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Article

Locally Homogeneous Manifolds Defined by Lie Algebra of Infinitesimal Affine Transformations

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Abstract: This article deals with Lie algebra $\mathfrak{G}$ of all infinitesimal affine transformations of the manifold $\mathcal{M}$ with an affine connection, its stationary subalgebra $\mathfrak{H} \subset \mathfrak{G}$, the Lie group $\mathcal{G}$ corresponding to the algebra $\mathfrak{G}$, and its subgroup $\mathcal{H} \subset \mathcal{G}$ corresponding to the subalgebra $\mathfrak{H} \subset \mathfrak{G}$. We consider the center $\mathfrak{Z} \subset \mathfrak{G}$ and the commutant $[\mathfrak{G}, \mathfrak{G}]$ of algebra $\mathfrak{G}$. The following condition for the closedness of the subgroup $\mathcal{H}$ in the group $\mathcal{G}$ is proved. If $\mathfrak{H} \cap (\mathfrak{Z} + [\mathfrak{G}; \mathfrak{G}]) = \mathfrak{H} \cap [\mathfrak{G}; \mathfrak{G}]$, then $\mathcal{H}$ is closed in $\mathcal{G}$. To prove it, an arbitrary group $\mathcal{G}$ is considered as a group of transformations of the set of left cosets $\mathcal{G}/\mathcal{H}$, where $\mathcal{H}$ is an arbitrary subgroup that does not contain normal subgroups of the group $\mathcal{G}$. Among these transformations, we consider right multiplications. The group of right multiplications coincides with the center of the group $\mathcal{G}$. However, it can contain the right multiplication by element $\overline{\hbar}$, belonging to normalizator of subgroup $\mathcal{H}$ and not belonging to the center of a group $\mathcal{G}$. In the case when $\mathcal{G}$ is in the Lie group, corresponding to the algebra $\mathfrak{G}$ of all infinitesimal affine transformations of the affine space $\mathcal{M}$ and its subgroup $\mathcal{H}$ corresponding to its stationary subalgebra $\mathfrak{H} \subset \mathfrak{G}$, we prove that such element $\overline{\hbar}$ exists if subgroup $\mathcal{H}$ is not closed in $\mathcal{G}$. Moreover $\overline{\hbar}$ belongs to the closures $\overline{\mathcal{H}}$ of subgroup $\mathcal{H}$ in $\mathcal{G}$ and does not belong to commutant $(\mathcal{G}, \mathcal{G})$ of group $\mathcal{G}$. It is also proved that $\mathcal{H}$ is closed in $\mathcal{G}$ if $(\mathfrak{P} + \mathfrak{Z}) \cap \mathfrak{H} = \mathfrak{P} \cap \mathfrak{H}$ for any semisimple algebra $\mathfrak{P} \in \mathfrak{G}$ for which $\mathfrak{P} + \mathfrak{R} = \mathfrak{G}$.

Keywords: affine connection; Lie group; Lie algebra; stationary subgroup; stationary subalgebra

MSC: 53C05

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1. Introduction

The topological space of left cosets $\mathcal{G}/\mathcal{H}$ of Lie group $\mathcal{G}$ by its Lie subgroup $\mathcal{H}$ is a manifold when and only when $\mathcal{H}$ is closed in $\mathcal{G}$. However, $\mathcal{H}$ may not be closed even if the group $\mathcal{G}$ is simply connected. Let $\mathfrak{G}$ be Lie algebra, Lie algebra $\mathfrak{H} \subset \mathfrak{G}$ be its subalgebra, $\mathcal{G}$ be the simply connected Lie group corresponding to algebra $\mathfrak{G}$, and $\mathcal{H} \subset \mathcal{G}$ be a subgroup corresponding to subalgebra $\mathfrak{H}$. If $\mathcal{H}$ is closed in $\mathcal{G}$, then for then non-closed Lie group $\mathcal{G}'$ with the same Lie algebra $\mathfrak{G}$ the subgroup $\mathcal{H}' \subset \mathcal{G}'$ corresponding to subalgebra $\mathfrak{H} \subset \mathfrak{G}$ may be non-closed in $\mathcal{G}'$. The question of the closedness of Lie subgroup $\mathcal{H}$ in a simply connected Lie group $\mathcal{G}$ is related to the existence of a homogeneous space of affine connection in the class of all locally equivalent analytic spaces of affine connection in the case that Lie algebra $\mathfrak{G}$ of the Lie group $\mathcal{G}$ is the algebra Lie of all infinitesimal transformations of the analytic manifold of the analytic affine connection. This question is related to the analytic extension of a locally given analytic manifold of affine connection to a homogeneous space with an analytic affine connection. Can it be performed by using only the properties of the Lie algebra $\mathfrak{G}$ of all analytic infinitesimal transformations of the analytic manifold with the analytic affine connection and its stationary subalgebra $\mathfrak{H}$? The characterization of non-closed Lie subgroups is contained in the classical work of A.I. Maltsev [1]. If Lie subgroup $\mathcal{H}$ of Lie group $\mathcal{G}$ is non-closed in $\mathcal{G}$, then the group $\mathcal{H}$ contains a one-parameter subgroup, the closure of which is not contained in $\mathcal{H}$. However, this property cannot be easily verified from the properties of Lie algebras $\mathfrak{G}$ and $\mathfrak{H}$. However, we are interested in deducing the closedness of subgroup $\mathcal{H}$ in simply connected group $\mathcal{G}$ from the properties of algebras $\mathfrak{G}$ and $\mathfrak{H}$.

