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Two Approximation Formulas for Bateman's G-Function with Bounded Monotonic Errors

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Abstract: Two new approximation formulas for Bateman's G-function are presented with strictly monotonic error functions and we deduced their sharp bounds. We also studied the completely monotonic (CM) degrees of two functions involving $G(r)$, deducing two of its inequalities and improving some of the recently published results.

Keywords: Bateman's G-function; approximation formula; error; CM degree; inequality

MSC: 33B15; 26A48; 26D15; 41A30

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Citation: Mahmoud, M.; Almuashi, H. Two Approximation Formulas for Bateman's G-Function with Bounded Monotonic Errors. *Mathematics* **2022**, *10*, 4787. https://doi.org/10.3390/math10244787

Academic Editors: Xiangmin Jiao, Clemente Cesarano and Sitnik Sergey

Received: 2 November 2022
Accepted: 14 December 2022
Published: 16 December 2022

1. Introduction and Preliminaries

Bateman's G-function is defined as [1]

$$G(r) = \psi((1+r)/2) - \psi(r/2), \quad r \in \mathbb{R} - \{0, -1, -2, \dots\} \tag{1}$$

where $\psi(r) = \frac{d}{dr} \ln \Gamma(r)$ is the digamma function and Γ is the Euler gamma function [2]. The function $G(r)$ has several inequalities, such as

Qiu and Vuorinen [3]: $$4(1.5 - \ln 4) r^{-2} < G(r) - r^{-1} < \tfrac{1}{2} r^{-2}, \quad r > \tfrac{1}{2} \tag{2}$$

Mortici [4]: $$0 < G(r) \le \gamma + \psi(1/2) + 3/2, \quad r \ge 2 \tag{3}$$

Mahmoud and et al. [5]: $$\ln\left(\tfrac{r+3}{r+2}\right) + \tfrac{2}{r(r+1)} < G(r) < \ln\left(\tfrac{r+2}{r+1}\right) + \tfrac{2}{r(r+1)} \quad r > 0 \tag{4}$$

Nantomah [6]: $$G(r) > \tfrac{1}{r} + \tfrac{1}{2(r+1)^2}, \quad r > 0 \tag{5}$$

$$2 - 2e^{\frac{1}{r+1}} < G(r) - \tfrac{2}{r} < 2e - 2\ln 2 - 2e^{\frac{1}{r+1}}, \quad r > 0 \tag{6}$$

where γ is the Euler–Mascheroni constant and the constants $c_1 = 3$ and $c_2 = \frac{e^4 - 16}{12}$ are the best possible.

Mahmoud and Almuashi [7] presented a generalization of Bateman's G-function by

$$G_\rho(r) = \psi\left(\frac{\rho + r}{2}\right) - \psi\left(\frac{r}{2}\right), \qquad r \ne -2m, -2m - \rho;\ 0 < \rho < 2;\ m = 0, 1, 2, \dots$$

and they proved the following inequality:

$$\ln\left(\frac{\rho}{r+\varepsilon} + 1\right) < G_\rho(r) - \frac{2\rho}{(r+\rho)r} < \ln\left(\frac{\rho}{r+\sigma} + 1\right), \quad r > 0;\ \rho \in (0, 2)$$

where $\varepsilon = \frac{\rho}{e^{\gamma + \frac{2}{\rho} + \psi\left(\frac{\rho}{2}\right)} - 1}$ and $\sigma = 1$ are the best possible.

