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Nonlinear Vibration of Electrostatically Actuated Microbeam

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Abstract: In this paper, an analytical technique based on the global residue harmonic balance method (GRHBM) is applied in order to obtain higher-order approximate analytical solutions of an electrostatically actuated micro-beam. To illustrate the applicability and accuracy of the method, a high level of accuracy was established for the analytical solutions by comparing the results of the solutions with the numerical solution as well as the already published literature, such as the variational approach (VA), Hamiltonian approach (HA), energy balance method (EBM), and homotopy analysis method (HAM). It is shown that the GRHB method can be easily applied to nonlinear problems and provides solutions with a higher precision than existing methods. The obtained analytical expressions are employed to study the effects of axial force, initial gape, and electrostatic load on nonlinear frequency.

Keywords: non-linear analysis; numerical methods; iteration/recursive method; electrostatically actuated microbeam

MSC: 34A34; 34A45; 34C15; 34C25; 70K60; 70K75; 74H45

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1. Introduction

Nonlinear oscillation problems play an important role in wavelet analysis, applied mathematics, physics, and mechanical engineering and have also attracted much interest due to their wide range of applicability. Most real systems are modeled by nonlinear differential equations, which play a crucial role in natural and physical simulations. In the last decades, significant progress has been made in using nonlinear equations to understand and accurately predict the stability of the motion of an oscillator [1–6].

Micro-electro-mechanical systems (MEMS) are intelligent structures with typically micron- or nanometer-sized systems. These tiny components, utilized in vibrators, sensors, switches, and other products, are a result of micro-electrical technology [7–11].

In general, nonlinear mathematical models are used to depict them. As a result, it is crucial to forecast these nonlinear models' approximate solutions in order to understand how they will behave dynamically. MEMS refers to the compact size of high-tech devices. They are developed based on micro-electronic technology. Consequently, they are used, for example, in the production of optical sensors, micro vibrators, and pressure sensors, and they also have many applications in engineering disciplines such as aerospace, optics, biomedicine, micro-switches, transistors, accelerometers, pressure sensors, micro-mirrors, micro-pumps, micro-grippers, bio-MEMS, and so on [1–11]. They are small integrated devices that combine electrical and mechanical components. They are also subject to electrically actuated MEMS devices, which require few mechanical components and low voltage levels for actuation and are continuously increasing their applications in modern technology. MEMS devices are commonly used as capacitive accelerometers [3], capacitive sensors [4], switches, and so on [5]; they are usually compared with traditional mechanical systems, and their largest size will not exceed one centimeter or will sometimes only be on the order of microns. Consequently, they can easily fabricate mechanical elements such as beams, gears, diaphragms, and springs for integrated circuits. In addition, MEMS devices can also be easily integrated into large layers and are highly resistant to vibrations and shocks [6]. For a large surface-to-volume ratio, integrated circuit (IC) technology in the modern industry facilitates the fabrication of huge numbers of MEMS devices, which increase reliability and reduced costs.

In recent years, many efficient analytical methods have been used to solve nonlinear differential equations, such as the variational iteration method [12], spreading residue harmonic balance method [13], energy balance method [14–16], frequency-amplitude formulation [17,18], homotopy perturbation method [19–23], modified harmonic balance method [24], differential quadrature method [25], parameter expansion method [26], variational approach [27,28], homotopy analysis method [29,30], higher-order Hamiltonian approach [2,31–33], and so on.

The main aim of this paper is to apply the global residue harmonic balance method introduced by Ju and Xue [34–37] and recently developed through [38–42], in order to obtain analytical approximate solutions to the large-amplitude vibration of electrostatically actuated micro-beams. The higher-order approximations (mainly second-order approximations) have been obtained for the large-amplitude vibration of electrostatically actuated micro-beams, providing the expected accuracy. A very simple solution procedure and high-accuracy results are the advantages in this paper.

2. Nonlinear Vibration of an Electrostatically Actuated Microbeam

In this section, we will consider a fully clamped microbeam with a uniform thickness h, length l, width $(b \gg 5h)$, effective modulus $\overline{E} = E/(1-v^2)$, Young's modulus E, Poisson's ratio v, and density ρ, as shown in Figure 1. Employing the classical beam theory and considering the mid-plane stretching effect as well as the distributed electrostatic force, the nonlinear vibration equation of an electrostatically actuated microbeam [2,16,28,30] is given by:

$$(a_1x^4 + a_2x^2 + a_3)\ddot{x} + a_4x + a_5x^3 + a_6x^5 + a_7x^7 = 0,\ x(0) = A,\ \dot{x}(0) = 0, \tag{1}$$

where x is the dimensionless deflection of the microbeam, and dot denotes the derivative with respect to the dimensionless time variable $t = \tau\sqrt{\overline{E}I/\rho bhl^4}$, with I and t being the second moment of area of the beam cross-section and time, respectively.

