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Some Inequalities of Hardy Type Related to Witten–Laplace Operator on Smooth Metric Measure Spaces

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Abstract: A complete Riemannian manifold equipped with some potential function and an invariant conformal measure is referred to as a complete smooth metric measure space. This paper generalizes some integral inequalities of the Hardy type to the setting of a complete non-compact smooth metric measure space without any geometric constraint on the potential function. The adopted approach highlights some criteria for a smooth metric measure space to admit Hardy inequalities related to Witten and Witten p-Laplace operators. The results in this paper complement in several aspect to those obtained recently in the non-compact setting.

Keywords: Riemannian manifold; Hardy inequality; uncertainty principle; elliptic operators; Rellich inequality

MSC: 22E30; 26D10; 46E30; 53C21; 58J05

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1. Introduction

A classical inequality due to Hardy [1] states that for any $f \in C_0^\infty(\mathbb{R}^m)$

$$\left(\frac{m-p}{p}\right)^p \int_{\mathbb{R}^m} \frac{|f(x)|^p}{|x|^p} dx \leq \int_{\mathbb{R}^m} |\nabla f(x)|^p dx, \quad (1)$$

where $p \in (1, m)$ and $((m-p)/p)^p$ is the best constant but never achieved. This class of inequalities has found numerous applications in geometric analysis, spectral theory, partial differential equations and so on. Owing to this, it has undergone several refinements, generalization and extensions in recent years. For various proofs and different applications, see [2–5], and for several extensions and generalizations, see [6–13] and the references cited therein.

A particular application of the above Hardy inequality (1) to singular problems was considered by García Azorero and Peral Alonso [4] in their study of the behavior of the solution to the following nonlinear critical p-heat equation:

$$\begin{cases} w_t = \Delta_p w + \frac{\Lambda}{|x|^p} w^{p-1}, & x \in \Omega,\ t > 0, \\ w(x,0) = h(x) \geq 0, & x \in \Omega, \\ w(x,t) = 0, & x \in \Omega,\ t > 0, \end{cases} \quad (2)$$

where Δ_p is the usual p-Laplacian, Ω is a bounded domain in $\mathbb{R}^m$ such that $0 \in \Omega$ and $p \in (0, m)$. Their work gave a relationship between Λ of (2) and the best constant $((m-p)/p)^p$ in the Hardy inequality (1). More precisely, suppose $\Lambda > 0$, then the above system (2) has solutions for $\Lambda \leq \Lambda_{m,p} = ((m-p)/p)^p$ and $p > 2$, while the solutions have finite time of extinction for $1 < p < 2$. There will be an instantaneous blow up for $\Lambda > \Lambda_{m,p}$ and $p > 2$. Thus, the best constant $\Lambda_{m,p}$ is the cut-off point for the nonexistence of positive solutions.

This problem was also studied by Aguilar Crespo and Peral Alonso [14] and Goldstein and Kombe [15].

Recently, researchers have been more interested in studying Hardy inequalities in curved spaces because of their potential applications in this setting too. In particular, the first extension to the Riemannian setting was due to Carron [16], where the inequalities are considered in the L^2 setting for noncompact manifold M with certain geometric conditions on the weight function. Carron's form of Hardy inequalities reads as

$$\left(\frac{C+\alpha-1}{2}\right)^2 \int_M \frac{|f|^2}{\rho^{2-\alpha}} dv \leq \int_M \rho^\alpha |\nabla f|^2 dv$$

for all $f \in C_0^\infty(M \backslash \rho^{-1}\{0\})$, $\alpha \in \mathbb{R}$, $C+\alpha > 1$, while the positive weight function ρ satisfying $|\nabla\rho| = 1$, $\Delta\rho \geq \frac{C}{\rho}$. Later, Kombe–Özaydin [17] (see also Kombe–Özaydin [18] and Kombe–Yener [19]) extended Carron's result to the general case $1 < p < \infty$. In these references, the authors proved several Hardy-type, Rellich-type and uncertainty principle inequalities on manifolds satisfying certain geometric restrictions. Yang, Su and Kong [20] applied the above ideas to obtain the following inequality of the Hardy type on a noncompact manifold of the Riemannian metric with a negative sectional curvature for $m \geq 3$, $1 < p < m-\alpha$, $\alpha \in \mathbb{R}$:

$$c_{m,\alpha,p} \int_M \frac{|f|^p}{\rho^{\alpha+p}} dv \leq \int_M \frac{|\nabla f|^p}{\rho^\alpha} dv,$$

where $f \in C_0^\infty(M)$, $|\nabla\rho| = 1$, $\Delta\rho \geq \frac{m-1}{\rho}$ and the constant $c_{m,\alpha,p} = ((m-p-\alpha)/p)^p$ is sharp. The integral inequalities described above have led to the derivation of Hardy-type, Rellich-type and uncertainty principle inequalities in various settings, such as the Poincaré model, Heisenberg groups, Cartan–Hadamard manifolds and so on [18,21,22]. Recently, this class of inequalities was studied in a much more general setting [23,24] of metric measure spaces, which may not possess a differentiable structure. More interestingly, a pair of weight functions was introduced in [24], where the weights have to satisfy certain compatibility conditions for the inequalities to hold true. In a related development, frank efforts from different authors (e.g., [25–27]) yielded various Hardy-type inequalities on Finslerian manifolds.

In [3] (see also [10]), the author started a program of obtaining inequalities of the Hardy type associated to some quasi-linear elliptic differential operator of the form $L_p f := \mathrm{div}_L(|\nabla_L f|^{p-2}\nabla_L f)$, where $p \in (1,\infty)$, ∇_L is a general vector field and $\mathrm{div} = -\nabla_L^*$, negative adjoint of ∇_L. In this program, which was built on the technique used in [11], several specific examples of L_p are considered, such as Heisenberg–Greiner operators, Grushin-type operators and sub-Laplacian on Carnot groups with respect to certain geometric conditions. In general, the main achievement of this program can be roughly stated as follows: for any positive weight function $\rho : \Omega \to \mathbb{R}$, $p > 1$ and $f \in C_0^1(\Omega)$, it holds that

$$c \int_\Omega \frac{|\nabla_L \rho|^p}{\rho^p} |f|^p dx \leq \int_\Omega |\nabla_L f|^p dx,$$

where $-L_p\rho \geq 0$ in the sense of distribution. This result generalizes some Euclidean cases, where ρ is just a distance function [11,28]. More interestingly, this program yields the best constant in several special cases [3,10]. The condition $-L_p\rho \geq 0$ helps in determining that the best constant is never attained. The knowledge about regularity of solutions to PDE involving the second-order elliptic operator helps to derive inequalities of the Hardy typ with sharp constants [4,28]. In fact, the author in [28] generalized this result to sub-Laplacian on the Heisenberg group, which is hypoelliptic, and p-Laplacian on compact manifolds.

