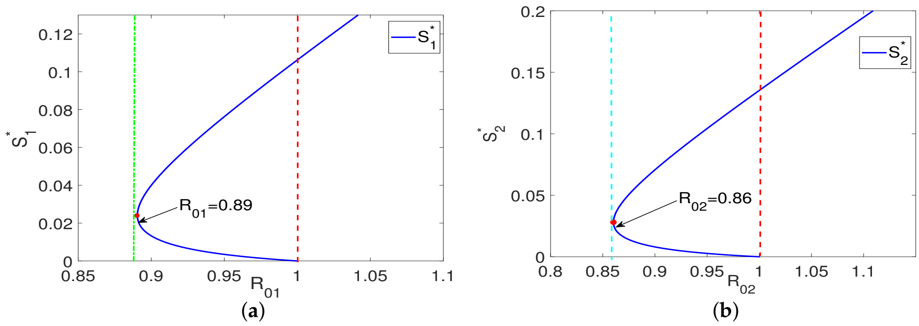


Figure 2. (**a**) The backward bifurcation graph of S_1^* and the basic reproduction number $\Re_{01}$. (**b**) The backward bifurcation graph of S_2^* and the basic reproduction number $\Re_{02}$.

2.3. Stability and Hopf Bifurcation of the Equilibria

In order to study the evolution tend of the solution of system (2), we need to analyze the stability of the equilibrium. Based on Theorem 1, system (2) may have multiple equilibria. It is very difficult to analyze the global stabilities of the rumor-free and rumor equilibria. Therefore, we will discuss the local stability and Hopf bifurcation of the equilibria.

Theorem 2. *For system (2), the rumor-free equilibrium E_0 is locally asymptotically stable when $\Re_0 < 1$ and unstable when $\Re_0 > 1$.*

Proof. The Jacobian matrix of system (2) at E_0 is given as

$$\mathcal{J}(E_0) = \begin{pmatrix} -(\mu_1+d) & 0 & -\frac{\alpha_1 B_1 \langle k \rangle}{d+\mu_1} & 0 \\ 0 & -(\mu_2+d) & 0 & -\frac{\alpha_2 B_2 \langle k \rangle}{d+\mu_2} \\ 0 & 0 & \frac{\alpha_1 B_1 \langle k \rangle}{d+\mu_1} - \frac{[r_1+\sigma_1(\beta_1+d+\rho)]}{\sigma_1} & 0 \\ 0 & 0 & \rho & \frac{\alpha_2 B_2 \langle k \rangle}{d+\mu_2} - \frac{[r_2+\sigma_2(\beta_2+d)]}{\sigma_2} \end{pmatrix}.$$

Then, the characteristic equation is equivalent to

$$\left[\lambda - \frac{[r_1+\sigma_1(\beta_1+d+\rho)]}{\sigma_1}(\Re_{01}-1)\right]\left[\lambda - \frac{[r_2+\sigma_2(\beta_2+d)]}{\sigma_2}(\Re_{02}-1)\right] \times (\lambda+\mu_1+d)(\lambda+\mu_2+d) = 0.$$

Thus, $\lambda_1 = -(\mu_1+d), \lambda_2 = -(\mu_2+d), \lambda_3 = \frac{[r_1+\sigma_1(\beta_1+d+\rho)]}{\sigma_1}(\Re_{01}-1), \lambda_4 = \frac{[r_2+\sigma_2(\beta_2+d)]}{\sigma_2} \times (\Re_{02}-1)$. Since $\Re_0 < 1$, it has $\lambda_3 < 0$ and $\lambda_4 < 0$. Based on the stability theory, we can conclude that E_0 is locally asymptotically stable. Otherwise, E_0 is unstable if $\Re_0 > 1$. □

Lemma 3 ([40])**.** *For any $a > 0$ and $b > 0$, if $\frac{\mathrm{d}x(t)}{\mathrm{d}t} \geq b - ax(t)$ for $t \geq 0$ and $x(0) > 0$, it has $\lim_{t\to+\infty} \inf x(t) \geq \frac{b}{a}$; if $\frac{\mathrm{d}x(t)}{\mathrm{d}t} \leq b - ax(t)$ for $t \geq 0$ and $x(0) > 0$, it has $\lim_{t\to+\infty} \sup x(t) \leq \frac{b}{a}$.*

Theorem 3. *For system (2), the rumor-free equilibrium E_0 is globally asymptotically stable if $\Re_{01} + \frac{r_1}{(1+\sigma_1)[r_1+\sigma_1(\beta_1+\rho+d)]} < 1$ and $\Re_{02} + \frac{r_2}{(1+\sigma_2)[r_2+\sigma_2(\beta_2+d)]} < 1$.*

Proof. Based on system (2), it has

$$\begin{aligned} \frac{\mathrm{d}I_1(t)}{\mathrm{d}t} &= B_1 - \langle k \rangle \alpha_1 I_1(t) S_1(t) - \mu_1 I_1(t) - dI_1(t) \\ &\leq B_1 - (\mu_1+d)I_1(t). \end{aligned}$$

By Lemma 3, one can obtain that

$$\lim_{t\to+\infty} \sup I_1(t) \leq \frac{B_1}{\mu_1+d}. \tag{10}$$

Hence, for any $\varepsilon_1 > 0$, there exists $t_1 > 0$ such that $I_1(t) \leq \frac{B_1}{\mu_1+d} + \varepsilon_1$ for $t > t_1$. Then, when $t > t_1$, one has

$$
\begin{aligned}
\frac{\mathrm{d}S_1(t)}{\mathrm{d}t} \leq &\Big(\langle k\rangle\alpha_1(\frac{B_1}{\mu_1+d}+\varepsilon_1)-(\frac{r_1}{\sigma_1+1}+\beta_1+\rho+d)\Big)S_1(t)\\
= &\ \frac{r_1+(\beta_1+d+\rho)\sigma_1}{\sigma_1}\Bigg[\Re_{01}-1+\frac{r_1}{(1+\sigma_1)[r_1+\sigma_1(\beta_1+\rho+d)]}\\
&+\frac{\sigma_1\langle k\rangle\alpha_1\varepsilon_1}{r_1+(\beta_1+d+\rho)\sigma_1}\Bigg]S_1(t).
\end{aligned}
$$

Denote $\omega(\varepsilon_1)=\Re_{01}-1+\frac{r_1}{(1+\sigma_1)[r_1+\sigma_1(\beta_1+\rho+d)]}+\frac{\sigma_1\langle k\rangle\alpha_1\varepsilon_1}{r_1+(\beta_1+d+\rho)\sigma_1}$. According to comparison theorem, one can obtain that

$$
S_1(t)\leq S_1(0)\exp\left\{\frac{r_1+(\beta_1+d+\rho)\sigma_1}{\sigma_1}\omega(\varepsilon_1)t\right\}.
$$

Because $\Re_{01}+\frac{r_1}{(1+\sigma_1)[r_1+\sigma_1(\beta_1+\rho+d)]}<1$ and $\varepsilon_1>0$ is an arbitrarily small real number, we can chose $\varepsilon_1>0$ such that $\omega(\varepsilon_1)<0$. Therefore, $\lim_{t\to+\infty}S_1(t)=0$.

Next, we prove that $\lim_{t\to+\infty}S_2(t)=0$. Similarly, for any $\varepsilon_2>0$, there exists $t_2>0$ such that $I_2(t)\leq\frac{B_2}{\mu_2+d}+\varepsilon_2$ and $S_2(t)\leq\varepsilon_2$ for $t>t_2$. Then, for $t>t_2$, it follows that

$$
\begin{aligned}
\frac{\mathrm{d}S_2(t)}{\mathrm{d}t} \leq &\Big(\langle k\rangle\alpha_2(\frac{B_2}{\mu_2+d}+\varepsilon_2)+\rho\varepsilon_2-(\frac{r_2}{\sigma_2+1}+\beta_2+d)\Big)S_2(t)\\
= &\ \frac{r_2+(\beta_2+d)\sigma_2}{\sigma_2}\Bigg[\Re_{02}-1+\frac{r_2}{(1+\sigma_2)[r_2+\sigma_2(\beta_2+d)]}\\
&+\frac{\sigma_2(\langle k\rangle\alpha_2+\rho)\varepsilon_2}{r_2+(\beta_2+d)\sigma_2}\Bigg]S_2(t).
\end{aligned}
$$

Denote $\varpi(\varepsilon_2)=\Re_{02}-1+\frac{r_2}{(1+\sigma_2)[r_2+\sigma_2(\beta_2+d)]}+\frac{\sigma_2(\langle k\rangle\alpha_2+\rho)\varepsilon_2}{r_2+(\beta_2+d)\sigma_2}$. By using comparison theorem, one can obtain that

$$
S_2(t)\leq S_2(0)\exp\left\{\frac{r_2+(\beta_2+d)\sigma_2}{\sigma_2}\varpi(\varepsilon_2)t\right\}.
$$

Since $\Re_{02}+\frac{r_2}{(1+\sigma_2)[r_2+\sigma_2(\beta_2+d)]}<1$ and $\varepsilon_2>0$ is an arbitrarily small real number, then we can chose $\varepsilon_2>0$ such that $\varpi(\varepsilon_2)<0$. Therefore, $\lim_{t\to+\infty}S_2(t)=0$.

Due to $\lim_{t\to+\infty}S_1(t)=0$, then for any $\varepsilon_3>0$, there exists $t_3>0$ such that $S_1(t)\leq\varepsilon_3$ for $t>t_3$. Therefore, it has

$$
\begin{aligned}
\frac{\mathrm{d}I_1(t)}{\mathrm{d}t} = &\ B_1-\langle k\rangle\alpha_1 I_1(t)S_1(t)-\mu_1 I_1(t)-dI_1(t)\\
\geq &B_1-(\langle k\rangle\alpha_1\varepsilon_3+\mu_1+d)I_1(t).
\end{aligned}
$$

By Lemma 3, one can get that $\lim_{t\to+\infty}I_1(t)\geq\frac{B_1}{\langle k\rangle\alpha_1\varepsilon_3+\mu_1+d}$. Let $\varepsilon_3\to0$, it follows that

$$
\lim_{t\to+\infty}\inf I_1(t)\geq\frac{B_1}{\mu_1+d}. \tag{11}
$$

In combination with (10) and (11), it is clear that $\lim_{t\to+\infty}I_1(t)=\frac{B_1}{\mu_1+d}$. Similar to the analysis of $I_1(t)$, we can obtain that $\lim_{t\to+\infty}I_2(t)=\frac{B_2}{\mu_2+d}$. This proves that the equilibrium E_0 is globally asymptotically stable. □

In order to facilitate the subsequent analysis process, we denote

$$\begin{aligned}
H_1(S_1^*) &= \alpha_1 S_1^*\langle k\rangle + \frac{r_1\sigma_1}{(\sigma_1+S_1^*)^2} + \Theta_1 + \Theta_3 - \alpha_1 I_1^*\langle k\rangle,\\
G_1(S_1^*) &= (\frac{r_1\sigma_1}{\sigma_1+S_1^*} + \Theta_1)(\alpha_1 S_1^*\langle k\rangle + \Theta_3) - \alpha_1 I_1^*\Theta_3\langle k\rangle,\\
H_2(S_2^*) &= \alpha_2 S_2^*\langle k\rangle + \frac{r_2\sigma_2}{(\sigma_2+S_2^*)^2} + \Theta_2 + \Theta_4 - \alpha_2 I_2^*\langle k\rangle,\\
G_2(S_2^*) &= (\frac{r_2\sigma_2}{\sigma_2+S_2^*} + \Theta_2)(\alpha_2 S_2^*\langle k\rangle + \Theta_4) - \alpha_2 I_2^*\Theta_4\langle k\rangle,
\end{aligned}$$

where $\Theta_1 = \beta_1 + d + \rho$, $\Theta_2 = \beta_2 + d$, $\Theta_3 = \mu_1 + d$, and $\Theta_4 = \mu_2 + d$.

Theorem 4. *Suppose that E_1^* is the rumor equilibrium of system (2). If $H_i(S_i^*) > 0$ and $G_i(S_i^*) > 0$, for $i = 1, 2$, then E_1^* is locally asymptotically stable. Otherwise, E_1^* is unstable.*

Proof. The Jacobian matrix of system (2) at E_1^* is expressed as

$$\mathcal{J}(E_1^*) = \begin{pmatrix} -\alpha_1 S_1^*\langle k\rangle - \Theta_3 & 0 & -\alpha_1 I_1^*\langle k\rangle & 0 \\ 0 & -\alpha_2 S_2^*\langle k\rangle - \Theta_4 & 0 & -\alpha_2 I_2^*\langle k\rangle \\ \alpha_1 S_1^*\langle k\rangle & 0 & \alpha_1 I_1^*\langle k\rangle - \frac{r_1\sigma_1}{(\sigma_1+S_1^*)^2} - \Theta_1 & 0 \\ 0 & \alpha_2 S_2^*\langle k\rangle & \rho & \alpha_2 I_2^*\langle k\rangle - \frac{r_2\sigma_2}{(\sigma_2+S_2^*)^2} - \Theta_2 \end{pmatrix}.$$

The characteristic equation for matrix $\mathcal{J}(E_1^*)$ is as follows:

$$(\lambda^2 + H_1(S_1^*)\lambda + G_1(S_1^*))(\lambda^2 + H_2(S_2^*)\lambda + G_2(S_2^*)) = 0. \tag{12}$$

Denote

$$\lambda^2 + H_1(S_1^*)\lambda + G_1(S_1^*) = 0, \tag{13}$$
$$\lambda^2 + H_2(S_2^*)\lambda + G_2(S_2^*) = 0. \tag{14}$$

According to the Routh–Hurwitz criterion, the rumor equilibrium E_1^* is locally asymptotically stable when $H_i(S_i^*) > 0$ and $G_i(S_i^*) > 0$, $i = 1, 2$. Otherwise, it is unstable. □

Remark 2. *In Theorem 4, we give a criterion for the local stability of the rumor equilibrium E_1^*. From Theorem 1, it can be seen that there are many cases above the rumor equilibrium in system (2). In either case, the criterion of Theorem 4 is applicable since it is of a general form. When system (2) has two equilibria, each point may be locally stable under different conditions. Therefore, it is very meaningful to give the threshold from stable to unstable for the equilibrium of system (2).*

Based on the definition of $H_i(S_i^*)$ and $G_i(S_i^*)$, $i = 1, 2$, we let

$$\begin{aligned}
\hat{r}_1 &= \frac{(\alpha_1 I_1^*\langle k\rangle - \alpha_1 S_1^*\langle k\rangle - \Theta_1 - \Theta_3)(\sigma_1+S_1^*)^2}{\sigma_1},\\
\check{r}_1 &= (\frac{\alpha_1 I_1^*\langle k\rangle\Theta_3}{\alpha_1 S_1^*\langle k\rangle + \Theta_3} - \Theta_1)\frac{(\sigma_1+S_1^*)}{\sigma_1},\\
\hat{r}_2 &= \frac{(\alpha_2 I_2^*\langle k\rangle - \alpha_2 S_2^*\langle k\rangle - \Theta_2 - \Theta_4)(\sigma_2+S_2^*)^2}{\sigma_2},\\
\check{r}_2 &= (\frac{\alpha_2 I_2^*\langle k\rangle\Theta_4}{\alpha_2 S_2^*\langle k\rangle + \Theta_4} - \Theta_2)\frac{(\sigma_2+S_2^*)}{\sigma_2},
\end{aligned}$$

and

$$\bar{r}_1 = \max\{\hat{r}_1, \check{r}_1\}, \bar{r}_2 = \max\{\hat{r}_2, \check{r}_2\}.$$

Then, the following theorem is given.