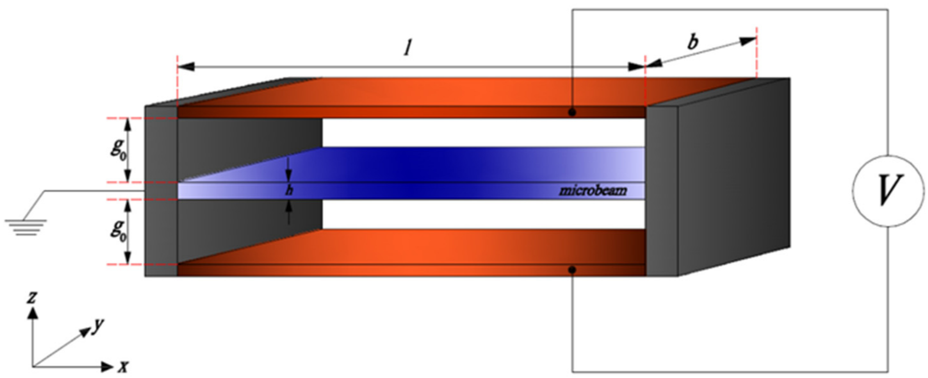

Figure 1. Schematics of a double-sided driven clamped-clamped microbeam-based electromechanical resonator [28].

The expressions of the governing physical parameters $a_i (i = 1, \ldots, 7)$ can be written as [16]:

$$\left.\begin{array}{l} a_1 = \int_0^1 \phi^6 d\zeta, \\ a_2 = -2\int_0^1 \phi^4 d\zeta, \\ a_3 = \int_0^1 \phi^2 d\zeta, \\ a_4 = \int_0^1 (\phi'''\phi - N\phi''\phi - V^2\phi^2) d\zeta, \\ a_5 = -\int_0^1 \left(2\phi''''\phi^3 - 2N\phi''\phi^3 + \alpha\phi''\phi\int_0^1 (\phi')^2 d\zeta\right) d\zeta, \\ a_6 = -\int_0^1 \left(\phi''''\phi^5 - N\phi''\phi^5 + 2\alpha\phi''\phi^3\int_0^1 (\phi')^2 d\zeta\right) d\zeta, \\ a_7 = -\int_0^1 \left(\alpha\phi''\phi^5\int_0^1 (\phi')^2 d\zeta\right) d\zeta, \end{array}\right\} \tag{2}$$

in which the following non-dimensional variables and parameters are introduced:

$$\alpha = \frac{6g_0^2}{h^2},\ \zeta = \frac{x}{l},\ N = \frac{\overline{N}l^2}{\overline{E}I},\ V^2 = \frac{24\varepsilon_0 l^4 \overline{V}^2}{\overline{E}h^3 g_0^3}. \tag{3}$$

The symbol $(')$ indicates the partial differentiation with respect to the coordinate variable x. Following the procedure presented in [16], the trial function ϕ in Equation (3) can be replaced by $\phi(\zeta) = 16\zeta^2(1-\zeta)^2$. The parameter $\overline{N}$ denotes the tensile or compressive axial load, g_0 is the initial gap between the microbeam and the electrode, V is the electrostatic load, and ε_0 is the vacuum permittivity. Other details of the problem can be found in [2,16,28,30].

With a new independent variable $\tau = \omega t$, Equation (1) can be transformed into:

$$(a_1x^4 + a_2x^2 + a_3)\omega^2 x'' + a_4x + a_5x^3 + a_6x^5 + a_7x^7 = 0,\ x(0) = A,\ x'(0) = 0, \tag{4}$$

where the prime denotes the derivative with respect to τ.

3. Application of the Global Residue Harmonic Balance Method (GRHBM)

In what follows, an analytical technique based on the global residue harmonic balance method (GRHBM) is employed to obtain the analytical approximate solutions of the problem discussed before. To obtain higher-order approximate solutions, all the residual errors are considered [33–36].

3.1. Zero-Order GRHBM Approximation

The initial approximation that satisfies the initial conditions of Equation (4) is given by:

$$x_0(\tau) = A\cos(\tau), \qquad \omega^2 = \omega_0^2. \tag{5}$$

Substituting Equation (5) into Equation (4) yields:

$$\begin{array}{l} \left(a_4 + \frac{48}{64}A^2a_5 + \frac{40}{64}A^4a_6 + \frac{35}{64}A^6a_7 - \frac{40}{64}A^4a_1\omega^2 - \frac{48}{64}A^2a_2\omega^2 - a_3\omega^2\right)A\cos(\tau) \\ +\left(\frac{16}{64}A^2a_5 + \frac{20}{64}A^4a_6 + \frac{21}{64}A^6a_7 - \frac{20}{64}A^4a_1\omega^2 - \frac{16}{64}A^2a_2\omega^2\right)\cos(3\tau) = 0. \end{array} \tag{6}$$

Inputting the coefficient of $\cos(\tau)$ to equal zero yields:

$$\omega_0 = \sqrt{\frac{64a_4 + 48A^2a_5 + 40A^4a_6 + 35A^6a_7}{40A^4a_1 + 48A^2a_2 + 64a_3}}. \tag{7}$$

At this time, we know the zero-order approximate periodic solution of Equation (4) in the form of Equation (7). Taking x_0 and ω_0 into the left-hand side of Equation (4), we have the residual error R_0 as follows:

$$R_0(\tau) = \frac{A^3}{128(5A^4a_1 + 6A^2a_2 + 8a_3)}\Delta\cos(3\tau), \tag{8}$$

where:

$$\Delta = -320A^2a_1a_4 - 256a_2a_4 - 80A^4a_1a_5 + 256a_3a_5 + 80A^4a_2a_6 + 320A^2a_3a_6 + 35A^8a_1a_7 + 112A^6a_2a_7 + 336A^4a_3a_7.$$