There are two natural questions waiting to be addressed:

(i). Is it possible to generalize those Hardy-type inequalities described above to the setting of complete smooth metric measure spaces with or without any constraint on the potential function? (ii). How can we find these inequalities on a complete smooth metric measure space endowed with a non-negative Bakry–Émery Ricci tensor?

These questions are natural and there are evidence that they are not trivial. For examples, Myers' compactness, Bishop–Gromov's volume comparison and Cheeger–Gromoll's splitting theorems of Riemannian manifolds do not hold trivially for the Bakry–Émery Ricci tensor bounded from below [29]. The author [30] gave a description of the sharp uncertainty principle on complete Riemannian manifolds of non-negative Ricci curvature and proved that such a manifold should be isometric to Euclidean space.

The main goal of this paper on one hand is to answer the first question in affirmative by performing analysis on some differential equations involving Witten–Laplacian and Witten p-Laplacian. This generalization is not trivial, as it involves a careful consideration of properties of the potential function and the choice of input data such that special conditions would not be required on the potential function.

It is convenient at this point to give brief descriptions of smooth metric measure spaces and Witten–Laplacian (see [31] for detail discussion and [32] for another application). Let M be a Riemannian manifold with Riemannian metric $g_{\mathbf{M}}$, a smooth potential (weight) function defined as $\phi : \mathbf{M} \to \mathbb{R}$ and the volume element induced by $g_{\mathbf{M}}$ as dv . A smooth metric measure space is defined by $M = (\mathbf{M}, g_{\mathbf{M}}, e^{-\phi}dv)$, where the measure $e^{-\phi}dv$ is a conformal transformation of the measure induced by metric $g_{\mathbf{M}}$. Associated with M are a symmetric self-adjoint elliptic differential operator, called Witten–Laplacian, defined by

$$L := \Delta_{\mathbf{M}} - \langle \nabla\phi, \nabla\cdot\rangle_{\mathbf{M}}$$

and the popularly called Bakry–Émery Ricci tensor, Ric_M^{ϕ}, defined by

$$Ric_M^{\phi} = Ric_{\mathbf{M}} + \text{Hess}\phi,$$

where $\Delta_{\mathbf{M}}$, ∇, $\langle\cdot,\cdot\rangle_{\mathbf{M}}$, $Ric_{\mathbf{M}}$ and $\text{Hess}\phi$ denote the Laplace–Beltrami operator, gradient, inner product on $(\mathbf{M}, g_{\mathbf{M}})$, Ricci tensor of $\mathbf{M}$ and Hessian of function ϕ, respectively. The symmetric property and self-adjointness of L yield the integration by parts formula

$$\int_M (Lf)he^{-\phi}dv = -\int_M \langle\nabla f, \nabla h\rangle_{\mathbf{M}} e^{-\phi}dv = \int_M f(Lh)e^{-\phi}dv$$

for all $f, h \in C_0^{\infty}(M)$. The subscript $\mathbf{M}$ will be dropped in the sequel and henceforth, $\Delta_{\mathbf{M}}$, $\langle\cdot,\cdot\rangle_{\mathbf{M}}$, etc., would respectively be written as Δ, $\langle\cdot,\cdot\rangle$, etc. Given a function $u \in C^2(M)$ and $p \in [1, \infty)$, Witten p-Laplacian is defined by

$$L_p u := \text{div}_{\phi}(e^{-\phi}|\nabla u|^{p-2}\nabla u) = \Delta_p u - |\nabla u|^{p-2}\langle\nabla\phi, \nabla u\rangle,$$

where $\text{div}_{\phi} := e^{\phi}\text{div}$, div is the divergence operator, the adjoint of gradient for the L^2-norm induced by the metric on the space of differential forms (div_{ϕ} will be called weighted divergence operator) and Δ_p is the so-called p-Laplacian, a degenerate quasilinear elliptic operator. Note that L_p coincides with the Witten–Laplacian when $p = 2$ and the usual p-Laplacian when ϕ is a constant.

In the references cited above, inequalities of the Hardy type have been specialized to the setting of hyperbolic manifolds and Cartan–Hadamard manifolds with several examples. There are few papers in the literature devoted to deriving these inequalities on non-negatively Ricci curved spaces and smooth metric measure spaces, even though there have been considerable efforts in extending them to the Riemannian setting. In [33], the authors showed that a metric measure space satisfying an inequality of the Hardy type and volume doubling condition has exactly m-dimensional ($m \geq 3$) volume growth. Application of this is that a complete non-compact M with a non-negative Bakry–Émery tensor satisfying Hardy-type inequalities with the best constant is not far to the Euclidean space of the same dimension. This agrees with [34] where the same results were established for a complete non-compact Riemannian manifold with a non-negative Ricci tensor. Recently, there was another attempt made by the authors in [35], where they extended some inequalities derived in [5] to the setting of smooth metric measure spaces. While in reprint [36], the authors adopted this approach on a closed (compact without boundary) pointed space with a non-negative Bakry–Émery Ricci tensor.Recently, many interesting papers have been written related to symmetry, molecular cluster geometry analysis, submanifold theory, singularity theory, eigen problems, etc. [37–39]. In our following works, we are going to study inequalities of the Hardy type for different queries and further improve the results in this paper, combined with the technics and results in [37–39].

The remaining part of this paper is as follows. Hardy-type inequalities and its improvement related to the Witten–Laplacian are discussed in Section 2. Hardy–Poincaré inequality and the Heisenberg–Pauli–Weyl uncertainty principle are also discussed. In Section 3, some integral inequalities related to Witten p-Laplacian are discussed via D'Ambrosio's program.