Recently, Ahfaf, Mahmoud, and Talat [8] introduced the following rational approximations

$$G(r) = \frac{1}{r} + \left[2r^2 + \sum_{m=1}^{s} 4A_m r^{2-2m}\right]^{-1} + O\left(\frac{1}{r^{2s+2}}\right),$$

with

$$A_1 = \frac{1}{4}, \text{ and } A_m = \frac{(1-2^{2m+2})B_{2m+2}}{1+m} + \sum_{s=1}^{m-1} \frac{(1-2^{2m-2s+2})B_{2m-2s+2}A_s}{1-s+m}, \ m > 1$$

where $B_j's$ are Bernoulli numbers. As a consequence, they presented the new bounds

$$\frac{r^4}{2\left(r^6 + \frac{1}{2}r^4 - \frac{3}{4}r^2 + \frac{27}{8}\right)} < G(r) - 1/r < \frac{r^2}{2\left(r^4 + \frac{1}{2}r^2 - \frac{3}{4}\right)}, \tag{7}$$

where the lower bound and the upper bound hold for $r > 0$ and $r > \frac{\sqrt{\sqrt{13}-1}}{2}$, respectively, and

$$\frac{1}{2r^2 M_1(r)} < G(r) - 1/r < \frac{1}{2r^2 M_2(r)}, \tag{8}$$

where the lower bound and the upper bound hold for $r > 0$ and $r > \frac{13}{5}$, respectively, with

$$M_1(r) = 1 + \frac{1}{2r^2} - \frac{3}{4r^4} + \frac{27}{8r^6} - \frac{423}{16r^8} + \frac{9927}{32r^{10}} - \frac{324423}{64r^{12}} + \frac{14098527}{128r^{14}}$$

and

$$M_2(r) = 1 + \frac{1}{2r^2} - \frac{3}{4r^4} + \frac{27}{8r^6} - \frac{423}{16r^8} + \frac{9927}{32r^{10}} - \frac{324423}{64r^{12}} + \frac{14098527}{128r^{14}} - \frac{787622823}{256r^{16}}.$$

Bateman's G-function is useful in summing certain numerical and algebraic series [9]. For example:

$$\sum_{m=0}^{\infty} \frac{(-1)^m}{\alpha + \beta m} = \frac{1}{2\beta} G\left(\frac{\alpha}{\beta}\right), \qquad \alpha \neq 0, -\beta, -2\beta, \ldots \tag{9}$$

and hence we obtain $\pi = G(1/2)$ and $\ln 4 = G(1)$. The function $G(r)$ and its generalization $G_\rho(r)$ are related to the generalized hypergeometric functions by the relations [7]

$$G(r) = r^{-1} \, {}_2F_1\left(1, 1; 1+r; \frac{1}{2}\right), \quad r > 0$$

and

$$G_\rho(r) = \frac{\rho}{r+\rho} \, {}_3F_2\left(1, 1, \frac{\rho+2}{2}; 2, \frac{r+\rho+2}{2}; 1\right), \quad r > 0; \ 0 < \rho < 2.$$

There is a relation between the function $G(r)$ and the Wallis's ratio $\frac{\Gamma(n+1/2)}{\sqrt{\pi}\,\Gamma(n+1)}$, $n \in \mathbb{N}$. Furthermore, the sequence

$$L_m = \frac{m-1}{2} \left[\int_0^{\pi/2} (\sin u)^{m-1} du \right], \quad m \in \mathbb{N},$$

which appears in the computation of the intersecting probability between a plane couple and a convex body [10], is related to the function $G(r)$ (see Ref. [11]).

The outline of the paper is as follows: Section 1 provides the definition of the Bateman's G-function with some of its inequalities. In Section 2, we studied the CM degrees of two functions involving $G(r)$ and, consequently, we presented two new inequalities of $G(r)$, which improve some recently published results. Additionally, we proved that the function

$$\theta_1(r) = r^3 \left[\frac{2}{r(1+r)} + \ln\left(\frac{2+r}{r+1}\right) - G(r) \right], \quad r > 0$$

is strictly increasing with the sharp bounds $0 < \theta_1(r) < \frac{1}{3}$, and the function

$$\theta_2(r) = r^4 \left[\frac{2}{r(1+r)} + \ln\left(\frac{2+r}{r+1}\right) - \frac{1}{3r^3} - G(r) \right], \quad r > 0$$

is strictly decreasing with the sharp bounds $\frac{-3}{2} < \theta_1(r) < 0$.