3.2. First-Order GRHBM Approximation

To obtain the first-order analytical approximation, the following relation is considered:

$$x(\tau) = x_0(\tau) + px_1(\tau), \quad \omega^2 = \omega_0^2 + \omega_1, \tag{9}$$

where p is a bookkeeping parameter. We assume:

$$x_1(\tau) = a_{31}(\cos(\tau) - \cos(3\tau)). \tag{10}$$

Substituting Equation (9) into Equation (4), one obtains all the coefficients of p as follows:

$$F_1(\tau,\omega_1,a_{31}) \triangleq \frac{\Delta_1\cos(\tau) + \Delta_2\cos(3\tau) + \Delta_3\cos(5\tau)}{(5A^4a_1 + 6A^2a_2 + 8a_3)}, \tag{11}$$

where:

$$\begin{aligned}\Delta_1 = {} & \tfrac{1}{128(5A^4a_1+6A^2a_2+8a_3)}A(1600A^3a_1a_4a_{31} + 1280Aa_2a_4a_{31} \\ & + 1680A^5a_1a_5a_{31} + 1536A^3a_2a_5a_{31} + 768Aa_3a_5a_{31} + 1600A^7a_1a_6a_{31}, \\ & + 1520A^5a_2a_6a_{31} + 960A^5a_3a_6a_{31} + 1505A^9a_1a_7a_{31} + 1456A^7a_2a_7a_{31} \\ & + 1008A^5a_3a_7a_{31} - 400A^8a_1^2\omega_1 - 960A^6a_1a_2\omega_1 - 576A^4a_2^2\omega_1 \\ & - 1280A^4a_1a_3\omega_1 - 1536A^2a_2a_3\omega_1 - 1024a_3^2\omega_1),\end{aligned}$$

$$\begin{aligned}\Delta_2 = {} & \tfrac{1}{128(5A^4a_1+6A^2a_2+8a_3)}(2752A^4a_1a_4a_{31} + 4096A^2a_2a_4a_{31} \\ & + 8192a_3a_4a_{31} + 2064A^6a_1a_5a_{31} + 3072A^4a_2a_5a_{31} + 6144A^2a_3a_5a_{31} \\ & + 1920A^8a_1a_6a_{31} + 2800A^6a_2a_6a_{31} + 5440A^4a_3a_6a_{31} + 1855A^{10}a_1a_7a_{31} \\ & + 2660A^8a_2a_7a_{31} + 5040A^6a_3a_7a_{31} - 200A^9a_1^2\omega_1 - 400A^7a_1a_2\omega_1 \\ & - 192A^5a_2^2\omega_1 - 320A^5a_1a_3\omega_1 - 256A^3a_2a_3\omega_1),\end{aligned}$$

$$\begin{aligned}\Delta_3 = {} & \tfrac{1}{128(5A^4a_1+6A^2a_2+8a_3)}A^2(3008A^2a_1a_4a_{31} + 2816a_2a_4a_{31} \\ & + 1776A^4a_1a_5a_{31} + 1536A^2a_2a_5a_{31} - 768a_3a_5a_{31} + 1280A^6a_1a_6a_{31} \\ & + 1040A^6a_1a_6a_{31} - 960A^2a_3a_6a_{31} + 1085A^8a_1a_7a_{31} + 868A^6a_2a_7a_{31} \\ & - 896A^4a_3a_7a_{31} - 40A^7a_1^2\omega_1 - 48A^5a_1a_2\omega_1 - 64A^3a_1a_3\omega_1).\end{aligned}$$

Using Equations (8) and (11), the following expression is constructed:

$$F_1(\tau,\omega_1,a_{13}) + R_0(\tau) = 0. \tag{12}$$

Equating the coefficients of $\cos(\tau)$ and $\cos(3\tau)$ as equal to zero in Equation (12), we can obtain the constants a_{31} and ω_1 as below:

$$\begin{aligned}a_{31} = {} & -(2A^3((5A^4a_1 + 6A^2a_2 + 8a_3)(-320A^2a_1a_4 - 256a_2a_4 - 80A^4a_1a_5) \\ & + 256a_3a_5 + 80A^4a_2a_6 + 320A^2a_3a_6 + 35A^8a_1a_7 + 112A^4a_2a_7 + 336A^4a_3a_7))/ \\ & (19520A^8a_1^2a_4 + 61184A^6a_1a_2a_4 + 44032A^4a_2^2a_4 + 125952A^4a_1a_3a_4 \\ & + 163840A^2a_2a_3a_4 + 131072a_3^2a_4 + 12240A^{10}a_1^2a_5 + 41088A^8a_1a_2a_5 \\ & + 11200A^{12}a_1^2a_6 + 37040A^{10}a_1a_2a_6 + 27520A^8a_2^2a_6 + 80320A^8a_1a_3a_6 \\ & + 26096A^{10}a_2^2a_7 + 75040^{10}a_1a_3a_7 + 99008A^8a_2a_3a_7 + 80640A^6a_3^2a_7),\end{aligned} \tag{13}$$