2. Hardy-Type Inequalities Related to Witten–Laplacian

Let M be a smooth metric measure space of dimension $m > 1$ as described above. The first result in this section is the following.

Theorem 1. *Let a complete non-compact be M, and denote a non-negative function on M by ρ satisfying $|\nabla\rho| = 1$ and $L\rho \geq \frac{C}{\rho} + V$, where $C > 0$ is a constant and V is a continuous function, in the distribution sense. Then the following inequality*

$$\int_M \rho^{\alpha}|\nabla f|^p e^{-\phi} dv \geq (\mathcal{A}_{\alpha,p})^p \int_M \frac{|f|^p}{\rho^{p-\alpha}} e^{-\phi} dv + (\mathcal{A}_{\alpha,p})^{p-1} \int_M V \frac{|f|^p}{\rho^{p-\alpha-1}} e^{-\phi} dv \tag{3}$$

holds for any $f \in C_0^{\infty}(M \backslash \rho^{-1}\{0\})$, $p \in (1,\infty)$, $\mathcal{A}_{\alpha,p} = (C+1+\alpha-p)/p$ with $C+1+\alpha > p$.

Proof. Let $f = \rho^{\beta} g$, where $\beta < 0$ is to be chosen later and $g \in C_0^{\infty}(M)$. Then

$$|\nabla f|^p = |\nabla(\rho^{\beta} g)|^p = |\beta\rho^{\beta-1} g\nabla\rho + \rho^{\beta}\nabla g|^p.$$

Applying the following elementary inequality with $\zeta_1, \zeta_2 \in \mathbb{R}^m$

$$|\zeta_1 + \zeta_2|^p \geq |\zeta_1|^p + p|\zeta_1|^{p-2}\langle\zeta_1, \zeta_2\rangle$$

which is known to be valid for all $p \in (1,\infty)$ [6,18]. Then (with $|\nabla\rho| = 1$),

$$|\nabla f|^p \geq |\beta|^p \rho^{(\beta-1)p}|g|^p + p|\beta|^{p-2}\beta\rho^{(\beta-1)(p-1)+\beta}|g|^{p-2} g\langle\nabla\rho, \nabla g\rangle. \tag{4}$$

Multiplying (4) by ρ^{α} and then integrating by parts with respect to the measure $e^{-\phi}dv$ gives

$$\begin{aligned} \int_M \rho^{\alpha}|\nabla f|^p e^{-\phi} dv \geq |\beta|^p \int_M \rho^{\beta p-p+\alpha}|g|^p e^{-\phi} dv \\ -\frac{|\beta|^{p-2}\beta}{\beta p-p+\alpha+2} \int_M L(\rho^{\beta p-p+\alpha+2})|g|^p e^{-\phi} dv. \end{aligned} \tag{5}$$

By an elementary differentiation, note that we obtain

$$\begin{aligned} L(\rho^{\beta p-p+\alpha+2}) &= \Delta(\rho^{\beta p-p+\alpha+2}) - \langle\nabla(\rho^{\beta p-p+\alpha+2}), \nabla\phi\rangle \\ &= (\beta p-p+\alpha+2)[(\beta p-p+\alpha+1)\rho^{\beta p-p+\alpha} + \rho^{\beta p-p+\alpha+1} L\rho]. \end{aligned}$$

Therefore, (5) becomes the following inequality (by using the condition $L\rho \geq \frac{C}{\rho} + V$ with $\beta < 0$), then we have

$$\begin{aligned}
\int_M \rho^\alpha |\nabla f|^p e^{-\phi} dv \geq |\beta|^p \int_M \rho^{\beta p - p + \alpha} |g|^p e^{-\phi} dv \\
- |\beta|^{p-2}\beta(\beta p - p + \alpha + 1) \int_M \rho^{\beta p - p + \alpha} |g|^p e^{-\phi} dv \\
- |\beta|^{p-2}\beta \int_M \rho^{\beta p - p + \alpha + 1} L\rho |g|^p e^{-\phi} dv \\
\geq |\beta|^p \int_M \rho^{\beta p - p + \alpha} |g|^p e^{-\phi} dv \\
- |\beta|^{p-2}\beta(\beta p - p + \alpha + 1 + C) \int_M \rho^{\beta p - p + \alpha} |g|^p e^{-\phi} dv \\
- |\beta|^{p-2}\beta \int_M V\rho^{\beta p - p + \alpha + 1} |g|^p e^{-\phi} dv.
\end{aligned}$$

Now choosing $\beta = -(\alpha + C + 1 - p)/p$ and using $g = \rho^{-\beta} f$, we arrive at

$$\begin{aligned}
\int_M \rho^\alpha |\nabla f|^p e^{-\phi} dv \geq \Big(\frac{\alpha + C + 1 - p}{p}\Big)^p \int_M \rho^{\alpha - p} |f|^p e^{-\phi} dv \\
+ \Big(\frac{\alpha + C + 1 - p}{p}\Big)^{p-1} \int_M V\rho^{\alpha - p + 1} |f|^p e^{-\phi} dv,
\end{aligned} \tag{6}$$

which is the required inequality. □

Remark 1. *In the special case $V \equiv 0$, (6) becomes*

$$\int_M \rho^\alpha |\nabla f|^p e^{-\phi} dv \geq \mathcal{A}_{\alpha,p}^p \int_M \rho^{\alpha - p} |f|^p e^{-\phi} dv \tag{7}$$

with the same hypothesis as in Theorem 1. Inequality (7) was recently mentioned in [35] and proved for the case $L\rho \leq C/\rho$ and $p < C + \alpha - 1$. If ϕ was a constant (7) reduces to Kombe-Özaydin's result ([18] Theorem 2.1). In the integral inequalities (3) and (7), the left-hand side is allowed to diverge.

The next result improves the inequality of Hardy-type on a bounded domain with smooth boundary in M. It thereby generalizes a result in [17].