2. Main Results

Recall that a function $H(r)$ on $r > 0$ is called CM if its derivatives exist for all orders, such that

$$(-1)^m H^{(m)}(r) \geq 0, \quad r > 0;\ m \in \mathbb{N}.$$

From Bernstein's well-known theory, the convergence of the following improper integral determines the necessary and sufficient condition for $H(r)$ to be CM on $r \geq 0$ [12]

$$H(r) = \int_0^\infty e^{-ur} d\xi(u), \qquad r \geq 0 \tag{10}$$

where $\xi(u)$ is non-decreasing and bounded for $u \geq 0$. Let $H(r)$ be a CM function for $r > 0$ and consider the notation $H(\infty) = \lim_{r\to\infty} H(r)$. If $r^\delta[H(r) - H(\infty)]$ is a CM function for $r > 0$ if and only if $\delta \in [0, \varpi]$; then, the number $\varpi \in R$ is called the CM degree of $H(r)$ for $r > 0$ and is denoted by $\deg^r_{\mathrm{CM}}[H(r)] = \varpi$. This concept gives more accuracy in measuring the complete monotonicity property [13,14].

Theorem 1. *The function*

$$F_1(r) = G(r) - \frac{2}{r(1+r)} - \ln\left(\frac{2+r}{r+1}\right) + \frac{1}{3r^3}, \quad r > 0 \tag{11}$$

satisfies that $2 \leq deg^r_{\mathrm{CM}}[F_1(r)] \leq 3$.

Proof. Using the relation [5]

$$G(r) = 2\int_0^\infty \frac{e^{-ru}}{1+e^{-u}} du, \quad r > 0 \tag{12}$$

we obtain

$$F_1(r) = \int_0^\infty \frac{e^{-2u}\kappa_2(u)}{6(e^u+1)u} e^{-ru} du, \quad r > 0$$

where

$$\begin{aligned}
\kappa_2(u) &= e^{2u}u^3 + e^{3u}u^3 + 12e^u u - 6e^{2u} + 6 \\
&= 3u^4 + \sum_{m=5}^{\infty} \frac{2^{m-3}(m^3 - 3m^2 + 2m - 48) + 3^{m-3}(m^3 - 3m^2 + 2m) + 12m}{m!} u^m \\
&> 0, \quad u > 0.
\end{aligned}$$

Then, $F_1(r)$ is the CM function. Furthermore, using the asymptotic formula [5]

$$G(r) - \frac{1}{r} \sim \sum_{m=1}^{\infty} \frac{(2^{2m}-1)B_{2m}}{mr^{2m}}, \qquad r \to \infty \tag{13}$$

we have

$$F_1(\infty) = \lim_{r\to\infty} F_1(r) = \lim_{r\to\infty}\left(\frac{3}{2r^4} - \frac{21}{5r^5} + O(r^{-6})\right) = 0.$$

Now,

$$r^2 F_1(r) = \int_0^\infty \frac{e^{-2u}}{3(e^u+1)^3 u^3} \chi_1(u) e^{-ru} du, \quad r > 0$$

where

$$\begin{aligned}\chi_1(u) &= e^{5u}u^3+e^{2u}\Big(19u^2+27u+18\Big)u+6\Big(2u^2+2u+1\Big)+3e^{4u}\Big(u^3-u^2-2u-2\Big)\\ &+ 3e^{u}\Big(2u^3+11u^2+10u+4\Big)+3e^{3u}\Big(9u^3+u^2-2u-4\Big)\\ &= 36u^4+\frac{378u^5}{5}+\frac{468u^6}{5}+\frac{6073u^7}{70}+\frac{133u^8}{2}+\frac{42001u^9}{945}+\frac{504139u^{10}}{18900}\\ &+ \sum_{m=11}^{\infty}\frac{f_m}{2400m!}u^m\end{aligned}$$

