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Reclamation of a Resource Extraction Site Model with Random Components

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Abstract: We compute the cooperative and the Nash equilibrium solutions for the discounted optimal control problem in a two-player differential game of reclamation of a resource extraction site, where each firm's planning horizon presents the period that extraction of the resources from their site is economically viable. Hence, the planning horizon is defined by a random duration determined on the infinite time horizon. The comparison of the cooperative and Nash solutions and also the comparative statics are provided numerically. We also define the concept of "normalized value of cooperation" and explain how this concept could help us to better characterize the losses the players will face if they continue to refrain from cooperation.

Keywords: differential game; random time horizon; open-loop strategies; resource extraction; reclamation; clean-up of extraction site

MSC: 91A23; 49N70

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1. Introduction

The extraction of natural resources, i.e., the withdrawal of materials (e.g., fossil fuels, rocks, timber, fish), have great impacts on the environment. Examples of such impacts are: soil degradation, destruction of natural habitats, water contamination, air pollution, deforestation, and solid waste. Moreover, as expectations for higher standards of living and the world's population continue to grow, the demand for resources will continue to grow. Hence, reclamation and clean-up of the resource extraction sites remain a top concern and priority for environmental regulation bodies and have given rise to a growing interest in reclamation and clean-up issues, especially over the past decade (see, e.g., [1–5]).

In the extant literature, the planning horizon is always assumed to be known. However, while the lease terms are usually known to the firms, what defines an extraction site's active lifespan is the duration when extraction remains economically viable. In other words, firms abandon their extraction sites when extraction becomes economically unprofitable. Firms make economic assessments about the availability of resources in each site; however, the exact amount of the resources and their economic profitability and hence, the economically viable resource extraction period, remain uncertain both due to factors related to the market (e.g., demand and extraction cost) and also technical limitations. These uncertainties could have important implications both for the regulators and firms. In this paper, we make a first exploratory attempt to embed uncertainty in the economic extraction period into a site reclamation model. For that purpose, we extend the paper by Marsiglio and Masoudi [1] by assuming that the extraction period (firm's planning horizon) is a random variable. Moreover, in this article, we define the concept of "normalized value of cooperation" and explain how this concept could help us to better characterize the losses players will face if they continue to refrain from cooperation.

The paper proceeds as follows. Section 2 presents our model. In Section 3, we focus on the cooperative solution of the problem, while Section 4 presents the results for the non-cooperative scenario. A comparison of the two scenarios is provided in Section 5. The concept of normalized value of cooperation is introduced in Section 6. Finally, Section 7 presents concluding remarks and highlights directions for future research.

2. Model Formulation

Consider a differential game with two players (indexed by z, where $z = \{i, j\}$) [6] devoted to resource management [7,8]. The model setting follows [1], except for the fact that in our model, firms' planning horizon presents the period that extraction of the resources from each site is economically viable and not the lease duration. Hence, while in [1], the extraction period is known, in our model this is defined by a random duration determined on the infinite time horizon, denoted T^j and T^i. For the models with a random time horizon, see [9–12].

Denote the level of environmental degradation or the pollution stock as p_t, which is the state variable of the process.

Assume that firm $z, z = \{i, j\}$ extracts resources at a rate $\gamma^z > 0$. For simplicity, we assume the extraction rate is given. However, the emissions due to extraction are not constant but increasing in the pollution stock due to, e.g., the decreasing returns of the extraction technologies. In other words, emissions, e_t^z, are given by $e_t^z = \epsilon^z \gamma^z p_t$, where $\epsilon^z > 0$ is an exogenous parameter defining the environmental inefficiency of extraction activity of the firm z.

Firms engage in reclamation or abatement activities throughout the entire planning horizon. Let the reclamation efforts of the firm be $a_t^z = \alpha^z \tau_t^z$, where $\alpha^z > 0$ is the efficiency of environmental reclamation. Therefore, the environmental degradation dynamics is given by:

$$\dot{p}_t = \left(\epsilon^i \gamma^i + \epsilon^j \gamma^j - \delta\right) p_t - \alpha^i \tau_t^i - \alpha^j \tau_t^j, \tag{1}$$

where $\delta > 0$ is the natural pollution decay rate. We assume that the growth rate of pollution in the absence of abatement is positive $\epsilon^i \gamma^i + \epsilon^j \gamma^j - \delta > 0$, pollution will increase over time, leading the firms to face a substantial reclamation fee on their lease termination date.