Theorem 2. *Let M be as defined above and $\Omega \subset M$ be a bounded domain with smooth boundary. Let ρ be a non-negative function on Ω and satisfying $L\rho \geq \frac{C}{\rho}, C > 0$ and $|\nabla\rho| = 1$ in the distribution sense. Then, there exists constant $\mathcal{B} = \mathcal{B}(m, p, |\Omega|) > 0$, where $|\Omega| = \int_M e^{-\phi} dv$ is the weighted volume of Ω, and the inequality*

$$\int_\Omega \rho^\alpha |\nabla f|^2 e^{-\phi} dv \geq (\mathcal{C}_\alpha)^2 \int_\Omega \frac{f^2}{\rho^{2-\alpha}} e^{-\phi} dv + \mathcal{B}\Big(\int_\Omega \rho^{\alpha p/2} |\nabla f|^p e^{-\phi} dv\Big)^{2/p}, \tag{8}$$

where $\mathcal{C}_\alpha := (C + \alpha - 1)/2$ holds for every compactly supported function $f \in C_0^\infty(\Omega)$, $\alpha \in \mathbb{R}$, $p \in (1, 2)$, and $C + \alpha - 1 > 0$.

Proof. Let $h = \rho^\beta$, where β is a negative number. It is straightforward to see that (Picone's identity [40])

$$|\nabla f|^2 - \Big\langle \nabla\Big(\frac{f^2}{h}\Big), \nabla h \Big\rangle = \Big|\nabla f - \frac{f}{h}\nabla h\Big|^2.$$

Then

$$\int_\Omega |\nabla f|^2 - \Big\langle \nabla\Big(\frac{f^2}{h}\Big), \nabla h\Big\rangle \rho^\alpha e^{-\phi} dv = \int_\Omega \rho^\alpha \Big|\nabla f - \frac{f}{h}\nabla h\Big|^2 e^{-\phi} dv$$
$$\geq \mathcal{B}_1 \Big(\int_\Omega \Big|\nabla f - \frac{f}{h}\nabla h\Big|^p \rho^{\alpha p/2} e^{-\phi} dv \Big)^{2/p} \tag{9}$$

by the application of Jensen's inequality with $\mathcal{B}_1 = |\Omega|^{1-2/p}$ (see ([5] Equation (2.17)). Obviously with the condition $L\rho \geq \frac{C}{\rho}$ and $|\nabla \rho| = 1$, one has

$$Lh = \Delta\rho^\beta - \langle \nabla\rho^\beta, \nabla\phi\rangle \leq \beta(\beta - 1 + C)\rho^{\beta-2}. \tag{10}$$

Hence, using integration by parts, (10) and the function $h = \rho^\beta$, we obtain

$$\begin{aligned}
&\int_\Omega |\nabla f|^2 - \Big\langle \nabla\Big(\frac{f^2}{h}\Big), \nabla h\Big\rangle \rho^\alpha e^{-\phi} dv \\
&= \int_\Omega |\nabla f|^2 \rho^\alpha e^{-\phi} dv - \int_\Omega \nabla\Big(\frac{f^2}{h}\Big) \nabla h \rho^\alpha e^{-\phi} dv \\
&= \int_\Omega |\nabla f|^2 \rho^\alpha e^{-\phi} dv + \int_\Omega \frac{f^2}{h}\Big(\alpha\rho^{\alpha-1}\nabla\rho\nabla h + \rho^\alpha L h\Big) e^{-\phi} dv \\
&\leq \int_\Omega |\nabla f|^2 \rho^\alpha e^{-\phi} dv + \int_\Omega \frac{f^2}{\rho^\beta}\Big(\alpha\beta\rho^{\alpha+\beta-2} + \beta(\beta-1+C)\rho^{\alpha+\beta-2}\Big) e^{-\phi} dv \\
&= \int_\Omega |\nabla f|^2 \rho^\alpha e^{-\phi} dv + \beta(\alpha+\beta-1+C) \int_\Omega \rho^{\alpha-2} f^2 e^{-\phi} dv.
\end{aligned}$$

Thus, (by combining the last inequality with (9)), we have

$$\int_\Omega |\nabla f|^2 \rho^\alpha e^{-\phi} dv \geq -\beta(\alpha+\beta-1+C) \int_\Omega \rho^{\alpha-2} f^2 e^{-\phi} dv.$$
$$+\mathcal{B}_1 \Big(\int_\Omega \Big|\nabla f - \frac{f}{h}\nabla h\Big|^p \rho^{\alpha p/2} e^{-\phi} dv \Big)^{2/p}. \tag{11}$$

Following the approach in [17] by using repeatedly the following elementary inequality for $\eta_1, \eta_2 \in \mathbb{R}^m$

$$c(p)|\eta_2|^p \geq |\eta_1 + \eta_2|^p - |\eta_1|p - p|\eta_2|^{p-2}\langle \eta_1, \eta_2\rangle, \quad c(p) > 0, \quad p \in (1,2), \tag{12}$$

Young inequality and Hardy inequality (6), we obtain

$$\int_\Omega \Big|\nabla f - \frac{f}{h}\nabla h\Big|^p \rho^{\alpha p/2} e^{-\phi} dv \geq \mathcal{B}_2 \int_\Omega |\nabla f|^p \rho^{\alpha p/2} e^{-\phi} dv. \tag{13}$$

Putting (13) into (11), we obtain

$$\int_\Omega |\nabla f|^2 \rho^\alpha e^{-\phi} dv \geq -\beta(\alpha+\beta-1+C) \int_\Omega \frac{f^2}{\rho^{2-\alpha}} e^{-\phi} dv + \mathcal{B}\Big(\int_\Omega |\nabla f|^p e^{-\phi} dv \Big)^{2/p}.$$

We can then choose $\beta = -\dfrac{C + \alpha - 1}{2} < 0$ and obtain the required inequality at once. □

Setting $\alpha = 0$ in Theorem 2, one can obtain the following easily.

Corollary 1. *With the assumptions of Theorem 2 it holds that*

$$\int_\Omega |\nabla f|^2 e^{-\phi} dv \geq \Big(\frac{C-1}{2}\Big)^2 \int_\Omega \frac{f^2}{\rho^2} e^{-\phi} dv + \mathcal{B}\Big(\int_\Omega |\nabla f|^p e^{-\phi} dv\Big)^{2/p},$$

where $C \neq 1$.

The next result presents a new version of a Hardy–Poincaré-type inequality on M with a distance function.