$$
\begin{aligned}
\omega_1 = &-(A^4(-320A^2a_1a_4 - 256a_2a_4 - 80A^4a_1a_5 + 256a_3a_5 + 80A^4a_2a_6 + 320A^2a_3a_6 \\
&+ 35A^8a_1a_7 + 122A^6a_2a_7 + 336A^4a_3a_7 + 1600A^2a_1a_4 + 1280a_2a_4 + 1680A^4a_1a_5 \\
&+ 1536A^2a_2a_5 + 768a_3a_7 + 1600A^6a_1a_6 + 1520A^4a_2a_6 + 960A^2a_3a_6 \\
&+ 1505A^8a_1a_7 + 1456A^6a_2a_7 + 1008A^4a_3a_7))/(8(5A^4a_1 + 6A^2a_2 + 8a_3) \\
&(19520A^8a_1^2a_4 + 61184A^6a_1a_2a_4 + 44032A^4a_2^2a_4 + 125952A^4a_1a_3a_4 \\
&+ 163840A^2a_2a_3a_4 + 131072a_3^2a_4 + 12240A^{10}a_1^2a_5 + 41088A^8a_2a_5 \\
&+ 30720A^6a_2^2a_5 + 90624A^6a_1a_3a_5 + 119808A^4a_2a_3a_5 + 98304A^2a_3^2a_5 \\
&+ 11200A^{12}a_1^2a_6 + 37040A^{10}a_1a_2a_6 + 27520A^8a_2^2a_6 + 80320A^8a_1a_3a_6 \\
&+ 106240A^6a_2a_3a_6 + 87040A^4a_3^2a_6 + 11025A^{14}a_1^2a_7 + 35560A^{12}a_1a_2a_7 \\
&+ 26096A^{10}a_2^2a_7 + 75040A^{10}a_1a_3a_7 + 99008A^8a_2a_3a_7 + 80640A^6a_3^2a_7)).
\end{aligned}
\tag{14}
$$

Regarding Equation (9), we obtain the first-order approximate frequency and periodic solution as follows:

$$x(\tau) = A\cos(\tau) + a_{31}(\cos(\tau) - \cos(3\tau)), \tag{15}$$

$$\omega = \sqrt{\frac{64a_4 + 48A^2a_5 + 40A^4a_6 + 35A^6a_7}{40A^4a_1 + 48A^2a_2 + 64a_3} + \omega_1}. \tag{16}$$

3.3. Second-Order GRHBM Approximation

To obtain the second-order analytical approximation, we consider:

$$x(\tau) = x_0(\tau) + x_1(\tau) + px_2(\tau),\ \omega^2 = \omega_0^2 + \omega_1 + p\omega_2, \tag{17}$$

in which:

$$x_2(\tau) = a_{32}(\cos(\tau) - \cos(3\tau)) + a_{52}(\cos(\tau) - \cos(5\tau)). \tag{18}$$

Substituting Equation (17) into Equation (4) and then equating the coefficients of $\cos(\tau)$, $\cos(3\tau)$, and $\cos(5\tau)$, the following three linear equations are obtained as follows:

$$
\begin{aligned}
\cos(\tau):\quad & \tfrac{7}{64}\alpha_7\big(14a_{31}A^6 + 42a_{31}^2A^5 + 100a_{31}^3A^4 + 175a_{31}^4A^3 + 210a_{31}^5A^2 + 154a_{31}^6A + 52a_{31}^7 + 5A^7\big) \\
&+ \tfrac{7}{64}a_{52}\alpha_7\big(132a_{31}A^5 + 375a_{31}^2A^4 + 700a_{31}^3A^3 + 840a_{31}^4A^2 + 588a_{31}^5A + 182a_{31}^6 + 28A^6\big) \\
&+ \tfrac{7}{32}a_{32}\alpha_7\big(42a_{31}A^5 + 150a_{31}^2A^4 + 350a_{31}^3A^3 + 525a_{31}^4A^2 + 462a_{31}^5A + 182a_{31}^6 + 7A^6\big) \\
&+ \tfrac{5}{16}a_{32}\alpha_6\big(24a_{31}A^3 + 60a_{31}^2A^2 + 80a_{31}^3A + 45a_{31}^4 + 5A^4\big) + \tfrac{3}{2}a_{32}\alpha_5\big(3a_{31}A + 3a_{31}^2 + A^2\big) \\
&+ \tfrac{5}{16}a_{52}\alpha_6\big(32a_{31}A^3 + 60a_{31}^2A^2 + 60a_{31}^3A + 25a_{31}^4 + 9A^4\big) + \tfrac{3}{4}a_{52}\alpha_5\big(6a_{31}A + 4a_{31}^2 + 3A^2\big) \\
&+ \tfrac{5}{16}\alpha_6\big(5a_{31}A^4 + 12a_{31}^2A^3 + 20a_{31}^3A^2 + 20a_{31}^4A + 9a_{31}^5 + 2A^5\big) + \alpha_4(a_{31} + a_{32} + a_{52} + A) \\
&+ \tfrac{3}{4}\alpha_5\big(2a_{31}A^2\alpha_5 + 3a_{31}^2A + 2a_{31}^3 + A^3\big) + [(\omega_0^2 + \omega_1)\tfrac{1}{4}\alpha_2(19a_{31}A^2 + 41a_{31}^2A + 48a_{31}^3 - A^3) \\
&+ \tfrac{1}{16}a_{32}\alpha_1(272a_{31}A^3 + 1164a_{31}^2A^2 + 2420a_{31}^3A + 1925a_{31}^4 + 53A^4) + \tfrac{1}{4}a_{32}\alpha_2(144a_{31}^2 + 19A^2 \\
&+82a_{31}A) + \tfrac{1}{16}a_{52}\alpha_1(232a_{31}A^3 + 582a_{31}^2A^2 + 680a_{31}^3A + 239a_{31}^4 + 91A^4) + \tfrac{1}{2}(12a_{52}A^2\alpha_2 \\
&+11a_{31}a_{52}A\alpha_2 - a_{31}^2a_{52}\alpha_2 + 18a_{31}\alpha_3 + 18a_{32}\alpha_3)] + \tfrac{1}{16}(\omega_0^2 + \omega_1 + \omega_2)(a_{31}\alpha_1(-92a_{31}A^3 \\
&-292a_{31}^2A^2 - 420a_{31}^3A - 245a_{31}^4 + 15A^4) - 10A^5\alpha_1) + \tfrac{1}{4}(a_{31}\alpha_2(-25a_{31}A - 30a_{31}^2 + 2A^2) \\
&-3A^3\alpha_2)\omega_2 - \alpha_3(a_{31} + A)\,\omega_2,
\end{aligned}
\tag{19}
$$