We assume that at the closure time T^z, by regulations, the firm z is responsible to pay a reclamation cost, referred to as the abandonment reclamation fee, proportional to the environmental costs of the unclaimed pollution stock at that time, given by: $f(p_{T^z}) = \phi^z \frac{p_{T^z}^2}{2}$. Here, $\phi^z \geq 0$ is determined by the regulator and quantifies the extent to which the firm z is effectively liable for the damage caused by the unclaimed pollution at its site dismissal. For any $\phi^z > 0$ (and finite) the firm needs to account for both its instantaneous losses and abandonment reclamation fee to determine its rehabilitation efforts. For the sake of simplicity, assume also that the reclamation cost of the firm z is quadratic: $\ell(\tau_t^z) = \frac{(\tau_t^z)^2}{2}$. Thus, the cost of the firm z is the following:

$$C^z = \int_0^{T^z} \frac{(\tau_t^z)^2}{2} e^{-\rho t} dt + \phi^z \frac{p_{T^z}^2}{2} e^{-\rho T^z} \to \min_{\tau_t^z}, \tag{2}$$

where ρ is the discount rate. We assume that T^j and T^i are **random** variables defined on the infinite interval $[0, \infty)$. Let T^j and T^i correspond to the exponential distribution with the cumulative distribution function [11]

$$F(t) = 1 - e^{-rt}, t \in [0, \infty).$$

In this paper, although the model is formulated for the general case with different random terminal times, we assume that decision-makers have the same exponential distribution. Considering two different distributions would result in a problem with time inconsistent preferences, which would require a different approach to the analysis.

Thus, each firm's payoff is the sum of the mathematical expectation of the integral payoff and the mathematical expectation of the terminal part according to the considered cumulative distribution function $F(t)$, i.e., we have:

$$C^z = E\left[\int_0^{T^z} \frac{(\tau_t^z)^2}{2} e^{-\rho t} dt\right] + E\left[\phi^z \frac{p_{T^z}^2}{2} e^{-\rho T^z}\right]. \tag{3}$$

We can simplify (3) using integration by parts, provided that the probability density function (p.d.f.) $f(t)$ exists and well-defined (a similar approach has been used in [13,14]):

$$\begin{aligned} C^z = E\left[\int_0^{T^z} \frac{(\tau_t^z)^2}{2} e^{-\rho t} dt\right] + E\left[\phi^z \frac{p_{T^z}^2}{2} e^{-\rho T^z}\right] = \\ \int_0^\infty \int_0^t \frac{(\tau_\theta^z)^2}{2} e^{-\rho\theta} d\theta dF(t) + \int_0^\infty \phi^z \frac{p_t^2}{2} e^{-\rho t} f(t) dt = \int_0^\infty \frac{(\tau_t^z)^2}{2} e^{-\rho t}(1 - F(t)) dt + \int_0^\infty \phi^z \frac{p_t^2}{2} e^{-\rho t} f(t) dt = \\ \int_0^\infty \left[\frac{(\tau_t^z)^2}{2} e^{-\rho t}(1 - F(t)) + \phi^z \frac{p_t^2}{2} e^{-\rho t} f(t)\right] dt = \int_0^\infty \left[\frac{(\tau_t^z)^2}{2} e^{-(\rho+r)t} + \phi^z r \frac{p_t^2}{2} e^{-(\rho+r)t}\right] dt = \\ \int_0^\infty \left[\frac{(\tau_t^z)^2}{2} + \phi^z r \frac{p_t^2}{2}\right] e^{-(\rho+r)t} dt, \quad z = \overline{i,j}. \end{aligned} \tag{4}$$

where $f(t) = F'(t)$ is a p.d.f. for T^i and T^j.

3. Cooperative Scenario

As our baseline, we first find the socially optimal levels of pollution and abatement activities of the two firms. In other words, we first solve the game assuming that the two firms cooperate with each other and minimize their joint reclamation cost. This solution gives us a benchmark to compare with the non-cooperative or the business as usual results. In the cooperative scenario, the two firms minimize their joint cost as follows:

$$\begin{aligned} C^i + C^j = \int_0^\infty \left[\frac{(\tau_t^i)^2}{2} + \phi^i r \frac{p_t^2}{2}\right] e^{-(\rho+r)t} dt + \int_0^\infty \left[\frac{\left(\tau_t^j\right)^2}{2} + \phi^j r \frac{p_t^2}{2}\right] e^{-(\rho+r)t} dt = \\ \int_0^\infty \left[\frac{(\tau_t^i)^2}{2} + \frac{\left(\tau_t^j\right)^2}{2} + (\phi^i + \phi^j) r \frac{p_t^2}{2}\right] e^{-(\rho+r)t} dt. \end{aligned} \tag{5}$$