Theorem 3. *Let* M *and* ρ *be as defined in Theorem 1. Then the following inequality*

$$\int_M \rho^{\alpha+p} |\langle \nabla\rho, \nabla f\rangle|^p e^{-\phi} dv \geq \mathcal{A}_\alpha^p \int_M \rho^\alpha |f|^p e^{-\phi} dv + \mathcal{A}_\alpha^{p-1} \int_M V\rho^{\alpha+1} |f|^p e^{-\phi} dv \quad (14)$$

holds for any $f \in C_0^\infty(M \backslash \rho^{-1}\{0\})$, $p \in (1, \infty)$, $\mathcal{A}_\alpha = (C+\alpha+1)/p$ *with* $C+\alpha+1 > 0$.

Proof. Using the conditions $|\nabla\rho| = 1$ and $L\rho \geq \frac{C}{\rho} + V$, we have

$$\mathrm{div}_\phi(e^{-\phi}\rho\nabla\rho) = e^\phi \mathrm{div}(e^{-\phi}\rho\nabla\rho) \geq (1 + C + V\rho). \quad (15)$$

Multiplying (15) by $\rho^\alpha|f|^p$, and integrating both sides over M gives

$$\begin{aligned}(1+C)\int_M \rho^\alpha |f|^p e^{-\phi} dv + \int_M V\rho^{\alpha+1}|f|^p e^{-\phi} dv &\leq \int_M \mathrm{div}(e^{-\phi}\rho\nabla\rho)\rho^\alpha|f|^p dv \\ &= -\alpha\int_M \rho^\alpha|f|^p e^{-\phi} dv - p\int_M |f|^{p-2} f \rho^{\alpha+1}\langle\nabla\rho, \nabla f\rangle e^{-\phi} dv,\end{aligned}$$

which implies

$$(1+C+\alpha)\int_M \rho^\alpha|f|^p e^{-\phi} dv + \int_M V\rho^{\alpha+1}|f|^p e^{-\phi} dv \leq -p\int_M |f|^{p-2} f\rho^{\alpha+1}\langle\nabla\rho, \nabla f\rangle e^{-\phi} dv. \quad (16)$$

Using Hölder and Young inequalities implies

$$\begin{aligned}&-p\int_M |f|^{p-2} f \rho^{\alpha+1}\langle\nabla\rho, \nabla f\rangle e^{-\phi} dv \\ &\leq p\Big(\int_M \rho^\alpha|f|^p e^{-\phi} dv\Big)^{\frac{p-1}{p}} \Big(\int_M \rho^{\alpha+p}|\langle\nabla\rho, \nabla f\rangle|^p e^{-\phi} dv\Big)^{\frac{1}{p}} \\ &\leq \frac{(p-1)}{\varepsilon^{\frac{p}{p-1}}}\int_M \rho^\alpha|f|^p e^{-\phi} dv + \varepsilon^p \int_M \rho^{\alpha+p}|\langle\nabla\rho, \nabla f\rangle|^p e^{-\phi} dv\end{aligned} \quad (17)$$

for any $\varepsilon > 0$ that will be determined later. Note that Young's inequality applied to obtain the second inequality in (17) can be described as follows: denoting

$$\Phi := \Big(\int_M \rho^\alpha|f|^p e^{-\phi} dv\Big)^{\frac{p-1}{p}} \quad \text{and} \quad \Psi := \Big(\int_M \rho^{\alpha+p}|\langle\nabla\rho, \nabla f\rangle|^p e^{-\phi} dv\Big)^{\frac{1}{p}},$$

then for any $\varepsilon > 0$

$$\Phi\Psi = \varepsilon\Phi\frac{\Psi}{\varepsilon} \leq \frac{1}{p}(\varepsilon\Phi)^p + \frac{1}{q}\Big(\frac{\Psi}{\varepsilon}\Big)^q \text{ with } q = \frac{p}{p-1} \text{ being the conjugate to } p.$$

Hence, putting (16) and (17) together, we obtain

$$\begin{aligned}\int_M \rho^{\alpha+p}|\langle\nabla\rho, \nabla f\rangle|^p e^{-\phi} dv &\geq \varepsilon^{-p}\Big(1 + C + \alpha - \frac{p-1}{\varepsilon^{\frac{p}{p-1}}}\Big)\int_M \rho^\alpha|f|^p e^{-\phi} dv \\ &+ \varepsilon^{-p}\int_M V\rho^{\alpha+1}|f|^p e^{-\phi} dv.\end{aligned} \quad (18)$$

We can now choose $\varepsilon = \Big(\dfrac{p}{1+C+\alpha}\Big)^{\frac{p-1}{p}}$ since the quantity $\varepsilon \longmapsto \Upsilon(\varepsilon) := \varepsilon^{-p}\Big(1 + C + \alpha - \dfrac{p-1}{\varepsilon^{\frac{p}{p-1}}}\Big)$ attains its maximum at this point. It is not difficult to obtain the attained maximum value: first and second derivatives of $\Upsilon(\varepsilon)$ are computed respectively, as

$$\Upsilon'(\varepsilon) = -p\varepsilon^{-(p+1)}\Big(1 + C + \alpha - p\varepsilon^{-\frac{p}{p-1}}\Big)$$

and

$$\Upsilon''(\varepsilon) = \varepsilon^{-(p+2)}\Big[p(p+1)(1+C+\alpha) - \frac{p^2(p+1)(p-1)+p^3}{p-1}\varepsilon^{-\frac{p}{p-1}}\Big].$$

Clearly, the critical points of the function $\Upsilon(\varepsilon)$ are at $\varepsilon_1 = 0$ and $\varepsilon_2 = \Big(\frac{1+C+\alpha}{p}\Big)^{-\frac{p-1}{p}}$. Finding the value of $\Upsilon''(\varepsilon)$ at both ε_1 and ε_2, we obtain

$$\Upsilon''(\varepsilon_1) = 0 \quad \text{and} \quad \Upsilon''(\varepsilon_2) = -\frac{p^2}{p-1}(1+C+\alpha)\Big(\frac{1+C+\alpha}{p}\Big)^{\frac{(p+1)(p-1)}{p}} \leq 0.$$

Consequently, ε_2 is the maximum point and obviously, the maximal value achieved is

$$\max_{\varepsilon} \Upsilon(\varepsilon) = \Upsilon(\varepsilon_2) = \Big(\frac{1+C+\alpha}{p}\Big)^{p}. \tag{19}$$