Theorem 5. *For system (2), the following conclusions are true.*

(1). If $r_1 > \bar{r}_1$ and $r_2 > \bar{r}_2$, the rumor equilibrium E_1^* is locally asymptotically stable, otherwise, E_1^* is unstable.

(2). System (2) has a Hopf bifurcation at $r_1 = \bar{r}_1$ or $r_2 = \bar{r}_2$.

Proof. Let $H_i(S_i^*) = 0$ for $i = 1, 2$, then one can get $r_i = \hat{r}_i$. By simple calculation, we can obtain that $H_i(S_i^*) > 0$ if $r_i > \hat{r}_i$. Similarly, let $G_i(S_i^*) = 0$ for $i = 1, 2$, then it follows $r_i = \check{r}_i$, and $G_i(S_i^*) > 0$ if $r_i > \check{r}_i$.

For the case of $\hat{r}_1 > \check{r}_1$, consider r_1 in the neighborhood of $\hat{r}_1$, then Equation (13) has two positive roots, which are given by

$$\lambda_{1,2} = \frac{-H_1(S_1^*) \pm (H_1(S_1^*)^2 - 4G_1(S_1^*))^{\frac{1}{2}}}{2}.$$

The derivative of r_1 can be obtained:

$$\frac{\mathrm{d}\lambda_{1,2}}{\mathrm{d}r_1} = -\frac{\sigma_1}{2(\sigma_1 + S_1^*)^2} \pm \frac{1}{4\sqrt{H_1(S_1^*)^2 - 4G_1(S_1^*)}}[2H_1(S_1^*)\frac{\sigma_1}{(\sigma_1 + S_1^*)^2} - 4\frac{\sigma_1(\alpha_1 S_1^* \langle k \rangle + \Theta_3)}{\sigma_1 + S_1^*}].$$

When $r_1 = \hat{r}_1$, one has

$$\frac{\mathrm{d}\lambda_{1,2}}{\mathrm{d}r_1} = -\frac{\sigma_1}{2(\sigma_1 + S_1^*)^2} \pm \frac{1}{2i\sqrt{G_1(S_1^*)}}\frac{\sigma_1(\alpha_1 S_1^* \langle k \rangle + \Theta_3)}{\sigma_1 + S_1^*}.$$

Obviously,

$$\frac{\mathrm{d}Re(\lambda_{1,2})}{\mathrm{d}r_1}|_{r_1=\hat{r}_1} = Re(\frac{\mathrm{d}\lambda_{1,2}}{\mathrm{d}r_1}|_{r_1=\hat{r}_1}) = -\frac{\sigma_1}{2(\sigma_1 + S_1^*)^2} \neq 0.$$

Therefore, there is a Hopf bifurcation at $r_1 = \hat{r}_1$.

Similarly, for the case of $\hat{r}_2 > \check{r}_2$, we can obtain

$$\frac{\mathrm{d}Re(\lambda_{3,4})}{\mathrm{d}r_2}|_{r_2=\hat{r}_2} = Re(\frac{\mathrm{d}\lambda_{3,4}}{\mathrm{d}r_2}|_{r_2=\hat{r}_2}) = -\frac{\sigma_2}{2(\sigma_2 + S_2^*)^2} \neq 0.$$

Hence, there is a Hopf bifurcation at $r_2 = \hat{r}_2$.

For the case of $\check{r}_1 > \hat{r}_1$, consider r_1 in the neighborhood of $\check{r}_1$, so Equation (13) has one positive root, which is presented as follows:

$$\lambda_5 = \frac{-H_1(S_1^*) + (H_1(S_1^*)^2 - 4G_1(S_1^*))^{\frac{1}{2}}}{2}.$$

The derivative of r_1 can be obtained:

$$\frac{\mathrm{d}\lambda_5}{\mathrm{d}r_1} = -\frac{\sigma_1}{2(\sigma_1 + S_1^*)^2} + \frac{1}{4\sqrt{H_1(S_1^*)^2 - 4G_1(S_1^*)}}[2H_1(S_1^*)\frac{\sigma_1}{(\sigma_1 + S_1^*)^2} - 4\frac{\sigma_1(\alpha_1 S_1^* \langle k \rangle + \Theta_3)}{\sigma_1 + S_1^*}].$$

When $r_1 = \check{r}_1$, one has $G_1(S_1^*) = 0$, $H_1(S_1^*) > 0$,

$$\frac{\mathrm{d}\lambda_5}{\mathrm{d}r_1}|_{r_1=\check{r}_1} = -\frac{\sigma_1}{H_1(S_1^*)(\sigma_1 + S_1^*)}(\alpha_1 S_1^* \langle k \rangle + \Theta_3) \neq 0.$$

Hence, there exists a Hopf bifurcation at $r_1 = \check{r}_1$.

Similarly, for the case of $\check{r}_2 > \hat{r}_2$, we can obtain

$$\frac{\mathrm{d}\lambda_6}{\mathrm{d}r_2}|_{r_2=\check{r}_2} = -\frac{\sigma_2}{H_2(S_2^*)(\sigma_2+S_2^*)}(\alpha_2 S_2^*\langle k\rangle + \Theta_4) \neq 0.$$

Therefore, there exists a Hopf bifurcation at $r_2 = \check{r}_2$. □

Remark 3. *In this paper, we introduce the isolation mechanism to the rumor-spreading model and study the local stability of the equilibrium. Our purpose is to give the condition of rumor extinction by choosing the control parameters. As we all know, the basic reproduction number $\Re_0$ is an important reference index. However, we find that system (2) may contain a rumor equilibrium and it is locally stable when $\Re_0 < 1$. Therefore, it is difficult to give the criterion of rumor extinction under any initial condition.*

3. Optimal Control Model

In system (2), the isolation mechanism is applied to suppress the spread of rumor, in which the isolation rates r_1 and r_2 are constants. In this section, we will consider the time-varying isolation mechanism. In order to reduce the control cost and achieve the desired control effect, we use the optimal control method to optimize the control cost. The model with the isolation mechanism is as follows:

$$\begin{cases} \frac{\mathrm{d}I_1(t)}{\mathrm{d}t} = B_1 - \langle k\rangle\alpha_1 I_1(t)S_1(t) - \mu_1 I_1(t) - dI_1(t), \\ \frac{\mathrm{d}I_2(t)}{\mathrm{d}t} = B_2 - \langle k\rangle\alpha_2 I_2(t)S_2(t) - \mu_2 I_2(t) - dI_2(t), \\ \frac{\mathrm{d}S_1(t)}{\mathrm{d}t} = \langle k\rangle\alpha_1 I_1(t)S_1(t) - \frac{r_1(t)S_1(t)}{\sigma_1+S_1(t)} - \beta_1 S_1(t) - dS_1(t) - \rho S_1(t), \\ \frac{\mathrm{d}S_2(t)}{\mathrm{d}t} = \langle k\rangle\alpha_2 I_2(t)S_2(t) + \rho S_1(t) - \frac{r_2(t)S_2(t)}{\sigma_2+S_2(t)} - \beta_2 S_2(t) - dS_2(t), \\ \frac{\mathrm{d}Q(t)}{\mathrm{d}t} = \frac{r_1(t)S_1(t)}{\sigma_1+S_1(t)} + \frac{r_2(t)S_2(t)}{\sigma_2+S_2(t)} - cQ(t) - dQ(t), \\ \frac{\mathrm{d}R(t)}{\mathrm{d}t} = \beta_1 S_1(t) + \beta_2 S_2(t) + \mu_1 I_1(t) + cQ(t) + \mu_2 I_2(t) - dR(t), \end{cases} \tag{15}$$

where the control $r_1(t)$ and $r_2(t)$ are isolation rates of the social platform. Considering the number of rumor spreaders and control cost, the following objective function is established:

$$J(r_1(t), r_2(t)) = \int_0^T [u_1 S_1(t) + u_2 S_2(t) + u_3 r_1^2(t) + u_4 r_2^2(t)]\mathrm{d}t. \tag{16}$$

where μ_1, μ_2, μ_3 and μ_4 are positive numbers, which are trade-off parameters among these items. T is the expected control time.

The feasible region of $r_1(t)$ and $r_2(t)$ is $U = \{(r_1(t), r_2(t))|0 \leq r_1(t) \leq 1, 0 \leq r_2(t) \leq 1, t \in (0, T]\}$. Optimal control r_1^* and r_2^* satisfy $J(r_1^*, r_2^*) = \min\{J(r_1(t), r_2(t)) : (r_1(t), r_2(t)) \in U\}$.

In order to obtain the optimal control, we construct the Lagrangian function:

$$L(S_1(t), S_2(t), r_1(t), r_2(t)) = u_1 S_1(t) + u_2 S_2(t) + u_3 r_1^2(t) + u_4 r_2^2(t).$$

The Hamiltonian function is defined as

$$H(I_i(t), S_i(t), Q(t), R(t), r_i(t), \lambda_j(t)) = L(S_i(t), r_i(t)) + \lambda(t)P(I_i(t), S_i(t), Q(t), R(t), r_i(t)),$$

where $i = 1, 2$, $j = 1, 2, \ldots, 6$. $\lambda(t) = (\lambda_1(t), \lambda_2(t), \ldots, \lambda_6(t))^T$, $P(I_i(t), S_i(t), Q(t), R(t), r_i(t)) = (\frac{\mathrm{d}I_1(t)}{\mathrm{d}t}, \frac{\mathrm{d}I_1(t)}{\mathrm{d}t}, \frac{\mathrm{d}I_1(t)}{\mathrm{d}t}, \frac{\mathrm{d}I_2(t)}{\mathrm{d}t}, \frac{\mathrm{d}S_1(t)}{\mathrm{d}t}, \frac{\mathrm{d}S_2(t)}{\mathrm{d}t})^T$.

Using Pontryagin's maximum principle, the following theorem is given.

Theorem 6. *Let $\hat{I}_1$, $\hat{I}_2$, $\hat{S}_1$, $\hat{S}_2$, $\hat{Q}$ and $\hat{R}$ be the optimal states of system (15) under the optimal control (r_1^*, r_2^*). Then, there exist adjoint variables $\lambda_j(t)$, $j = 1, \cdots, 6$ satisfying*

$$
\begin{cases}
\dfrac{\mathrm{d}\lambda_1(t)}{\mathrm{d}t} = \lambda_1(t)(\langle k\rangle\alpha_1\hat{S}_1 + \mu_1 + d) - \lambda_3(t)\langle k\rangle\alpha_1\hat{S}_1 - \lambda_6(t)\mu_1, \\
\dfrac{\mathrm{d}\lambda_2(t)}{\mathrm{d}t} = \lambda_2(t)(\langle k\rangle\alpha_2\hat{S}_2 + \mu_2 + d) - \lambda_4(t)\langle k\rangle\alpha_2\hat{S}_2 - \lambda_6(t)\mu_2, \\
\dfrac{\mathrm{d}\lambda_3(t)}{\mathrm{d}t} = -u_1 + \lambda_1\langle k\rangle\alpha_1\hat{I}_1 - \lambda_3(t)[\langle k\rangle\alpha_1\hat{I}_1 - \dfrac{r_1(t)\sigma_1}{(\sigma_1+\hat{S}_1)^2} - d - \beta_1 - \rho] \\
\qquad - \lambda_4(t)\rho - \lambda_5(t)\dfrac{r_1(t)\sigma_1}{(\sigma_1+\hat{S}_1)^2} - \lambda_6(t)\beta_1, \\
\dfrac{\mathrm{d}\lambda_4(t)}{\mathrm{d}t} = -u_2 + \lambda_2\langle k\rangle\alpha_2\hat{I}_2 - \lambda_4(t)[\langle k\rangle\alpha_2\hat{I}_2 - \dfrac{r_2(t)\sigma_2}{(\sigma_2+\hat{S}_2)^2} - \beta_2 - d] \\
\qquad - \lambda_5(t)\dfrac{r_2(t)\sigma_2}{(\sigma_2+\hat{S}_2)^2} - \lambda_6(t)\beta_2, \\
\dfrac{\mathrm{d}\lambda_5(t)}{\mathrm{d}t} = \lambda_5(t)(c+d) - \lambda_6(t)c, \\
\dfrac{\mathrm{d}\lambda_6(t)}{\mathrm{d}t} = \lambda_6(t)d,
\end{cases}
$$

with the transversality conditions $\lambda_j(T) = 0$. Optimal control r_1^* and r_2^* are given by

$$
r_1^* = \max\{\min\{\frac{(\lambda_3-\lambda_5)\hat{S}_1}{2u_3(\sigma_1+\hat{S}_1)}, 0\}, r_1^{\max}\},
$$
$$
r_2^* = \max\{\min\{\frac{(\lambda_4-\lambda_5)\hat{S}_2}{2u_4(\sigma_2+\hat{S}_2)}, 0\}, r_2^{\max}\}.
$$

