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Differential Subordination and Differential Superordination for Classes of Admissible Multivalent Functions Associated with a Linear Operator

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Abstract: In this paper, we first introduce a linear integral operator $\Im_p(a,c,\mu)$ $(\mu > 0;\ a,c \in \mathbb{R};\ c > a > -\mu p;\ p \in \mathbb{N}^+ := \{1,2,3,\ldots\})$, which is somewhat related to a rather specialized form of the Riemann–Liouville fractional integral operator and its varied form known as the Erdélyi–Kober fractional integral operator. We then derive some differential subordination and differential superordination results for analytic and multivalent functions in the open unit disk $\mathbb{U}$, which are associated with the above-mentioned linear integral operator $\Im_p(a,c,\mu)$. The results presented here are obtained by investigating appropriate classes of admissible functions. We also obtain some Sandwich-type results.

Keywords: analytic functions; univalent; multivalent functions; differential subordination; differential superordination; sandwich-type theorems; admissible function classes; linear operator

MSC: 30C45; 30C80

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1. Introduction

Let $\mathcal{H}(\mathbb{U})$ be the class of functions that are analytic in the open unit disk

$$\mathbb{U} = \{z : z \in \mathbb{C} \quad \text{and} \quad |z| < 1\}.$$

Additionally, let $\mathcal{H}[a_0, n]$ be the subclass of $\mathcal{H}(\mathbb{U})$, which consists of functions of the following form:

$$f(z) = a_0 + a_n z^n + a_{n+1} z^{n+1} + \ldots$$
$$\left(a_0 \in \mathbb{C};\ n \in \mathbb{N}^+ := \{1,2,3,\ldots\}\right).$$

Clearly, for the familiar class $\mathcal{A}(p)$ of analytic and multivalent (or p-valent) functions in $\mathbb{U}$, with the power-series expansion given by

$$f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n \qquad (p \in \mathbb{N}^+;\ z \in \mathbb{U}), \tag{1}$$

we have

$$\mathcal{A}(p) := \mathcal{H}[0,p] \quad (a_p \equiv 1).$$

In the theory and widespread applications of fractional calculus (see, for example, [1,2]; see also the recent survey-cum-expository review article [3]), one of the most popular operators happens to be the Riemann–Liouville fractional integral operator of order $\alpha \in \mathbb{C}$ $\big(\Re(\alpha) > 0\big)$ defined by

$$\big(I_{0+}^{\alpha} f\big)(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-\tau)^{\alpha-1} f(\tau)\, \mathrm{d}\tau \qquad \big(x > 0;\ \Re(\alpha) > 0\big) \tag{2}$$

in terms of the familiar (Euler's) Gamma function $\Gamma(\alpha)$. An interesting variant of the Riemann–Liouville operator I_{0+}^{α}, which is known as the Erdélyi–Kober fractional integral operator of order $\alpha \in \mathbb{C}$ $\big(\Re(\alpha) > 0\big)$ defined by

$$\big(I_{0+;\sigma,\eta}^{\alpha} f\big)(x) = \frac{\sigma x^{-\sigma(\alpha+\eta)}}{\Gamma(\alpha)} \int_0^x \tau^{\sigma(\eta+1)-1} \big(x^{\sigma} - \tau^{\sigma}\big)^{\alpha-1} f(\tau)\, \mathrm{d}\tau \tag{3}$$

$$\big(x > 0;\ \Re(\alpha) > 0\big),$$

which corresponds essentially to (2) when $\sigma - 1 = \eta = 0$, since

$$\big(I_{0+;1,0}^{\alpha} f\big)(x) = x^{-\alpha} \big(I_{0+}^{\alpha} f\big)(x) \qquad \big(x > 0;\ \Re(\alpha) > 0\big).$$

Motivated essentially by the special case of the definition (3) when $x = \sigma = 1$, $\eta = a - 1$, and $\alpha = c - a$, here we consider a linear integral operator $\Im_p(a,c,\mu)$ defined for a function $f \in \mathcal{A}(p)$ by (see [4])

$$\Im_p(a,c,\mu)f(z) = \frac{\Gamma(c+\mu p)}{\Gamma(a+\mu p)\Gamma(c-a)} \int_0^1 \tau^{a-1} (1-\tau)^{c-a-1} f\big(z\tau^{\mu}\big)\, \mathrm{d}\tau$$

$$(\mu > 0;\ a, c \in \mathbb{R};\ c > a > -\mu p;\ p \in \mathbb{N}^+).$$

When evaluated by means of the Eulerian Beta-function integral:

$$\mathrm{B}(\alpha,\beta) := \begin{cases} \displaystyle\int_0^1 \tau^{\alpha-1} (1-\tau)^{\beta-1}\, \mathrm{d}\tau & \big(\min\{\Re(\alpha), \Re(\beta)\} > 0\big) \\[2ex] \dfrac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} & \big(\alpha, \beta \in \mathbb{C} \setminus \mathbb{Z}_0^-\big), \end{cases}$$

we readily find that

$$\Im_p(a,c,\mu)f(z) = \begin{cases} z^p + \dfrac{\Gamma(c+\mu p)}{\Gamma(a+\mu p)} \displaystyle\sum_{n=p+1}^{\infty} \frac{\Gamma(a+\mu n)}{\Gamma(c+\mu n)}\, a_n z^n & (c > a) \\[2ex] f(z) & (c = a), \end{cases} \tag{4}$$

$\mathbb{Z}_0^-$ being the set of nonpositive integers. It is easy to deduce from (4) that

$$z\big(\Im_p(a,c,\mu)f(z)\big)' = \left(\frac{a}{\mu} + p\right) \Im_p(a+1,c,\mu)f(z) - \frac{a}{\mu}\, \Im_p(a,c,\mu)f(z). \tag{5}$$

We also note that the linear operator $\Im_p(a,c,\mu)$ is a generalization of many other integral operators, which were considered in earlier works. For example, for $f \in \mathcal{A}(p)$ we have the following special cases:

(i) Putting $p = 1$,, we obtain the operator $\tilde{I}(a,c;\mu)$ studied by Raina and Sharma (see [5] with $m = 0$);

(ii) Putting $a = \beta$, $c = \beta + 1$ and $\mu = 1$, we obtain the operator $\Im_p^\beta$ $(\beta > -p)$, which was studied by Saitoh et al. [6];
(iii) Putting $a = \beta$, $c = \alpha + \beta - \gamma + 1$ and $\mu = 1$, we obtain the operator $\mathfrak{R}_{\beta,p}^{\alpha,\gamma}$ $(\gamma > 0;\ \alpha \geqq \gamma - 1;\ \beta > -p)$, which was studied by Aouf et al. [7];
(iv) Putting $a = \beta$, $c = \alpha + \beta$ and $\mu = 1$, we obtain the operator $\chi_{\beta,p}^\alpha$ $(\alpha \geqq 0;\ \beta > -p)$, which was studied by Liu and Owa [8];
(v) Putting $p = 1$, $a = \beta$, $c = \alpha + \beta$ and $\mu = 1$, we obtain the operator $\mathfrak{R}_\beta^\alpha$ $(\alpha \geqq 0;\ \beta > -1)$, which was studied by Jung et al. [9];
(vi) Putting $p = 1$, $a = \alpha - 1$, $c = \beta - 1$ and $\mu = 1$, we obtain the operator $L(\alpha, \beta)$ $\left(\alpha, \beta \in \mathbb{C} \setminus \mathbb{Z}_0^-\right)$, which was studied by Carlson and Shaffer [10];
(vii) Putting $p = 1$, $a = a - 1$, $c = v$ and $\mu = 1$, we obtain the operator $I_{a,v}$ $(a > 0;\ v \geqq -1)$, which was studied by Choi et al. [11];
(viii) Putting $p = 1$, $a = \alpha$, $c = 0$ and $\mu = 1$, we obtain the operator $\mathfrak{D}^\alpha$ $(\alpha > -1)$, which was studied by Ruscheweyh [12];
(ix) Putting $p = 1$, $a = \alpha = 1$, $c = m$ and $\mu = 1$, we obtain the operator I_m $(m \in \mathbb{N}_0^+ := \mathbb{N}^+ \cup \{0\})$, which was studied by Noor [13];
(x) Putting $p = 1$, $a = \beta$, $c = \beta + 1$ and $\mu = 1$, we obtain the operator $\mathfrak{I}_\beta$, which was studied by Bernardi [14];
(xi) Putting $p = 1$, $a = 1$, $c = 2$ and $\mu = 1$, we obtain $\mathfrak{I}$, which was studied by Libera [15].