$$
\begin{aligned}
\cos(3\tau):\quad & \tfrac{7}{64}\alpha_7\big(-18a_{31}^2A^5 - 75a_{31}^3A^4 - 175a_{31}^4A^3 - 252a_{31}^5A^2 - 210a_{31}^6A - 78a_{31}^7 + 3A^7\big) \\
&- \tfrac{7}{64}a_{31}a_{32}\alpha_7\big(225a_{31}A^4 + 700a_{31}^2A^3 + 1260a_{31}^3A^2 + 1260a_{31}^4A + 546a_{31}^5 + 36A^5\big) \\
&- \tfrac{21}{32}\big(a_{31}a_{52}\alpha_7\big(25a_{31}A^4 + 70a_{31}^2A^3 + 105a_{31}^3A^2 + 84a_{31}^4A + 28a_{31}^5 + 3A^5\big) - a_{52}A^6\alpha_7\big) \\
&+ \tfrac{5}{16}a_{52}\alpha_6(a_{31} + A)\,\big(-9a_{31}A^2 - 21a_{31}^2A - 19a_{31}^3 + A^3\big) - \tfrac{5}{16}a_{32}\alpha_6(16a_{31}A^3 + 60a_{31}^2A^2 \\
&+100a_{31}^3A + 65a_{31}^4 + A^4) + \tfrac{5}{16}\alpha_6\big(-a_{31}A^4 - 8a_{31}^2A^3 - 20a_{31}^3A^2 - 25a_{31}^4A - 13a_{31}^5 + A^5\big) \\
&- \tfrac{3}{4}\alpha_5\big(a_{32}A^2 + 6a_{31}a_{32}A + 2a_{31}a_{52}A + 8a_{31}^2a_{32} + 2a_{31}^2a_{52}\big) + \tfrac{1}{4}\alpha_5(A^3 - 3a_{31}A^2 - 9a_{31}^2A \\
&-8a_{31}^3) - (a_{31} + a_{32})\,\alpha_4 - \tfrac{1}{4}\omega_2\big(-19a_{31}A^2\alpha_2 - 41a_{31}^2A\alpha_2 - 48a_{31}^3\alpha_2 - 36a_{31}\alpha_3 + A^3\alpha_2\big) \\
&+(\omega_0^2 + \omega_1)[\tfrac{1}{16}a_{52}\alpha_1(232a_{31}A^3 + 582a_{31}^2A^2 + 680a_{31}^3A + 239a_{31}^4 + 91A^4) + \tfrac{1}{4}a_{32}\alpha_2(19A^2 \\
&+82a_{31}A + 144a_{31}^2) + \tfrac{1}{16}a_{32}\alpha_1\big(272a_{31}A^3 + 1164a_{31}^2A^2 + 2420a_{31}^3A + 1925a_{31}^4 + 53A^4\big) \\
&+ \tfrac{1}{4}\alpha_2(19a_{31}A^2 + 41a_{31}^2A + 48a_{31}^3 - A^3) + \tfrac{1}{2}a_{52}\alpha_2\big(11a_{31}A - a_{31}^2 + 12A^2\big) + 9(a_{31} + a_{32})\,\alpha_3] \\
&- \tfrac{1}{16}\alpha_1(\omega_0^2 + \omega_1 + \omega_2)(-53a_{31}A^4 - 136a_{31}^2A^3 - 388a_{31}^3A^2 - 605a_{31}^4A - 385a_{31}^5 + 5A^5),
\end{aligned}
\tag{20}
$$