Proof. Let $I_1(t) = \hat{I}_1$, $I_2(t) = \hat{I}_2$, $S_1(t) = \hat{S}_1$, $S_2(t) = \hat{S}_2$, $Q(t) = \hat{Q}$ and $R(t) = \hat{R}$, using Pontryagin's maximum principle [41], we obtain the following adjoint equations:

$$
\begin{aligned}
\frac{\mathrm{d}\lambda_1(t)}{\mathrm{d}t} &= -\frac{\partial H}{\partial I_1(t)} = \lambda_1(t)(\langle k\rangle\alpha_1\hat{S}_1 + \mu_1 + d) - \lambda_3(t)\langle k\rangle\alpha_1\hat{S}_1 - \lambda_6(t)\mu_1, \\
\frac{\mathrm{d}\lambda_2(t)}{\mathrm{d}t} &= -\frac{\partial H}{\partial I_2(t)} = \lambda_2(t)(\langle k\rangle\alpha_2\hat{S}_2 + \mu_2 + d) - \lambda_4(t)\langle k\rangle\alpha_2\hat{S}_2 - \lambda_6(t)\mu_2, \\
\frac{\mathrm{d}\lambda_3(t)}{\mathrm{d}t} &= -\frac{\partial H}{\partial S_1(t)} = -u_1 + \lambda_1\langle k\rangle\alpha_1\hat{I}_1 - \lambda_3(t)[\langle k\rangle\alpha_1\hat{I}_1 - \frac{r_1(t)\sigma_1}{(\sigma_1+\hat{S}_1)^2} - d - \beta_1 - \rho] \\
&\qquad - \lambda_4(t)\rho - \lambda_5(t)\frac{r_1(t)\sigma_1}{(\sigma_1+\hat{S}_1)^2} - \lambda_6(t)\beta_1, \\
\frac{\mathrm{d}\lambda_4(t)}{\mathrm{d}t} &= -\frac{\partial H}{\partial S_2(t)} = -u_2 + \lambda_2\langle k\rangle\alpha_2\hat{I}_2 - \lambda_4(t)[\langle k\rangle\alpha_2\hat{I}_2 - \frac{r_2(t)\sigma_2}{(\sigma_2+\hat{S}_2)^2} - \beta_2 - d] \\
&\qquad - \lambda_5(t)\frac{r_2(t)\sigma_2}{(\sigma_2+\hat{S}_2)^2} - \lambda_6(t)\beta_2, \\
\frac{\mathrm{d}\lambda_5(t)}{\mathrm{d}t} &= -\frac{\partial H}{\partial Q(t)} = \lambda_5(t)(c+d) - \lambda_6(t)c, \\
\frac{\mathrm{d}\lambda_6(t)}{\mathrm{d}t} &= -\frac{\partial H}{\partial R(t)} = \lambda_6(t)d.
\end{aligned}
$$

By the optimality condition, the derivative of the Hamiltonian function with respect to $r_1(t)$ and $r_2(t)$ is as follows:

$$\frac{\partial H}{\partial r_1(t)}\Big|_{r_1(t)=r_1^*} = 2u_3 r_1(t) + \frac{(\lambda_5(t)-\lambda_3(t))\hat{S}_1}{\sigma_1+\hat{S}_1} = 0,$$
$$\frac{\partial H}{\partial r_2(t)}\Big|_{r_2(t)=r_2^*} = 2u_4 r_2(t) + \frac{(\lambda_5(t)-\lambda_4(t))\hat{S}_2}{\sigma_2+\hat{S}_2} = 0.$$

Then, we obtain the optimal control

$$r_1^* = \frac{(\lambda_3(t)-\lambda_5(t))\hat{S}_1}{2u_3(\sigma_1+\hat{S}_1)}, \quad r_2^* = \frac{(\lambda_4(t)-\lambda_5(t))\hat{S}_2}{2u_4(\sigma_2+\hat{S}_2)}.$$

By combining the properties of bounded set U, the interval of r_1^* and r_2^* are shown in the following form:

$$r_1^* = \max\left\{ \min\{\frac{(\lambda_3(t)-\lambda_5(t))\hat{S}_1}{2u_3(\sigma_1+\hat{S}_1)}, 0\}, r_1^{\max} \right\},$$
$$r_2^* = \max\left\{ \min\{\frac{(\lambda_4(t)-\lambda_5(t))\hat{S}_2}{2u_4(\sigma_2+\hat{S}_2)}, 0\}, r_2^{\max} \right\}.$$

□

Remark 4. *In the optimal control, the control $r_1(t)$ and $r_2(t)$ are time-varying and associated with objective function (14). For different expected control time T and objective function, the control rates $r_1(t)$ and $r_2(t)$ may be different. This can be verified by the simulation in the later section.*

4. 2I2SQR Rumor Model with Time-Delays

4.1. Model Formulation

In general, when the influence of rumor becomes greater, the social platform will take control measures to isolate the rumor disseminators. Because it takes a certain time to search for rumor disseminators on the network platform, the implementation of this measure sometimes lags behind the rumor-spreading process. Therefore, system (1) incorporating time-delays is formulated as follows:

$$\begin{cases} \frac{\mathrm{d}I_1(t)}{\mathrm{d}t} = B_1 - \langle k\rangle\alpha_1 I_1(t)S_1(t) - \mu_1 I_1(t) - dI_1(t), \\ \frac{\mathrm{d}I_2(t)}{\mathrm{d}t} = B_2 - \langle k\rangle\alpha_2 I_2(t)S_2(t) - \mu_2 I_2(t) - dI_2(t), \\ \frac{\mathrm{d}S_1(t)}{\mathrm{d}t} = \langle k\rangle\alpha_1 I_1(t)S_1(t) - \frac{r_1 S_1(t-\tau_1)}{\sigma_1+S_1(t-\tau_1)} - \beta_1 S_1(t) - dS_1(t) - \rho S_1(t), \\ \frac{\mathrm{d}S_2(t)}{\mathrm{d}t} = \langle k\rangle\alpha_2 I_2(t)S_2(t) + \rho S_1(t) - \frac{r_2 S_2(t-\tau_2)}{\sigma_2+S_2(t-\tau_2)} - \beta_2 S_2(t) - dS_2(t), \\ \frac{\mathrm{d}Q(t)}{\mathrm{d}t} = \frac{r_1 S_1(t-\tau_1)}{\sigma_1+S_1(t-\tau_1)} + \frac{r_2 S_2(t-\tau_2)}{\sigma_2+S_2(t-\tau_2)} - cQ(t) - dQ(t), \\ \frac{\mathrm{d}R(t)}{\mathrm{d}t} = \beta_1 S_1(t) + \beta_2 S_2(t) + \mu_1 I_1(t) + cQ(t) + \mu_2 I_2(t) - dR(t). \end{cases} \tag{17}$$

where $\tau_1, \tau_2 > 0$ represent the time delays, and other parameters and variables are consistent with the explanation in system (1). The initial conditions of system (17) are given by

$$I_i(\theta) > 0,\ S_i(\theta) > 0,\ Q(\theta) \geq 0,\ R(\theta) \geq 0,\ \theta \in [-\tau, 0],\ \ i = 1, 2,$$

where $\tau = \max\{\tau_1, \tau_2\}$.

Similar to the previous analysis, we simplify system (17) as follows:

$$\begin{cases} \dfrac{\mathrm{d}I_1(t)}{\mathrm{d}t} = B_1 - \langle k\rangle\alpha_1 I_1(t)S_1(t) - \mu_1 I_1(t) - dI_1(t), \\ \dfrac{\mathrm{d}I_2(t)}{\mathrm{d}t} = B_2 - \langle k\rangle\alpha_2 I_2(t)S_2(t) - \mu_2 I_2(t) - dI_2(t), \\ \dfrac{\mathrm{d}S_1(t)}{\mathrm{d}t} = \langle k\rangle\alpha_1 I_1(t)S_1(t) - \dfrac{r_1 S_1(t-\tau_1)}{\sigma_1 + S_1(t-\tau_1)} - \beta_1 S_1(t) - dS_1(t) - \rho S_1(t), \\ \dfrac{\mathrm{d}S_2(t)}{\mathrm{d}t} = \langle k\rangle\alpha_2 I_2(t)S_2(t) + \rho S_1(t) - \dfrac{r_2 S_2(t-\tau_2)}{\sigma_2 + S_2(t-\tau_2)} - \beta_2 S_2(t) - dS_2(t). \end{cases} \tag{18}$$

Because the equilibria of system (18) are the same as that of system (2), which does not depend on the time delays, then Theorem 1 is still established in this section. Hence, we omit the theoretical analysis of the existence of equilibria.

4.2. Stability and Hopf Bifurcation of Equilibria

In this subsection, we will first discuss the stability and Hopf bifurcation of rumor-free equilibrium E_0, and then analyze the dynamics of the rumor equilibrium.

The Jacobian matrix of system (18) at E_0 is expressed as follows:

$$\mathcal{J}(E_0) = \begin{pmatrix} -(\mu_1+d) & 0 & -\frac{\alpha_1 B_1\langle k\rangle}{d+\mu_1} & 0 \\ 0 & -(\mu_2+d) & 0 & -\frac{\alpha_2 B_2\langle k\rangle}{d+\mu_2} \\ 0 & 0 & -\frac{r_1 e^{-\lambda\tau_1}}{\sigma_1} - A_1 & 0 \\ 0 & 0 & \rho & -\frac{r_2 e^{-\lambda\tau_2}}{\sigma_2} - A_2 \end{pmatrix},$$

where $A_1 = (\beta_1 + d + \rho) - \frac{\alpha_1 B_1\langle k\rangle}{d+\mu_1}$, $A_2 = \beta_2 + d - \frac{\alpha_2 B_2\langle k\rangle}{d+\mu_2}$. Accordingly, the characteristic equation is

$$(\lambda + \mu_1 + d)(\lambda + \mu_2 + d)(\lambda + A_1 + \frac{r_1}{\sigma_1}e^{-\lambda\tau_1})(\lambda + A_2 + \frac{r_2}{\sigma_2}e^{-\lambda\tau_2}) = 0. \tag{19}$$

It is clear that $\lambda_1 = -(\mu_1 + d) < 0$, $\lambda_2 = -(\mu_2 + d) < 0$, and other eigenvalues are determined by the following equations:

$$\lambda + A_1 + \frac{r_1}{\sigma_1}e^{-\lambda\tau_1} = 0, \ \ \lambda + A_2 + \frac{r_2}{\sigma_2}e^{-\lambda\tau_2} = 0.$$

Let

$$f(\lambda) = \lambda + A_1 + \frac{r_1}{\sigma_1}e^{-\lambda\tau_1} = 0, \tag{20}$$

$$g(\lambda) = \lambda + A_2 + \frac{r_2}{\sigma_2}e^{-\lambda\tau_2} = 0. \tag{21}$$

By Theorem 2, we know that all solutions of Equation (19) are negative when $\tau_1 = \tau_2 = 0$ and $\Re_0 < 1$. When $\tau_1, \tau_2 > 0$, assume that $\lambda = i\omega(\omega > 0)$ is a solution of (19). Substituting $\lambda = i\omega$ into (19), the following two cases are discussed:
Case (1). If $f(\lambda) = 0$, then it yields

$$i\omega + A_1 + \frac{r_1}{\sigma_1}(\cos\omega\tau_1 - i\sin\omega\tau_1) = 0.$$

Separating the real parts and imaginary parts, one has

$$A_1 + \frac{r_1}{\sigma_1}\cos\omega\tau_1 = 0, \ \ \omega - \frac{r_1}{\sigma_1}\sin\omega\tau_1 = 0.$$

Squaring and adding the above equations gives

$$\omega^2 = (\frac{r_1}{\sigma_1})^2 - A_1^2 = \frac{r_1 + \sigma_1(\beta_1 + d + \rho)}{\sigma_1}(1 - \Re_{01})\left(\frac{r_1}{\sigma_1} + \frac{\langle k\rangle \alpha_1 B_1}{d + \mu_1} - (\beta_1 + d + \rho)\right). \quad (22)$$

Assume that $\frac{r_1}{\sigma_1} + \frac{\langle k\rangle \alpha_1 B_1}{d+\mu_1} > \beta_1 + d + \rho$, then there exist a positive real number ω_{10} such that Equation (20) has purely imaginary roots $\lambda = \pm i\omega_{10}(\omega_{10} > 0)$. Then, according to the above analysis, we have

$$\cos(\omega_{10}\tau_1) = -\frac{A_1\sigma_1}{r_1} = \frac{\sigma_1}{r_1}\left[\frac{\langle k\rangle \alpha_1 B_1}{d + \mu_1} - (\beta_1 + d + \rho)\right],$$

and $\tau_{1i} = \frac{1}{\omega_{10}} \arccos(\frac{\sigma_1}{r_1}[\frac{\langle k\rangle \alpha_1 B_1}{d+\mu_1} - (\beta_1 + d + \rho)]) + \frac{2\pi i}{\omega_{10}}, i = 0, 1, 2, \cdots$.