By the principle of subordination between analytic functions, given $f, g \in \mathcal{H}(\mathbb{U})$, the function $f(z)$ is said to be subordinate to $g(z)$ or, equivalently, the function $g(z)$ is said to be superordinate to $f(z)$, if there exists a Schwarz function $w(z)$, which (by definition) is analytic in $\mathbb{U}$ with

$$w(0) = 0 \quad \text{and} \quad |w(z)| < 1 \qquad (z \in \mathbb{U}),$$

such that $f(z) = g\big(w(z)\big)$. In such a case, we write $f(z) \prec g(z)$. Furthermore, if the function $g(z)$ is univalent in $\mathbb{U}$, then we have the following equivalence (see, for example, [16]; see also [17,18]):

$$f(z) \prec g(z) \iff f(0) = g(0) \quad \text{and} \quad f(\mathbb{U}) \subset g(\mathbb{U}).$$

We denote by $\wp$ the set of all functions χ that are injective on $\overline{\mathbb{U}} \setminus E(\chi)$, where

$$E(\chi) = \{\varsigma : \varsigma \in \partial\mathbb{U} \quad \text{and} \quad \lim_{z \to \varsigma} f(z) = \infty\}$$

and are such that $\chi'(\varsigma) \neq 0$ for $\varsigma \in \partial\mathbb{U} \setminus E(\chi)$. We also denote by $\wp(\mathfrak{a})$ the subclass of $\wp$ for which $\chi(0) = \mathfrak{a}$ and let

$$\wp(0) = \wp_0 \quad \text{and} \quad \wp(1) = \wp_1.$$

Definition 1 (see ([18], p. 27, Definition 2.3a)). *Let Ω be a set in $\mathbb{C}$, $\chi \in \wp$ and $n \in \mathbb{N}^+$. The class $\Psi_n[\Omega, \chi]$ of admissible functions consists of the functions $\psi : \mathbb{C}^3 \times \mathbb{U} \to \mathbb{C}$ that satisfy the admissibility condition $\psi(r, s, t; z) \notin \Omega$ whenever*

$$r = \chi(\varsigma), \quad s = k\varsigma\chi'(\varsigma) \quad \text{and} \quad \Re\left(\frac{t}{s} + 1\right) \geqq k\Re\left(1 + \frac{\varsigma\chi''(\varsigma)}{\chi'(\varsigma)}\right),$$

where $z \in \mathbb{U}$, $\varsigma \in \partial\mathbb{U} \setminus E(\chi)$ and $k \geqq n$. For simplicity, we write $\Psi_1[\Omega, \chi]$ as $\Psi[\Omega, \chi]$. In particular, if we set

$$\chi(z) = \left(\frac{Mz + \mathfrak{a}}{M + \bar{\mathfrak{a}}z}\right)M \qquad (M > 0;\ |\mathfrak{a}| < M),$$

then

$$\chi(\mathbb{U}) = U_M = \{w : |w| < M\}, \quad \chi(0) = \mathfrak{a}, \quad E(\chi) = \phi \quad \text{and} \quad \chi \in \wp(\mathfrak{a}).$$

In this case, we set

$$\Psi_n[\Omega, M, \mathfrak{a}] = \Psi_n[\Omega, \chi]$$

and, in the special case when the set $\Omega = U_M$, the resulting class is simply denoted by $\Psi_n[M, \mathfrak{a}]$.

Definition 2 (see ([19], p. 817, Definition 3))**.** *Let Ω be a set in $\mathbb{C}$ and $\chi \in \mathcal{H}[\mathfrak{a}, n]$ with $\chi'(z) \neq 0$. The class $\Psi'_n[\Omega, \chi]$ of admissible functions consists of the functions $\psi : \mathbb{C}^3 \times \overline{\mathbb{U}} \to \mathbb{C}$ that satisfy the admissibility condition $\psi(r, s, t; \varsigma) \in \Omega$ whenever*

$$r = \chi(z), \quad s = \frac{z\chi'(z)}{m} \quad \text{and} \quad \Re\left(\frac{t}{s} + 1\right) \leqq \frac{1}{m}\,\Re\left(1 + \frac{z\chi''(z)}{\chi'(z)}\right),$$

where $z \in \mathbb{U}$, $\varsigma \in \partial\mathbb{U}$ and $m \geqq n \geqq 1$. In particular, we write $\Psi'_1[\Omega, \chi]$ as $\Psi'[\Omega, \chi]$.

Here, in our present investigation, we need the following lemmas, which are proved by Miller and Mocanu (see [18,19]).

Lemma 1 (see ([18], p. 28, Theorem 2.3b))**.** *Let $\psi \in \Psi_n[\Omega, \chi]$ with $\chi(0) = \mathfrak{a}$. If the function $\omega(z)$ given by*

$$\omega(z) = \mathfrak{a} + a_n z^n + a_{n+1} z^{n+1} + \ldots$$

is analytic in $\mathbb{U}$ and satisfies the following inclusion relation:

$$\psi\big(\omega(z), z\omega'(z), z^2\omega''(z); z\big) \in \Omega,$$

then $\omega(z) \prec \chi(z)$.

Lemma 2 (see ([19], p. 818, Theorem 1))**.** *Let $\psi \in \Psi'_n[\Omega, \chi]$ with $\chi(0) = \mathfrak{a}$. If $\omega(z) \in \wp(\mathfrak{a})$ and the function $\psi\big(\omega(z), z\omega'(z), z^2\omega''(z); z\big)$ is univalent in $\mathbb{U}$, then the following set inclusion:*

$$\Omega \subset \left\{\psi\big(\omega(z), z\omega'(z), z^2\omega''(z); z\big) : z \in \mathbb{U}\right\}$$

implies that $\chi(z) \prec \omega(z)$.

In this paper, we determine the sufficient conditions for certain specified classes of admissible functions of analytic and multivalent (or p-valent) functions that are associated with the linear operator $\Im_p(a, c, \mu)$ so that

$$\chi_1(z) \prec \left[\Im_p(a, c, \mu) f(z)\right]^{\gamma} \prec \chi_2(z)$$

and

$$\chi_1(z) \prec \left(\frac{\Im_p(a, c, \mu) f(z)}{z^{p-1}}\right)^{\gamma} \prec \chi_2(z),$$

where the functions $\chi_1(z)$ and $\chi_2(z)$ are univalent in $\mathbb{U}$. We also derive some differential sandwich-type results. Similar problems for subordination or superordinations for analytic functions were studied by Aghalary et al. [20], Ali et al. [21], Kim and Srivastava [22], Shanmugam et al. [23], Frasin [24], and other authors.

2. Subordination Results Involving the Operator $\Im_p(a, c, \mu)$

Unless otherwise mentioned, we suppose throughout this paper that

$$\gamma > 0, \quad \mu > 0, \quad a, c \in \mathbb{R} \ \ \big(c > a > -\mu p;\ p \in \mathbb{N}^+\big) \quad \text{and} \quad z \in \mathbb{U}.$$

Moreover, all powers are assumed to be the principal values.

Definition 3. *Let Ω be a set in $\mathbb{C}$ and $\chi \in \wp_0 \cap \mathcal{H}[0, \gamma p]$. The class $\Phi_1[\Omega, \chi, \gamma]$ of admissible functions consists of the functions $\varphi : \mathbb{C}^3 \times \mathbb{U} \to \mathbb{C}$ that satisfy the following admissibility condition:*

$$\varphi(u, v, w; z) \notin \Omega$$

whenever

$$u = \chi(\varsigma), \quad v = \frac{k\varsigma\chi'(\varsigma) + \frac{\gamma\mathfrak{a}}{\mu}\chi(\varsigma)}{\gamma\left(\frac{\mathfrak{a}}{\mu} + p\right)}$$

and

$$\Re\left(\frac{\left(\frac{\mathfrak{a}}{\mu} + p\right)^2 w - \left(\frac{\mathfrak{a}}{\mu} + p\right)\left(\frac{2\gamma\mathfrak{a}+1}{\mu}\right)v + \gamma\left(\frac{\mathfrak{a}}{\mu}\right)^2 u}{\frac{\mathfrak{a}}{\mu}(v-u) + pv}\right)$$
$$\geqq k\Re\left(1 + \frac{\varsigma\chi''(\varsigma)}{\chi'(\varsigma)}\right) \qquad (k > 0), \tag{6}$$

where $z \in \mathbb{U}$, $\varsigma \in \partial\mathbb{U} \setminus E(\chi)$ *and* $k \geqq \gamma p$.

We now state and prove our first result as Theorem 1 below.