For simplicity, denote $d = \left(\epsilon^i \gamma^i + \epsilon^j \gamma^j - \delta\right)$. We then transform the problem to a maximization problem as follows:

$$\begin{cases} \int_0^\infty (-1)\left[\frac{(\tau_t^i)^2}{2} + \frac{\left(\tau_t^j\right)^2}{2} + (\phi^i + \phi^j) r \frac{p_t^2}{2}\right] e^{-(\rho+r)t} dt \to \max_{\tau_t^i, \tau_t^j}, \\ \dot{p}_t = d p_t - \alpha^i \tau_t^i - \alpha^j \tau_t^j, \\ p_0 \text{ is given.} \end{cases} \tag{6}$$

To solve the problem, we proceed by using a modification of Pontryagin's Maximum Principle [15] for an infinite interval [16]. The Hamiltonian function corresponding to our problem is defined as

$$H = \psi_t\left[dp_t - \alpha^i\tau_t^i - \alpha^j\tau_t^j\right] - \left[\frac{(\tau_t^i)^2}{2} + \frac{\left(\tau_t^j\right)^2}{2} + (\phi^i + \phi^j)r\frac{p_t^2}{2}\right]. \tag{7}$$

Hence,

$$\frac{\partial H}{\partial \tau_t^z} = -\alpha^z\psi_t - \tau_t^z = 0, \quad z = \overline{i,j}. \tag{8}$$

The optimal control of the player z is given by:

$$\tau_t^{z*} = -\alpha^z\psi_t, \quad z = \overline{i,j}. \tag{9}$$

The adjoint variable equation (according to maximum principle modification on infinite interval) is:

$$\dot{\psi}_t = (\rho + r)\psi_t - d\psi_t + r(\phi^i + \phi^j)p_t. \tag{10}$$

At the same time, using (9) we have:

$$\dot{p}_t = dp_t + ((\alpha^i)^2 + (\alpha^j)^2)\psi_t. \tag{11}$$

Hence, we obtain the following system of differential equations:

$$\begin{cases} \dot{\psi}_t = (\rho + r - d)\psi_t + r(\phi^i + \phi^j)p_t, \\ \dot{p}_t = dp_t + ((\alpha^i)^2 + (\alpha^j)^2)\psi_t. \end{cases} \tag{12}$$

In the matrix form, we have:

$$\begin{bmatrix} \dot{p}_t \\ \dot{\psi}_t \end{bmatrix} = \begin{bmatrix} d & (\alpha^i)^2 + (\alpha^j)^2 \\ r(\phi^i + \phi^j) & \rho + r - d \end{bmatrix} \begin{bmatrix} p_t \\ \psi_t \end{bmatrix} = A^C \begin{bmatrix} p_t \\ \psi_t \end{bmatrix}. \tag{13}$$

Note that for the optimal control problem defined on an infinite horizon, the optimal solution is a trajectory that converges to the equilibrium point (assuming there is only one equilibrium point; otherwise, more analysis is needed). If the canonical system is linear, this problem can be solved relatively easily, as a linear system has only one equilibrium point. To find a stable trajectory to this point, we need to identify all negative eigenvalues of the matrix A^C.

A^C has two eigenvalues:

$$\Lambda(A^C) = \left\{\sigma_1^C, \sigma_2^C\right\} = \left\{ \ \frac{r}{2} + \frac{\rho}{2} - \frac{\sqrt{D^C}}{2}, \quad \frac{r}{2} + \frac{\rho}{2} + \frac{\sqrt{D^C}}{2} \ \right\},$$

where $D^C = (\rho + r - 2d)^2 + 4r(\phi^i + \phi^j)((\alpha^i)^2 + (\alpha^j)^2) > 0$.

Since the second eigenvalue $\sigma_2^C = \frac{r}{2} + \frac{\rho}{2} + \frac{\sqrt{D^C}}{2}$ is positive, it can not produce a stable solution. However, the first eigenvalue $\sigma_1^C = \frac{r}{2} + \frac{\rho}{2} - \frac{\sqrt{D^C}}{2}$ is negative if $D^C - (r+\rho)^2 = 4r(\phi_i + \phi_j)(\alpha_i^2 + \alpha_2^2) + 4d(d - r - \rho) > 0$. Then its corresponding eigenvector is

$$v^C = \begin{bmatrix} 1 \\ \frac{2(\phi_i + \phi_j)r}{2d - r - \rho - \sqrt{D^C}} \end{bmatrix}.$$

Note that any trajectory that starts from $p_0 + span(v)$ will converge to the equilibrium point. There is only one stable equilibrium, so the initial value p_0 is uniquely determined.