Additionally, the derivative of Equation (20) with respect to τ_1 yields

$$(\frac{d\lambda}{d\tau_1})^{-1} = \frac{1}{\lambda \frac{r_1}{\sigma_1} e^{-\lambda\tau_1}} - \frac{\tau_1}{\lambda} = \frac{1}{-\lambda(\lambda + A_1)} - \frac{\tau_1}{\lambda}. \quad (23)$$

Taking $\lambda = i\omega_{10}$, $\tau_1 = \tau_{10}$ into (23), and separating the real part and the imaginary part, one has

$$Re(\frac{d\lambda}{d\tau_1})^{-1}\Big|_{\lambda = i\omega_{10}, \tau_1 = \tau_{10}} = \frac{1}{A_1^2 + \omega_{10}^2} > 0.$$

Case (2). If $g(\lambda) = 0$, the analysis process is similar to Case (1). Hence, we obtain that

$$\omega^2 = (\frac{r_2}{\sigma_2})^2 - A_2^2 = \frac{r_2 + \sigma_2(\beta_2 + d)}{\sigma_2}(1 - \Re_{02})\left(\frac{r_2}{\sigma_2} + \frac{\langle k\rangle \alpha_2 B_2}{d + \mu_2} - (\beta_2 + d)\right). \quad (24)$$

When $\frac{r_2}{\sigma_2} + \frac{\langle k\rangle \alpha_2 B_2}{d+\mu_2} > \beta_2 + d$, based on (24), there exists a positive real number equation ω_{20} such that Equation (21) has a purely imaginary root $\lambda = i\omega_{20}$. Through the analysis, it has

$$\tau_{2j} = \frac{1}{\omega_{20}} \arccos\left(\frac{\sigma_2}{r_2}\left[\frac{\langle k\rangle \alpha_2 B_2}{d + \mu_2} - (\beta_2 + d)\right]\right) + \frac{2\pi j}{\omega_{20}}, \; j = 0, 1, 2, \cdots . \quad (25)$$

and

$$Re(\frac{d\lambda}{d\tau_2})^{-1}\Big|_{\lambda = i\omega_{20}, \tau_2 = \tau_{20}} = \frac{1}{A_2^2 + \omega_{20}^2} > 0.$$

In summary, the following theorem is given.

Theorem 7. *For system (18), if $\Re_0 < 1$, $\frac{r_1}{\sigma_1} + \frac{\langle k\rangle \alpha_1 B_1}{d+\mu_1} > \beta_1 + d + \rho$ and $\frac{r_2}{\sigma_2} + \frac{\langle k\rangle \alpha_2 B_2}{d+\mu_2} > \beta_2 + d$ are satisfied, the following statements hold.*

(1). When $\tau_1 \in [0, \tau_{10})$ and $\tau_2 \in [0, \tau_{20})$, the rumor-free equilibrium E_0 is locally asymptotically stable.

(2). When $\tau_1 > \tau_{10}$ or $\tau_2 > \tau_{20}$, the rumor-free equilibrium E_0 is unstable. Moreover, system (16) has a Hopf bifurcation at E_0 when $\tau_1 = \tau_{10}$ or $\tau_2 = \tau_{20}$.

Next, we will discuss the stability and Hopf bifurcation of the rumor equilibrium E_1^*. The Jacobian matrix of system (18) at E_1^* is given as follows:

$$\mathcal{J}(E_1^*) = \begin{pmatrix} -\Omega_1 & 0 & -\alpha_1 I_1^*\langle k\rangle & 0 \\ 0 & -\Omega_2 & 0 & -\alpha_2 I_2^*\langle k\rangle \\ \alpha_1 S_1^*\langle k\rangle & 0 & -\Omega_3 - \frac{r_1\sigma_1 e^{-\lambda\tau_1}}{(\sigma_1+S_1^*)^2} & 0 \\ 0 & \alpha_2 S_2^*\langle k\rangle & \rho & -\Omega_4 - \frac{r_2\sigma_2 e^{-\lambda\tau_2}}{(\sigma_2+S_2^*)^2} \end{pmatrix},$$

where $\Omega_1 = \alpha_1 S_1^*\langle k\rangle + \mu_1 + d$, $\Omega_2 = \alpha_2 S_2^*\langle k\rangle + \mu_2 + d$, $\Omega_3 = -\alpha_1 I_1^*\langle k\rangle + (\beta_1 + d + \rho)$, and $\Omega_4 = -\alpha_2 I_2^*\langle k\rangle + (\beta_2 + d)$.

The characteristic equation for matrix $\mathcal{J}(E_1^*)$ is expressed as

$$\begin{aligned} &\left(\lambda^2 + (\Omega_1+\Omega_3)\lambda + C + (\lambda+\Omega_1)U_1 e^{-\lambda\tau_1}\right) \\ \times &\left(\lambda^2 + (\Omega_2+\Omega_4)\lambda + D + (\lambda+\Omega_2)U_2 e^{-\lambda\tau_2}\right) = 0, \end{aligned}$$

where $C = (\alpha_1 S_1^*\langle k\rangle + \mu_1 + d)(\beta_1 + d + \rho) - \alpha_1 I_1^*\langle k\rangle(\mu_1 + d)$, $D = (\alpha_2 S_2^*\langle k\rangle + \mu_2 + d)(\beta_2 + d) - \alpha_2 I_2^*\langle k\rangle(\mu_2 + d)$, $U_1 = \frac{r_1\sigma_1}{(\sigma_1+S_1^*)^2}$, $U_2 = \frac{r_2\sigma_2}{(\sigma_2+S_2^*)^2}$. The stability of the matrix $\mathcal{J}(E_1^*)$ is determined by the solution of the following equations

$$\lambda^2 + (\Omega_1+\Omega_3)\lambda + C + (\lambda+\Omega_1)U_1 e^{-\lambda\tau_1} = 0, \tag{26}$$

$$\lambda^2 + (\Omega_2+\Omega_4)\lambda + D + (\lambda+\Omega_2)U_2 e^{-\lambda\tau_2} = 0. \tag{27}$$

For the case of $\tau_1 = \tau_2 = 0$, according to the proof of Theorem 4, the rumor equilibrium E_1^* is locally asymptotically stable under the conditions of $H_i(S_i^*) > 0$ and $G_i(S_i^*) > 0$.

For the case of $\tau_1, \tau_2 > 0$, assuming that the solutions of Equations (26) and (27) are $\lambda = i\omega_1$ and $\lambda = i\omega_2$, respectively. Then, separating the real part and the imaginary part of Equations (26) and (27), the following equations can be obtained:

$$\begin{cases} \omega_1 U_1 \sin\omega_1\tau_1 + \Omega_1 U_1 \cos\omega_1\tau_1 = \omega_1^2 - C, \\ \omega_1 U_1 \cos\omega_1\tau_1 - \Omega_1 U_1 \sin\omega_1\tau_1 = -(\Omega_1+\Omega_3)\omega_1, \end{cases} \tag{28}$$

$$\begin{cases} \omega_2 U_2 \sin\omega_2\tau_2 + \Omega_2 U_2 \cos\omega_2\tau_2 = \omega_2^2 - D, \\ \omega_2 U_2 \cos\omega_2\tau_2 - \Omega_2 U_2 \sin\omega_2\tau_2 = -(\Omega_2+\Omega_4)\omega_2. \end{cases} \tag{29}$$

By adding the squares of the two equations in Equations (28) and (29), and letting $x = \omega_1^2$ and $y = \omega_2^2$, it has

$$x^2 + M_1 x + M_2 = 0, \tag{30}$$

$$y^2 + L_1 y + L_2 = 0, \tag{31}$$

where $M_1 = (\Omega_1+\Omega_3)^2 - 2C - U_1^2$, $M_2 = C^2 - \Omega_1^2 U_1^2$, $L_1 = (\Omega_2+\Omega_4)^2 - 2D - U_2^2$, $L_2 = D^2 - \Omega_2^2 U_2^2$. Denote $G(x) = x^2 + M_1 x + M_2$, $W(y) = y^2 + L_1 y + L_2$.

For Equation (30), the following conclusions are true:

$(\mathcal{H}_1)$. If $M_2 < 0$, Equation (30) has a positive root.

$(\mathcal{H}_2)$. If $M_2 \geq 0$, and $M_1 \geq 0$, Equation (30) does not have any positive roots.

$(\mathcal{H}_3)$. If $M_2 > 0$, $M_1 < 0$, and $M_1^2 - 4M_2 \geq 0$, Equation (30) has two positive roots.

$(\mathcal{H}_4)$. If $M_2 > 0$, $M_1 < 0$, and $M_1^2 - 4M_2 < 0$, Equation (30) does not have any positive roots.

Similar to Equation (31), the following conclusions are true:

$(\mathcal{K}_1)$. If $L_2 < 0$, Equation (31) has a positive root.

$(\mathcal{K}_2)$. If $L_2 > 0$, and $L_1 < 0$, and $L_1^2 - 4L_2 < 0$, Equation (31) does not have any positive roots.

$(\mathcal{K}_3)$. If $L_2 > 0$, $L_1 < 0$, and $L_1^2 - 4L_2 \geq 0$, Equation (31) has two positive roots.

$(\mathcal{K}_4)$. If $L_2 \geq 0$, and $L_1 \geq 0$, Equation (31) does not have any positive roots.

For the case of $(\mathcal{H}_1)$, Equation (30) has a positive root $x_0 = \omega_{10}^2$. By Equation (28), we have

$$\cos(\omega_1\tau_1) = \frac{\omega_1^2 - C - (\Omega_1 + \Omega_3)\omega_1}{(\omega_1 U_1 + \Omega_1 U_1)},$$

then

$$\tau_{\mathcal{H}_{10}}^j = \frac{1}{\omega_{10}} \arccos(\frac{\omega_{10}^2 - C - (\Omega_1 + \Omega_3)\omega_{10}}{\omega_{10} U_1 + \Omega_1 U_1}) + \frac{2\pi j}{\omega_{10}}, \tag{32}$$

where $j = 0, 1, 2, \ldots$, $\pm\omega_{10}$ is a pair of pure imaginary roots of Equation (30). Further, taking the derivative of Equation (26) with respect to τ_1, we have

$$(\frac{\mathrm{d}\lambda}{\mathrm{d}\tau_1})^{-1} = \frac{2\lambda + \Omega_1 + \Omega_3}{-[\lambda^2 + (\Omega_1 + \Omega_3)\lambda + C]\lambda} + \frac{1}{\lambda(\lambda + \Omega_1)} - \frac{\tau_1}{\lambda}.$$

Then

$$Re(\frac{\mathrm{d}\lambda}{\mathrm{d}\tau_1})^{-1}|_{\lambda = i\omega_{10}, \tau_1 = \tau_{\mathcal{H}_{10}}^0} = \frac{2(\omega_{10})^2 - 2C + (\Omega_1 + \Omega_3)^2 - U_1^2}{(\omega_{10}^2 + \Omega_1^2)U_1^2} = \frac{\mathcal{G}(\omega_{10})}{(\omega_{10}^2 + \Omega_1^2)U_1^2}. \tag{33}$$

Since $M_2 < 0$, $x_0 = \omega_{10}^2 > 0$ is a positive root of (30), it has $\mathcal{G}(\omega_{10}) > 0$. Hence

$$Re(\frac{\mathrm{d}\lambda}{\mathrm{d}\tau_1})^{-1}\Big|_{\lambda = i\omega_{10}, \tau_1 = \tau_{\mathcal{H}_{10}}^0} = \frac{\mathcal{G}(\omega_{10})}{(\omega_{10}^2 + \Omega_1^2)U_1^2} > 0.$$

Similarly, for the case of $(\mathcal{K}_1)$, if $L_2 < 0$, Equation (31) has a positive root $y_0 = \omega_{20}^2$. Then, we can obtain

$$\tau_{\mathcal{K}_{10}}^j = \frac{1}{\omega_{20}} \arccos(\frac{\omega_{20}^2 - D - (\Omega_2 + \Omega_4)\omega_{20}}{\omega_{20} U_2 + \Omega_2 U_2}) + \frac{2\pi j}{\omega_{20}}, \ j = 0, 1, 2, \cdots, \tag{34}$$

$$Re(\frac{\mathrm{d}\lambda}{\mathrm{d}\tau_2})^{-1}\Big|_{\lambda = i\omega_{20}, \tau_2 = \tau_{\mathcal{K}_{10}}^0} = \frac{2(\omega_{20})^2 - 2D + (\Omega_2 + \Omega_4)^2 - U_2^2}{((\omega_{20})^2 + \Omega_2^2)U_2^2} = \frac{\mathcal{W}(\omega_{20})}{((\omega_{20})^2 + \Omega_2^2)U_2^2}. \tag{35}$$

Similar to the analysis of $(\mathcal{H}_1)$, one has

$$Re(\frac{\mathrm{d}\lambda}{\mathrm{d}\tau_2})^{-1}|_{\lambda = i\omega_{20}, \tau_2 = \tau_{\mathcal{K}_{10}}^0} = \frac{\mathcal{W}(\omega_{20})}{((\omega_{20})^2 + \Omega_2^2)U_2^2} > 0.$$

Therefore, the following theorem is given.

Theorem 8. *For system (18), if $H_i(S_i^*) > 0$ and $G_i(S_i^*) > 0$, $i = 1, 2$, the following results are satisfied.*

(1). If one of $(\mathcal{H}_i)$ and one of $(\mathcal{K}_i)$ for $i = 2, 4$ are satisfied, the rumor equilibrium point E_1^* is locally asymptotically stable for all $\tau_1, \tau_2 \geq 0$.