Theorem 1. *Let* $\varphi \in \Phi_1[\Omega, \chi, \gamma]$. *If the function* $f(z) \in \mathcal{A}(p)$ *satisfies the following set inclusion*:

$$\begin{aligned}\Big\{\varphi\Big(&\left[\Im_p(a,c,\mu)f(z)\right]^\gamma, \left[\Im_p(a,c,\mu)f(z)\right]^{\gamma-1}\Im_p(a+1,c,\mu)f(z),\\ &(\gamma-1)\left[\Im_p(a,c,\mu)f(z)\right]^{\gamma-2}\cdot\left[\Im_p(a+1,c,\mu)f(z)\right]^2 + \frac{a+1+\mu p}{a+\mu p}\\ &\cdot\left[\Im_p(a,c,\mu)f(z)\right]^{\gamma-1}\,\Im_p(a+2,c,\mu)f(z); z\Big) : z \in \mathbb{U}\Big\} \subset \Omega,\end{aligned} \tag{7}$$

then

$$\left[\Im_p(a,c,\mu)f(z)\right]^\gamma \prec \chi(z).$$

Proof. Define the analytic function $\omega(z)$ in $\mathbb{U}$ by

$$\omega(z) = \left[\Im_p(a,c,\mu)f(z)\right]^\gamma. \tag{8}$$

Differentiating (8) with respect to z and using the identity (5), we obtain

$$\left[\Im_p(a,c,\mu)f(z)\right]^{\gamma-1}\Im_p(a+1,c,\mu)f(z) = \frac{z\omega'(z) + \frac{\gamma a}{\mu}\omega(z)}{\gamma\left(\frac{a}{\mu} + p\right)}. \tag{9}$$

Further computations show that

$$\begin{aligned}&(\gamma-1)\left[\Im_p(a,c,\mu)f(z)\right]^{\gamma-2}\left(\Im_p(a+1,c,\mu)f(z)\right)^2\\ &\qquad + \frac{a+1+\mu p}{a+\mu p}\left[\Im_p(a,c,\mu)f(z)\right]^{\gamma-1}\Im_p(a+2,c,\mu)f(z)\\ &= \frac{z^2\omega''(z) + \left(1 + \frac{2\gamma a+1}{\mu}\right)z\omega'(z) + \frac{\gamma a(\gamma a+1)}{\mu^2}\omega(z)}{\gamma\left(\frac{a}{\mu} + p\right)^2}.\end{aligned} \tag{10}$$

We now define the following transformations for $\varphi : \mathbb{C}^3 \times \mathbb{U} \to \mathbb{C}$:

$$u(r,s,t) = r, \quad v(r,s,t) = \frac{s + \frac{\gamma a}{\mu}r}{\gamma\left(\frac{a}{\mu} + p\right)}$$

and

$$w(r,s,t) = \frac{t + \left(1 + \frac{2\gamma a+1}{\mu}\right)s + \frac{\gamma a(\gamma a+1)}{\mu^2}r}{\gamma\left(\frac{a}{\mu} + p\right)^2}. \tag{11}$$

We also set

$$\begin{aligned} \psi(r,s,t;z) &= \varphi(u,v,w;z) \\ &= \varphi\left(r, \frac{s + \frac{\gamma a}{\mu}r}{\gamma\left(\frac{a}{\mu} + p\right)}, \frac{t + \left(1 + \frac{2\gamma a+1}{\mu}\right)s + \frac{\gamma a(\gamma a+1)}{\mu^2}r}{\gamma\left(\frac{a}{\mu} + p\right)^2}; z\right). \end{aligned} \tag{12}$$

Then, by using the Equations (8)–(12), we obtain

$$\begin{aligned} &\psi(\omega(z), z\omega'(z), z^2\omega''(z); z) \\ &= \varphi\Big(\left[\Im_p(a,c,\mu)f(z)\right]^{\gamma}, \left[\Im_p(a,c,\mu)f(z)\right]^{\gamma-1} \Im_p(a+1,c,\mu)f(z), \\ &(\gamma-1)\left[\Im_p(a,c,\mu)f(z)\right]^{\gamma-2} \cdot \left[\Im_p(a+1,c,\mu)f(z)\right]^2 \\ &+ \frac{a+1+\mu p}{a+\mu p}\left[\Im_p(a,c,\mu)f(z)\right]^{\gamma-1} \Im_p(a+2,c,\mu)f(z); z\Big). \end{aligned} \tag{13}$$

Thus, clearly, the Equation (7) becomes

$$\psi(\omega(z), z\omega'(z), z^2\omega''(z); z) \in \Omega.$$

The proof of Theorem 1 is completed if it can be shown that the admissibility condition for $\varphi \in \Phi_1[\Omega, \chi, \gamma]$ is equivalent to the admissibility condition for ψ as given in Definition 1. For this purpose, we note that

$$\frac{t}{s} + 1 = \frac{\left(\frac{a}{\mu} + p\right)^2 w - \left(\frac{a}{\mu} + p\right)\left(\frac{2\gamma a+1}{\mu}\right)v + \gamma\left(\frac{a}{\mu}\right)^2 u}{\frac{a}{\mu}(v-u) + pv},$$

and hence that $\psi \in \Psi_{\gamma p}[\Omega, \chi]$. Consequently, by applying Lemma 1, we have

$$\omega(z) \prec \chi(z) \qquad \text{or} \qquad \left[\Im_p(a,c,\mu)f(z)\right]^{\gamma} \prec \chi(z),$$

which proves Theorem 1. □

In the case when $\Omega \neq \mathbb{C}$ is a simply-connected domain, then $\Omega = h(\mathbb{U})$ for some conformal mapping $h(z)$ of $\mathbb{U}$ onto Ω. In this case, the class $\Phi_1[h(\mathbb{U}), \chi, \gamma]$ is written as $\Phi_1[h, \chi, \gamma]$.

The following result is an immediate consequence of Theorem 1.

Theorem 2. *Let $\varphi \in \Phi_1[h, \chi, \gamma]$. If the function $f(z) \in \mathcal{A}(p)$ satisfies the following subordination relation:*

$$\varphi\left(\left[\Im_p(a,c,\mu)f(z)\right]^\gamma,\ \left[\Im_p(a,c,\mu)f(z)\right]^{\gamma-1}\Im_p(a+1,c,\mu)f(z),\right.$$
$$(\gamma-1)\left[\Im_p(a,c,\mu)f(z)\right]^{\gamma-2}\left[\Im_p(a+1,c,\mu)f(z)\right]^2+\frac{a+1+\mu p}{a+\mu p} \tag{14}$$
$$\left.\cdot\left[\Im_p(a,c,\mu)f(z)\right]^{\gamma-1}\Im_p(a+2,c,\mu)f(z);z\right)\prec h(z),$$

then

$$\left[\Im_p(a,c,\mu)f(z)\right]^\gamma\prec\chi(z).$$

Our next result is an extension of Theorem 1 to the case when the behavior of $\chi(z)$ on $\partial\mathbb{U}$ is not known.

Theorem 3. *Let $\Omega\subset\mathbb{C}$ and suppose that the function $\chi(z)$ is univalent in $\mathbb{U}$ with $\chi(0)=0$. Additionally, let $\varphi\in\Phi_1[\Omega,\chi_\rho,\gamma]$ for some $\rho\in(0,1)$, where*

$$\chi_\rho(z)=\chi(\rho z).$$

If $f\in\mathcal{A}(p)$ and

$$\left\{\varphi\left(\left[\Im_p(a,c,\mu)f(z)\right]^\gamma,\left[\Im_p(a,c,\mu)f(z)\right]^{\gamma-1}\Im_p(a+1,c,\mu)f(z),\right.\right.$$
$$(\gamma-1)\left[\Im_p(a,c,\mu)f(z)\right]^{\gamma-2}\left[\Im_p(a+1,c,\mu)f(z)\right]^2+\frac{a+1+\mu p}{a+\mu p} \tag{15}$$
$$\left.\left.\cdot\left[\Im_p(a,c,\mu)f(z)\right]^{\gamma-1}\Im_p(a+2,c,\mu)f(z);z\right):z\in\mathbb{U}\right\}\subset\Omega,$$

then

$$\left[\Im_p(a,c,\mu)f(z)\right]^\gamma\prec\chi(z).$$

Proof. Theorem 1 yields the following subordination relation:

$$\left[\Im_p(a,c,\mu)f(z)\right]^\gamma\prec\chi_\rho(z).$$

The result is now deduced from the subordination hypothesis:

$$\chi_\rho(z)\prec\chi(\rho z)$$

for some $\rho\in(0,1)$. □

Theorem 4. *Let the functions $h(z)$ and $\chi(z)$ be univalent in $\mathbb{U}$ with $\chi(0)=0$ and set*

$$\chi_\rho(z)=\chi(\rho z)\qquad\text{and}\qquad h_\rho(z)=h(\rho z).$$

Suppose also that the mapping $\varphi:\mathbb{C}^3\times\mathbb{U}\to\mathbb{C}$ satisfies one of the following conditions:

(1) $\varphi\in\Phi_1[h,\chi_\rho,\gamma]$ for some $\rho\in(0,1)$ or
(2) $\rho_0\in(0,1)$ exists such that $\varphi\in\Phi_1[h_\rho,\chi_\rho,\gamma]$ for all $\rho\in(\rho_0,1)$.