To save in notation, let us denote $\sigma_1^C = \sigma^C$. Given the initial condition p_0, we can write the solution as follows:

$$p_t^C = p_0 e^{\sigma^C t}, \tag{14}$$

$$\psi_t = -\frac{2(\phi_i + \phi_j)r}{r - 2d + \rho + \sqrt{D^C}} p_0 e^{\sigma^C t}. \tag{15}$$

and consequently, the optimal controls are:

$$(\tau_t^z)^C = \alpha^z \frac{2(\phi_i + \phi_j)r}{r - 2d + \rho + \sqrt{D^C}} p_0 e^{\sigma^C t}. \tag{16}$$

Proposition 1 summarizes this result.

Proposition 1. *The cooperative rule for the reclamation effort for firm $z = \{i, j\}$, $(\tau_t^z)^C$, and the cooperative time path of pollution, p_t^{*C}, are respectively given by:*

$$(\tau_t^z)^C = \alpha^z \frac{2(\phi^i + \phi^j)r}{r - 2d + \rho + \sqrt{D^C}} p_0 e^{\sigma t}, \tag{17}$$

$$p_t^C = p_0 e^{\sigma t}, \tag{18}$$

where $\sigma^C = \frac{r}{2} + \frac{\rho}{2} - \frac{\sqrt{D^C}}{2} < 0$, and $D^C = (\rho + r - 2d)^2 + 4r(\phi^i + \phi^j)((\alpha^i)^2 + (\alpha^j)^2) > 0$.

From Proposition 1 we can see that, since under the cooperative scenario firms take into account their joint abandonment reclamation fees, the only factor that differentiates their optimal reclamation efforts is their reclamation efficiency, $\alpha^z, z = \{i, j\}$, so much so that the firm with higher reclamation efficiency is asked to implement higher efforts and when $\alpha^i = \alpha^j$, regardless of any other heterogeneity among the two firms, they will implement the same levels of reclamation efforts over time. Moreover, as expected, higher initial pollution stock and abandonment reclamation fees leads to higher reclamation efforts for the two firms.

4. Nash Equilibrium

Now, we turn to solve the problem under the business as usual scenario, that is, when firms do not cooperate on their reclamation efforts and tend to focus on minimizing their own individual costs. This leads us to seek for the open-loop Nash Equilibrium, where we are facing the following optimal control problem:

$$\begin{cases} C^i = (-1)\frac{1}{2}\int_0^\infty e^{-(\rho+r)t}\left[\left(\tau_t^i\right)^2 + \phi^i r p_t^2\right]dt \to \max_{\tau_t^i}, \\ C^j = (-1)\frac{1}{2}\int_0^\infty e^{-(\rho+r)t}\left[\left(\tau_t^j\right)^2 + \phi^j r p_t^2\right]dt \to \max_{\tau_t^j}, \\ \dot{p}_t = d p_t - \alpha^i \tau_t^i - \alpha^j \tau_t^j, \\ p_0 \text{ is given.} \end{cases} \tag{19}$$

To find the Nash equilibrium, we define two (current state) Hamiltonian functions:

$$H^z(p, \psi_t^z, \tau_t^i, \tau_t^j) = -\frac{\phi^z\, r\, p^2}{2} - \frac{\tau_t^{z2}}{2} + \psi_t^z\left(d p_t - \alpha^i \tau_t^i - \alpha^j \tau_t^j\right). \tag{20}$$

Note that the calculations are similar to the ones under the cooperative scenario. Hence, the optimal controls will be $\tau_i^* = -\alpha^i \psi_t^i$. The respective canonical system written for (p, ψ_t^i, ψ_t^j) is:

$$\begin{bmatrix} \dot{p} \\ \dot{\psi}^i \\ \dot{\psi}^j \end{bmatrix} = \begin{bmatrix} d & {\alpha^i}^2 & {\alpha^j}^2 \\ \phi^i r & r-d+\rho & 0 \\ \phi^j r & 0 & r-d+\rho \end{bmatrix} \begin{bmatrix} p \\ \psi^i \\ \psi^j \end{bmatrix} = A^N \begin{bmatrix} p \\ \psi_1 \\ \psi_2 \end{bmatrix}. \tag{21}$$

The matrix A^N has three eigenvalues:

$$\Lambda(A^N) = \left\{\sigma_1^N, \sigma_2^N, \sigma_3^N\right\} = \left\{\ r-d+\rho,\quad \tfrac{r}{2}+\tfrac{\rho}{2}-\tfrac{\sqrt{D^N}}{2},\quad \tfrac{r}{2}+\tfrac{\rho}{2}+\tfrac{\sqrt{D^N}}{2}\ \right\},$$

where $D^N = 4\phi^i(\alpha^i)^2 r + 4\phi^j(\alpha^j)^2 r + 4d^2 - 4dr - 4d\rho + r^2 + 2r\rho + \rho^2 > 0$.