(2). If $(\mathcal{H}_1)$ and $(\mathcal{K}_1)$ are satisfied, the rumor equilibrium point E_1^* is locally asymptotically stable when $\tau_1 \in [0, \tau_{\mathcal{H}_{10}}^0)$ and $\tau_2 \in [0, \tau_{\mathcal{K}_{10}}^0)$, and the rumor equilibrium point E_1^* is unstable when $\tau_1 > \tau_{\mathcal{H}_{10}}^0$ or $\tau_2 > \tau_{\mathcal{K}_{10}}^0$. Therefore, system (18) has a Hopf bifurcation at E_1^* when $\tau_1 = \tau_{\mathcal{H}_{10}}^0$ or $\tau_2 = \tau_{\mathcal{K}_{10}}^0$.

For the case $(\mathcal{H}_3)$, when $L_2 \geq 0$, $L_1 \geq 0$, and $L_1^2 - 4L_2 \geq 0$, Equation (30) has two positive roots, defined by $x_1 = \omega_{11}^2$, $x_2 = \omega_{12}^2$, and $x_1 > x_2$. Choose $\omega_{11} > 0, \omega_{12} > 0$ then $\omega_{11} > \omega_{12}$, $\mathcal{G}(\omega_{11}) > 0$, $\mathcal{G}(\omega_{12}) < 0$. According to Equation (32), we obtain that

$$\tau_{\mathcal{H}_3 m}^{\nu} = \frac{1}{\omega_{1m}} \arccos\left(\frac{\omega_{1m}^2 - C - (\Omega_1 + \Omega_3)\omega_{1m}}{\omega_{1m}U_1 + \Omega_1 U_1}\right) + \frac{2\pi\nu}{\omega_{1m}}, \ \ \nu = 0, 1, \cdots, m = 1, 2. \quad (36)$$

Let $\tau_1^* = \tau_{\mathcal{H}_{3m}}^0 = \min\{\tau_{\mathcal{H}_{3m}}^{\nu} : m = 1, 2, \ \ \nu = 0, 1, 2, \cdots\}$ and assume that $\lambda_{11} = v_{11} + i\omega_{11}$, $\lambda_{12} = v_{12} + i\omega_{12}$, then

$$Re(\frac{\mathrm{d}\lambda}{\mathrm{d}\tau_1})^{-1}|_{\lambda = i\omega_{11}, \tau_1 = \tau_{\mathcal{H}_{31}}^{\nu}} > 0, \quad \nu = 0, 1, 2, \cdots.$$

$$Re(\frac{\mathrm{d}\lambda}{\mathrm{d}\tau_1})^{-1}|_{\lambda = i\omega_{12}, \tau_1 = \tau_{\mathcal{H}_{32}}^{\nu}} < 0, \quad \nu = 0, 1, 2, \cdots.$$

Similarly, for the case of $(\mathcal{K}_3)$, defined $y_1 = \omega_{21}^2$, $y_2 = \omega_{22}^2$, and $y_1 > y_2$, then $\omega_{21} > \omega_{22}$, $\mathcal{W}(\omega_{21}) > 0$, $\mathcal{W}(\omega_{22}) < 0$. According to (36), it has

$$\tau_{\mathcal{K}_3 m}^{p} = \frac{1}{\omega_{2m}} \arccos\left(\frac{\omega_{2m}^2 - D - (\Omega_2 + \Omega_4)\omega_{2m}}{\omega_{2m}U_2 + \Omega_2 U_2}\right) + \frac{2\pi p}{\omega_{2m}}, \ \ p = 0, 1, \cdots, m = 1, 2 \quad (37)$$

and assume that $\lambda_{21} = v_{21} + i\omega_{21}$, $\lambda_{22} = v_{22} + i\omega_{22}$, then

$$Re(\frac{\mathrm{d}\lambda}{\mathrm{d}\tau_2})^{-1}|_{\lambda = i\omega_{21}, \tau_2 = \tau_{\mathcal{K}_{31}}^{p}} > 0, \quad p = 0, 1, 2, \cdots.$$

$$Re(\frac{\mathrm{d}\lambda}{\mathrm{d}\tau_2})^{-1}|_{\lambda = i\omega_{22}, \tau_2 = \tau_{\mathcal{K}_{32}}^{p}} < 0, \quad p = 0, 1, 2, \cdots.$$

Lemma 4. *If $2\omega_1 B_i + \alpha_i\langle k\rangle B_i I_i^* > (B_i)^2 I_i^* + \alpha_i\langle k\rangle(\mu_i + d)$, and $\omega_{i1} > \omega_{i2}$ for $i = 1, 2$, then $\tau_{\mathcal{H}_{31}}^{\nu} < \tau_{\mathcal{H}_{32}}^{\nu}$ and $\tau_{\mathcal{K}_{31}}^{\nu} < \tau_{\mathcal{K}_{32}}^{\nu}$, $\nu = 1, 2, \cdots$.*

Proof. Define

$$g(\omega) = \frac{\omega_1^2 - C - (\Omega_1 + \Omega_3)\omega_1}{(\omega_1 U_1 + \Omega_1 U_1)}, \ \ \bar{g}(\omega) = \arccos g(\omega), \ \ \bar{G}(\omega) = \frac{1}{\omega}\bar{g}(\omega).$$

Then, the derivative of $g(\omega)$ is

$$\begin{aligned} g'(\omega) &= \frac{(\omega_1^2 + 2\omega_1\Omega_1 - (\Omega_1 + \Omega_3)\Omega_1 + C)U_1}{(\omega_1 U_1 + \Omega_1 U_1)^2} \\ &= \frac{[\omega_1^2 + (2\omega_1 - B_1 I_1^* + \alpha_1 I_1^*\langle k\rangle)B_1 I_1^* - \alpha_1 I_1^*\langle k\rangle(\mu_1 + d)]U_1}{(\omega_1 U_1 + \Omega_1 U_1)^2} \\ &\geq \frac{[2\omega_1 B_1 + \alpha_1 I_1^*\langle k\rangle B_1 - (B_1)^2 I_1^* - \alpha_1\langle k\rangle(\mu_1 + d)]I_1^*}{U_1(\omega_1 + \Omega_1)^2} \\ &> 0. \end{aligned}$$

Since $g(\omega)$ is always an increasing function about ω, it is obtained that $\bar{g}(\omega) = \arccos g(\omega)$ is a decreasing function about ω. Therefore, we can obtain that

$$\bar{G}'(\omega) = -\frac{1}{\omega^2}\bar{g}(\omega) + \frac{1}{\omega}\bar{g}'(\omega) < 0.$$

By conclusion, $\tau^{\nu}_{\mathcal{H}_{31}} < \tau^{\nu}_{\mathcal{H}_{32}}$ when $\omega_{11} > \omega_{12}$. Similarly, we obtain that $\tau^{\nu}_{\mathcal{K}_{31}} < \tau^{\nu}_{\mathcal{K}_{32}}$ for $\omega_{21} > \omega_{22}$, where $\nu = 1, 2, \cdots$. Hence, the following theorem is given. □

Theorem 9. *For system (18), if Lemma 4 and $H_i(S_i^*) > 0$, $G_i(S_i^*) > 0$ for $i = 1, 2$, are satisfied, the following results can be obtained:*

(1). If both $(\mathcal{H}_3)$ and $(\mathcal{K}_3)$ are satisfied, the rumor equilibrium point E_1^* is locally asymptotically stable for $\tau_1 \in [0, \tau^0_{\mathcal{H}_{31}})$ and $\tau_2 \in [0, \tau^0_{\mathcal{K}_{31}})$, and unstable for $\tau_1 > \tau^0_{\mathcal{H}_{31}}$ or $\tau_2 > \tau^0_{\mathcal{K}_{31}}$. Hence, system (18) has a Hopf bifurcation at E_1^* when $\tau_1 = \tau^0_{\mathcal{H}_{31}}$ or $\tau_2 = \tau^0_{\mathcal{K}_{31}}$.

(2). If $(\mathcal{H}_1)$ and $(\mathcal{K}_3)$ are satisfied, the rumor equilibrium point E_1^* is locally asymptotically stable for $\tau_1 \in [0, \tau^0_{\mathcal{H}_{10}})$ and $\tau_2 \in [0, \tau^0_{\mathcal{K}_{31}})$, and unstable for $\tau_1 > \tau^0_{\mathcal{H}_{10}}$ or $\tau_2 > \tau^0_{\mathcal{K}_{31}}$. Therefore, system (18) has a Hopf bifurcation at E_1^* when $\tau_1 = \tau^0_{\mathcal{H}_{10}}$ or $\tau_2 = \tau^0_{\mathcal{K}_{31}}$.

(3). If $(\mathcal{H}_3)$ and $(\mathcal{K}_1)$ are satisfied, the rumor equilibrium point E_1^* is locally asymptotically stable when $\tau_1 \in [0, \tau^0_{\mathcal{H}_{31}})$ and $\tau_2 \in [0, \min\{\tau^0_{\mathcal{K}_{10}})$, and the rumor equilibrium point E_1^* is unstable when $\tau_1 > \tau^0_{\mathcal{H}_{31}}$ or $\tau_2 > \tau^0_{\mathcal{K}_{10}}$. Therefore, system (18) has a Hopf bifurcation at E_1^* when $\tau_1 = \tau^0_{\mathcal{H}_{31}}$ or $\tau_2 = \tau^0_{\mathcal{K}_{10}}$.

(4). If one of $(\mathcal{H}_i)(i = 1, 3)$ and one of $(\mathcal{K}_j)(j = 2, 4)$ are satisfied, the rumor equilibrium point E_1^* is locally asymptotically stable for $\tau_1 \in [0, \tau^0_{\mathfrak{H}_{i0}})$, and unstable for $\tau_1 > \tau^0_{\mathcal{H}_{i0}}$. Hence, system (18) has a Hopf bifurcation at E_1^* when $\tau_1 = \tau^0_{\mathcal{H}_{i0}}$.

(5). If one of $(\mathcal{H}_i)(i = 2, 4)$ and one of $(\mathcal{K}_j)(j = 1, 3)$ are satisfied, the rumor equilibrium point E_1^* is locally asymptotically stable for $\tau_2 \in [0, \tau^0_{\mathcal{K}_{j0}})$, and unstable for $\tau_2 > \tau^0_{\mathcal{K}_{j0}}$. Hence, system (18) has a Hopf bifurcation at E_1^* when $\tau_2 = \tau^0_{\mathcal{K}_{j0}}$.

Remark 5. *From the proof of the above theorems, we can see that the stability analysis of rumor equilibrium in the model with time delay is more complex than that in the model without time delays. This shows that the dynamic behavior of the rumor equilibrium is affected by the time delay.*

Remark 6. *In existing works [36–39], some rumor-spreading models were proposed on OSNs in a multilingual environment. In these models, the isolation mechanism, immune mechanism and time delay were not considered. However, in this paper, we comprehensively consider these factors and propose a new rumor-spreading model on OSNs. The dynamic behavior is carefully studied. Moreover, the optimal control method is applied to suppress the spreading of the rumor.*

5. Numerical Simulations

In this section, we verify the effectiveness of the theoretical results by selecting different parameters. In Table 1, based on the range of parameters in the model, we randomly select seven groups of parameters to simulate the evolution of the rumor under different threshold conditions, and verify the correctness of the theoretical results.

5.1. The Stability of Rumor-Free Equilibrium

Example 1. *For system (1) with the parameters in Set 1, we obtain $\Re_{01} = 0.61 < 1$ and $\Re_{02} = 0.25 < 1$. Then, it has a unique rumor-free equilibrium $E_0 = (0.48, 0.2, 0, 0, 0, 0.32)$. According to Theorem 2, E_0 is locally asymptotically stable. For different initial values, the dynamics of system (1) is shown in Figure 3. In Figure 3a, we can see that $S_1(t)$ and $S_2(t)$ will converge asymptotically to 0 under different initial values. In Figure 3b, $Q(t)$ gradually tends to 0. $I_1(t)$, $I_2(t)$ and $R(t)$ converge to 0.48, 0.2 and 0.32, respectively. It can be verified that E_0 is locally asymptotically stable. In other words, the rumor will disappear automatically. According to the expression of E_0, one can find that the stable states of $I_1(t)$ and $I_2(t)$ are $\frac{B_1}{d+\mu_1}$ and $\frac{B_2}{d+\mu_2}$, respectively. In Set 1, we choose $B_1 = B_2$, then the main factor affecting $I_1(t)$ and $I_2(t)$ are the parameters μ_1 and μ_2. Because $\mu_1 = 0.01$ and $\mu_2 = 0.3$, the stable value of $I_1(t)$ is greater than the stable value of $I_2(t)$ in the numerical simulation. Moreover, as $\mu_2 > \mu_1$, the descending speed and amplitude of $I_2(t)$ are greater than $I_1(t)$. In addition, the evolutionary processes of $I_1(t)$ and $I_2(t)$ are different under different initial values.*

Table 1. The set of parametric values for simulation.