$$
\begin{aligned}
\cos(5\tau): \quad & -\tfrac{3}{4}a_{32}A\alpha_5(2a_{31}+A)+\tfrac{1}{16}\alpha_6\left(-40a_{31}^2A^3-50a_{31}^3A^2-25a_{31}^4A+a_{31}^5+A^5\right)-\tfrac{15}{16}a_{31}A^4\alpha_6 \\
& -\tfrac{7}{64}a_{52}\alpha_7(13A^6+108a_{31}A^5+360a_{31}^2A^4+700a_{31}^3A^3+840a_{31}^4A^2+588a_{31}^5A+188a_{31}^6) \\
& -\tfrac{5}{16}a_{32}\alpha_6\left(16a_{31}A^3+30a_{31}^2A^2+20a_{31}^3A-a_{31}^4+3A^4\right)-\tfrac{3}{4}a_{52}\alpha_5\left(6a_{31}A+5a_{31}^2+2A^2\right) \\
& +\tfrac{7}{64}\alpha_7(A^7-33a_{31}^2A^5-75a_{31}^3A^4-105a_{31}^4A^3-84a_{31}^5A^2-28a_{31}^6A+2a_{31}^7)+25a_{52}\alpha_3 \\
& -\tfrac{7}{64}a_{32}\alpha_7(66a_{31}A^5+225a_{31}^2A^4+420a_{31}^3A^3+420a_{31}^4A^2+168a_{31}^5A-14a_{31}^6+8A^6) \\
& -\tfrac{5}{16}a_{52}\alpha_6\left(28a_{31}A^3+60a_{31}^2A^2+64a_{31}^3A+29a_{31}^4+5A^4\right)-\tfrac{3}{4}a_{31}A\alpha_5(a_{31}+A)-a_{52}\alpha_4 \\
& -\tfrac{7}{8}a_{31}A^6\alpha_7+(\omega_0^2+\omega_1)[\tfrac{1}{16}a_{32}\alpha_1\left(144a_{31}A^3+294a_{31}^2A^2+100a_{31}^3A-245a_{31}^4+47A^4\right) \\
& +\tfrac{1}{4}a_{52}\alpha_2\left(130a_{31}A+143a_{31}^2+54A^2\right)+\tfrac{1}{16}a_{52}\alpha_1 588a_{31}A^3+1452a_{31}^2A^2+1952a_{31}^3A \\
& +1137a_{31}^4+169A^4)+\tfrac{1}{4}\alpha_2\left(11a_{31}A^2+3a_{31}^2A-8a_{31}^3\right)+\tfrac{1}{4}a_{32}\alpha_2\left(6a_{31}A-24a_{31}^2+11A^2\right)] \\
& +\tfrac{1}{16}\left(\omega_0^2+\omega_1+\omega_2\right)\left(\tfrac{1}{16}a_{31}\alpha_1\left(72a_{31}A^3+98a_{31}^2A^2+25a_{31}^3A-49a_{31}^4+47A^4\right)-A^5\alpha_1\right) \\
& +\tfrac{1}{4}\alpha_2\omega_2\left(11a_{31}A^2+3a_{31}^2A-8a_{31}^3\right).
\end{aligned}
\tag{21}
$$

Solving Equations (19)–(21), three unknown constants (a_{32}, a_{52}, and ω_2) are determined. The Mathematica command software was employed for this purpose. The expressions obtained for a_{32}, a_{52}, and ω_2 take too much space and cannot be presented here. However, their numerical values will be reported in the results and discussion section.

Regarding Equation (17), the second-order approximation solution of Equation (4) is obtained as follows:

$$
\begin{aligned}
x(\tau) &= (A+a_{31}+a_{32}+a_{52})\cos(\tau)-(a_{31}+a_{32})\cos(3\tau)-a_{52}\cos(5\tau), \\
\omega &= \sqrt{\omega_0^2+\omega_1+\omega_2}.
\end{aligned}
\tag{22}
$$

4. Results and Discussion

In this work, the effectiveness and convenience of the global residue harmonic balance method for the nonlinear vibration of an electrostatically actuated micro-beam have been displayed. The obtained approximate analytical solution was verified through a comparison with the fourth-order Runge–Kutta method. From Figure 2, one can observe that for all different values of amplitude A, the second-order approximate solutions match extremely well with the numerical solutions. In Table 1, the approximate angular frequencies obtained using the GRHB method, i.e., ω_{GRHBM}, are compared with those achieved using the higher-order Hamiltonian approach ω_{HA} [2], energy balance method ω_{EBM} [16], He's variational approach ω_{VA} [28], and the homotopy analysis method ω_{HAM} [30]. In Table 1, the exact values ω_{exact} are also reported. The comparison reveals the correctness and accuracy of the proposed method.

Table 1. Comparison of the approximated frequencies with corresponding exact frequencies based on the various parameters in Equation (1).

A	N	α	V	ω_{HA} (Error %)	ω_{EBM} (Error %)	ω_{VA} (Error %)	ω_{HAM} (Error %)	ω_{GRHBM2} (Error %)	ω_{Exact}
Constant Parameters				[2]	[16]	[28]	[30]	Present	[30]
0.3	10	24	0	26.3669	26.3867	26.3644	26.8372	26.8372	26.8372
				(1.7837)	(1.7073)	(1.7933)	(0.0000)	(0.0000)	
0.3	10	24	20	16.3547	16.3829	16.3556	16.6486	16.6486	16.6486
				(1.7970)	(1.6218)	(1.7914)	(0.0000)	(0.0000)	
0.6	10	24	10	26.3562	26.5324	26.1671	28.5368	28.5378	28.5382
				(8.2789)	(7.5598)	(9.0614)	(0.0049)	(0.0014)	
0.6	10	24	20	17.3013	17.5017	17.0940	18.5902	18.5902	18.5902
				(7.4497)	(6.2194)	(8.7528)	(0.0000)	(0.0000)	

The analytical expression obtained in Equation (17) can be used to study the effects of the parameters given in Equation (4) on the nonlinear frequency. Figure 3 shows the effect of parameter N on the nonlinear frequency with respect to the amplitude for $\alpha = 24$ and $V = 10$. The important point is that the effect of parameter N on the nonlinear frequency is dependent on the amplitude. As can be seen, for small amplitudes, the nonlinear frequency increases when increasing the axial tensile load. However, for amplitudes close to unity, the frequency is independent of the axial load. Moreover, for each value of the axial load, there is an amplitude in which the frequency of the electrostatically actuated micro-beam exhibits the maximum value.