Finally, the required inequality can be determined by substituting (19) into (18) as follows:

$$\int_M \rho^{\alpha+p}|\langle\nabla\rho,\nabla f\rangle|^p e^{-\phi}dv \geq \Big(\frac{1+C+\alpha}{p}\Big)^{p}\int_M \rho^{\alpha}|f|^p e^{-\phi}dv + \Big(\frac{1+C+\alpha}{p}\Big)^{p-1}\int_M V\rho^{\alpha+1}|f|^p e^{-\phi}dv.$$

This completes the proof. □

Remark 2. *As suggested in [18], one can apply the Cauchy–Schwarz inequality to the quantity $|\langle\nabla\rho,\nabla f\rangle|$ and replace α with $\alpha - p$ using the condition $|\nabla\rho| = 1$ to obtain (3) and (7) for the special case $V \equiv 0$. Note that the case ϕ is a constant that was obtained in [18] (Theorem 2.1).*

Heisenberg–Pauli–Weyl Uncertainty Principle on Smooth Metric Measure Spaces

Another application of our result is the derivation of Heisenberg–Pauli–Weyl uncertainty principles on smooth metric measure space. Recall that the classical Heisenberg uncertainty arising from quantum mechanics says that the position and momentum of a given particle cannot be simultaneously determined.

Theorem 4. *Let M and ρ be as defined before such that ρ satisfies $|\nabla\rho| = 1$ and $L\rho \geq \frac{C}{\rho}$, $C > 0$, in distribution sense. Then the following inequality*

$$\Big(\int_M \rho^p|f|^p e^{-\phi}dv\Big)^{p/q}\Big(\int_M |\nabla f|^p e^{-\phi}dv\Big) \geq \mathcal{D}_p\int_M \frac{|f|^p}{\rho^{2-p}}e^{-\phi}dv \tag{20}$$

holds for any compactly supported function $f \in C_0^\infty(M\backslash\rho^{-1}\{0\})$, where $1/p + 1/q = 1$, $p \in (1,\infty)$, $0 < q \leq p$ and $\mathcal{D}_p = ((C+p-1)/p)^p$ with $C + p - 1 > 0$.

Proof. Using the conditions $|\nabla\rho| = 1$ and $L\rho \geq \frac{C}{\rho}$, we have

$$L(\rho^p) \geq p(C+p-1)\rho^{p-2}$$

and then

$$\int_M L(\rho^p)|f|^p e^{-\phi} dv \geq p(C+p-1) \int_M |f|^p e^{-\phi} dv \rho^{p-2}. \tag{21}$$

Applying integration by parts and Hölder's inequality with conjugates p and q, we have

$$\begin{aligned}\int_M L(\rho^p)|f|^p e^{-\phi} dv &= -p^2 \int_M (\rho f)^{p-1} \langle \nabla\rho, \nabla f \rangle e^{-\phi} dv \\ &\leq p^2 \int_M \rho^{p-1} |f|^{p-1} |\nabla\rho||\nabla f| e^{-\phi} dv \\ &\leq p^2 \Big(\rho^p |f|^p e^{-\phi} dv\Big)^{\frac{p-1}{p}} \Big(|\nabla f|^p e^{-\phi} dv\Big)^{\frac{1}{p}}.\end{aligned}$$

Substituting the last inequality into (21) directly yields

$$\Big(\rho^p |f|^p e^{-\phi} dv\Big)^{p-1} \Big(|\nabla f|^p e^{-\phi} dv\Big) \geq \Big(\frac{C+p-1}{p}\Big)^p \int_M \rho^{p-2} |f|^p e^{-\phi} dv,$$

which is the required inequality. □

3. Integral Inequalities Related to Witten p-LAPLACIAN

Let $\Omega \subset M$. The first main theorem of this section is stated as follows.

Theorem 5. *Given $\Omega \subset M$ and $\rho \in W_0^{1,p}(\Omega)$, a non-negative function on Ω, satisfying $|\nabla\rho| \in L_{loc}^{p-1}(\Omega)$ and $-L_p\rho \geq 0$ on Ω in distribution sense, then $\rho^{-p}|\nabla\rho|^p \in L_{loc}^1(\Omega)$ and*

$$\int_\Omega |\nabla f|^p e^{-\phi} dv \geq \Big(\frac{p-1}{p}\Big)^p \int_\Omega \frac{|f|^p}{\rho^p} |\nabla\rho|^p e^{-\phi} dv \tag{22}$$

holds, where f is any non-negative function $f \in C_0^\infty(\Omega)$.

The condition $-L_p\rho \geq 0$ says that ρ is weighted p-superharmonic in the sense that

$$\int_\Omega |\nabla\rho|^{p-2} \langle \nabla\rho, \nabla\varphi \rangle e^{-\phi} dv \geq 0$$

for every test non-negative function $\varphi \in C_0^1(\Omega)$. Suppose Ω is open in M and $\mathcal{D}^{1,p}(\Omega)$ denotes the completion of $C_0^\infty(\Omega)$ with respect to the norm $\|f\|_{\mathcal{D}^{1,p}} = \Big(\int_\Omega |\nabla f|^p e^{-\phi} dv\Big)^{1/p}$. Theorem 5 can be extended to the whole of M as a result of the embedding $\mathcal{D}^{1,p}(M) \subset \mathcal{D}^{1,p}(\Omega)$ see (Appendix A of [10]) .

Corollary 2. *Let $\rho \in W_0^{1,p}(M)$ be a non-negative function on M fulfilling the condition of Theorem 5 such that $\mathcal{D}^{1,p}(M) \subset \mathcal{D}^{1,p}(\Omega)$. Then $\rho^{-p}|\nabla\rho|^p \in L_{loc}^1(M)$ and*

$$\int_M |\nabla f|^p e^{-\phi} dv \geq \Big(\frac{p-1}{p}\Big)^p \int_M \frac{|f|^p}{\rho^p} |\nabla\rho|^p e^{-\phi} dv \tag{23}$$

holds, where f is any non-negative function in $C_0^\infty(M)$.