with

$$\begin{aligned}f_m &= -144000(2(3^m)+4^m-2)+192(m-1)(m-2)m5^m+125m(576(2m+3)(m+1)\\ &+ 9(-26+(m-7)m)4^m+64(m(3m-8)-1)3^m+3(m(19m-3)+56)2^{m+3}\Big)\\ &= 144000m^3+360000m^2+216000m+288000+5^m\Big(192m^3-576m^2+384m\Big)\\ &+ 2^{2m}\Big(1125m^3-7875m^2-29250m-144000\Big)\\ &+ 3^m\Big(24000m^3-64000m^2-8000m-288000\Big)\\ &+ 2^m\Big(57000m^3-9000m^2+168000m\Big)>0,\quad m\geq 10.\end{aligned}$$

Then, $2\leq \deg^r_{\mathrm{CM}}[F_1(r)]$. However,

$$r^3F_1(r)=\int_0^\infty \frac{e^{-2u}}{(e^u+1)^4u^4}\chi_2(u)e^{-ru}du,\quad r>0$$

where

$$\begin{aligned}\chi_2(u) &= -2e^{4u}\Big(8u^4+2u^3-6u-9\Big)+e^{5u}\Big(u^3+3u^2+6u+6\Big)-2\Big(4u^3+6u^2+6u+3\Big)\\ &\quad -4e^{2u}\Big(2u^4+11u^3+15u^2+12u+3\Big)-e^{u}\Big(2u^4+31u^3+45u^2+42u+18\Big)\\ &\quad -2e^{3u}\Big(5u^4+13u^3+15u^2+6u-6\Big)\end{aligned}$$

with $\chi_2(0.5)=2.08162$ and $\chi_2(0.9)=-21.5214$. Then, $r^3F_1(r)$ is not a CM function; hence, $\deg^r_{\mathrm{CM}}[F_1(r)]<3$. □

From Theorem 1, the function $F_1(r)$ is a decreasing function and $F_1(\infty)=0$; then, we obtain the following result:

Corollary 1. *The function $G(r)$ satisfies that*

$$\frac{2}{r(1+r)}+\ln\left(\frac{2+r}{r+1}\right)-\frac{1}{3r^3}<G(r),\quad r>0. \tag{14}$$

Theorem 2. *The function*

$$F_2(r)=\frac{2}{r(1+r)}+\ln\left(\frac{2+r}{r+1}\right)-\frac{1}{3r^3}+\frac{3}{2r^4}-G(r),\quad r>0 \tag{15}$$

satisfies that $3\leq deg^r_{\mathrm{CM}}[F_2(r)]\leq 4$.

Proof. Using the relation (12), we have

$$F_2(r)=\int_0^\infty \frac{\kappa_1(u)}{12(e^u+1)u}e^{-ru}du,\quad r>0$$

where

$$\begin{aligned}\kappa_1(u) &= 3u^4 + e^u(3u-2)u^3 - 2u^3 - 24e^{-u}u - 12e^{-2u} + 12 \\ &= \frac{21u^5}{5} + \sum_{m=6}^{\infty} \frac{b_m}{108m!} u^m > 0, \quad u > 0\end{aligned}$$

with

$$\begin{aligned}b_m &= 2^{m-2}\Big(81m^4 - 594m^3 + 1215m^2 - 702m + 5184\Big) \\ &\quad +3^m\Big(4m^4 - 32m^3 + 68m^2 - 40m\Big) - 2592m \\ &> (m-2)\Big(85m^3 - 456m^2 + 371m - 2592\Big) > 0, \quad m \geq 6.\end{aligned}$$

Then, $F_2(r)$ is a CM function. Furthermore, using the asymptotic Formula (13), we obtain

$$F_2(\infty) = \lim_{r\to\infty} F_2(r) = \lim_{r\to\infty}\left(\frac{21}{5r^5} - \frac{9}{r^6} + O(r^{-7})\right) = 0.$$