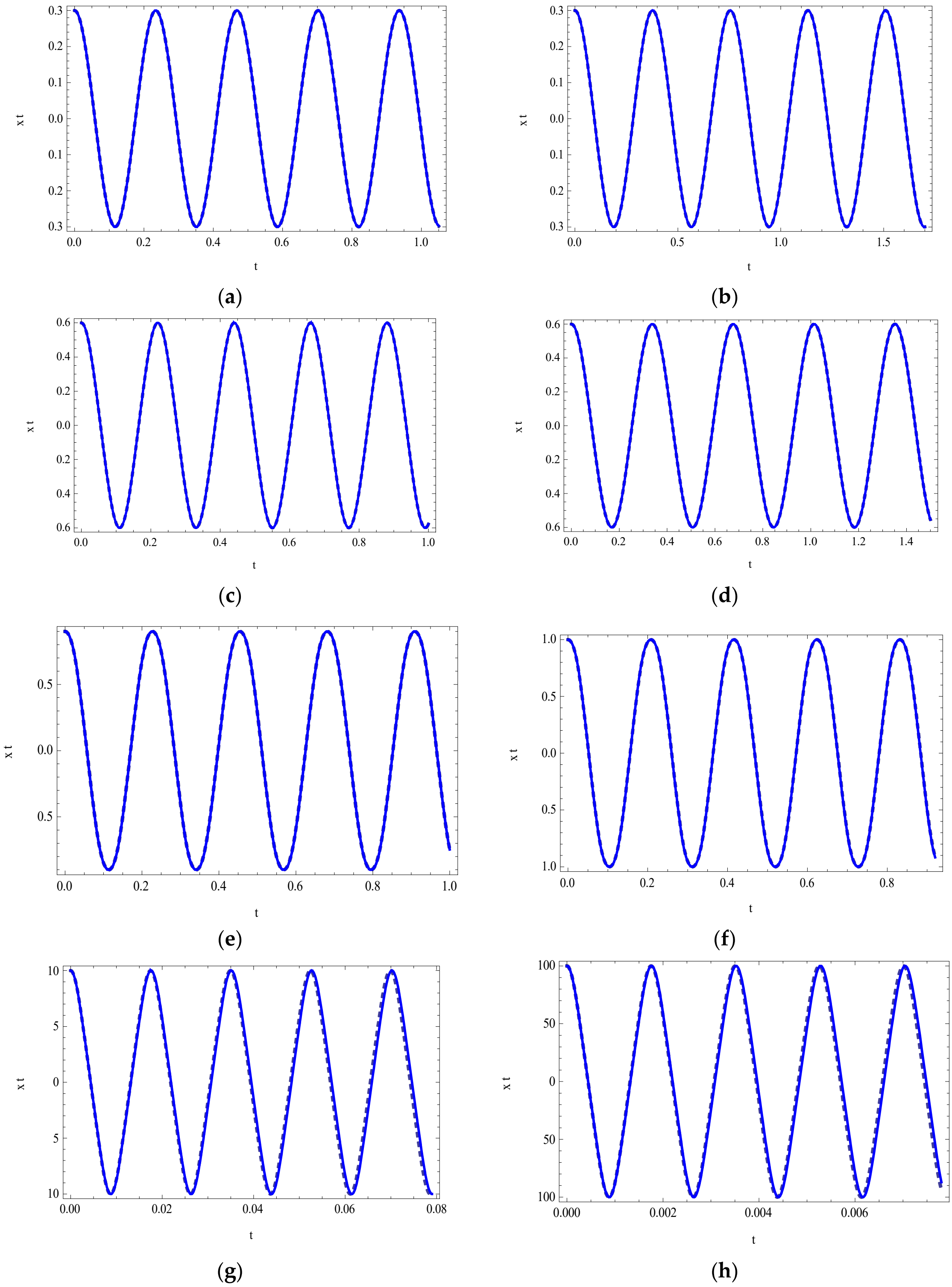

Figure 2. The comparison between analytical approximate solutions using the GRHBM (dashed line) and numerical solution (solid line). (**a**) A = 0.3, V = 0, N = 10, α = 24. (**b**) A = 0.3, V = 20, N = 10, α = 24. (**c**) A = 0.6, V = 10, N = 10, α = 24. (**d**) A = 0.6, V = 20, N = 10, α = 24. (**e**) A = 0.9, V = 10, N = 10, α = 24. (**f**) A = 1, V = 0, N = 10, α = 24. (**g**) A = 10, V = 10, N = 10, α = 24. (**h**) A = 100, V = 20, N = 10, α = 24.