The optimal solution corresponds to a stable solution to (21). To determine the stable solution, we need to analyze the eigenvalues. Note that $\sigma_3^N > 0$ is positive, so it can not produce a stable solution. However, the second one $\sigma_2^N = \frac{1}{2}(r+\rho-\sqrt{D})$ is negative if

$$D^N - (r+\rho)^2 = 4\phi^i(\alpha^i)^2 r + 4\phi^j(\alpha^j)^2 r + 4d(d-r-\rho) > 0.$$

and then its corresponding eigenvector is:

$$v_2^N = \begin{bmatrix} \frac{2d-r-\rho-\sqrt{D}}{2r} \\ \phi_1 \\ \phi_2 \end{bmatrix}.$$

Note that any trajectory of (21) that initiates from a point along this vector, will be described by $\dot{p} = \sigma_2 p$, and $\psi_i = p_t \frac{2r\phi^i}{2d-r-\rho-\sqrt{D^N}}$. It is important to note that $2d-r-\rho-\sqrt{D^N} < 0$, so the optimal values of the adjoint variables that correspond to $p_t > 0$ are strictly negative and hence, the respective optimal controls $\tau_t^{i*} = -\alpha_i \psi_t^{i*}$ are positive.

One special case occurs when $d > r+\rho$, i.e., the growth rate of pollution stock exceeds the depreciation rate $r+\rho$ (which is very realistic). In this case, the system (21) has 2 stable eigenvalues. Note that $d > r+\rho$ immediately implies $\sigma_2^N < 0$.

The eigenvector corresponding to $\sigma_1^N = r+\rho-d$ is:

$$v_1^N = \begin{bmatrix} 0 \\ -(\alpha^j)^2 \\ (\alpha_i)^2 \end{bmatrix}.$$

However, since the components corresponding to ψ_t^i and ψ_t^j are of different signs, in a neighborhood of the equilibrium point one of the adjoint variables will turn positive, which does not make sense. Thus, we dismiss this eigenvector and the respective eigenvector.

Finally, we obtain the optimal adjoint variables in the following form:

$$\psi_t^{i*} = -\frac{2p_0\phi^i r e^{\sigma_2^N t}}{r-2d+\rho+\sqrt{D^N}}.$$

Now, to save in notation, let us denote $\sigma_2^N = \sigma^N$. Proposition 2 summarizes the results for the Nash Equilibrium.

Proposition 2. *The Nash rule for the reclamation effort for firm* $z = \{i, j\}$, $(\tau_t^z)^C$, *and the cooperative time path of pollution,* p_t^{*C}, *are respectively given by:*

$$(\tau_t^z)^N = \alpha^z \frac{2p_0\phi^i r e^{\sigma^N t}}{r - 2d + \rho + \sqrt{D^N}}, \tag{22}$$

$$p_t^N = p_0 e^{\sigma^N t}, \tag{23}$$

where $\sigma^N = \frac{r}{2} + \frac{\rho}{2} - \frac{\sqrt{D^N}}{2} < 0$, *and* $D^N = 4\phi^i(\alpha^i)^2 r + 4\phi^j(\alpha^j)^2 r + 4d^2 - 4dr - 4d\rho + r^2 + 2r\rho + \rho^2 > 0$.

Proposition 2 suggests that, like the cooperative case, under the non-cooperative scenario, the firm with higher reclamation effort efficiency will implement higher efforts, ceteris paribus. However, unlike the cooperative scenario, in this case, since firms are taking only their private costs into account, another factor causes asymmetry in the firms' reclamation effort trajectory: their individual abandonment reclamation fees. Indeed, ceteris paribus, the firm with higher abandonment liability engages in more reclamation activities. Note that other sources of heterogeneity will not cause asymmetries in the two firms choices.

5. Comparison Analysis

In this section, we compare the results under cooperative and non-cooperative scenarios. However, due to the nature of our model, analytical comparison is not feasible and hence, we need to resort to numerical illustrations. The base parameter values we use for our numerical analysis are as follows: $p_0 = 1$, $r = 0.0001$, $\rho = 0.0001$, $d = 0.01$, $\phi_1 = 5$, $\phi_2 = 7$, $\alpha_1 = 20$, $\alpha_2 = 30$. We used these values in order to make sure we have an interior solution, but we report the results for other values in our analysis.