Parameters	Set 1	Set 2	Set 3	Set 4	Set 5	Set 6	Set 7
B_1	0.1	0.11	0.11	0.11	0.09	0.03	0.1
B_2	0.1	0.1	0.1	0.1	0.13	0.18	0.1
α_1	0.27	0.25	0.25	0.23	0.2	0.32	0.23
α_2	0.26	0.14	0.25	0.15	0.25	0.25	0.15
μ_1	0.01	0.01	0.01	0.01	0.01	0.01	0.01
μ_2	0.3	0.01	0.01	0.01	0.01	0.3	0.01
β_1	0.13	0.13	0.13	0.13	0.24	0.08	0.13
β_2	0.12	0.15	0.15	0.14	0.15	0.08	0.14
c	0.1	0.1	0.1	0.1	0.1	0.1	0.1
σ_1	0.003	0.52	0.52	0.41	0.5	0.58	0.5
σ_2	0.0028	0.51	0.51	0.4	0.52	0.5	0.5
r_1	0.001	0.02	0.02	0.05	0.1	0.1	0.06
r_2	0.001	0.05	0.03	0.05	0.08	0.0005	0.06
d	0.2	0.21	0.21	0.21	0.21	0.21	0.2
ρ	0.08	0.03	0.03	0.08	0.05	0.08	0.08
$\langle k \rangle$	3.5	7	7	7	7	3.5	7

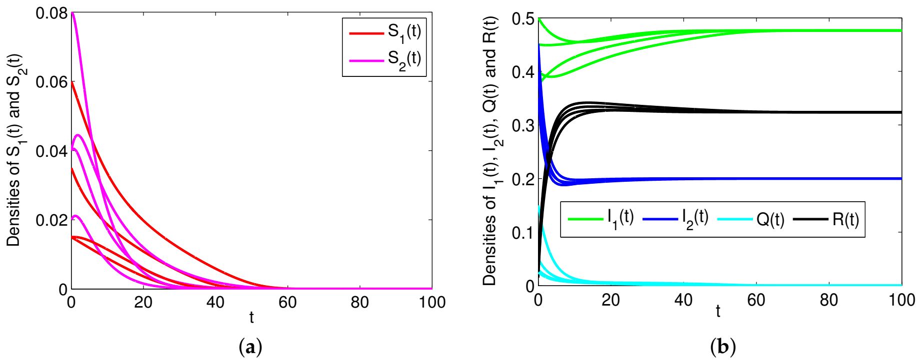


Figure 3. The dynamics of system (1) with $\Re_0 < 1$. (**a**) The trajectories of $S_1(t)$ and $S_2(t)$. (**b**) The trajectories of $I_1(t)$, $I_2(t)$, $Q(t)$ and $R(t)$.

For the case with time delays, we can verify that the conditions of Theorem 7 hold. By calculation, $\tau_{10} = 4.38$ and $\tau_{20} = 5.12$. As shown in Figure 4, we find that the time-delay affects the convergence rate and the stability of the rumor-free equilibrium point. When $\tau_1 \in [0, \tau_{10})$ and $\tau_2 \in [0, \tau_{20})$, E_0 is locally asymptotically stable. When $\tau_1 > \tau_{10}$ or $\tau_2 > \tau_{20}$, E_0 is unstable. This is consistent with the results of Theorem 7.

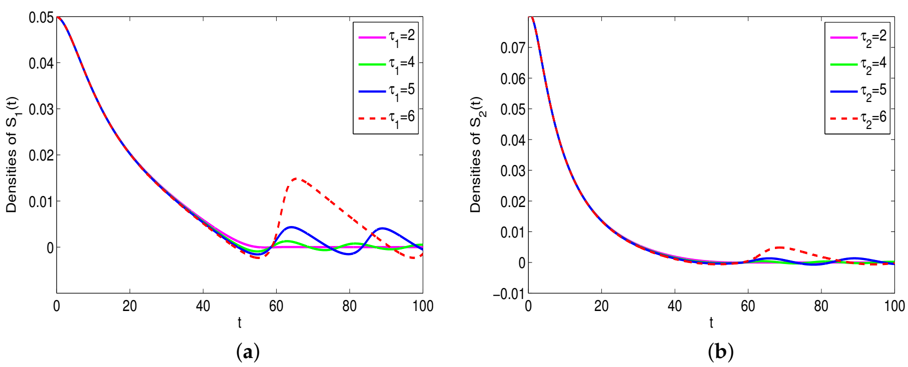
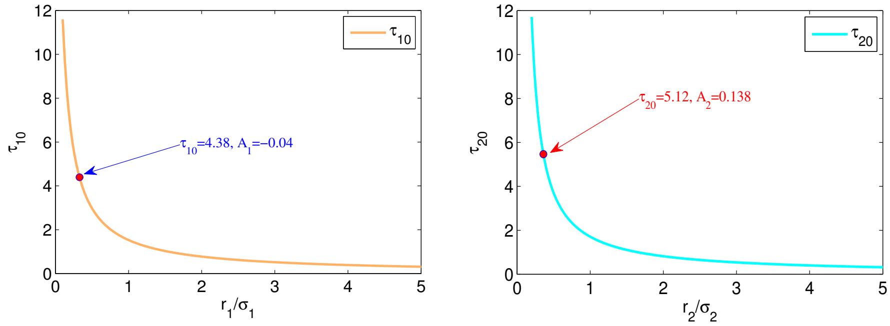


Figure 4. (**a**) The trajectories of $S_1(t)$ with different time delays. (**b**) The trajectories of $S_2(t)$ with different time delays.

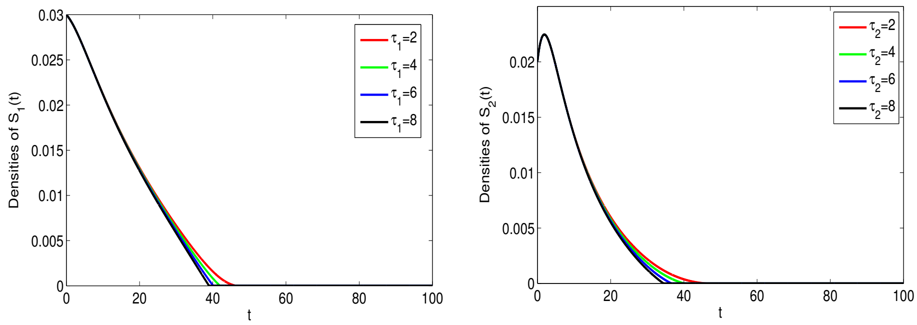
In the analysis of Theorem 7, based on (22), we find that the ratios of r_i to σ_i, $i = 1, 2$ have an important influence on the threshold condition of the time delay. From Figure 5, one can find that with the increase in the ratio, the time-delay threshold decreases, and they are inversely proportional.



(**a**) (**b**)

Figure 5. (**a**) The evolution between τ_{10} and $\frac{r_1}{\sigma_1}$. (**b**) The evolution between τ_{20} and $\frac{r_2}{\sigma_2}$.

However, the Hopf bifurcation exists at E_0 when the time delay is greater than the threshold in theoretical. In Figure 4, it is easy to find that there exist $S_1(t) < 0$ and $S_2(t) < 0$. This is inconsistent with the reality. In real life, $S_i(t) \geq 0$ for $i = 1, 2$. Therefore, the Hopf bifurcation will not exist. We redrew the dynamic behavior of $S_i(t)$ in combination with $S_i(t) \geq 0$ for $i = 1, 2$. It can be seen from Figure 6 that the time delay will inhibit the spread of the rumor.

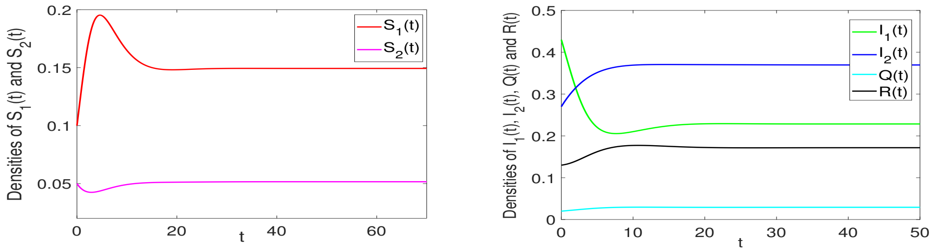


(**a**) (**b**)

Figure 6. (**a**) The trajectories of $S_1(t)$ with different time delays. (**b**) The trajectories of $S_2(t)$ with different time delays.

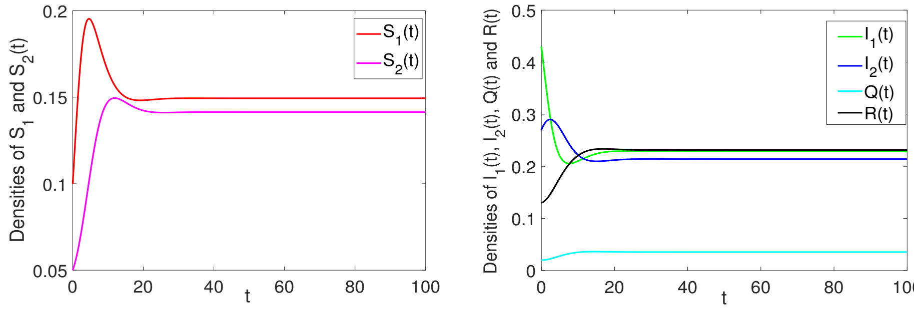
5.2. The Stability of Rumor Equilibrium

Example 2. *Consider system (1) with parameters in Set 2 of Table 1. It can be obtained that* $\Re_{01} = 2.14 > 1$ *and* $\Re_{02} = 0.97 < 1$, $\alpha_2\sigma_2\langle k\rangle = 0.26 > \mu_2 + d = 0.22$, *then system (1) has a unique rumor equilibrium* E_1^*. *As shown in Figure 7, rumor equilibrium* E_1^* *is locally asymptotically stable. By calculating the parameters in Set 3 of Table 1, the results of* $\Re_{01} = 2.14 > 1$ *and* $\Re_{02} = 1.9 > 1$, $\alpha_2\sigma_2\langle k\rangle = 0.26 > \mu_2 + d = 0.22$ *are gained. Figure 8 shows that the system (1) achieves local asymptotic stability. Similarly, Figure 9 shows the locally asymptotically stability of the equilibrium point* E_1^* *drawn with the parameters in Set 4 of Table 1. By comparing with the results, we find that when* $\Re_{01} > 1$, *the value of* S_2 *at the equilibrium point increases with the increase in* $\Re_{02}$. *Figure 10 describes the asymptotical stability of the equilibrium* E_1^*. *These simulation results show that when* $\Re_{01} > 1$, *the rumor in the network will continue to prevail in both languages.*



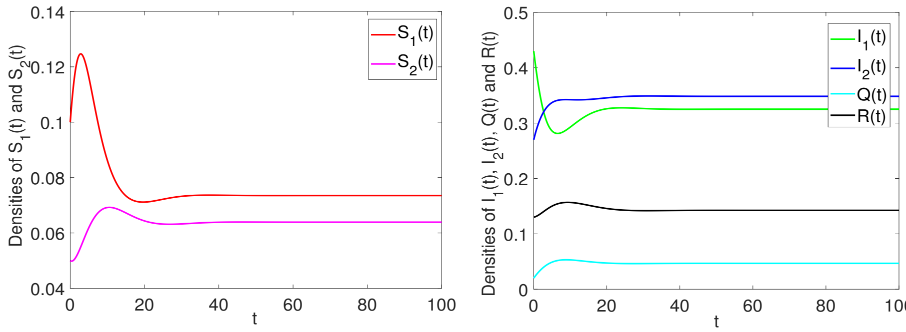
(**a**) (**b**)

Figure 7. The dynamics of system (1) with $\Re_{01} = 2.14 > 1$ and $\Re_{02} = 0.97 < 1$. (**a**) The trajectories of $S_1(t)$ and $S_2(t)$. (**b**) The trajectories of $I_1(t)$, $I_2(t)$, $Q(t)$ and $R(t)$.



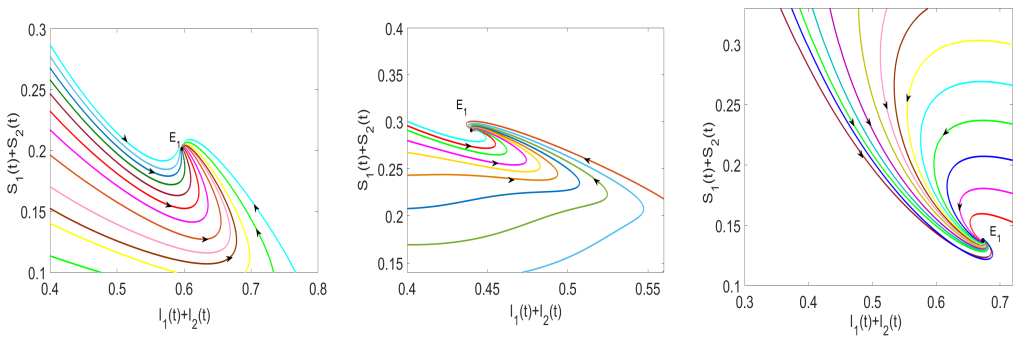
(**a**) (**b**)

Figure 8. The dynamics of system (1) with $\Re_{01} = 2.14 > 1$ and $\Re_{02} = 1.9 > 1$. (**a**) The trajectories of $S_1(t)$ and $S_2(t)$. (**b**) The trajectories of $I_1(t)$, $I_2(t)$, $Q(t)$ and $R(t)$.



(**a**) (**b**)

Figure 9. The dynamics of system (1) with $\Re_{01} = 1.48 > 1$ and $\Re_{02} = 1$. (**a**) The trajectories of $S_1(t)$ and $S_2(t)$. (**b**) The trajectories of $I_1(t)$, $I_2(t)$, $Q(t)$ and $R(t)$.



(**a**) (**b**) (**c**)

Figure 10. The phase diagram of $S^1(t) + S^2(t)$ and $I^1(t) + I^2(t)$. (**a**) $\Re_{01} = 2.14 > 1$ and $\Re_{02} = 0.97 < 1$. (**b**) $\Re_{01} = 2.14 > 1$ and $\Re_{02} = 1.9 > 1$. (**c**) $\Re_{01} = 1.48 > 1$ and $\Re_{02} = 1$.

Furthermore, Figure 11 shows the states of $S_1(t)$ and $S_2(t)$ with parameters in Set 5 and Set 6 of Table 1, respectively. One can find that when $\Re_{01} < 1$, the rumor will only continue to spread in the official language.