The following results on the closedness of the Lie subgroup in the Lie group based on the properties of their Lie algebras are well known. Let $\mathcal{G}$ be a connected Lie group and $\mathcal{H}$ be its analytic subgroup. Let $\mathfrak{G}$ and $\mathfrak{H}$ denote the corresponding Lie algebras:

1. Assume $\mathcal{G}$ is simply connected. If $\mathfrak{H}$ is an ideal in $\mathfrak{G}$ then $\mathcal{H}$ is closed in $\mathcal{G}$ [2].
2. Assume $\mathcal{G}$ is simply connected. If $\mathfrak{H}$ is semisimple then $\mathcal{H}$ is closed in $\mathcal{G}$ [3].
3. Assume $\mathcal{G}$ is compact. If $\mathfrak{H}$ is semisimple then $\mathcal{H}$ is closed in $\mathcal{G}$ [3].
4. Assume $\mathcal{G}$ is solvable and simply connected: then $\mathcal{H}$ is closed in $\mathcal{G}$ [2].
5. Let $\mathcal{G}$ be simply connected and $\dim \mathfrak{G} - \dim \mathfrak{H} < 5$: then $\mathcal{H}$ is closed in $\mathcal{G}$ [3].

In this study, investigate the following problem: whether Lie subgroup $\mathcal{H}$ in a simply connected Lie group $\mathcal{G}$ is closed or non-closed in the case when the algebra $\mathfrak{G}$ is the Lie algebra of all analytic infinitesimal affine transformations of an analytic manifold with an analytic affine connection. This question is equivalent to the question of the possibility of an analytic extension of a locally given analytic affine connection of a locally homogeneous space to be analytically extended to an affine connection of homogeneous space. It would be interesting to study the analytic extension of an arbitrary locally given manifold with an affine connection using the analyticity of functions $\Gamma^i_{j,k}$ defining affine connection. The notion of extendability was mentioned in classical monographs [4,5], but for any locally given analytic affine connection there are a lot of unnatural analytic extensions to the unextendable manifold. Constructing the most symmetric and most complete analytic extension is difficult even in the case of Riemannian manifolds. Certain progress in this direction was made in works [6–8].

The study of the analytic extension of Riemannian analytic manifolds whose Lie algebra of all Killing vector fields has no center leads to the proof of the following result. The construction of the so-called quasicomplete or, in other words, compressed manifold was given in [6]. It is a universally attracting object in the following category. Objects of this category are locally isometric Riemannian analytic manifolds; morphisms are locally isometric analytic maps $f : \mathcal{M} \setminus \mathcal{S} \to \mathcal{N}$, where $\mathcal{S}$ is the set of fixed points of all local isometries that preserve orientation and Killing vector fields. $\mathcal{S}$ is an analytic subset of codimension less than two.

The construction of a quasicomplete manifold is based on the analytic extension of a small ball $\mathcal{M}$ with marked point p. In Lie algebra $\mathfrak{G}$ we define stationary subalgebra $\mathfrak{H}$. Subalgebra $\mathfrak{H}$ consists of all vector fields $\mathcal{X} \in \mathfrak{G}$ which equal zero at a fixed point, $\mathcal{X}(p) = 0$. If we mark another point, $q \in \mathcal{M}$, then stationary subalgebra $\mathfrak{H}_{\cdot q}$ defined by point q is conjugate to algebra $\mathfrak{H}$.

The study of quasicomplete manifolds leads to the fact that if the Lie algebra $\mathfrak{G}$ of all Killing vector fields of some Riemannian analytic manifolds has no center, then its stationary subalgebra $\mathfrak{H}$ defines a closed subgroup in the simply connected group corresponding to the algebra $\mathfrak{G}$ [6]. The study of analytic extension of an arbitrary locally given analytic space with an analytic affine connection is also connected with the closedness of the subgroup $\mathcal{H}$ in $\mathcal{G}$. In an arbitrary case such study is difficult, but in the case of a locally homogeneous space with an analytic affine connection the study of analytic extension is easier. If Lie algebra $\mathfrak{G}$ of all infinitesimal affine transformations has no center, then its stationary subalgebra $\mathfrak{H}$ defines a closed subgroup in the simply connected group corresponding to the algebra $\mathfrak{G}$ [6,9]. We consider here the Lie algebra $\mathfrak{G}$ of all infinitesimal affine transformations with centers and give a sufficient condition for the closedness of Lie subgroup $\mathcal{H}$ corresponding to stationary subalgebra $\mathfrak{H} \subset \mathfrak{G}$ in simply connected Lie group $\mathcal{G}$ corresponding to Lie algebra $\mathfrak{G}$, expressed in terms of properties of the algebra $\mathfrak{G}$, its stationary subalgebra $\mathfrak{H}$, the center $\mathfrak{Z}$, and the commutant $[\mathfrak{G}, \mathfrak{G}]$.

To begin with, we give some definitions and statements concerning the analytic extension of a locally given analytic space with an affine connection.

Definition 1. *An analytic extension of a connected analytic manifold $\mathcal{M}$with an affine connection is a connected analytic manifold $\mathcal{N}$with analytic affine connection and imbedding analytic affine transformation $f : \mathcal{M} \to \mathcal{N}$, where $f(\mathcal{M})$is a proper open subset of $\mathcal{N}$. Any manifold that does not admit analytic extension is called non-extendable.*

Definition 2. *A local affine mapping between two connected analytic manifolds $\mathcal{M}$ and $\mathcal{N}$ is one to one analytic affine mapping $\varphi : \mathcal{U} \to \mathcal{V}$ between open subsets $\mathcal{U} \subset \mathcal{M}, \mathcal{V} \subset \mathcal{N}$. Manifolds between which there is a local affine mapping are called locally equivalent.*

Definition 3. *Let $\mathcal{M}$ be an analytic manifold with an analytic affine connection, $\mathfrak{G}$ be Lie algebra of all its infinitesimal affine transformations, and $\mathfrak{H} \subset \mathfrak{G}$ be stationary subalgebra. $\mathcal{M}$ is called locally homogeneous if it satisfies the following condition: $dim\mathfrak{G} - dim\mathfrak{H} = dim\mathcal{M}$.*

Any two open subsets $\mathcal{U} \subset \mathcal{M}$, $\mathcal{V} \subset \mathcal{M}$ of locally homogeneous manifold $\mathcal{M}$ are locally equivalent.