Figure 1 compares the trajectory of the reclamation efforts of player i under cooperative and non-cooperative scenarios. As expected, at the beginning, the reclamation efforts are considerably higher under the cooperative scenario than the non-cooperative case. However, as the gap between pollution stock under these two scenarios increases (see Figure 2), interestingly, this behavior reverses and the cooperative reclamation efforts become less than non-cooperative. Note that both reclamation efforts and pollution stock converge to zero under both scenarios, even though the rate of convergence is faster under cooperation.

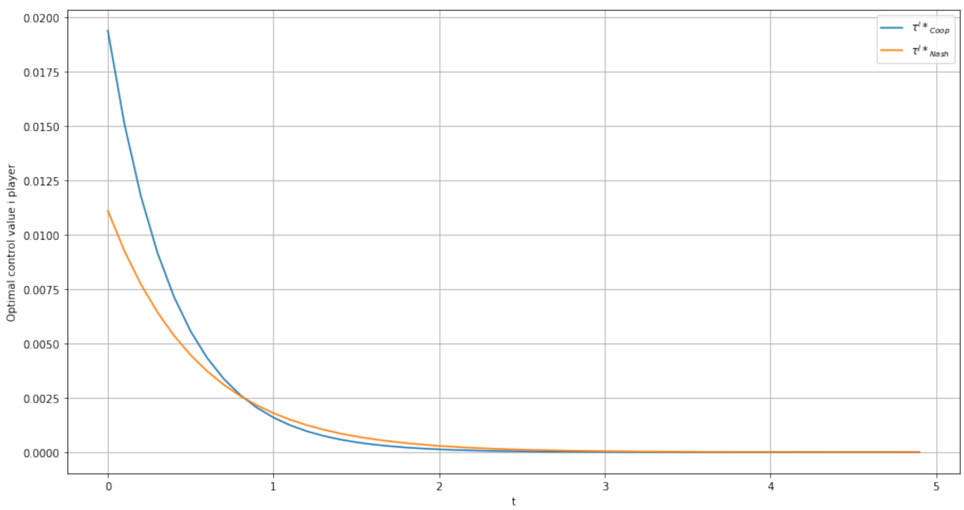

Figure 1. The optimal controls of player i under cooperation VS Nash equilibrium.

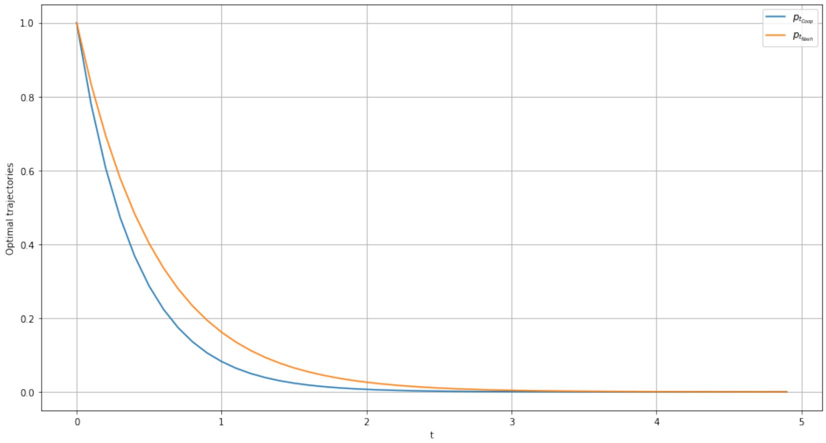

Figure 2. The optimal trajectory of pollution stock under cooperation VS Nash equilibrium.

5.1. Nash Equilibrium Analysis

Now, we turn to analyzing the impact of our key model parameters on the behavior of the players under a non-cooperative scenario. To clearly see the impact of the variables, we plot the optimal controls from two different perspectives: we show the figures for two heterogeneous players, and also we present the results for a representative player considering different values for the parameter, ceteris-paribus.

First, let us focus on the impact of the environmental reclamation efficiency, α. The analytical presentation of this impact is presented by Equation (24), which is too complicated to allow analytical comparisons. Hence, we use numerical analysis by plotting the reclamation effort trajectories under different values for α.

$$\frac{\partial(\tau_t^z)^N}{\partial\alpha^z} = \frac{2p_0\phi^z re^{\sigma^N t}}{(r-2d+\rho+\sqrt{D^N})^2}\left[(1-2(\alpha^z)^2\phi^z r\frac{1}{\sqrt{D^N}})(r-2d+\rho+\sqrt{D^N})-4(\alpha^z)^2\phi^z r\frac{1}{\sqrt{D^N}}\right]. \tag{24}$$

From Figure 3, we can see that, as expected, the player with higher α implements higher reclamation efforts throughout the entire non-cooperative game. However, from Figure 4, that presents the representative player i's optimal controls assuming different values of α for this player, we see a rather interesting result. That is, the impact of α on the trajectory of the optimal control of the representative player is not monotonic. In fact, while the higher α encourages the player to implement higher efforts, as the pollution stock drops quickly, the player reduces its efforts faster in compare to a situation with lower α, so much so, after some point of time, the case with lower α performs higher reclamation.