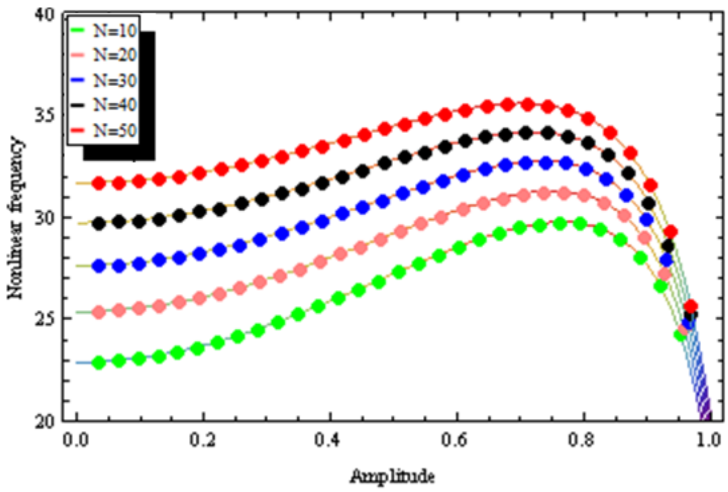

Figure 3. Effect of parameter N on the nonlinear frequency of electrostatically actuated micro-beam.

Figure 4 illustrates the effect of the parameter V on the nonlinear frequency of an electrostatically actuated microbeam for $\alpha = 24$ and $N = 10$. It can be seen that for a specific amplitude value, the nonlinear frequency decreases when increasing the electrostatic load. Additionally, for a specific value of parameter V, the nonlinear frequency increases and then decreases again. This means that for each value of the electrostatic load, there is an amplitude in which the frequency of the electrostatically actuated micro-beam exhibits the maximum value.

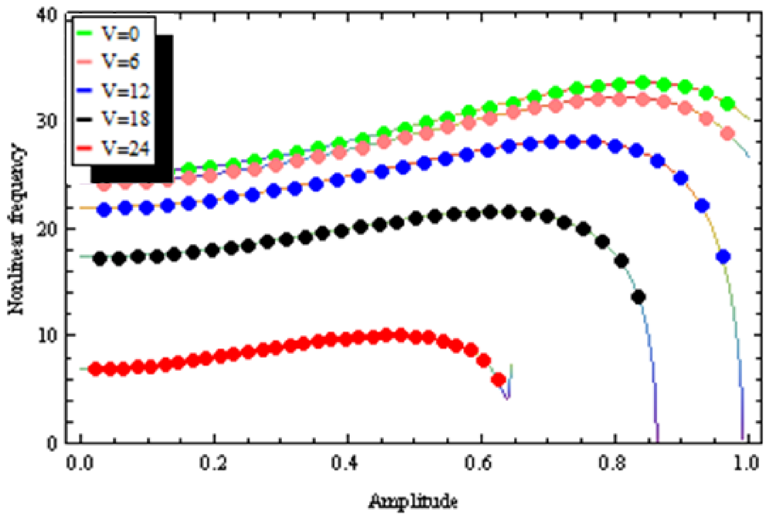

Figure 4. Effect of parameter V on nonlinear frequency of electrostatically actuated micro beam.

Figure 5 shows the effect of the parameter α on the nonlinear frequency of the electrostatically actuated microbeam for $N = V = 10$. Figure 5 reveals that for small amplitude values, the nonlinear frequency is independent of the parameter α. According to Equation (3), the non-dimensional parameter α reflects the ratio of the initial gap g_0 to the beam thickness h. Hence, it may be concluded that for a microbeam with specific dimensions, the frequency is not influenced by the change of the initial gap between the nanobeam and electrode for small amplitudes. This observation is also seen when the amplitude is close to unity. Additionally, for each value of the parameter α, the maximum frequency occurs when the amplitude is almost 0.8.

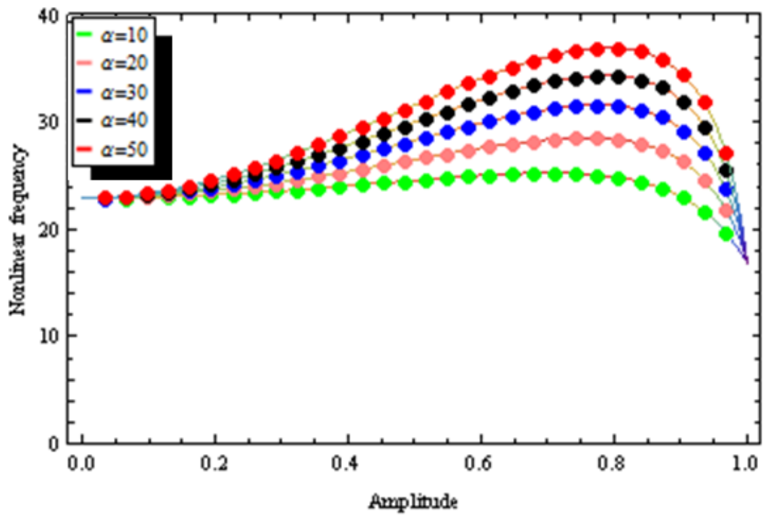

Figure 5. Effect of parameter α on nonlinear frequency of electrostatically actuated microbeam.

5. Conclusions

In this paper, the global residue harmonic balance method has been successfully applied to investigate the large-amplitude vibration of electrostatically actuated microbeams. An excellent agreement has been found between the approximated and numerical solutions obtained from exact ones. It is remarkably important that the obtained results are valid for small as well as large values of an initial oscillation amplitude with a high accuracy. The GRHB method is very easy to apply, straightforward, concise, and highly efficient in finding approximate analytical solutions for wide classes of nonlinear ordinary differential equations. The global residue harmonic balance method is applied as a powerful technique for solving many other sophisticated nonlinear problems arising in physics and engineering applications.

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