It is possible to extend analytically any local isometry f of the compressed Riemannian analytic manifold $\mathcal{M}$ into itself to isometry $f : \mathcal{M} \approx \mathcal{M}$, [6]. It is also possible to extend analytically any local analytic affine mapping φ of locally homogeneous analytic manifold $\mathcal{M}$ with an analytic affine connection into itself to a diffeomorphic analytic affine transformation of a homogeneous manifold with an affine connection in the case that Lie algebra $\mathfrak{G}$ of all infinitesimal affine transformations has no center [9]: $f : \mathcal{M} \approx \mathcal{M}$ [6]. However, it is impossible to analytically extend an arbitrary local analytic affine mapping φ of analytic affine manifold $\mathcal{M}$ into itself to the analytic affine diffeomorphism $\varphi : \mathcal{M} \approx \mathcal{M}$. An insurmountable obstacle to this is non-closedness of the subgroup $\mathcal{H} \subset \mathcal{G}$, where $\mathcal{G}$ is the simply connected group corresponding to the Lie algebra $\mathfrak{G}$ of all infinitesimal transformations of the analytic manifold $\mathcal{M}$ of affine connection and $\mathcal{H} \subset \mathcal{G}$ is the subgroup corresponding to the stationary subalgebra $\mathfrak{H} \subset \mathfrak{G}$. However, infinitesimal analytic affine transformation can be extended analytically to the whole $\mathcal{M}$.

Preposition 1. *Let $\mathcal{M}$ be an analytic manifold with an analytic affine connection, $\mathcal{X}$ be an analytic tensor of infinitesimal affine transformation defined on an open connected set $\mathcal{U} \subset \mathcal{M}$, and let $\alpha(t)$, $0 \le t \le 1$, be a continuous curve in $\mathcal{M}$, $\alpha(0) \in \mathcal{U}$. Then, the vector field $\mathcal{X}$ is analytically extendable along α. If the curves $\alpha(t)$ and $\beta(\int)$, $0 \le t \le 1$, $0 \le \int \le 1$, $\alpha(0) = \beta(0)$, $\alpha(1) = \beta(1) = x_1$ are homotopic, then the extensions of the vector fields to the point x_1 along these curves coincide* [6].

It follows from preposition that all locally equivalent analytic manifolds with an affine connection have the same Lie algebra of infinitesimal affine transformations. Thus, one can speak of Lie algebra of infinitesimal affine transformations of a locally given analytic affine connection. Analytic extensions of Riemannian analytic manifolds without Killing vector fields were studied in [7]. A so-called compressed Riemannian analytic manifold in the case of Lie algebra of all Killing vector fields of given a Riemannian analytic metric that has no center was constructed in [6,8]. Every local isometry of such manifold can be analytically extended to global isometry. The study of analytic extensions of an arbitrary manifold with an affine connection is difficult, but locally homogeneous analytic manifolds with analytic affine connections were investigated. It was proved in [9] that any small enough open subsets of a locally homogeneous analytic manifold of affine connection can be analytically extended to the homogeneous space of an affine connection if the Lie algebra of all its infinitesimal affine transformations has no center. We investigate here the analytic extension of locally homogeneous manifolds with an analytic affine connection whose Lie algebra of all infinitesimal affine transformations has an untrivial center. In addition, we give sufficient conditions for the analytic extendibility of a locally homogeneous analytic manifold $\mathcal{M}$ with an affine connection to a homogeneous space, in terms of properties of the algebra $\mathfrak{G}$ of all infinitesimal affine transformations of a manifold $\mathcal{M}$ and its stationary subalgebra $\mathfrak{H}$.

Therefore, we study a small enough analytic manifold $\mathcal{M}$ with an analytic affine connection, Lie algebra $\mathfrak{G}$ of all infinitesimal affine transformations of this manifold, and stationary subalgebra $\mathfrak{H}$ defining some marked point, $p \in \mathcal{M}$. Let $\mathcal{U}$ be a normal neighborhood of point p. Then, in normal coordinates $\mathcal{U}$ can be identified with the connected neighborhood of 0 in Euclidian space $\mathfrak{G}/\mathfrak{H}$. Thus, we consider $\mathfrak{G}$ as the Lie algebra of all infinitesimal affine transformations of manifold $\mathcal{U}$ with marked point $p \in \mathcal{U}$, and Lie algebra $\mathfrak{H} \subset \mathfrak{G}$ as the stationary subalgebra. We can identify manifold $\mathcal{U}$ with marked point $p \in \mathcal{U}$ with some neighborhoods of point $0 \in \mathbb{R}^n$.

2. Conditions for Closedness of a Stationary Subgroup

In order to investigate the closedness of the above-mentioned Lie subgroup $\mathcal{H}$ in a Lie group $\mathcal{G}$, we introduce the notion of so-called right multiplication for an abstract group and for the non-closed Lie subgroup $\mathcal{H}$ in a simply connected group $\mathcal{G}$. We constructed one parameter group $Exp(tZ)$ of a local affine transformation on an open subset $\mathcal{U} \subset \mathcal{M}$ which consists of non-trivial right multiplications. The infinitesimal affine transformation Z satisfies the following conditions: $Z \in \mathfrak{Z}\ \mathfrak{H} \cap (Z + [\mathfrak{G};\mathfrak{G}]) = \mathfrak{H} \cap [\mathfrak{G};\mathfrak{G}]$, where $\mathfrak{G}$ is the algebra of all infinitesimal affine transformations of manifold $\mathcal{M}$, $\mathfrak{H}$ is the stationary subalgebra of $\mathfrak{G}$, $\mathfrak{Z}$ is center of $\mathfrak{G}$, and $[\mathfrak{G};\mathfrak{G}]$ is the commutant of $\mathfrak{G}$.