It is in order to make some notation at this point so as to follow the language of [3,10]. Let $X \in L_{loc}^1(\Omega)$ be a vector field; obviously, for any smooth function ϕ, a weighted vector field $X_\phi = e^{-\phi}X$ can be defined such that $X_\phi \in L_{loc}^1(\Omega)$. Note that the distribution $\mathrm{div}_\phi(X_\phi)$ in the weak sense, via the weighted divergence, is defined by the following identity:

$$\int_\Omega \varphi \mathrm{div}_\phi(X_\phi) e^{-\phi} dv = -\int_\Omega \langle \nabla\varphi, X \rangle e^{-\phi} dv, \quad \forall \varphi \in C_0^\infty(\Omega). \tag{24}$$

Let $h_X \in L^1_{loc}(\Omega)$ be a positive function, then we say that $h_X \leq \text{div}_\phi(X_\phi)$ in the weak sense if

$$\int_\Omega \varphi h_X e^{-\phi} dv \leq \int_\Omega \varphi \text{div}_\phi(X_\phi) e^{-\phi} dv \quad \forall \varphi \in C_0^\infty(\Omega). \tag{25}$$

By using Equations (24) and (25), and the direct differentiation one obtains for $p > 1$, then

$$\int_\Omega |\varphi|^p h_X e^{-\phi} dv \leq \int_\Omega |\varphi|^p \text{div}_\phi(e^{-\phi} X) e^{-\phi} dv = -p \int_M |\varphi|^{p-1} \langle \nabla \varphi, X \rangle e^{-\phi} dv. \tag{26}$$

Following the program of D'Ambrosio [3,10], the next lemma is fundamental in proving Theorem 1. The proof of the lemma is also included for the sake of completeness. Identities (25) and (26) play crucial roles in the proof.

Lemma 1. *Suppose M is a complete non-compact smooth metric measure space. Let $\Omega \subset M$ and $X, X_\phi \in L^1_{loc}(\Omega)$ be vector fields and $h_X \in L^1_{loc}(\Omega)$ be a non-negative function where (i) $h_X \leq div_\phi(e^{-\phi} X)$, (ii) $h_X^{1-p} |X_\phi|^p \in L^1_{loc}(\Omega)$. Then the inequality*

$$\int_\Omega \frac{|X|^p}{h_X^{p-1}} |\nabla f|^p e^{-\phi} dv \geq p^{-p} \int_\Omega |f|^p h_X e^{-\phi} dv \tag{27}$$

holds for any $f \in C_0^\infty(\Omega)$.

Proof. Note that the RHS of (27) is finite since $f \in C_0^\infty(\Omega)$. Now applying the identities (25) and (26), then, the dominated convergence and the Hölder's inequality imply

$$\begin{aligned}
\int_\Omega |f|^p h_X e^{-\phi} dv &\leq \int_\Omega |f|^p \text{div}_\phi(X_\phi) e^{-\phi} dv \\
&= -p \int_\Omega |f|^{p-2} f \langle \nabla f, X \rangle e^{-\phi} dv \\
&\leq p \int_\Omega |f|^{p-1} h_X^{\frac{p-1}{p}} |X| h_X^{-\frac{p-1}{p}} |\nabla f| e^{-\phi} dv \\
&\leq p \Big(|f|^p h_X e^{-\phi} dv \Big)^{\frac{p-1}{p}} \Big(h_X^{1-p} |X|^p |\nabla f|^p e^{-\phi} dv \Big)^{\frac{1}{p}}.
\end{aligned}$$

The proof is concluded by raising both sides to the power of p and then collecting the like quantities on one side. □

Remark 3. *Suppose one chooses $h_X = div_\phi(X_\phi)$, then inequality (27) becomes*

$$\int_\Omega \frac{|X|^p}{|div_\phi(X_\phi)|^{p-1}} |\nabla f|^p e^{-\phi} dv \geq p^{-p} \int_\Omega |f|^p div_\phi(X_\phi) e^{-\phi} dv. \tag{28}$$

Define a function V in $L^1_{loc}(\Omega)$ whose weak partial derivatives of order up to two are in $L^1_{loc}(\Omega)$ and $LV \geq 0$. The choice $X_\phi = e^{-\phi} \nabla V$ implies

$$h_X = \text{div}_\phi(X_\phi) = e^\phi \text{div}(e^{-\phi} \nabla V) = \Delta V - \langle \nabla \phi, \nabla V \rangle = LV \geq 0,$$

and then (28) yields

$$\int_\Omega \frac{|\nabla V|^p}{|LV|^{p-1}} |\nabla f|^p e^{-\phi} dv \geq p^{-p} \int_\Omega |f|^p |LV| e^{-\phi} dv. \tag{29}$$

Another choice of a vector field $X_\phi = e^{-\phi}|\nabla V|^{p-2}\nabla V$, whenever $L_p V = \mathrm{div}_\phi(e^{-\phi}|\nabla V|^{p-2}\nabla V) \geq 0$, yields the following:

$$\int_\Omega \frac{|\nabla V|^{p(p-1)}}{|L_p V|^{p-1}}|\nabla f|^p e^{-\phi} dv \geq p^{-p}\int_\Omega |f|^p |L_p V| e^{-\phi} dv. \tag{30}$$

The above therefore extends the results in [9] to complete smooth metric measure spaces. We obtain a replicate of results in [10] if the potential is constant or zero. Note that one can derive from the last two expressions specific inequalities of the form

$$\int_M |\nabla f|^p e^{-\phi} dv \geq c_p \int_M \frac{|f|^p}{\rho^p} e^{-\phi} dv \tag{31}$$

with a sharp constant by choosing a suitable function V. For instance, a choice of

$$V := \begin{cases} \rho^{2-p}, & 1 < p < 2 \\ \ln \rho, & p = 2 \\ -\rho^{2-p}, & 2 < p < m \end{cases}$$

while requiring $LV \geq C/\rho$ and $|\nabla \rho| = 1$ will yield (31) with constant $c_p = ((C+1-p)/p)^p$. This is a special case of Theorem 5 (cf. inequality (7); see also [3,10]).