If the function $f(z)\in\mathcal{A}(p)$ satisfies the condition (14), then

$$\left[\Im_p(a,c,\mu)f(z)\right]^\gamma\prec\chi(z).$$

Proof. The proof is similar to the proof of a known result ([18], p. 30, Theorem 2.3d) and it is, therefore, omitted here. □

The next theorem yields the best dominant of the differential subordination (14).

Theorem 5. *Let the function $h(z)$ be univalent in $\mathbb{U}$ and let $\varphi : \mathbb{C}^3 \times \mathbb{U} \to \mathbb{C}$. Suppose that the following second-order differential equation*:

$$\varphi\big(\omega(z), z\omega'(z), z^2\omega''(z); z\big) = h(z) \tag{16}$$

has a solution $\chi(z)$, with $\chi(0) = 0$, satisfying one of the following conditions:

(1) $\chi(z) \in \wp_0$ and $\varphi \in \Phi_1[h, \chi, \gamma]$ or
(2) $\chi(z)$ is univalent in $\mathbb{U}$ and $\varphi \in \Phi_1[h, \chi_\rho, \gamma]$ for some $\rho \in (0,1)$ or
(3) $\chi(z)$ is univalent in $\mathbb{U}$ and $\rho_0 \in (0,1)$ exists such that $\varphi \in \Phi_1[h_\rho, \chi_\rho, \gamma]$ for all $\rho \in (\rho_0, 1)$.

If the function $f(z) \in \mathcal{A}(p)$ satisfies (14), then

$$\big[\Im_p(a,c,\mu)f(z)\big]^\gamma \prec \chi(z)$$

and $\chi(z)$ is the best dominant.

Proof. Using the technique in proving the known result ([18], p. 31, Theorem 2.3e), we deduce that $\chi(z)$ is a dominant from Theorems 1 and 2. Moreover, since $\chi(z)$ satisfies (16), it is also a solution of (14). Therefore, $\chi(z)$ will be dominated by all dominants. Hence, $\chi(z)$ is the best dominant. □

In the particular case when $\chi(z) = Mz \quad (M > 0)$ and, in view of Definition 3, the class $\Phi_1[\Omega, \chi, \gamma]$ of admissible functions, which we denote by $\Phi_1[\Omega, M, \gamma]$, is described below.

Definition 4. *Let Ω be a set in $\mathbb{C}$ and $M > 0$. The class $\Phi_1[\Omega, M, \gamma]$ of admissible functions consists of the functions $\varphi : \mathbb{C}^3 \times \mathbb{U} \to \mathbb{C}$ such that*

$$\varphi\left(Me^{i\theta}, \frac{k + \frac{\gamma a}{\mu}}{\gamma\left(\frac{a}{\mu} + p\right)} Me^{i\theta}, \frac{L + \left[\left(1 + \frac{2\gamma a + 1}{\mu}\right)k + \frac{\gamma a(\gamma a + 1)}{\mu^2}\right]Me^{i\theta}}{\gamma\left(\frac{a}{\mu} + p\right)^2}; z\right) \notin \Omega$$

whenever $z \in \mathbb{U}$, $\theta \in \mathbb{R}$ and

$$\Re\left(Le^{-i\theta}\right) \geqq (k-1)kM \qquad (\forall\, \theta \in \mathbb{R};\ k \geqq \gamma p).$$

Corollary 1. *Let $\varphi \in \Phi_1[\Omega, M, \gamma]$. If the function $f(z) \in \mathcal{A}(p)$ satisfies the following inclusion relation*:

$$\begin{aligned}\varphi\Bigg(&\big[\Im_p(a,c,\mu)f(z)\big]^\gamma,\ \big[\Im_p(a,c,\mu)f(z)\big]^{\gamma-1}\Im_p(a+1,c,\mu)f(z),\\ &(\gamma-1)\big[\Im_p(a,c,\mu)f(z)\big]^{\gamma-2}\big[\Im_p(a+1,c,\mu)f(z)\big]^2 + \frac{a+1+\mu p}{a+\mu p}\\ &\cdot\big[\Im_p(a,c,\mu)f(z)\big]^{\gamma-1}\Im_p(a+2,c,\mu)f(z); z\Bigg) \in \Omega,\end{aligned} \tag{17}$$

then

$$\big|\Im_p(a,c,\mu)f(z)\big|^\gamma < M.$$

Proof. From Definition 4

$$\chi(z) = Mz\,.$$

The result is now deduced:

$$\big|\Im_p(a,c,\mu)f(z)\big|^\gamma < M.$$

□

In the special case when

$$\Omega = \chi(\mathbb{U}) = \{w : |w| < M\},$$

the class $\Phi_1[\Omega, M, \gamma]$ is simply denoted by $\Phi_1[M, \gamma]$.

Corollary 2. *Let $\varphi \in \Phi_1[M, \gamma]$. If the function $f(z) \in \mathcal{A}(p)$ satisfies the following inequality*:

$$\begin{aligned} \Bigg|\varphi\Bigg(& \left[\Im_p(a,c,\mu)f(z)\right]^{\gamma}, \left[\Im_p(a,c,\mu)f(z)\right]^{\gamma-1} \Im_p(a+1,c,\mu)f(z), \\ & (\gamma-1)\left[\Im_p(a,c,\mu)f(z)\right]^{\gamma-2} \left[\Im_p(a+1,c,\mu)f(z)\right]^2 + \frac{a+1+\mu p}{a+\mu p} \\ & \cdot \left[\Im_p(a,c,\mu)f(z)\right]^{\gamma-1} \Im_p(a+2,c,\mu)f(z);\ z\Bigg)\Bigg| < M, \end{aligned} \tag{18}$$

then

$$\left|\Im_p(a,c,\mu)f(z)\right|^{\gamma} < M.$$

Corollary 3. *If $k > \gamma p$ and if the function $f(z) \in \mathcal{A}(p)$ satisfies the following condition*:

$$\left|\left[\Im_p(a,c,\mu)f(z)\right]^{\gamma-1} \Im_p(a+1,c,\mu)f(z)\right| < M,$$

then

$$\left|\Im_p(a,c,\mu)f(z)\right|^{\gamma} < M.$$

Proof. The proof follows from Corollary 1 by taking

$$\varphi(u,v,w;z) = v = \frac{k + \frac{\gamma a}{\mu}}{\gamma\left(\frac{a}{\mu} + p\right)}\, Me^{i\theta}.$$

□

Example 1. *If the function $f(z) \in \mathcal{A}(p)$, $a = c = 0$ and $\gamma = \mu = 1$, then we see that*

(i) $\Im_p(0,0,1)f(z) = f(z)$;

(ii) $\Im_p(1,0,1)f(z) = \frac{zf'(z)}{p}$;

(iii) $\Im_p(2,0,1)f(z) = \frac{z^2 f''(z)}{p(p+1)} + 2\frac{zf'(z)}{p(p+1)}$.

Thus, upon substituting in Corollary 3 from the above relations, we obtain

$$\left|\frac{zf'(z)}{p}\right| < M \implies |f(z)| < M.$$

Definition 5. *Let Ω be a set in $\mathbb{C}$ and suppose that $\chi(z) \in \wp_0 \cap \mathcal{H}[0,\gamma]$. The class $\Phi_2[\Omega, \chi, \gamma]$ of admissible functions consists of the functions $\varphi : \mathbb{C}^3 \times \mathbb{U} \to \mathbb{C}$ that satisfy the following admissibility condition*:

$$\varphi(u,v,w;z) \notin \Omega$$

whenever

$$u = \chi(\varsigma), \quad v = \frac{k\varsigma\chi'(\varsigma) + \gamma\left(\frac{a}{\mu} + (p-1)\right)\chi(\varsigma)}{\gamma\left(\frac{a}{\mu} + p\right)}$$

and

$$\Re\left(\frac{\frac{(a+\mu p)^2}{\mu(a+\mu(p-1))}w-\left(\frac{\gamma a+1}{\mu}+\gamma(p-1)\right)v+\gamma\left(\frac{a}{\mu}+(p-1)\right)u}{\frac{a+\mu p}{a+\mu(p-1)}v-u}\right) \geqq k\Re\left(1+\frac{\varsigma\chi''(\varsigma)}{\chi'(\varsigma)}\right), \tag{19}$$

where $z\in\mathbb{U}$, $\varsigma\in\partial\mathbb{U}\setminus E(\chi)$ *and* $k\geqq\gamma$.