Let us consider an arbitrary group $\mathcal{G}$ and its arbitrary subgroup $\mathcal{H} \subset \mathcal{G}$ not containing normal subgroups of the group $\mathcal{G}$ and let $\mathcal{G}\ /\mathcal{H}$ be the set of left cosets. The group $\mathcal{G}$ is considered as a group of one-to-one transformations of the set $\mathcal{G}\ /\mathcal{H}$. These transformations are defined by multiplication in the group $\mathcal{G}$, $x \to gx$. Since the group $\mathcal{H}$ does not contain normal subgroups of the group $\mathcal{G}$, different elements of the group $\mathcal{G}$ define different transformations of the set $\mathcal{G}\ /\mathcal{H}$, i.e., for different elements $g_1, g_2 \in \mathcal{G}$ the transformations $x \to g_1x$ and $x \to g_2x$ are different. Let us prove it. Let the elements $g_1, g_2 \in \mathcal{G}$ define the same transformation on the set of left cosets $\mathcal{G}\ /\mathcal{H}$. Then, $\forall x \in \mathcal{G}\ \exists h \in \mathcal{H}$ such that $g_1x = g_2xh$. Therefore, $g_2^{-1}g_1x = xh$. Substituting $x = e$ we obtain $g_2^{-1}g_1 = h$. In this case, $\forall x \in \mathcal{G}\ hx = xh\ \forall x \in \mathcal{G}\ x^{-1}hx = x$. Therefore, the element $h \in \mathcal{H}$ generates a subgroup that is a normal subgroup of the group $\mathcal{G}$. This contradicts the assumption.

Let us define a subgroup $\widetilde{\mathcal{N}} \subset \mathcal{G}$ consisting of the so-called right multiplications; $\tilde{n} \in \widetilde{\mathcal{N}}$ if and only if $\forall x \in \mathcal{G}/\mathcal{H}\ \exists n \in \mathcal{G}$ such that $\tilde{n}x = xn$ i.e., $\forall g \in x\ \tilde{n}g = xnh_g$, and $h_g \in \mathcal{H}$. Such multiplication by element $\tilde{n}$ is called the right multiplication by element $n \in \mathcal{G}$. It is easy to prove that the group $\widetilde{\mathcal{N}}$ coincides with the center of the group $\mathcal{G}$. Indeed, $\tilde{n}gx = gxn = g\tilde{n}x, \forall \tilde{n} \in \widetilde{\mathcal{N}}\ \forall g \in \mathcal{G}\ \forall x \in \mathcal{G}/\mathcal{H}$. Therefore, $\tilde{n}g = g\tilde{n}$.

Elements $n \in \mathcal{G}$ such that $\forall g \in x\ \tilde{n}g = xnh_g\ h_g \in \mathcal{H}$ form a subgroup $\mathcal{N} \subset \mathcal{G}$, containing the center of the group $\mathcal{G}$ and the subgroup $\mathcal{H}$. Since right multiplications by elements $n \in \mathcal{G}$ define transformations on the set of left cosets $\mathcal{G}\ /\mathcal{H}$, then $\forall x \in \mathcal{G}\ \forall h \in \mathcal{H}$ $\exists h_1 \in \mathcal{H}$ such that $ghn = gnh_1$. Therefore, $hn = nn_1$, $n^{-1}hn = h_1$, i.e., the subgroup $\mathcal{H}$ is a normal subgroup of the group $\mathcal{N}$, $\mathcal{H} \lhd \mathcal{N}$. Right multiplication by any element $n \in \mathcal{N}$ in group $\mathcal{G}$ defines element $\tilde{n} \in \widetilde{\mathcal{N}} \subset \mathcal{G}$. Right multiplications by elements $h \in \mathcal{H}$. define identical transformation $e \in \widetilde{\mathcal{N}} \subset \mathcal{G}$. Therefore, $\widetilde{\mathcal{N}} = \mathcal{N}/\mathcal{H}$. We are most interested in the elements $n \in \mathcal{N}$ such that $n \notin \mathcal{Z}\mathcal{H}$, where $\mathcal{Z}$ is the center of the group $\mathcal{G}$.

Let us consider the Lie algebra $\mathfrak{G}$ of all infinitesimal affine transformations of a locally homogeneous space $\mathcal{M}$ of affine connection with a marked point $p \in \mathcal{M}$ and its stationary subalgebra $\mathfrak{H} \subset \mathfrak{G}$, $\mathcal{X} \in \mathfrak{H} \Leftrightarrow \mathcal{X}(p) = 0$. The groups $\mathfrak{G}$ and $\mathfrak{H}$ and point p are the same as defined at the end of introduction. Let $\mathcal{G}$ be a simply connected Lie group corresponding to the algebra $\mathfrak{G}$, $\mathcal{H} \subset \mathcal{G}$ be its subgroup corresponding to the subalgebra $\mathfrak{H} \subset \mathfrak{G}$, $[\mathfrak{G},\mathfrak{E}]$ be a commutant of the algebra $\mathfrak{G}$, and $\mathfrak{Z}$ be center of the algebra $\mathfrak{G}$: $dim\mathcal{M} = n$, $dim\mathcal{H} = k$, $dim\mathcal{G} = n + k$.

Preposition 2. *Let $\mathcal{H}$ be the Lie subgroup of the group $\mathcal{G}$ corresponding to subalgebra $\mathfrak{H} \subset \mathfrak{G}$. Let the subgroup $\mathcal{H} \subset \mathcal{G}$ be non-closed in the group $\mathcal{G}$ and $\overline{\mathcal{H}} \subset \mathcal{G}$ be the closure of the group $\mathcal{H}$ in $\mathcal{G}$. Let $\overline{\mathfrak{H}} \subset \mathfrak{G}$ be the Lie algebra of subgroup $\overline{\mathcal{H}} \subset \mathcal{G}$. Then, $\exists \mathcal{Y} \in \overline{\mathfrak{H}}$, $\mathcal{Y} \notin \mathfrak{H}$, such that the group $\mathcal{G}$*

contains right multiplications by elements of the local one-parameter subgroup $Exp(t\mathcal{Y})$ generated by the vector field $\mathcal{Y}$ for all sufficiently small t, $|t| < \delta$.