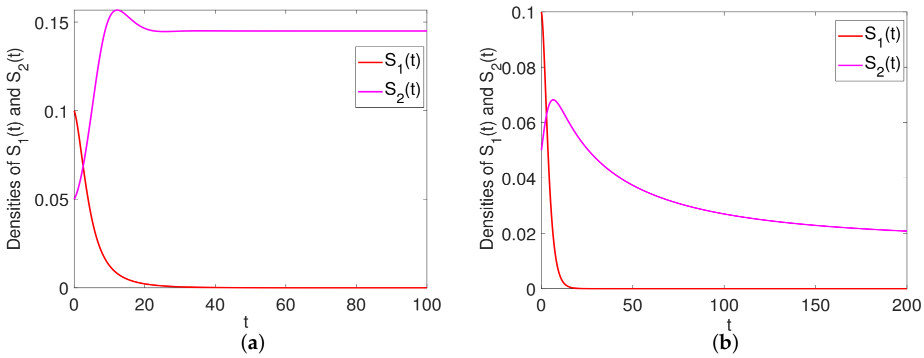


Figure 11. The dynamics of $S_1(t)$ and $S_2(t)$. (**a**) $\Re_{01} = 0.95 < 1$ and $\Re_{02} = 1.5$. (**b**) $\Re_{01} = 0.28 < 1$ and $\Re_{02} = 1$.

5.3. The Hopf Bifurcation of Rumor Equilibrium

Example 3. *In order to verify the influence of time delays on rumor spreading, we select the parameters in Set 7 of Table 1 in model (15). By calculating, we can obtian that $(\mathcal{H}_1)$ and $(\mathcal{K}_1)$ are satisfied. In particular, if $\tau_1 = \tau_2 = \tau$, it has $\min\{\tau_{10}^0, \tau_{20}^0\} = 11.2549$. As shown in Figure 12, $S_1(t)$ and $S_2(t)$ are locally asymptotically stable if $\tau < 11.2549$, and Figure 13 shows that $S_1(t)$ and $S_2(t)$ are unstable when $\tau = 11.264$. Figure 14 describes the phase diagram of the equilibrium E_1^*. Therefore, it can be seen that the equilibrium E_1^* becomes unstable as the time delay increasing. When the time delay is greater than the threshold, the stability of the equilibrium will disappear. This means that the rumor disseminator will exist continuously, and the density of the disseminator will fluctuate within a certain range over time.*

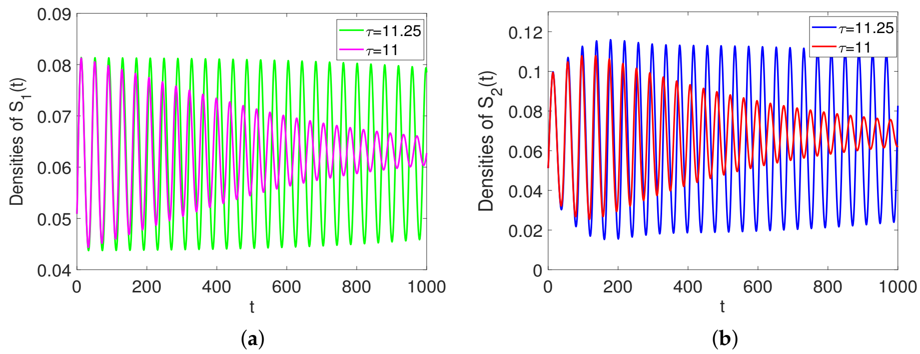


Figure 12. (**a**) The trajectories of $S_1(t)$ with $\tau = 11$ and $\tau = 11.25$. (**b**) The trajectories of $S_2(t)$ with $\tau = 11$ and $\tau = 11.25$.

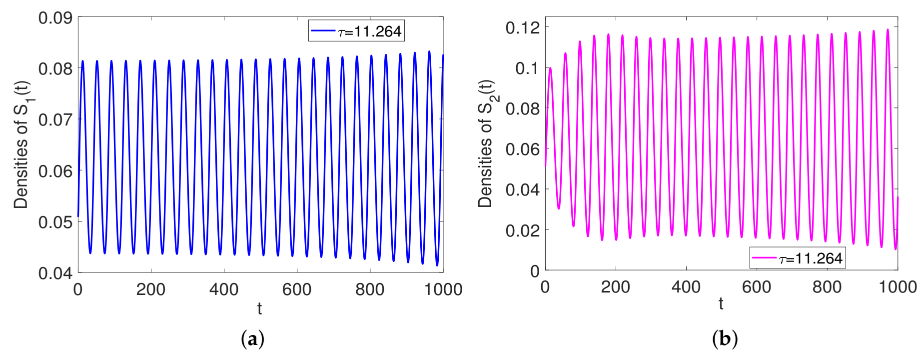
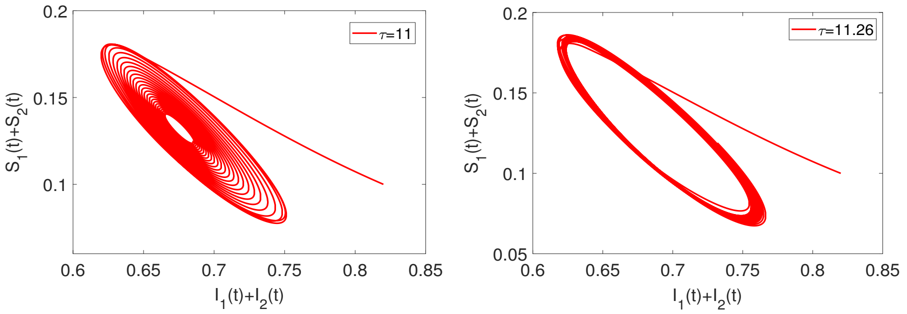


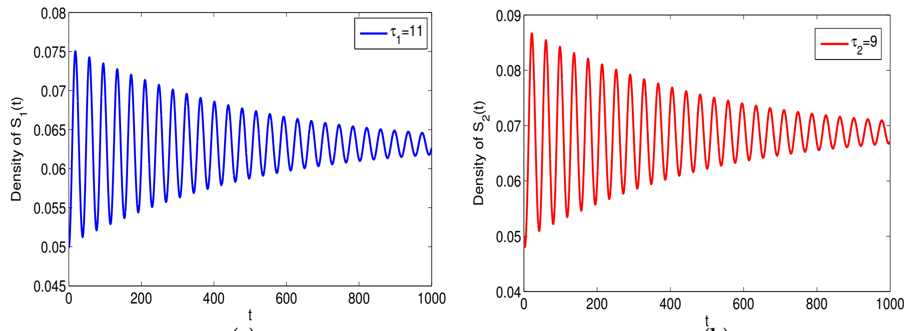
Figure 13. (**a**) The trajectory of $S_1(t)$ with $\tau = 11.264$. (**b**) The trajectory of $S_2(t)$ with $\tau = 11.264$.



(**a**) (**b**)

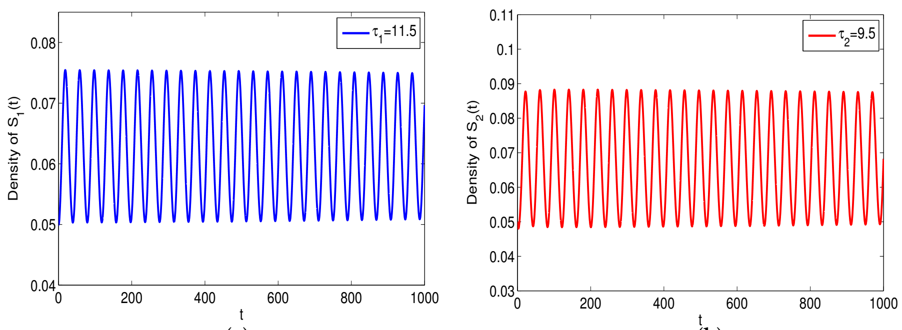
Figure 14. (**a**) The phase diagram of $S^1(t) + S^2(t)$ with $\tau = 11$. (**b**) The phase diagram of $S^1(t) + S^2(t)$ with $\tau = 11.26$.

In model (15), when $\tau_1 \neq \tau_2 > 0$, the parameters are selected as the ones in Set 7 of Table 1. Through verification, $(\mathcal{H}_1)$ and $(\mathcal{K}_1)$ are satisfied in Theorem 8, and $\tau^0_{\mathcal{H}_{10}} = 11.5$, $\tau^0_{\mathcal{K}_{10}} = 9.5$. Figures 15–17 show the densities of $S_1(t)$ and $S_2(t)$ with different time delays. We observe that the system is stable at the rumor equilibrium if the time delays are less than their threshold values in Figure 15. However, the system undergoes Hopf bifurcation when the time delays are equal to their threshold values. This situation can be seen in Figure 16. In addition, the system is unstable at the rumor equilibrium if the time delays are great than their threshold values, shown in Figure 17.



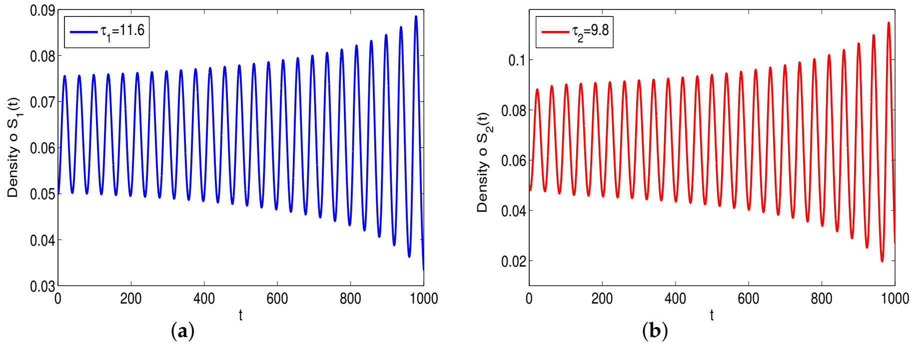
(**a**) (**b**)

Figure 15. (**a**) The trajectory of $S_1(t)$ with $\tau_1 = 11$. (**b**) The trajectory of $S_2(t)$ with $\tau_2 = 9$.



(**a**) (**b**)

Figure 16. (**a**) he trajectory of $S_1(t)$ with $\tau_1 = 11.5$. (**b**) The trajectory of $S_2(t)$ with $\tau_2 = 9.5$.



(**a**) (**b**)

Figure 17. (**a**) The trajectory of $S_1(t)$ with $\tau_1 = 11.6$. (**b**) The trajectory of $S_2(t)$ with $\tau_2 = 9.8$.

5.4. Feasibility of Optimal Control

Example 4. *For optimal control system (15), the parameters are shown in Set 3 of Table 1. In the objective function (16), the parameters are $u_1 = 3$, $u_2 = 2$, $u_3 = 5$ and $u_4 = 1$, respectively. In order to explore the influence of optimal control on the system (15), the trajectories of the system (15) with and without optimal control are simulated, as shown in Figure 18. Through observation, we find that the values of $S_i(t)(i = 1, 2)$ and $R(t)$ will decrease under the optimal control, and the densities of $I_i(t)(i = 1, 2)$ and $Q(t)$ will be more under optimal control.*

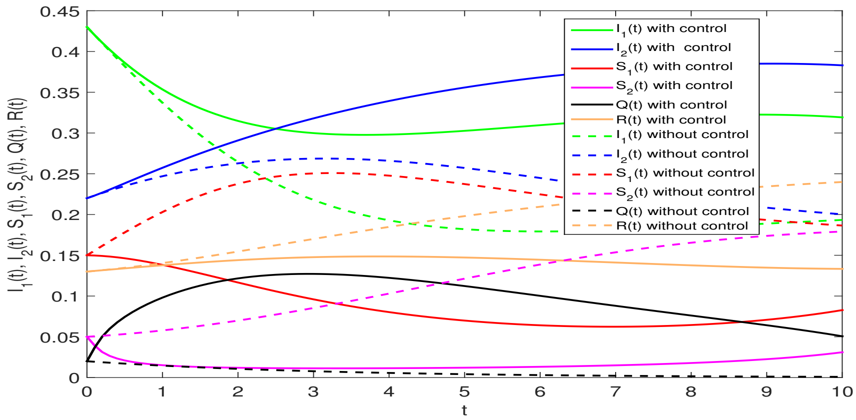
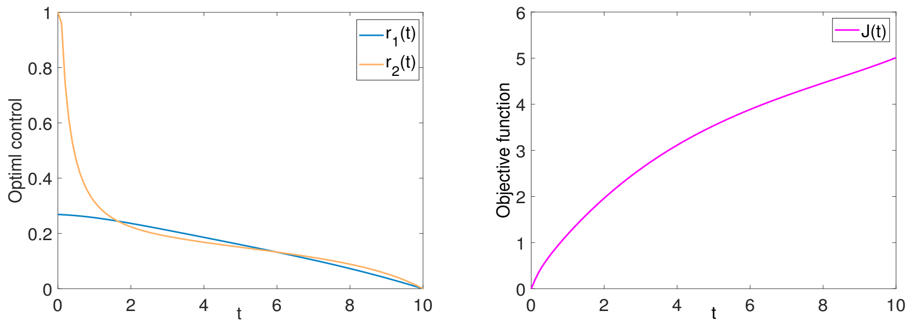


Figure 18. Densities of individuals with and without optimal control.