Now,

$$r^3 F_2(r) = \int_0^{\infty} \frac{e^{-2u}}{2(e^u+1)^4 u^4} \chi_3(u) e^{-ru} du, \quad r > 0$$

where

$$\begin{aligned}\chi_3(u) &= 3e^{6u}u^4 + e^{4u}\Big(50u^4 + 8u^3 - 24u - 36\Big) + 4\Big(4u^3 + 6u^2 + 6u + 3\Big) \\ &+ e^u\Big(4u^4 + 62u^3 + 90u^2 + 84u + 36\Big) + 2e^{5u}\Big(6u^4 - u^3 - 3u^2 - 6u - 6\Big) \\ &+ 4e^{3u}\Big(8u^4 + 13u^3 + 15u^2 + 6u - 6\Big) + e^{2u}\Big(19u^4 + 88u^3 + 120u^2 + 96u + 24\Big) \\ &= \frac{672u^5}{5} + \frac{1968u^6}{5} + \frac{68512u^7}{105} + \frac{82507u^8}{105} + \frac{717838u^9}{945} \\ &+ \sum_{m=10}^{\infty} \frac{h_m}{6480000m!} u^m\end{aligned}$$

with

$$\begin{aligned}h_m &= 2560000m^4 3^m + 1265625m^4 4^m + 124416m^4 5^m + 625m^4 2^{m+3} 3^{m+1} + 961875m^4 2^{m+3} \\ &+ 25920000m^4 - 850176m^3 5^m - 625m^3 2^{m+4} 3^{m+1} - 625m^3 2^{m+5} 3^{m+1} - 320000m^3 3^{m+2} \\ &+ 1569375m^3 2^{m+4} - 3391875m^3 2^{2m+1} + 246240000m^3 + 33920000m^2 3^m \\ &+ 11491875m^2 4^m + 124416m^2 5^m + 625m^2 2^{m+6} 3^{m+1} + 625m^2 2^{m+3} 3^{m+2} \\ &+ 8150625m^2 2^{m+3} - 336960000m^2 - 14950656m 5^m + 6080000m 3^{m+1} \\ &- 625m 2^{m+4} 3^{m+2} + 13314375m 2^{m+4} - 22426875m 2^{2m+1} + 609120000m \\ &- (124416)5^{m+4} - (640000)3^{m+5} + (151875)2^{m+10} - (455625)2^{2m+9} + 233280000 \\ &= 25920000m^4 + 246240000m^3 - 336960000m^2 + 609120000m + 233280000 \\ &+ 5^m\Big(124416m^4 - 850176m^3 + 124416m^2 - 14950656m - 77760000\Big) \\ &+ 2^{2m}\Big(1265625m^4 - 6783750m^3 + 11491875m^2 - 44853750m - 233280000\Big) \\ &+ 2^m\Big(7695000m^4 + 25110000m^3 + 65205000m^2 + 213030000m + 155520000\Big) \\ &+ 6^m\Big(15000m^4 - 90000m^3 + 165000m^2 - 90000m\Big) \\ &+ 3^m\Big(2560000m^4 - 2880000m^3 + 33920000m^2 + 18240000m - 155520000\Big) > 0, \quad m \geq 9.\end{aligned}$$

Then, $3 \leq \deg^{r}_{CM}[F_2(r)]$. However,

$$r^4 F_2(r) = \int_0^\infty \frac{e^{-2u}}{(e^u+1)^5 u^5} \chi_4(u) e^{-ru} du, \quad r > 0$$