Proof of Theorem 5. In order to apply Lemma 1, define a vector field $X_\phi = -e^{-\phi}\rho^{1-p}|\nabla\rho|^{p-2}\nabla\rho$ and a function $h_X = (p-1)\rho^{-p}|\nabla\rho|^p$. Let $0 < \varepsilon < 1$ and $\rho_\varepsilon = \rho + \varepsilon$. Obviously, since $\frac{1}{\rho_\varepsilon} \leq \frac{1}{\varepsilon}$ and $\rho \in W^{1,p}_{loc}(\Omega)$ one has that $X^\varepsilon_\phi = -e^{-\phi}\rho_\varepsilon^{1-p}|\nabla\rho_\varepsilon|^{p-2}\nabla\rho_\varepsilon \in L^1_{loc}(\Omega)$, $h_{X^\varepsilon} = (p-1)\rho_\varepsilon^{-p}|\nabla\rho_\varepsilon|^p \in L^1_{loc}(\Omega)$ and $h^{1-p}_{X^\varepsilon}|X^\varepsilon_\phi|^p = (p-1)^{1-p}|e^{-\phi}|^p \in L^1_{loc}(\Omega)$, which implies that condition (ii) in Lemma 1 holds. Condition (i) of the Lemma is fulfilled, provided that for every non-negative function $\varphi \in C^1_0(\Omega)$

$$\begin{aligned} (p-1)\int_\Omega \frac{|\nabla\rho_\varepsilon|^p}{\rho_\varepsilon^p}\varphi e^{-\phi} dv &\leq -\int_\Omega \varphi \mathrm{div}\Big(\frac{|\nabla\rho_\varepsilon|^{p-2}\nabla\rho_\varepsilon}{\rho_\varepsilon^{p-1}}\Big)e^{-\phi} dv \\ &= \int_\Omega \Big\langle \frac{|\nabla\rho_\varepsilon|^{p-2}\nabla\rho_\varepsilon}{\rho_\varepsilon^{p-1}}, \nabla\varphi \Big\rangle e^{-\phi} dv \end{aligned} \tag{32}$$

holds. Note that the last inequality is equivalent to the condition $-L_p(\rho_\varepsilon) \geq 0$ in the sense of distribution, that is, for $\varphi \in C^1_0(\Omega)$

$$\int_M |\nabla\rho_\varepsilon|^{p-2}\langle\nabla\rho_\varepsilon, \nabla\varphi\rangle e^{-\phi} dv \geq 0. \tag{33}$$

Choosing $\varphi := \rho_\varepsilon^{-(p-1)}\psi$ into (33), we obtain

$$0 \leq \int_\Omega \Big\langle \frac{|\nabla\rho_\varepsilon|^{p-2}\nabla\rho_\varepsilon}{\rho_\varepsilon^{p-1}}, \nabla\psi \Big\rangle e^{-\phi} dv - (p-1)\int_\Omega \frac{|\nabla\rho_\varepsilon|^p}{\rho_\varepsilon^p}\psi e^{-\phi} dv,$$

which is the same as inequality (32) by replacing ϕ by ψ. Thus, to show that condition (i) of Lemma 1 is fulfilled, it is enough to prove inequality (32). The proof of the identity follows by mimicking the proof of identity (2.19) in [10]. Finally, the required inequality (22) follows from the application of (27) in Lemma 1 and sending $\varepsilon \to 0$. □

Lastly, in this section, we give two possible generalizations of Theorem 5. The first one is due to [10] while the second is due to [3], the proofs of both are similar to the proof of Theorem 5, depending on the choice of the vector field X viz-a-viz X_ϕ and function h_X in Lemma 1. The results are the following.

Theorem 6. *Let Ω and M be as defined in Theorem 5. For $p \in (1, \infty)$, $\alpha \in \mathbb{R}$ and $\rho \in W^{1,p}_{loc}(\Omega)$ satisfying $\rho^{\alpha-p}|\nabla\rho|^p, \rho^\alpha \in L^1_{loc}(\Omega)$ and $-tL_p(\rho) \geq 0$, where $t := (p-1-\alpha) \neq 0$, on Ω in the distribution sense. Then for any $f \in C^1_0(\Omega)$, the inequality*

$$\int_M \rho^\alpha |\nabla f|^p e^{-\phi} dv \geq (\mathcal{D}_{\alpha,p})^p \int_\Omega \rho^{\alpha-p} |f|^p |\nabla\rho|^p e^{-\phi} dv \tag{34}$$

holds on Ω, where $\mathcal{D}_{\alpha,p} := (|p-1-\alpha|)/p$.

Theorem 7. *Let Ω and M be as defined in Theorem 5. For $p \in (1, \infty)$, $\alpha, \gamma \in \mathbb{R}$, $\alpha \neq 0$ and $\rho \in W^{1,p}_{loc}(\Omega)$ satisfying $\rho^{p+\gamma}, \rho^\gamma|\nabla\rho|^p, \rho^{(\alpha-1)(p-1)}|\nabla\rho|^{p-1} \in L^1_{loc}(\Omega)$ and $-L_p(t\rho^\alpha) \geq 0$, where $t := \alpha[(\alpha-1)(p-1) - (\gamma+1)] \geq 0$, on Ω in the distribution sense. Then for any $f \in C^1_0(\Omega)$, the inequality*

$$\int_M \rho^{p+\gamma} |\nabla f|^p e^{-\phi} dv \geq (\mathcal{D}_{\alpha,\gamma,p})^p \int_\Omega \rho^\gamma |f|^p |\nabla\rho|^p e^{-\phi} dv \tag{35}$$

holds on Ω, where $\mathcal{D}_{\alpha,\gamma,p} := t/p$.

Brief Discussion on the Proof of the Last Two Results

For Theorem 6, one can argue as in [10] that by choosing the vector field $X := -(p-1-\alpha)\rho^{\alpha+1-p}|\nabla\rho|^{p-2}\nabla\rho$ and function $h_X := (p-1-\alpha)^2\rho^{\alpha-p}|\nabla\rho|^p$ and later $\varphi = \rho^{-(p-1-\alpha)}\psi \in C^\infty_0(\Omega)$, one will be able to apply Lemma 1.

Similarly for Theorem 7, one can define the vector field by $X := -\alpha\rho^{\gamma+1}|\nabla\rho|^{p-2}\nabla\rho$ and function $h_X := t\rho^\gamma|\nabla\rho|^p$, $t = \alpha[(\alpha-1)(p-1) - (\gamma+1)] \geq 0$ and later the test function $\varphi = \rho^{-t/\alpha}\psi$, in order to be able to apply Lemma 1 as in [3].

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