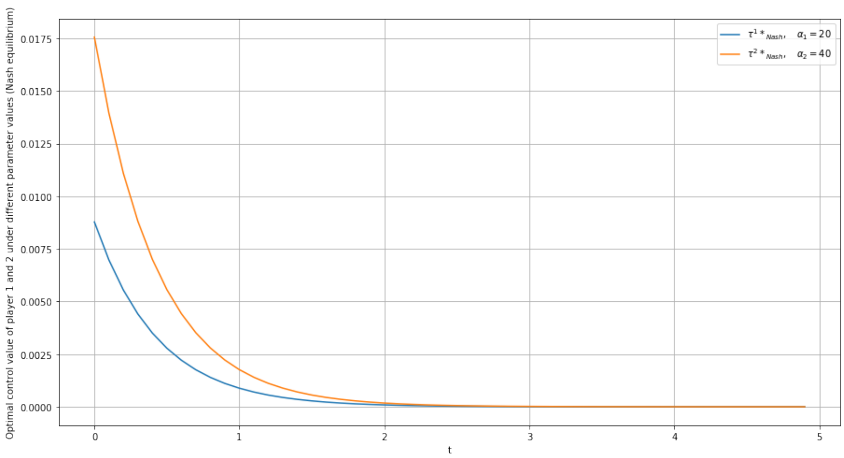

Figure 3. The optimal controls of two players under Nash equilibrium and different α.

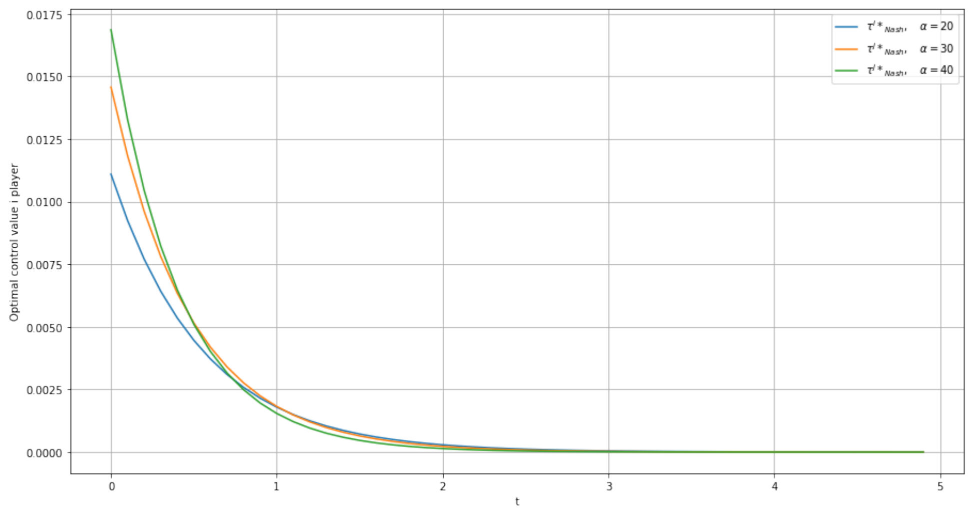

Figure 4. The optimal controls of player i under Nash equilibrium and different α.

Equation (25) presents the impact of the abandonment reclamation fee parameter ϕ on the players' choices. In Figures 5 and 6, we demonstrate this numerically.

$$\frac{\partial(\tau_t^z)^N}{\partial \phi^z} = \frac{2p_0\alpha^z r e^{\sigma^N t}}{(r - 2d + \rho + \sqrt{D^N})^2}\left[(1 - (\alpha^z)^2\phi^z r\frac{1}{\sqrt{D^N}})(r - 2d + \rho + \sqrt{D^N}) - 2(\alpha^z)^2\phi^z r\frac{1}{\sqrt{D^N}}\right]. \tag{25}$$

From Figure 5, we can see that, again as expected, the player with higher abandonment fees puts more efforts into cleaning-up the site throughout the entire planning horizon. However, the effect of the abandonment fees remains strong and monotonically increasing, i.e., as seen in Figure 6, the higher the player's abandonment environmental obligations is, the higher is their reclamation efforts at any point of time.