where

$$\begin{aligned}
\chi_4(u) &= e^{6u}\left(u^4+4u^3+12u^2+24u+24\right) - 8\left(2u^4+4u^3+6u^2+6u+3\right) \\
&- 2e^{3u}u\left(11u^4+75u^3+140u^2+180u+120\right) + 2e^{4u}\left(u^5-35u^4-60u^3-60u^2+60\right) \\
&- 5e^{2u}\left(2u^5+31u^4+60u^3+84u^2+72u+24\right) \\
&- e^{u}\left(2u^5+79u^4+156u^3+228u^2+216u+96\right) \\
&+ e^{5u}\left(-32u^5-11u^4-12u^3+12u^2+72u+96\right)
\end{aligned}$$

with $\chi_4(1.2) = 1268.84$ and $\chi_4(1.3) = -1981.21$. Then, $r^4 F_2(r)$ is not a CM function and, hence, $\deg^{r}_{CM}[F_1(r)] < 4$. □

From Theorem 2, the function $F_2(r)$ is a decreasing function and $F_2(\infty) = 0$; then, we obtain the following result:

Corollary 2. *The function $G(r)$ satisfies that*

$$G(r) < \frac{2}{r(1+r)} + \ln\left(\frac{2+r}{r+1}\right) - \frac{1}{3r^3} + \frac{3}{2r^4}, \quad r > 0. \tag{16}$$

Lemma 1. *The function*

$$\theta_1(r) = r^3\left[\frac{2}{r(1+r)} + \ln\left(\frac{2+r}{r+1}\right) - G(r)\right], \quad r > 0 \tag{17}$$

is a strictly increasing function with sharp bounds $0 < \theta_1(r) < \frac{1}{3}$.

Proof. Using the relation (12), we have

$$\theta_1(r) = -r^3 \int_0^\infty \frac{e^{-2u}\left(2e^u u - e^{2u} + 1\right)}{(e^u+1)u} e^{-ru} du, \quad r > 0$$

and

$$\frac{d}{dr}\theta_1(r) = r^2 \int_0^\infty \frac{\kappa_2(u)}{\left(e^u + e^{2u}\right)^2 u} e^{-ru} du$$

where

$$\begin{aligned}
\kappa_2(u) &= -\left(2u^2+7u+3\right)e^u + \left(3-4u-4u^2\right)e^{2u} + (3+u)e^{3u} - 2u - 3 \\
&= \sum_{m=4}^{\infty} \frac{m+9}{m!}\left[3^{m-1} - \frac{2^m\left(m^2+m-3\right) + (1+m)(3+2m)}{9+m}\right]u^m, \quad u > 0.
\end{aligned}$$

Using the induction, we obtain

$$3^{m-1} > \frac{\left(m^2+m-3\right)2^m + (m+1)(3+2m)}{9+m}, \quad m \geq 4$$

with the aid of the relation

$$\frac{3\left(2^m\left(m^2+m-3\right) + (2m+3)(1+m)\right)}{m+9} - \frac{2^{m+1}\left(m+(m+1)^2-2\right) + (m+2)(5+2m)}{m+10}$$

$$= \frac{4m(m^2+12m+17) + 2^m(m^3+9m^2-31m-72)}{(m+9)(m+10)} > 0, \qquad m \geq 4.$$

Then, $\theta_1(r)$ is a strictly increasing function on $r > 0$. Furthermore,

$$\lim_{r\to 0} \theta_1(r) = \lim_{r\to 0}\left[(-\gamma - 2 + \ln(2) - \psi(1/2))r^3 + \left(\frac{3}{2} - \frac{\pi^2}{6}\right)r^4 + O(r^5)\right] = 0,$$

and

$$\lim_{r\to\infty} \theta_1(r) = \lim_{r\to\infty}\left[\frac{1}{3} - \frac{3}{2}r^{-1} + O(r^{-2})\right] = \frac{1}{3},$$

where $\gamma = -\Gamma'(1)$ is the Euler–Mascheroni constant. Hence, $0 < \theta_1(r) < \frac{1}{3}$ with sharp bounds. □

Lemma 2. *The function*

$$\theta_2(r) = r^4\left[\frac{2}{r(1+r)} + \ln\left(\frac{2+r}{r+1}\right) - \frac{1}{3r^3} - G(r)\right], \quad r > 0 \tag{18}$$

is a strictly decreasing function with sharp bounds $\frac{-3}{2} < \theta_1(r) < 0$.