Theorem 6. *Let* $\varphi\in\Phi_2[\Omega,\chi,\gamma]$. *If the function* $f(z)\in\mathcal{A}(p)$ *satisfies the following set inclusion*:

$$\left\{\varphi\left(\left[\frac{\Im_p(a,c,\mu)f(z)}{z^{p-1}}\right]^\gamma,\left[\frac{\Im_p(a,c,\mu)f(z)}{z^{p-1}}\right]^\gamma\frac{\Im_p(a+1,c,\mu)f(z)}{\Im_p(a,c,\mu)f(z)},\left[\frac{\Im_p(a,c,\mu)f(z)}{z^{p-1}}\right]^\gamma\left[(\gamma-1)\left(\frac{\Im_p(a+1,c,\mu)f(z)}{\Im_p(a,c,\mu)f(z)}\right)^2+\frac{a+1+\mu p}{a+\mu p}\frac{\Im_p(a+2,c,\mu)f(z)}{\Im_p(a,c,\mu)f(z)}\right];z\right):z\in\mathbb{U}\right\}\subset\Omega, \tag{20}$$

then

$$\left[\frac{\Im_p(a,c,\mu)f(z)}{z^{p-1}}\right]^\gamma\prec\chi(z).$$

Proof. We define an analytic function $g(z)$ in $\mathbb{U}$ by

$$g(z)=\left[\frac{\Im_p(a,c,\mu)f(z)}{z^{p-1}}\right]^\gamma. \tag{21}$$

By making use of (5) and (21), we obtain

$$\left[\frac{\Im_p(a,c,\mu)f(z)}{z^{p-1}}\right]^\gamma\frac{\Im_p(a+1,c,\mu)f(z)}{\Im_p(a,c,\mu)f(z)}=\frac{zg'(z)+\gamma\left(\frac{a}{\mu}+(p-1)\right)g(z)}{\gamma\left(\frac{a}{\mu}+p\right)}. \tag{22}$$

Further computations show that

$$\left[\frac{\Im_p(a,c,\mu)f(z)}{z^{p-1}}\right]^\gamma\left[(\gamma-1)\left(\frac{\Im_p(a+1,c,\mu)f(z)}{\Im_p(a,c,\mu)f(z)}\right)^2+\frac{a+1+\mu p}{a+\mu p}\frac{\Im_p(a+2,c,\mu)f(z)}{\Im_p(a,c,\mu)f(z)}\right]$$
$$=\frac{z^2g''(z)+\left(1+\frac{2\gamma a+1}{\mu}+2\gamma(p-1)\right)zg'(z)+(\frac{\gamma a}{\mu}+\gamma(p-1))\left(\frac{\gamma a+1}{\mu}+\gamma(p-1)\right)g(z)}{\gamma\left(\frac{a}{\mu}+p\right)^2}.$$

We next define the transformations from $\mathbb{C}^3$ to $\mathbb{C}$ by

$$u=r,\quad v=\frac{s+\gamma\left(\frac{a}{\mu}+(p-1)\right)r}{\gamma\left(\frac{a}{\mu}+p\right)},$$

and

$$w = \frac{t + \left(1 + \frac{2\gamma a+1}{\mu} + 2\gamma(p-1)\right)s + (\frac{\gamma a}{\mu} + \gamma(p-1))\left(\frac{\gamma a+1}{\mu} + \gamma(p-1)\right)r}{\gamma\left(\frac{a}{\mu} + p\right)^2}.$$

Additionally, let

$$\begin{aligned}
&\psi(r,s,t;z) = \varphi(u,v,w;z) \\
&= \varphi\left(r, \frac{s + \gamma\left(\frac{a}{\mu} + (p-1)\right)r}{\gamma\left(\frac{a}{\mu} + p\right)},\right. \\
&\left.\frac{t + \left(1 + \frac{2\gamma a+1}{\mu} + 2\gamma(p-1)\right)s + (\frac{\gamma a}{\mu} + \gamma(p-1))\left(\frac{\gamma a+1}{\mu} + \gamma(p-1)\right)r}{\gamma\left(\frac{a}{\mu} + p\right)^2}; z\right).
\end{aligned} \tag{23}$$

Thus, by using Equations (21)–(23), we obtain

$$\begin{aligned}
&\psi(g(z), zg'(z), z^2g''(z); z) \\
&= \varphi\left(\left[\frac{\Im_p(a,c,\mu)f(z)}{z^{p-1}}\right]^\gamma, \left[\frac{\Im_p(a,c,\mu)f(z)}{z^{p-1}}\right]^\gamma \frac{\Im_p(a+1,c,\mu)f(z)}{\Im_p(a,c,\mu)f(z)},\right. \\
&\left[\frac{\Im_p(a,c,\mu)f(z)}{z^{p-1}}\right]^\gamma \left[(\gamma-1)\left(\frac{\Im_p(a+1,c,\mu)f(z)}{\Im_p(a,c,\mu)f(z)}\right)^2\right. \\
&\left.\left. + \frac{a+1+\mu p}{a+\mu p}\frac{\Im_p(a+2,c,\mu)f(z)}{\Im_p(a,c,\mu)f(z)}\right]; z\right).
\end{aligned} \tag{24}$$

Hence, (20) implies that

$$\psi(g(z), zg'(z), z^2g''(z); z) \in \Omega.$$

The proof of Theorem 6 is completed if it can be shown that the admissibility condition for $\varphi \in \Phi_2[\Omega, \chi, \gamma]$ is equivalent to the admissibility condition for ψ as given in Definition 1. For this purpose, we note that

$$\frac{t}{s} + 1 = \frac{\frac{(a+\mu p)^2}{\mu(a+\mu(p-1))}w - \left(\frac{\gamma a+1}{\mu} + \gamma(p-1)\right)v + \gamma\left(\frac{a}{\mu} + (p-1)\right)u}{\frac{a+\mu p}{a+\mu(p-1)}v - u},$$

and hence that $\psi \in \Psi[\Omega, \chi, \gamma]$. Therefore, by applying Lemma 1, we conclude that $g(z) \prec \chi(z)$ or, equivalently, that

$$\left[\frac{\Im_p(a,c,\mu)f(z)}{z^{p-1}}\right]^\gamma \prec \chi(z),$$

which completes the proof of Theorem 6. □

We next consider the case when $\Omega \neq \mathbb{C}$ is a simply-connected domain, with $\Omega = h(\mathbb{U})$, for some conformal mapping $h(z)$ of $\mathbb{U}$ onto Ω. In this case, $\Phi_2[h(\mathbb{U}), \chi, \gamma]$ is written as $\Phi_2[h, \chi, \gamma]$. In the particular case when $\chi(z) = Mz \quad (M > 0)$, we denote the class $\Phi_2[\Omega, \chi, \gamma]$ of admissible functions by $\Phi_2[\Omega, M]$.

Proceeding similarly as in the previous section, the following result is an immediate consequence of Theorem 6.

Theorem 7. *Let $\varphi \in \Phi_2[h, \chi, \gamma]$. If the function $f(z) \in \mathcal{A}(p)$ satisfies the following subordination relation*:

$$\varphi\Bigg(\left[\frac{\Im_p(a,c,\mu)f(z)}{z^{p-1}}\right]^{\gamma}, \left[\frac{\Im_p(a,c,\mu)f(z)}{z^{p-1}}\right]^{\gamma} \frac{\Im_p(a+1,c,\mu)f(z)}{\Im_p(a,c,\mu)f(z)},$$
$$\left[\frac{\Im_p(a,c,\mu)f(z)}{z^{p-1}}\right]^{\gamma} \cdot \left[(\gamma-1)\left(\frac{\Im_p(a+1,c,\mu)f(z)}{\Im_p(a,c,\mu)f(z)}\right)^2 \right. \tag{25}$$
$$\left. + \frac{a+1+\mu p}{a+\mu p} \frac{\Im_p(a+2,c,\mu)f(z)}{\Im_p(a,c,\mu)f(z)}\right]; z\Bigg) \prec h(z),$$

then

$$\left[\frac{\Im_p(a,c,\mu)f(z)}{z^{p-1}}\right]^{\gamma} \prec \chi(z).$$

Definition 6. *Let Ω be a set in $\mathbb{C}$ and $M > 0$. The class $\Phi_2[\Omega, M, \gamma]$ of admissible functions consists of the functions $\varphi : \mathbb{C}^3 \times \mathbb{U} \to \mathbb{C}$ such that*

$$\varphi\Bigg(Me^{i\theta}, \frac{k+\gamma\left(\frac{a}{\mu}+(p-1)\right)}{\gamma\left(\frac{a}{\mu}+p\right)} Me^{i\theta},$$
$$\frac{L+\left[\left(1+\frac{2\gamma a+1}{\mu}+2\gamma(p-1)\right)k+\left(\frac{\gamma a}{\mu}+\gamma(p-1)\right)\left(\frac{\gamma a+1}{\mu}+\gamma(p-1)\right)\right]Me^{i\theta}}{\gamma\left(\frac{a}{\mu}+p\right)^2}; z\Bigg) \tag{26}$$
$$\notin \Omega$$

whenever $z \in \mathbb{U}$, $\theta \in \mathbb{R}$ and

$$\Re\left(Le^{-i\theta}\right) \geqq (k-1)kM \qquad (\forall\, \theta \in \mathbb{R};\ p \in \mathbb{N}^+;\ k \geqq \gamma).$$