Proof. Let us consider a normal neighborhood $\mathcal{U} \subset \mathcal{M}$ of the point $\wp \in \mathcal{M}$, which, in normal coordinates, we identify with the ball $\mathcal{B} \subset \mathbb{R}^n$ centered at 0 and with a of radius 2δ. Let us consider also the neighborhood $\mathcal{U}_1 \subset \mathcal{U}$ of the point $\wp$, which we identify in the same normal coordinates with the ball $\mathcal{B}_1 \subset \mathcal{B} \subset \mathbb{R}^n$ with center at 0 and radius δ. Let us consider a sufficiently small neighborhood $\mathcal{V}$ of 0 of the vector space $\mathfrak{G} = \mathbb{R}^{n+k}$, which under the exponential mapping is identified with the neighborhood of the identity $\mathcal{V} \subset \mathcal{G}$. We assume that the elements $g \in \mathcal{V}$ are so close to the identity element that they define affine transformations of the set $\mathcal{U}_1$ into the set $\mathcal{U}$. Let $\mathcal{W} = \mathcal{V} \cap \mathcal{H}$. $\mathcal{W}$ is everywhere dense in the set $\overline{\mathcal{W}} = \mathcal{V} \cap \overline{\mathcal{H}}$, because $\mathcal{H}$ is everywhere dense in $\overline{\mathcal{H}}$. Let $\mathcal{W}_0 \subset \mathcal{W}$ be the connected component of the identity of the set $\mathcal{W} = \mathcal{V} \cap \mathcal{H}$ in the set $\overline{\mathcal{W}} = \mathcal{V} \cap \overline{\mathcal{H}}$. In normal coordinates, subset $\mathcal{W}_0 = \mathcal{V} \cap \mathfrak{H}$.

Let $\mathcal{Y} \in \overline{\mathfrak{H}}$, $\mathcal{Y} \notin \mathfrak{H}$. Since the set $\overline{\mathcal{H}}$ is the closure of the set $\mathcal{H}$, then for an arbitrary element $\hbar_t = Exp(t\mathcal{Y}) \in \mathcal{W}$ there exists a sequence $\hbar_n \in \mathcal{V} \cap \mathcal{H}$ converging to $\hbar_t$. It was shown in the classical work of A.I. Maltsev [1] that the group $\mathcal{H}$ is a normal subgroup of the group $\overline{\mathcal{H}}$, and the algebra $\mathfrak{H}$ is a normal subalgebra of the algebra $\overline{\mathfrak{H}}$. Therefore, right multiplication by $\hbar_t$ defines a local analytic transformation of the manifold $\mathcal{M}$. Let $\hbar_t = Exp(t_0\mathcal{X})$ for some t_0 and $\mathcal{X} \in \mathcal{W} = \mathcal{V} \cap \mathcal{H}$. Let us consider a neighborhood $\mathcal{U}_0 \subset \mathcal{U} \subset \mathcal{M}$ of a point $\wp$ (for example, a ball in normal coordinates), such that $\hbar_n^{-1}\mathcal{U}_0\hbar_n = Exp(-t_0\mathcal{X})\mathcal{U}_0 Exp(t_0\mathcal{X}) \in \mathcal{U}$ if $\hbar_n = Exp(t_0\mathcal{X}) \in \mathcal{V}$. In addition, let us consider a neighborhood $\mathcal{V}_0$ of the identity element $\mathcal{V}_0 \subset \mathcal{V} \subset \mathcal{G}$ such that $\hbar_n^{-1}\mathcal{U}_0\hbar_n = Exp(-t_0\mathcal{X})\mathcal{V}_0 Exp(t_0\mathcal{X}) \in \mathcal{V}$. Let us consider also a neighborhood $\mathcal{W}_0 \subset \mathcal{W}$, which is a connected component of the identity of the set $\mathcal{V}_0 \cap \mathcal{H}$. For all $g \in \mathcal{V}_0$, the sequence $\hbar_n^{-1} g \hbar_n$ converges to $\overline{\hbar}_t^{-1} g \overline{\hbar}_t \in \mathcal{V}$. Therefore, the sequence of local affine mappings $\hbar_n \in \mathcal{H}$ of the space $\mathcal{M}$ converges to the mapping $\widetilde{\hbar}_t$ defined by the inner automorphisms $g \mapsto \hbar_t^{-1} g \hbar_t$. Since these inner automorphisms are identical at unit $e \in \mathcal{G}$, local diffeomorphisms from $\mathcal{W}_0$ to $\mathcal{W}$ are defined by them. However, the analytic transformation, which is the limit of a sequence of affine transformations, is itself an affine transformation $\widetilde{\hbar}_t(x) \mapsto \hbar_t^{-1} x \hbar_t$ $\forall x \in \mathcal{V}_0$. Since $\hbar_t^{-1}\hbar\hbar_t \in \mathcal{H}$, then $\widetilde{\hbar}_t \in \mathcal{H}$.