Proof. Using the relation

$$\theta_2(r) = -r^4 F_1(r), \quad r > 0$$

we have

$$\frac{d}{dr}\theta_2(r) = -r^3 \int_0^\infty \frac{\kappa_3(u)}{6(e^u+1)^2 u} e^{-ru} du$$

where

$$\begin{aligned}
\kappa_3(u) &= e^{4u}u^3 + 2e^{3u}\left(u^3 - 3u - 12\right) + 12(u+2) + 6e^u(u+4)(2u+1) \\
&+ e^{2u}(u(u(u+24)+36) - 24) \\
&= 870912u^5 + 11197440u^6 + 93747456u^7 + 645470208u^8 + 3970944000u^9 \\
&+ \sum_{m=10}^{\infty} \frac{a_m}{1728m!}u^m \quad u > 0
\end{aligned}$$

with

$$\begin{aligned}
a_m &= 4^m\left(27m^3 - 81m^2 + 54m\right) + 3^m\left(128m^3 - 384m^2 - 3200m - 41472\right) \\
&\quad + 2^m\left(216m^3 + 9720m^2 + 21168m - 41472\right) + 72576m + 41472 + 20736m^2 \\
&> 0, \quad m \geq 10.
\end{aligned}$$

Then, $\theta_2(r)$ is a strictly decreasing function on $r > 0$. Furthermore,

$$\lim_{r\to 0} \theta_2(r) = \lim_{r\to 0}\left[r^4(-\gamma - 2 + \ln(2) - \psi(1/2)) - \frac{r}{3} + O(r^5)\right] = 0,$$

and

$$\lim_{r\to\infty} \theta_2(r) = \lim_{r\to\infty}\left[-\frac{3}{2} + \frac{21}{5r} + O(r^{-2})\right] = -\frac{3}{2}.$$

Hence, $\frac{-3}{2} < \theta_1(r) < 0$ with sharp bounds. □

Remark 1. *The lower bound of (14) is better than the lower bound of (4) for* $r > 1.62$. *Furthermore, the upper bound of (16) is better than the lower bound of (4) for* $r > \frac{9}{2}$.

Remark 2. *The lower bound of (14) is better than the lower bound of (5) for* $r > 1.2$.

Remark 3. *The lower bound of (14) is better than the lower bound of (6) for $r > 0.73$. Furthermore, the upper bound of (16) is better than the upper bound of (6) for $r > 0.97$.*

Remark 4. *The lower bound of (14) is better than the lower bound of (8) for $0.86745 < r < 2.45$.*

Remark 5. *The Upper bound of (16) is better than the upper bound of (8) for $0 < r < 2.77879$.*

3. Conclusions

The main conclusions of this paper are stated in Lemmas 1 and 2. Concretely speaking, the authors studied two approximations for Bateman's G-function. The approximate formulas are characterized by one strictly increasing towards $G(r)$ as a lower bound, and the other strictly decreasing as an upper bound with the increases in r values. Furthermore, our new two-sided inequality for $G(r)$ improved some of the recently published results. The results enable us to obtain the bounds of some alternating series, some generalized hypergeometric functions, Wallis's ratio, and some other functions.

Author Contributions: Writing to Original draft, M.M. and H.A. All authors contributed equally to the writing of this paper. All authors have read and agreed to the published version of the manuscript.

Funding: This research work was funded by Institutional Fund Projects under grant no. (IFPIP:719-130-1443). The authors gratefully acknowledge technical and financial support provided by the Ministry of Education and King Abdulaziz University, DSR, Jeddah, Saudi Arabia.

Data Availability Statement: Not applicable.

Conflicts of Interest: The authors declare no conflict of interest.

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