Corollary 4. *Let $\varphi \in \Phi_2[\Omega, M, \gamma]$. If the function $f(z) \in \mathcal{A}(p)$ satisfies the following inclusion relation*:

$$\varphi\Bigg(\left[\frac{\Im_p(a,c,\mu)f(z)}{z^{p-1}}\right]^{\gamma}, \left[\frac{\Im_p(a,c,\mu)f(z)}{z^{p-1}}\right]^{\gamma} \frac{\Im_p(a+1,c,\mu)f(z)}{\Im_p(a,c,\mu)f(z)},$$
$$\left[\frac{\Im_p(a,c,\mu)f(z)}{z^{p-1}}\right]^{\gamma} \left[(\gamma-1)\left(\frac{\Im_p(a+1,c,\mu)f(z)}{\Im_p(a,c,\mu)f(z)}\right)^2 \right. \tag{27}$$
$$\left. + \frac{a+1+\mu p}{a+\mu p} \frac{\Im_p(a+2,c,\mu)f(z)}{\Im_p(a,c,\mu)f(z)}\right]; z\Bigg) \in \Omega,$$

then

$$\left|\frac{\Im_p(a,c,\mu)f(z)}{z^{p-1}}\right|^{\gamma} < M.$$

In the special case when

$$\Omega = \chi(\mathbb{U}) = \{w : w \in \mathbb{C} \quad \text{and} \quad |w| < M\},$$

the class $\Phi_2[\Omega, M, \gamma]$ is simply denoted by $\Phi_2[M, \gamma]$.

Corollary 5. *Let $\varphi \in \Phi_2[M,\gamma]$. If the function $f(z) \in \mathcal{A}(p)$ satisfies the following inequality*:

$$\left|\varphi\left(\left[\frac{\Im_p(a,c,\mu)f(z)}{z^{p-1}}\right]^\gamma, \left[\frac{\Im_p(a,c,\mu)f(z)}{z^{p-1}}\right]^\gamma \frac{\Im_p(a+1,c,\mu)f(z)}{\Im_p(a,c,\mu)f(z)}, \left[\frac{\Im_p(a,c,\mu)f(z)}{z^{p-1}}\right]^\gamma \left[(\gamma-1)\left(\frac{\Im_p(a+1,c,\mu)f(z)}{\Im_p(a,c,\mu)f(z)}\right)^2 + \frac{a+1+\mu p}{a+\mu p}\frac{\Im_p(a+2,c,\mu)f(z)}{\Im_p(a,c,\mu)f(z)}\right]; z\right)\right| < M, \tag{28}$$

then

$$\left|\frac{\Im_p(a,c,\mu)f(z)}{z^{p-1}}\right|^\gamma < M.$$

Corollary 6. *If $k > \gamma$ and if the function $f(z) \in \mathcal{A}(p)$ satisfies the following condition*:

$$\left|\left(\frac{\Im_p(a,c,\mu)f(z)}{z^{p-1}}\right)^\gamma \frac{\Im_p(a+1,c,\mu)f(z)}{\Im_p(a,c,\mu)f(z)}\right| < M,$$

then

$$\left|\frac{\Im_p(a,c,\mu)f(z)}{z^{p-1}}\right|^\gamma < M.$$

Proof. The proof follows from Corollary 5 by taking

$$\varphi(u,v,w;z) = v = \frac{k+\gamma\left(\frac{a}{\mu}+(p-1)\right)}{\gamma\left(\frac{a}{\mu}+p\right)} Me^{i\theta}.$$

□

Example 2. *rmFor $f(z) \in \mathcal{A}(p)$, $a = c = 0$ and $\gamma = \mu = 1$, if we ubstitute in Corollary 6 from Example 1 (i) and (ii), we obtain*

$$\left|\frac{f'(z)}{pz^{p-2}}\right| < M \implies \left|\frac{f(z)}{z^{p-1}}\right| < M.$$

3. Superordination and Sandwich-Type Results Involving $\Im_p(a,c,\mu)$

In this section, we investigate differential superordination and sandwich-type results for the linear operator $\Im_p(a,c,\mu)$. For this purpose, the class of admissible functions is defined as follows.

Definition 7. *Let Ω be a set in $\mathbb{C}$ and suppose that $\chi(z) \in \mathcal{H}[0,\gamma p]$ with $z\chi'(z) \neq 0$. The class $\Phi_1'[\Omega,\chi,\gamma]$ of admissible functions consists of the functions $\varphi : \mathbb{C}^3 \times \overline{\mathbb{U}} \to \mathbb{C}$ that satisfy the following admissibility condition*:

$$\varphi(u,v,w;\varsigma) \in \Omega$$

whenever

$$u = \chi(z), \quad v = \frac{z\chi'(z) + m\frac{\gamma a}{\mu}\chi(z)}{m\gamma\left(\frac{a}{\mu}+p\right)}$$

and

$$\Re\left(\frac{\left(\frac{a}{\mu}+p\right)^2 w-\left(\frac{a}{\mu}+p\right)\left(\frac{2\gamma a+1}{\mu}\right)v+\gamma\left(\frac{a}{\mu}\right)^2 u}{\frac{a}{\mu}(v-u)+pv}\right)$$
$$\leqq \frac{1}{m}\,\Re\left(1+\frac{z\chi''(z)}{\chi'(z)}\right),$$

where $z \in \mathbb{U}$, $\varsigma \in \partial\mathbb{U}$ *and* $m \geqq \gamma p$.

Theorem 8. *Let* $\varphi \in \Phi_1'[\Omega,\chi,\gamma]$. *If* $f(z) \in \mathcal{A}(p)$, $[\Im_p(a,c,\mu)f(z)]^\gamma \in \wp_0$ *and the function* φ *given by*

$$\begin{aligned}\varphi\Big(& [\Im_p(a,c,\mu)f(z)]^\gamma, [\Im_p(a,c,\mu)f(z)]^{\gamma-1}\Im_p(a+1,c,\mu)f(z),\\ & (\gamma-1)[\Im_p(a,c,\mu)f(z)]^{\gamma-2}\,(\Im_p(a+1,c,\mu)f(z))^2+\frac{a+1+\mu p}{a+\mu p}\\ & \cdot[\Im_p(a,c,\mu)f(z)]^{\gamma-1}\,\Im_p(a+2,c,\mu)f(z);z\Big)\end{aligned} \tag{29}$$

is univalent in $\mathbb{U}$, *then the following set inclusion*:

$$\begin{aligned}\Omega\subset\Big\{\varphi\Big(& [\Im_p(a,c,\mu)f(z)]^\gamma, [\Im_p(a,c,\mu)f(z)]^{\gamma-1}\Im_p(a+1,c,\mu)f(z),\\ & (\gamma-1)[\Im_p(a,c,\mu)f(z)]^{\gamma-2}\,(\Im_p(a+1,c,\mu)f(z))^2+\frac{a+1+\mu p}{a+\mu p}\\ & \cdot[\Im_p(a,c,\mu)f(z)]^{\gamma-1}\,\Im_p(a+2,c,\mu)f(z);z\Big):z\in\mathbb{U}\Big\}\end{aligned} \tag{30}$$

implies that

$$\chi(z) \prec [\Im_p(a,c,\mu)f(z)]^\gamma.$$

Proof. From (13) and (30), we have

$$\Omega \subset \left\{\psi(\omega(z), z\omega'(z), z^2\omega''(z);\ z) : z \in \mathbb{U}\right\}.$$

Moreover, we see from (11) that the admissibility condition for $\varphi \in \Phi_1'[\Omega,\chi,\gamma]$ is equivalent to the admissibility condition for ψ as given in Definition 2. Hence $\psi \in \Psi_{\gamma p}'[\Omega,\chi]$ and, by Lemma 2, we obtain

$$\chi(z) \prec \omega(z) \qquad \text{or} \qquad \chi(z) \prec [\Im_p(a,c,\mu)f(z)]^\gamma.$$

□

If $\Omega \neq C$ is a simply-connected domain, then $\Omega = h(\mathbb{U})$ for some conformal mapping $h(z)$ for $\mathbb{U}$ onto Ω. In this case, the class $\Phi_1'[h(\mathbb{U}),\chi,\gamma]$ is written simply as $\Phi_1'[h,\chi,\gamma]$.

Proceeding as in Section 2, the following result is seen to be an immediate consequence of Theorem 8.