Thus, the affine transformation $\widetilde{m}_t = \overline{\hbar}_t\widetilde{\hbar}_t$ given by formula $x \mapsto x\overline{\hbar}_1$ is a right multiplication by $\overline{\hbar}_t \notin \mathcal{H}$. Since the mappings $\overline{\hbar}_t$ and $\widetilde{m}_t$ are defined for all sufficiently small t and form local one-parameter transformation groups, they are generated by infinitesimal affine transformations $\mathcal{Y}$ and $\mathcal{Z}$. Moreover $\mathcal{Z} \in \mathfrak{Z}$, $\mathcal{Y} \in \overline{\mathfrak{H}}$, and $\mathcal{Y} \notin \mathfrak{H}$. Therefore, Preposition 2 is proved. □

Theorem 1. *Let $\mathfrak{G}$ be the Lie algebra of all infinitesimal affine transformations on a locally homogeneous analytic manifold $\mathcal{M}$ of an affine connection. Let $\mathfrak{H}$ be its stationary subalgebra, and $\mathfrak{Z}$ be the center of the algebra $\mathfrak{G}$. Let $\mathcal{G}$ be a simply connected subgroup generated by the algebra $\mathfrak{G}$ and $\mathcal{H}$ be its subgroup generated by the subalgebra $\mathfrak{H}$. If $\mathfrak{H} \cap (\mathfrak{Z} + [\mathfrak{G};\mathfrak{G}]) = \mathfrak{H} \cap [\mathfrak{G};\mathfrak{G}]$, then $\mathcal{H}$ is closed in $\mathcal{G}$.*

Proof. Let us assume the opposite. Consider the closure $\overline{\mathcal{H}}$ of the group $\mathcal{H}$ in $\mathcal{G}$ and the subalgebra $\overline{\mathfrak{H}} \subset \mathfrak{G}$ corresponding to the subgroup $\overline{\mathcal{H}} \subset \mathcal{G}$. The subalgebra $\mathfrak{H}$ is a normal subalgebra of the algebra $\overline{\mathfrak{H}}$ [1]. From the definition of subgroup $\mathfrak{H}$ a marked point $\wp \in \mathcal{M}$ given at the end of introduction point $\wp \in \mathcal{M}$, $\mathcal{X} \in \mathfrak{H} \Longleftrightarrow \mathcal{X}(\wp) = 0$. Let us consider a one-parameter subgroup $\overline{\hbar}_t \in \overline{\mathcal{H}}$, $\overline{\hbar}_t \notin \mathcal{H}$, defined by the vector field $\mathcal{Y} \in \overline{\mathfrak{H}}$, $\mathcal{Y} \notin \mathfrak{H}$. As proved in [1], there exists a torus $\mathcal{T}$ in a simple compact subgroup $\mathcal{P} \subset \mathcal{G}$ such that $\mathcal{H} \cap \mathcal{T}$ is everywhere dense winding of the torus $\mathcal{T}$. Therefore, we can choose $\mathcal{Y} \in \overline{\mathfrak{H}}$ such that $\overline{\hbar}_t \in \mathcal{T} \subset \mathcal{P}$. Thus, the one-parameter group generated by the vector field $\mathcal{Y}$ is a circle. Then, the infinitesimal affine transformation $\mathcal{Y}$ of tangent vectors to orbits of the local one-parameter group $\overline{\hbar}_t$ belongs to the algebra $\mathfrak{T}$ of the group $\mathcal{T}$ and hence $\mathcal{Y} \in \mathfrak{T} \subset \mathfrak{P}$, where $\mathfrak{P}$ is the Lie algebra of the group $\mathcal{P}$.

Since the vector field $\mathcal{Y}$ generating the local one-parameter group $\overline{\hbar}_t$ belongs to a compact subalgebra of the algebra $\mathfrak{G}$, it follows that $\mathcal{Y}$ belongs to the commutant $[\mathfrak{G}, \mathfrak{G}]$ of the algebra $\mathfrak{G}$ and does not belong to the center $\mathfrak{Z}$ of the algebra $\mathfrak{G}$. Let $\mathcal{K}$ be a maximal compact subgroup of a simply connected group $\mathcal{G}$ and $\overline{\hbar}_t \in \mathcal{K}$. Then, $\mathcal{G}$ is diffeomorphic to the direct product of $\mathcal{K}$ and a Euclidean space. Let $\widetilde{m}_t = \overline{\hbar}_t\widetilde{\hbar}_t$ be right multiplication by $\overline{\hbar}_t$ that is the same as in the proof of the previous preposition. The vector field $\mathcal{Z}$ of tangent vectors to the orbits of the local one-parameter group $\widetilde{m}_t$ of right multiplications by $\overline{\hbar}_t$ is an infinitesimal affine transformation and belongs to the center of the algebra of all infinitesimal affine transformations on $\mathcal{M}$, $\mathcal{Z} \in \mathfrak{Z}$. Let us prove that $\mathcal{Z} \notin [\mathfrak{G}, \mathfrak{G}]$. The vector field $\mathcal{Y}$ can be chosen so that the one-parameter group $Exp(t\mathcal{Y})$ generated by the vector field $\mathcal{Y}$ is diffeomorphic to a circle. Then, the one-parameter subgroup of right multiplications $Exp(t\mathcal{Z})$ is also diffeomorphic to the circle. If we assume that the center element $\mathcal{Z}$ belongs to the subgroup corresponding to commutant $[\mathfrak{G}, \mathfrak{G}]$, then $\mathcal{Z} \in \mathfrak{R}$, where $\mathfrak{R}$ is the maximal solvable subalgebra of the algebra $\mathfrak{G}$. In this case, the one-parameter subgroup $Exp(t\mathcal{Z})$ acting locally on the manifold $\mathcal{M}$ transforms the orbit of the radical $\mathcal{R}$ of the group $\mathcal{G}$ into itself and belongs to radial $\mathcal{R}$ of the group $\mathcal{G}$.