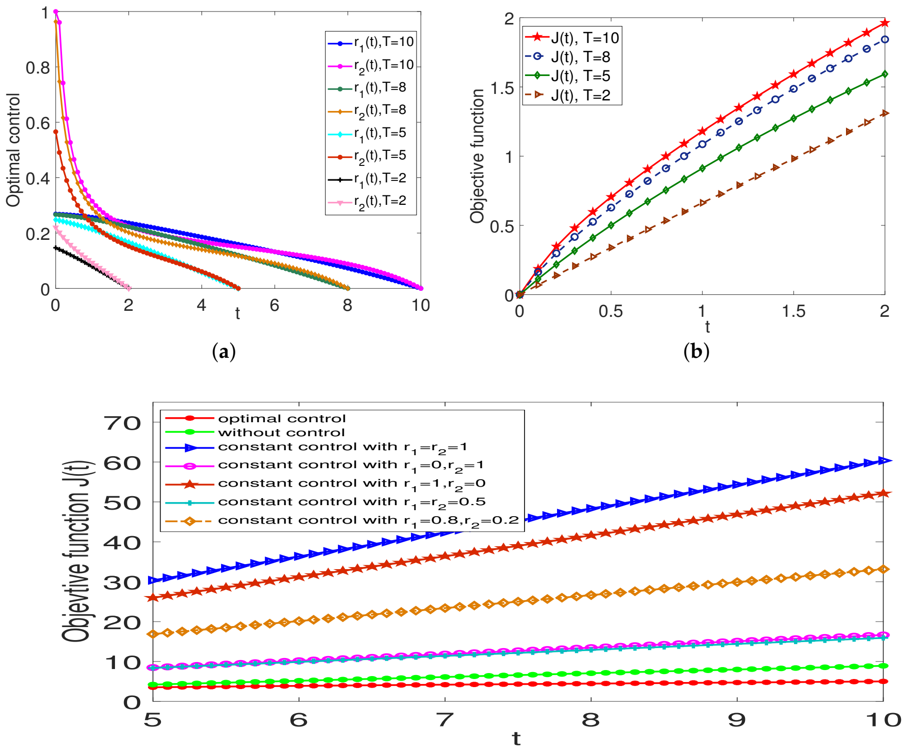
In (16), we choose $T = 10$, then the control strength $r_1(t)$ and $r_2(t)$ are shown in Figure 19a. It can be observed that the control $r_1(t)$ and $r_2(t)$ decreases with time and gradually tends to 0 at $t = 10$. Figure 19b shows the performance of the objective function. It can be seen that $J(t)$ is monotone and nondecreasing. Obviously, in a certain period of time, the control intensity of the system gradually decreases to 0, while the control consumption gradually increases to the maximum.



(**a**) (**b**)

Figure 19. (**a**) The trajectories of $r_i(t)$ $i = 1, 2$. (**b**) The trajectory of control cost $J(t)$.

In order to facilitate the control, the control time in the optimal control will be given in advance. In the specific simulation, we give the trajectories of $r_1(t)$ and $r_2(t)$ when $T = 2$, $T = 5$, $T = 8$ and $T = 10$, as shown in Figure 20a. Figure 20b shows the consumption at $T = 2$, $T = 5$, $T = 8$ and $T = 10$, respectively. Through observation, we find that the larger the given time T, the greater the consumption of $J(t)$. Figure 20c shows the trajectories of $J(t)$ under different control forces. We found that among all these controls, $r_1 = 1$ and $r_2 = 1$ have the largest consumption, while the consumption of optimal control is minimal. That means that the optimal control proposed in this paper not only can control the rumor within the specified time, but also consumes the least resources.



(**c**)

Figure 20. (**a**) The trajectories of optimal control $r_i(t)$ $i = 1, 2$. (**b**) The objective function. $J(t)$ with $T = 2, 5, 8, 10$. (**c**) The consumption of $J(t)$ under different $r_i(t)$, $i = 1, 2$.

6. Conclusions

In this paper, we studied the modeling, dynamic analysis and control of rumor propagation with quarantine control on multilingual OSNs. Firstly, we proposed a 2I2SQR rumor-spreading model without considering propagation delay. We gave the rumor-free equilibrium and calculated the basic reproduction number by using the next generation matrix method. Based on backward bifurcation theory, we discussed the existence of a rumor equilibrium, which is somewhat more complicated. It was found that the number of rumor equilibria is different under different parameters, including no rumor equilibrium, only one rumor equilibrium and the case with two rumor equilibria. By using the Jacobian matrix method and a differential inequality lemma, the local and global asymptotic stabilities of rumor free-equilibrium were proved. The local stability and bifurcation of the rumor equilibrium were discussed, and some related theorems were given. Secondly, in order to suppress the spreading of rumor with the lowest cost in an expected period, a continuous optimal control measure was proposed to curb the propagation of rumor. According to Pontryagin's maximum principle, an optimal isolation strength was obtained. Moreover, considering the time delays from rumor spreading to isolation control, we proposed a 2I2SQR rumor propagation model with time delays on multilingual OSNs. The local stability of rumor-free and rumor equilibria was investigated. It was found that, different from the model without delay, both the rumor-free equilibrium and the rumor equilibrium are unstable and have Hopf bifurcations when the propagation delay is greater than some certain threshold conditions. Finally, the theory results were verified by some numerical simulations. In this research, the uniform network structure was considered in the model, and a continuous control strategy was proposed. In our future work, we will explore a more general heterogeneous network structure and propose some new control strategies to suppress the spread of rumors.

Author Contributions: Conceptualization, S.Y. and Z.Y.; methodology, S.Y.; software, Z.Y.; validation, S.Y., Z.Y. and H.J.; formal analysis, S.Y.; investigation, S.Y.; resources, Z.Y.; data curation, S.Y.; writing—original draft preparation, S.Y.; writing—review and editing, Z.Y.; visualization, S.Y.; supervision, H.J.; project administration, H.J.; funding acquisition, Z.Y. and H.J. All authors have read and agreed to the published version of the manuscript.

Funding: This work was supported in part by the National Natural Science Foundation of China (Grant No. U1703262), in part by the Special Project for Local Science and Technology Development Guided by the Central Government (Grant No. ZYYD2022A05), and in part by Xinjiang Key Laboratory of Applied Mathematics (Grant No. XJDX1401).

Data Availability Statement: Not applicable.

Conflicts of Interest: The authors declare no conflict of interest.

Abbreviations

The following abbreviations are used in this manuscript:

OSNs	online social networks
DK	Daley and Kendall
MT	Maki and Thompson
2I2SQR model	the model with 2 ignorants, 2 spreaders, quarantined and recovered users

References

1. Doer, B.; Fouz, M.; Friedrich, T. Why rumors spread so quickly in social networks. *Commun. ACM* **2012**, *55*, 70–75. [CrossRef]
2. Centola, D. The spread of behavior in an online social network experiment. *Science* **2010**, *329*, 1194–1197. [CrossRef] [PubMed]
3. Haeupler, B. Simple, fast and deterministic gossip and rumor spreading. *J. ACM* **2015**, *62*, 47. [CrossRef]
4. Chierichetti, F.; Lattanzi, S.; Panconesi, A. Rumor spreading in social networks. *Theor. Comput. Sci.* **2011**, *412*, 2601–2610. [CrossRef]
5. Zhang, Z.; Zhang, Z. An interplay model for rumor spreading and emergency development. *Phys. A* **2009**, *388*, 4159–4166. [CrossRef]
6. Thomas, S. Lies, damn lies, and rumors: An analysis of collective efficacy, rumors, and fear in the wake of Katrina. *Sociol. Spectr.* **2007**, *27*, 679–703. [CrossRef]
7. Daley, D.; Kendall, D. Epidemics and rumours. *Nature* **1964**, *204*, 1118. [CrossRef]
8. Daley, D.; Kendall, D. Stochastic rumours. *IMA J. Appl. Math.* **1965**, *1*, 42–55. [CrossRef]
9. Maki-Thompson, D.P. *Mathematical Models and Applications, with Emphasis on Social, Life, and Management Sciences*; Pearson College Div: Englewood Cliffs, NJ, USA, 1973; pp. 10–12.
10. Zanette, D. Criticality behavior of propagation on small-world networks. *Phys. Rev. E* **2001**, *64*, 050901. [CrossRef]
11. Zanette, D. Dynamics of rumor propagation on small-world networks. *Phys. Rev. E* **2002**, *65*, 041908. [CrossRef] [PubMed]
12. Xing, Q.; Zhang, Y.; Liang, Z.; Zhang, F. Dynamics of organizational rumor communication on connecting multi-small-world networks. *Chin. Phys. B* **2011**, *20*, 120204. [CrossRef]
13. Mehrabian, A.; Pourmiri, A. Randomized rumor spreading in poorly connected small-world networks. *Random Struct. Algorithms* **2014**, *8784*, 346–360.
14. Moreno, Y.; Nekovee, M.; Pacheco, A. Dynamic of rumor spreading im complex networks. *Phys. Rev. E* **2004**, *69*, 066130. [CrossRef] [PubMed]
15. Isham, V.; Harden, S.; Nekovee, M. Stchastic epidemics and rumors on finite random network. *Phys. A* **2010**, *389*, 561–576. [CrossRef]
16. Nekovee, M.; Moreno, Y.; Bianconi, G.; Marsili, M. Theory of rumor spreading in complex social networks. *Phys. A* **2007**, *374*, 457–470. [CrossRef]
17. Moreno, Y.; Nekovee, M.; Vespignani, A. Efficiency and reliability of epidemic data dissemination in complex networks. *Phys. Rev. E* **2004**, *69*, 055101. [CrossRef]
18. Trpevski, D.; Tang, W.; Kocarev, L. Model for rumor spreading over network. *Phys. Rev. E* **2010**, *81*, 909–912. [CrossRef]

19. Jia, P.; Wang, C.; Zhang, G.; Ma, J. A rumor spreading model based on two propagation channels in social networks. *Phys. A* **2019**, *524*, 342–353. [CrossRef]
20. Zhang, Y.; Zhu, J. Dynamic behavior of an I2S2R rumor propagation model on weighted contract networks. *Phys. A* **2019**, *536*, 120981. [CrossRef]
21. Han, Q.; Wen, H.; Miao, F. Rumor spreading in interdependent social networks. *Peer-to-Peer Netw. Appl.* **2018**, *11*, 955–965. [CrossRef]
22. Xia, L.; Jiang, G.; Song, B.; Yu, R. Rumor spreading model considering hesitating mechanism in conplex social networks. *Phys. A* **2015**, *437*, 295–303. [CrossRef]
23. Zhao, L.; Xie, W.; Gao, H.; Wang, X.; Zhang, S. A rumor spreading model with variable forgetting rate. *Phys. A* **2013**, *392*, 6146–6154. [CrossRef]
24. Schneider, B. The reference model simpan agent-based modeling of human behavior in panic situations. In Proceedings of the Tenth International Conference on Computer Modeling and Simulation, Cambridge, UK, 1–3 April 2008; pp. 599–604.
25. Jia, J.; Wu, W. A rumor transmission model with incubation in social networks. *Phys. A* **2018**, *491*, 453–462. [CrossRef]
26. Liu, Q.; Li, T.; Sun, M. The analysis of an SEIR rumor propagation model on heterogeneous network. *Phys. A* **2017**, *469*, 372–380. [CrossRef]
27. Zhao, L.; Cui, H.; Qiu, X.; Wang, X.; Wang, J. SIR rumor spreading model in the new media age. *Phys. A* **2013**, *392*, 995–1003. [CrossRef]
28. Huo, L.; Wang, L.; Song, G. Global stability of a two-mediums rumor spreading model with media coverage. *Phys. A* **2017**, *482*, 757–781. [CrossRef]
29. Zhang, N.; Huang, H.; Su, B.; Zhao, J.; Zhang, B. Dynamic 8-state ICSAR rumor propagation model considering official rumor refutation. *Phys. A* **2014**, *415*, 333–346. [CrossRef]
30. Chen, J.; Song, Q.; Zhou, Z. Agent-based simulation of rumor propagation on social network based on active immune mechanism. *J. Syst. Sci. Inf.* **2017**, *5*, 571–584. [CrossRef]
31. Afassinou, K. Analysis of the impact of education rate on the rumor spreading mechanism. *Phys. A* **2014**, *414*, 43–52. [CrossRef]
32. Jain, A.; Dhar, J.; Gupta, V. Rumor model on homogeneous social network incorporating delay in expert intervention and government action. *Commun. Nonlinear Sci. Numer. Simul.* **2020**, *84*, 105189. [CrossRef]
33. Ding, L.; Hu, P.; Guan, Z.; Li, T. An efficient hybrid control strategy for restraining rumor spreading. *IEEE Trans. Syst. Man Cybern. Syst.* **2021**, *51*, 6779–6791. [CrossRef]
34. Zhu, L.; Wang, B. Stability analysis of a SAIR rumor spreading model with control strategies in onlinear social network. *Inf. Sci.* **2020**, *526*, 1–19. [CrossRef]
35. Zhu, L.; Liu, W.; Zhang, Z. Delay differential equations modeling of rumor propagation in both homogeneous and heterogeneous networks with a forced silence function. *Appl. Math. Comput.* **2020**, *370*, 124925. [CrossRef]
36. Li, J.; Jiang, H.; Yu, Z.; Hu, C. Dynamical analysis of rumor spreading model in homogeneous complex networks. *Appl. Math. Comput.* **2019**, *359*, 374–385. [CrossRef]
37. Wang, J.; Jiang, H.; Ma, T.; Hu, C. Global dynamics of the multi-lingual SIR rumor spreading model with cross-transmitted mechanism. *Chaos Solitons Fractals* **2019**, *126*, 148–157. [CrossRef]
38. Li, J.; Jiang, H.; Mei, X.; Hu, C.; Zhang, G. Dynamical analysis of rumor spreading model in multi-lingual environment and heterogeneous complex networks. *Inf. Sci.* **2020**, *536*, 391–408. [CrossRef]
39. Yu, S.; Yu, Z.; Jiang, H.; Mei, X.; Li, J. The spread and control of rumors in a multilingual environment. *Nonlinear Dyn.* **2020**, *100*, 2933–2951. [CrossRef]
40. Chen, F. On a nonlinear nonautonomous predator-prey model with diffusion and distributed delay. *J. Comput. Appl. Math.* **2005**, *180*, 33–49. [CrossRef]
41. Grass, D.; Vienna, A.; Caulkins, J.; Feichtinger, G.; Tragler, G. *Optimal Control of Nonlinear Processes*; Springer: Berlin, Germany, 2008.