Theorem 9. *Let $\chi(z) \in \mathcal{H}[0,\gamma]$, the function $h(z)$ is analytic in $\mathbb{U}$ and $\varphi \in \Phi'_1[h,\chi,\gamma]$. If $f(z) \in \mathcal{A}(p)$ and $\left[\Im_p(a,c,\mu)f(z)\right]^\gamma \in \wp_0$, and if the function φ given by*

$$\begin{aligned}\varphi\Bigg(&\left[\Im_p(a,c,\mu)f(z)\right]^\gamma, \left[\Im_p(a,c,\mu)f(z)\right]^{\gamma-1}\Im_p(a+1,c,\mu)f(z), \\ &(\gamma-1)\left[\Im_p(a,c,\mu)f(z)\right]^{\gamma-2}\left(\Im_p(a+1,c,\mu)f(z)\right)^2 \\ &+\frac{a+1+\mu p}{a+\mu p}\left[\Im_p(a,c,\mu)f(z)\right]^{\gamma-1}\Im_p(a+2,c,\mu)f(z);z\Bigg)\end{aligned} \tag{31}$$

is univalent in $\mathbb{U}$, then the following subordination relation:

$$\begin{aligned}h(z) \prec \varphi\Bigg(&\left[\Im_p(a,c,\mu)f(z)\right]^\gamma, \left[\Im_p(a,c,\mu)f(z)\right]^{\gamma-1}\Im_p(a+1,c,\mu)f(z), \\ &(\gamma-1)\left[\Im_p(a,c,\mu)f(z)\right]^{\gamma-2}\left(\Im_p(a+1,c,\mu)f(z)\right)^2 \\ &+\frac{a+1+\mu p}{a+\mu p}\left[\Im_p(a,c,\mu)f(z)\right]^{\gamma-1}\Im_p(a+2,c,\mu)f(z);z\Bigg)\end{aligned} \tag{32}$$

implies that

$$\chi(z) \prec \left[\Im_p(a,c,\mu)f(z)\right]^\gamma.$$

Theorems 8 and 9 can only be used to obtain the subordinants of differential superordination of the form (30) or (32). The following theorem proves the existence of the best subordinant of (32) for some function φ.

Theorem 10. *Let the function $h(z)$ be analytic in $\mathbb{U}$ and let $\varphi : \mathbb{C}^3 \times \overline{\mathbb{U}} \to \mathbb{C}$. Suppose that the following differential equation*:

$$\varphi\Big(\chi(z), z\chi'(z), z^2\chi''(z); z\Big) = h(z)$$

has a solution $\chi(z) \in \wp_0$. If

$$\varphi \in \Phi'_1[h,\chi,\gamma], \quad f(z) \in \mathcal{A}(p) \quad \text{and} \quad \left[\Im_p(a,c,\mu)f(z)\right]^\gamma \in \wp_0,$$

and the function φ given by

$$\begin{aligned}\varphi\Bigg(&\left[\Im_p(a,c,\mu)f(z)\right]^\gamma, \left[\Im_p(a,c,\mu)f(z)\right]^{\gamma-1}\Im_p(a+1,c,\mu)f(z), \\ &(\gamma-1)\left[\Im_p(a,c,\mu)f(z)\right]^{\gamma-2}\left(\Im_p(a+1,c,\mu)f(z)\right)^2 \\ &+\frac{a+1+\mu p}{a+\mu p}\left[\Im_p(a,c,\mu)f(z)\right]^{\gamma-1}\Im_p(a+2,c,\mu)f(z);z\Bigg)\end{aligned} \tag{33}$$

is univalent in $\mathbb{U}$, then the following subordination relation:

$$\begin{aligned}h(z) \prec \varphi\Bigg(&\left[\Im_p(a,c,\mu)f(z)\right]^\gamma, \left[\Im_p(a,c,\mu)f(z)\right]^{\gamma-1}\Im_p(a+1,c,\mu)f(z), \\ &(\gamma-1)\left[\Im_p(a,c,\mu)f(z)\right]^{\gamma-2}\left(\Im_p(a+1,c,\mu)f(z)\right)^2 \\ &+\frac{a+1+\mu p}{a+\mu p}\left[\Im_p(a,c,\mu)f(z)\right]^{\gamma-1}\Im_p(a+2,c,\mu)f(z);\ z\Bigg)\end{aligned} \tag{34}$$

implies that

$$\chi(z) \prec \left[\Im_p(a,c,\mu)f(z)\right]^\gamma$$

and $\chi(z)$ is the best subordinant.

Proof. The proof of Theorem 10 is similar to that of Theorem 4 and is, therefore, omitted here. □

Combining Theorems 2 and 9, we obtain the following sandwich-type theorem.

Theorem 11. *Let the functions $h_1(z)$ and $\chi_1(z)$ be analytic in $\mathbb{U}$, and let the function $h_2(z)$ be univalent in $\mathbb{U}$, $\chi_2(z) \in \wp_0$ with $\chi_1(0) = \chi_2(0) = 0$ and*

$$\varphi \in \Phi_1[h_2, \chi_2, \gamma] \cap \Phi_1'[h_1, \chi_1, \gamma].$$

If $f(z) \in \mathcal{A}(p)$,

$$\left[\Im_p(a,c,\mu)f(z)\right]^\gamma \in \mathcal{H}[0, \gamma p] \cap \wp_0$$

and the function φ given by

$$\varphi\Bigg(\left[\Im_p(a,c,\mu)f(z)\right]^\gamma, \left[\Im_p(a,c,\mu)f(z)\right]^{\gamma-1}\Im_p(a+1,c,\mu)f(z), (\gamma-1)\left[\Im_p(a,c,\mu)f(z)\right]^{\gamma-2}\left(\Im_p(a+1,c,\mu)f(z)\right)^2 + \frac{a+1+\mu p}{a+\mu p}\left[\Im_p(a,c,\mu)f(z)\right]^{\gamma-1}\Im_p(a+2,c,\mu)f(z); z\Bigg)$$

is univalent in $\mathbb{U}$, then the following subordination relation:

$$h_1(z) \prec \varphi\Bigg(\left[\Im_p(a,c,\mu)f(z)\right]^\gamma, \left[\Im_p(a,c,\mu)f(z)\right]^{\gamma-1}\Im_p(a+1,c,\mu)f(z), (\gamma-1)\left[\Im_p(a,c,\mu)f(z)\right]^{\gamma-2}\left(\Im_p(a+1,c,\mu)f(z)\right)^2 + \frac{a+1+\mu p}{a+\mu p}\left[\Im_p(a,c,\mu)f(z)\right]^{\gamma-1}\Im_p(a+2,c,\mu)f(z); z\Bigg) \prec h_2(z)$$

implies that

$$\chi_1(z) \prec \left[\Im_p(a,c,\mu)f(z)\right]^\gamma \prec \chi_2(z).$$

Definition 8. *Let Ω be a set in $\mathbb{C}$ and $\chi(z) \in \mathcal{H}[0,\gamma]$ with $z\chi'(z) \neq 0$. The class $\Phi_2'[\Omega, \chi, \gamma]$ of admissible functions consists of the functions $\varphi : \mathbb{C}^3 \times \mathbb{U} \to \mathbb{C}$ that satisfy the following admissibility condition:*

$$\varphi(u, v, w; \varsigma) \in \Omega \tag{35}$$

whenever

$$u = \chi(z), \quad v = \frac{z\chi'(z) + m\gamma\left(\frac{a}{\mu} + (p-1)\right)\chi(z)}{m\gamma\left(\frac{a}{\mu} + p\right)},$$

and

$$\Re\left(\frac{\frac{(a+\mu p)^2}{\mu\left(a+\mu(p-1)\right)}w - \left(\frac{\gamma a+1}{\mu} + \gamma(p-1)\right)v + \gamma\left(\frac{a}{\mu} + (p-1)\right)u}{\frac{a+\mu p}{a+\mu(p-1)}v - u}\right) \leqq \frac{1}{m}\Re\left(1 + \frac{z\chi''(z)}{\chi'(z)}\right),$$

where $z \in \mathbb{U}$, $\varsigma \in \partial\mathbb{U} \setminus E(\chi)$ *and* $m \geqq \gamma$.

We now state and prove the dual result of Theorem 6 for differential superordination.