As $\mathcal{R}$ is the normal subgroup of group $\mathcal{G}$, then $Exp(-t\mathcal{Y})\mathcal{R}Exp(t\mathcal{Y}) = \mathcal{R}$. Therefore, $\mathcal{R} = Exp(t\mathcal{Z})\mathcal{R} = \mathcal{R}Exp(t\mathcal{Y}) = Exp(t\mathcal{Y})Exp(-t\mathcal{Y})\mathcal{R}Exp(t\mathcal{Y}) = Exp(t\mathcal{Y})$. Therefore, $Exp(t\mathcal{Y}) \in \mathcal{R}$, and this contradicts the Levi–Maltsev decomposition $\mathcal{G} = \mathcal{R}\mathcal{P}$, where $\mathcal{P}$ is the maximal semisimple subgroup containing $Exp(t\mathcal{Y})$. Thus, $\mathcal{Z} \notin [\mathfrak{G}, \mathfrak{G}]$. Therefore, $\mathfrak{H} \cap (\mathfrak{Z} + [\mathfrak{G}; \mathfrak{G}]) \neq \mathfrak{H} \cap [\mathfrak{G}; \mathfrak{G}]$. This completes the proof of the theorem by contradiction. □

Theorem 2. *Let $\mathfrak{G}$ be the Lie algebra of all infinitesimal affine transformations on a locally homogeneous analytic manifold $\mathcal{M}$ of an affine connection; $\mathfrak{H}$ is its stationary subalgebra, $\mathfrak{Z}$ is the center of the algebra $\mathfrak{G}$, and $\mathfrak{R}$ is its radical (maximal solvable subalgebra) of the group $\mathcal{G}$. Let $\mathcal{G}$ be a simply connected subgroup generated by the algebra $\mathfrak{G}$ and $\mathcal{H}$ be its subgroup generated by the subalgebra $\mathfrak{H}$. Then, for any maximal semisimple algebra $\mathfrak{P} \subset \mathfrak{G}$ ($\mathfrak{P} + \mathfrak{R} = \mathfrak{G}$ by Levi-Maltsev decomposition), the following condition of closedness of a subgroup $\mathcal{H}$ takes place. If $(\mathfrak{P} + \mathfrak{Z}) \cap \mathfrak{H} = \mathfrak{P} \cap \mathfrak{H}$, then $\mathcal{H}$ is closed in $\mathcal{G}$.*

Proof. Assume the opposite. Let subgroup $\mathcal{H}$ be unclosed in $\mathcal{G}$ and let us consider the closure $\overline{\mathcal{H}}$ of the subgroup $\mathcal{H}$ in $\mathcal{G}$. Let us also consider the one-parameter subgroup $\overline{\hbar}_t \in \overline{\mathcal{H}}$, $\overline{\hbar}_t \notin \mathcal{H}$ and one-parameter subgroup $\widetilde{m}_t = \overline{\hbar}_t\widetilde{\hbar}_t$ of right multiplications by the elements of the one-parameter group of local affine transformation $\overline{\hbar}_t$ (as in the proof of the previous preposition) generated by right multiplication in the group $\mathcal{G}$ the same way as in the proof of Theorem 1,. Let $\overline{\mathcal{X}}$ be the vector field (infinitesimal affine transformation) of tangent vectors to the orbits' local one-parameter local affine transformation group $\overline{\hbar}_t^{-1}$ and $\mathcal{Z}$ be the vector field of the one-parameter local affine transformations group $\widetilde{m}_t$.

Let $\mathfrak{P}$ be a maximal semisimple subalgebra of $\mathfrak{G}$ containing a vector field $\overline{\mathcal{X}}$, $\overline{\mathcal{X}} \in \mathfrak{P} \subset \mathfrak{G}$. Let us prove that $\mathcal{Z} + \overline{\mathcal{X}} \in \mathfrak{H}$ and $\mathcal{Z} + \overline{\mathcal{X}} \notin \mathfrak{P}$. Let $\mathcal{R}$ be a radical (maximal solvable subgroup) of a simply connected Lie group $\mathcal{G}$. The subgroup $\mathcal{R}$ corresponds to the subalgebra $\mathfrak{R}$ and the semisimple subgroup $\mathcal{P}$ corresponds to the subalgebra $\mathfrak{P}$. Then, $\mathcal{R}$ is a normal subgroup of the group $\mathcal{G}$, $\mathfrak{P}$ is a normal subalgebra of the algebra $\mathfrak{G}$, and $\mathcal{R} \cap \mathcal{P} = e$, $\mathfrak{R} \cap \mathfrak{P} = 0$. Levi–Maltsev decomposition $\mathcal{G} = \mathcal{R}\mathcal{P}$ takes place. We assume that a semisimple algebra $\mathfrak{P}$ has no center, and the center of the group $\mathcal{G}$ is contained in $\mathfrak{R}$. The group $\mathcal{G}$ contains an open neighborhood of the identity (chunk of a group) acting as a local group of local affine transformations in a neighborhood of the marked point $p \in \mathcal{M}$. Since $\widetilde{m}_t$ belongs to the center of the group $\mathcal{G}$, $\widetilde{m}_t \in \mathcal{R}$, and since the subgroup $\mathcal{H}$ is a normal subgroup of the group $\overline{\mathcal{H}}$, ([3]), then $\overline{\hbar}_t^{-1} z_t\mathcal{H} = \overline{\hbar}_t^{-1}\mathcal{H}\overline{\hbar}_t = \mathcal{H}$. Consequently, the local affine transformations $\overline{\hbar}_t^{-1} z_t$ leave the point p fixed and, therefore, transformations $\overline{\hbar}_t^{-1} z_t$ belong to the stationary subgroup $\mathcal{H}$. It follows that infinitesimal affine transformation $\mathcal{Z} + \overline{\mathcal{X}}$ which generates local group $\overline{\hbar}_t^{-1} z_t$ belongs to stationary subalgebra $\mathfrak{H}$. However, since $\overline{\mathcal{X}} \in \mathfrak{P}$ and $\mathcal{Z} \notin \mathfrak{P}$, then $\left(\mathcal{Z} + \overline{\mathcal{X}}\right) \notin \mathfrak{P}$, and since $\left(\mathcal{Z} + \overline{\mathcal{X}}\right) \in \mathfrak{H}$, we can conclude that the statement $(\mathfrak{P} + \mathfrak{Z}) \cap \mathfrak{H} \neq \mathfrak{P} \cap \mathfrak{H}$ is true for the chosen maximal semisimple algebra $\mathfrak{P}$. This proves the theorem by contradiction. $\square$