Theorem 12. *Let* $\varphi \in \Phi_2'[\Omega, \chi, \gamma]$. *If*

$$f(z) \in \mathcal{A}(p) \quad \text{and} \quad \left[\frac{\Im_p(a,c,\mu)f(z)}{z^{p-1}}\right]^{\gamma} \in \wp_0,$$

and if the function φ *given by*

$$\begin{aligned}\varphi\Bigg(&\left[\frac{\Im_p(a,c,\mu)f(z)}{z^{p-1}}\right]^{\gamma}, \left[\frac{\Im_p(a,c,\mu)f(z)}{z^{p-1}}\right]^{\gamma} \frac{\Im_p(a+1,c,\mu)f(z)}{\Im_p(a,c,\mu)f(z)}, \\ &\left[\frac{\Im_p(a,c,\mu)f(z)}{z^{p-1}}\right]^{\gamma} \cdot \left[(\gamma-1)\left(\frac{\Im_p(a+1,c,\mu)f(z)}{\Im_p(a,c,\mu)f(z)}\right)^2 \right. \\ &\left. + \frac{a+1+\mu p}{a+\mu p} \frac{\Im_p(a+2,c,\mu)f(z)}{\Im_p(a,c,\mu)f(z)}\right]; z\Bigg)\end{aligned} \tag{36}$$

is univalent in $\mathbb{U}$, *then the following set inclusion*:

$$\begin{aligned}\Omega \subset \Bigg\{\varphi\Bigg(&\left[\frac{\Im_p(a,c,\mu)f(z)}{z^{p-1}}\right]^{\gamma}, \left[\frac{\Im_p(a,c,\mu)f(z)}{z^{p-1}}\right]^{\gamma} \frac{\Im_p(a+1,c,\mu)f(z)}{\Im_p(a,c,\mu)f(z)}, \\ &\left[\frac{\Im_p(a,c,\mu)f(z)}{z^{p-1}}\right]^{\gamma} \left[(\gamma-1)\left(\frac{\Im_p(a+1,c,\mu)f(z)}{\Im_p(a,c,\mu)f(z)}\right)^2 \right. \\ &\left. + \frac{a+1+\mu p}{a+\mu p} \frac{\Im_p(a+2,c,\mu)f(z)}{\Im_p(a,c,\mu)f(z)}\right]; z\Bigg) : z \in \mathbb{U}\Bigg\}\end{aligned} \tag{37}$$

implies that

$$\chi(z) \prec \left[\frac{\Im_p(a,c,\mu)f(z)}{z^{p-1}}\right]^{\gamma}.$$

Proof. From (24) and (37), we have

$$\Omega \subset \left\{\psi(\omega(z), z\omega'(z), z^2\omega''(z); z) : z \in \mathbb{U}\right\}.$$

We also observe from (23) that the admissibility condition for $\varphi \in \Phi_2'[\Omega, \chi, \gamma]$ is equivalent to the admissibility condition for ψ as given in Definition 2. Hence, $\psi \in \Psi'[\Omega, \chi]$ and, by Lemma 2, we find that

$$\chi(z) \prec \omega(z) \quad \text{or} \quad \chi(z) \prec \left[\frac{\Im_p(a,c,\mu)f(z)}{z^{p-1}}\right]^{\gamma},$$

which completes the proof of Theorem 12. □

If $\Omega \neq \mathbb{C}$ is a simply-connected domain, then $\Omega = h(\mathbb{U})$ for some conformal mapping $h(z)$ of $\mathbb{U}$ onto Ω. In this case, the class $\Phi_2'[h(\mathbb{U}), \chi, \gamma]$ is written, for convenience, as $\Phi_2'[h, \chi, \gamma]$.

The following result is an immediate consequence of Theorem 12.

Theorem 13. *Let $\chi(z) \in \mathcal{H}[0,\gamma]$, the function $h(z)$ is analytic in $\mathbb{U}$ and $\varphi \in \Phi_2'[h,\chi,\gamma]$. If*

$$f(z) \in \mathcal{A}(p) \quad \textit{and} \quad \left[\frac{\Im_p(a,c,\mu)f(z)}{z^{p-1}}\right]^{\gamma} \in \wp_0,$$

and if the function φ given by

$$\begin{aligned}\varphi\Bigg(&\left[\frac{\Im_p(a,c,\mu)f(z)}{z^{p-1}}\right]^{\gamma}, \left[\frac{\Im_p(a,c,\mu)f(z)}{z^{p-1}}\right]^{\gamma}\frac{\Im_p(a+1,c,\mu)f(z)}{\Im_p(a,c,\mu)f(z)},\\ &\left[\frac{\Im_p(a,c,\mu)f(z)}{z^{p-1}}\right]^{\gamma}\cdot\left[(\gamma-1)\left(\frac{\Im_p(a+1,c,\mu)f(z)}{\Im_p(a,c,\mu)f(z)}\right)^2\right.\\ &\left.+\frac{a+1+\mu p}{a+\mu p}\frac{\Im_p(a+2,c,\mu)f(z)}{\Im_p(a,c,\mu)f(z)}\right];z\Bigg)\end{aligned} \tag{38}$$

is univalent in $\mathbb{U}$, then the following set inclusion:

$$\begin{aligned}h(z) \prec \varphi\Bigg(&\left[\frac{\Im_p(a,c,\mu)f(z)}{z^{p-1}}\right]^{\gamma}, \left[\frac{\Im_p(a,c,\mu)f(z)}{z^{p-1}}\right]^{\gamma}\frac{\Im_p(a+1,c,\mu)f(z)}{\Im_p(a,c,\mu)f(z)},\\ &\left[\frac{\Im_p(a,c,\mu)f(z)}{z^{p-1}}\right]^{\gamma}\left[(\gamma-1)\left(\frac{\Im_p(a+1,c,\mu)f(z)}{\Im_p(a,c,\mu)f(z)}\right)^2\right.\\ &\left.+\frac{a+1+\mu p}{a+\mu p}\;\frac{\Im_p(a+2,c,\mu)f(z)}{\Im_p(a,c,\mu)f(z)}\right];z\Bigg)\end{aligned} \tag{39}$$

implies that

$$\chi(z) \prec \left[\frac{\Im_p(a,c,\mu)f(z)}{z^{p-1}}\right]^{\gamma}.$$

Finally, upon combining Theorems 7 and 13, we are led to the following sandwich-type theorem.

Theorem 14. *Let the functions $h_1(z)$ and $\chi_1(z)$ be analytic in $\mathbb{U}$, and let the function $h_2(z)$ be univalent in $\mathbb{U}$. Suppose also that $\chi_2(z) \in \wp_0$ with*

$$\chi_1(0) = \chi_2(0) = 0 \quad \textit{and} \quad \varphi \in \Phi_2[h_2,\chi_2,\gamma] \cap \Phi_2'[h_1,\chi_1,\gamma].$$

If

$$f(z) \in \mathcal{A}(p) \quad \textit{and} \quad \left[\frac{\Im_p(a,c,\mu)f(z)}{z^{p-1}}\right]^{\gamma} \in \mathcal{H}[0,\gamma] \cap \wp_0,$$

and if the function φ given by

$$\begin{aligned}\varphi\Bigg(&\left[\frac{\Im_p(a,c,\mu)f(z)}{z^{p-1}}\right]^{\gamma}, \left[\frac{\Im_p(a,c,\mu)f(z)}{z^{p-1}}\right]^{\gamma}\;\frac{\Im_p(a+1,c,\mu)f(z)}{\Im_p(a,c,\mu)f(z)},\\ &\left[\frac{\Im_p(a,c,\mu)f(z)}{z^{p-1}}\right]^{\gamma}\cdot\left[(\gamma-1)\left(\frac{\Im_p(a+1,c,\mu)f(z)}{\Im_p(a,c,\mu)f(z)}\right)^2\right.\\ &\left.+\frac{a+1+\mu p}{a+\mu p}\;\frac{\Im_p(a+2,c,\mu)f(z)}{\Im_p(a,c,\mu)f(z)}\right];z\Bigg)\end{aligned} \tag{40}$$

is univalent in $\mathbb{U}$, *then the following set inclusion*:

$$h_1(z) \prec \varphi\left(\left[\frac{\Im_p(a,c,\mu)f(z)}{z^{p-1}}\right]^\gamma, \left[\frac{\Im_p(a,c,\mu)f(z)}{z^{p-1}}\right]^\gamma \frac{\Im_p(a+1,c,\mu)f(z)}{\Im_p(a,c,\mu)f(z)}, \left[\frac{\Im_p(a,c,\mu)f(z)}{z^{p-1}}\right]^\gamma \cdot \left[(\gamma-1)\left(\frac{\Im_p(a+1,c,\mu)f(z)}{\Im_p(a,c,\mu)f(z)}\right)^2 + \frac{a+1+\mu p}{a+\mu p}\frac{\Im_p(a+2,c,\mu)f(z)}{\Im_p(a,c,\mu)f(z)}\right]; z\right) \prec h_2(z) \tag{41}$$

implies that

$$\chi_1(z) \prec \left[\frac{\Im_p(a,c,\mu)f(z)}{z^{p-1}}\right]^\gamma \prec \chi_2(z).$$

4. Conclusions

By using a rather specialized version of the Riemann–Liouville fractional integral operator and its varied form known as the Erdélyi–Kober fractional integral operator, we have first introduced the following linear integral operator:

$$\Im_p(a,c,\mu) \quad (\mu > 0;\ a,c \in \mathbb{R};\ c > a > -\mu p;\ p \in \mathbb{N}^+ := \{1,2,3,\ldots\}),$$

which was considered earlier by El-Ashwah and Drbuk [4]. We have then derived several results involving the differential subordination and the differential superordination for the admissible classes $\Phi_1[\Omega,\chi,\delta]$ and $\Phi_1'[\Omega,\chi,\delta]$ of multivalent (or p-valent) functions associated with operator $\Im_p(a,c,\mu)$.

The various results, which also include sandwich-type theorems, which we have presented in this paper, are new and would motivate further research in the field of the geometric function theory of complex analysis.

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