3. Discussion

The properties of the Lie algebra indicated in Theorem 1 are sufficient conditions for the closedness of Lie subgroup $\mathcal{H}$ in the Lie group $\mathcal{G}$. In the case that $\mathcal{G}$ is a simply connected Lie group corresponding to the Lie algebra $\mathfrak{G}$ of all infinitesimal affine transformations of a locally homogeneous analytic manifold $\mathcal{M}$ with an analytic affine connection, $\mathcal{H} \subset \mathcal{G}$ is a subgroup corresponding to the stationary subalgebra $\mathfrak{H} \subset \mathfrak{G}$. The problem is to find the necessary condition for closedness of $\mathcal{H}$ in $\mathcal{G}$. Therefore, there is a problem of proving the necessity of the conditions formulated in Theorems 1 and 2 or of finding a really necessary and sufficient condition for the closedness of subgroup $\mathcal{H}$ in group $\mathcal{G}$. This question has not been solved not only for a locally homogeneous manifold of an affine connection, but also not for a Riemannian analytic locally homogeneous manifold. In the case of a Riemannian manifold, this problem is connected with the investigation of a Riemannian metric, which is left invariant with respect to the action of the group $\mathcal{G}$ on a locally homogeneous manifold and right invariant under the action of the normalizer $\mathcal{N}$ of the group $\mathcal{H}$.

Another subject of further research is the study of the regular analytic extension of a locally given affine connection. Such extension should lead to generalization of the notion of completeness. For a locally given Riemannian metric there is a normal extension to the so-called pseudocomplete manifold. It is an attractive object in a category whose objects are locally isometric, simply connected, oriented Riemannian analytic manifolds, and morphisms of this category are locally isometric, preserving orientation covering mappings $f : \mathcal{M} \to \mathcal{N}$ whose images $f(\mathcal{M}) \subset \mathcal{N}$ are proper open subsets of $\mathcal{N}$ [4]. This definition is also applicable to analytic manifolds with an analytic affine connection. However, the study of pseudocomplete manifolds in the general case is difficult.

In the case of a Riemannian metric, for which the Lie algebra of Killing vector fields has no center, there exists an analytic extension to the so-called quasicomplete (compressed) manifold, which is unique in the class of all locally isometric manifolds. A compressed manifold has the maximum possible symmetry, i.e., any local isometry of this manifold into itself can be analytically extended to a global isometry [4]. The construction of a quasicomplete manifold is carried out by using the factorization of a Riemannian analytic manifold whose Lie algebra of Killing vector fields has no center by a pseudogroup of local isometries, preserving orientation and all Killing fields. More precisely, we factorize a manifold without fixed points of local isometries, which preserves the orientation and all Killing vector fields. Such factorization is well defined, since the set of fixed points of such local isometries is an analytic subset of a codimension no less than two. This factorization is followed by analytic extension. In the case of an analytic manifold with an affine connection, such a factorization is impossible, since the set of fixed points of a local orientation preserving affine transformation can be an analytic subset of codimension one.

It seems possible to use the factorization by the pseudogroup of local isometries of preserving orientation and Killing vector fields to manifolds whose Lie algebra $\mathfrak{G}$ of all Killing vector fields satisfies the condition of Theorem 1: $\mathfrak{H} \cap (\mathfrak{Z} + [\mathfrak{G};\mathfrak{G}]) = \mathfrak{H} \cap [\mathfrak{G};\mathfrak{G}]$, where $\mathfrak{H}$ is the stationary subalgebra of algebra $\mathfrak{G}$, $\mathfrak{Z}$ is the center of $\mathfrak{G}$, and $[\mathfrak{G};\mathfrak{G}]$ is the commutant of $\mathfrak{G}$. As a result of gluing all locally isometric manifolds obtained after factorization, we obtain a manifold K (which is not Riemannian). We consider all possible fiber bundles over a manifold K whose fibers are diffeomorphic to the product of a Euclidean space and a torus and an analytic extension of a locally given Riemannian metric to the metric on these bundles. In this way, it is probably possible to construct a Riemannian analytic manifold having the property of analytic extensibility of any local isometry into itself to the global isometry of this manifold.

It is also appropriate to consider specific Riemannian locally given analytic metrics and analytic affine connections.

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References

1. Maltsev, M.A. On the theory of Lie groups in the large. *Mathem. Sb.* **1945**, *16*, 163–190.
2. Chevalley, C. *Theory of Lie Groups*; Princeton University Press: Princeton, NJ, USA, 1946; Volume 1.
3. Mostow, G.D. Extensibility of Local Lie Groups of Transformations and Groups on Surfaces. *Ann. Math.* **1950**, *52*, 606–636. [CrossRef]
4. Helgason, S. *Differential Geometry, Lie Groups and Symmetric Spaces*; Academic Press, Inc.: Cambridge, MA, USA, 1978.
5. Kobayashi, S.; Nomidzu, K. *Foundations of Differential Geometry*; Interscience Publisher: New York, NY, USA, 1969.
6. Popov, V.A. Analytic Extension of Riemannian Manifolds and Local Isometries. *Mathematics* **2020**, *8*, 1855. [CrossRef]
7. Smith, G.H. Analytic extension of Riemannian manifolds. *Bull. Austral. Math. Soc.* **1978**, *18*, 147–148. [CrossRef]
8. Popov, V.A. Extendability of Locally Defined Isometries of a Pseudo-Riemannian Manifold. *J. Math. Sb.* **1988**, *135*, 45–64. [CrossRef]
9. Popov, V.A. On Closeness of Stationary Subgroup of Affine Transformation Groups. *Lobachevskii J. Math.* **2017**, *38*, 724–729. [CrossRef]