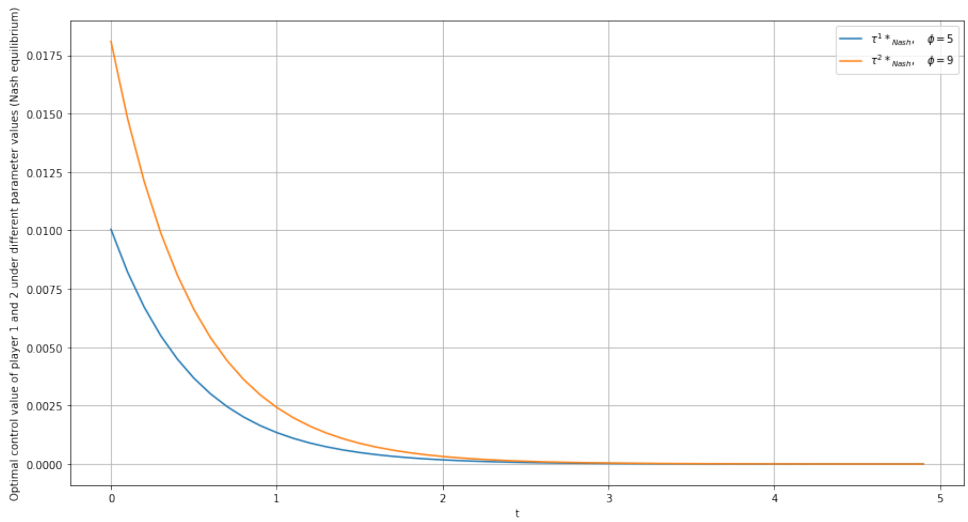

Figure 5. The optimal controls of two players under Nash equilibrium and different ϕ.

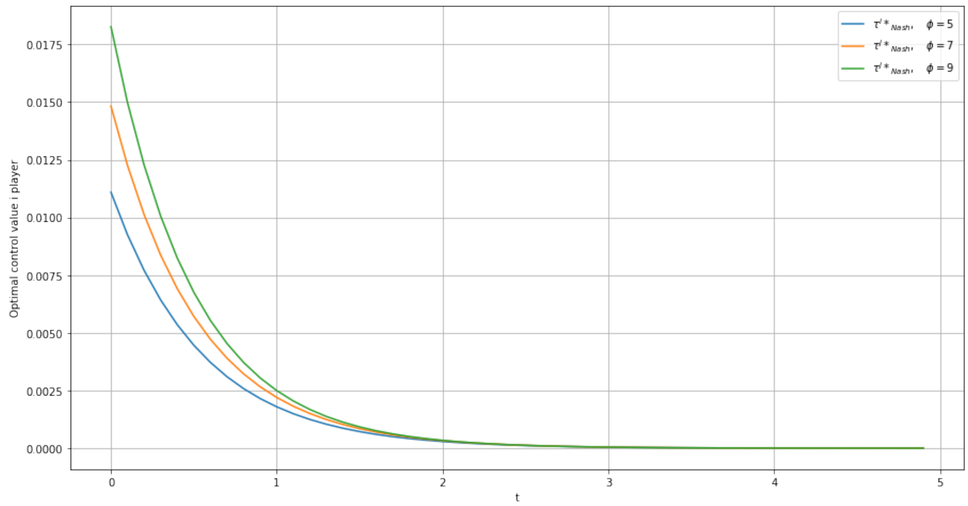

Figure 6. The optimal controls of player i under Nash equilibrium and different ϕ.

5.2. Cooperative Scenario Analysis

In this subsection, we focus on the impact of our key variables under the cooperative scenario. First, let us discuss the impact of reclamation effort efficiency parameter α, as presented in Equation (26).

$$\frac{\partial(\tau_t^z)^C}{\partial\alpha^z} = \frac{2p_0(\phi^i+\phi^j)re^{\sigma^C t}}{(r-2d+\rho+\sqrt{D^C})^2}\left[(1-2(\alpha^z)^2(\phi^i+\phi^j)r\frac{1}{\sqrt{D^C}})(r-2d+\rho+\sqrt{D^C}) -4(\alpha^z)^2(\phi^i+\phi^j)r\frac{1}{\sqrt{D^C}}\right]. \tag{26}$$

From Figures 7 and 8, it is clear that the impact of environmental efficiency parameter on players' reclamation effort is similar to what we observed under the non-cooperative scenario. That is, higher efficiency calls for higher reclamation efforts. However, as this leads to faster fall in pollution stock (see Figure 9), this relationship reverses over time.

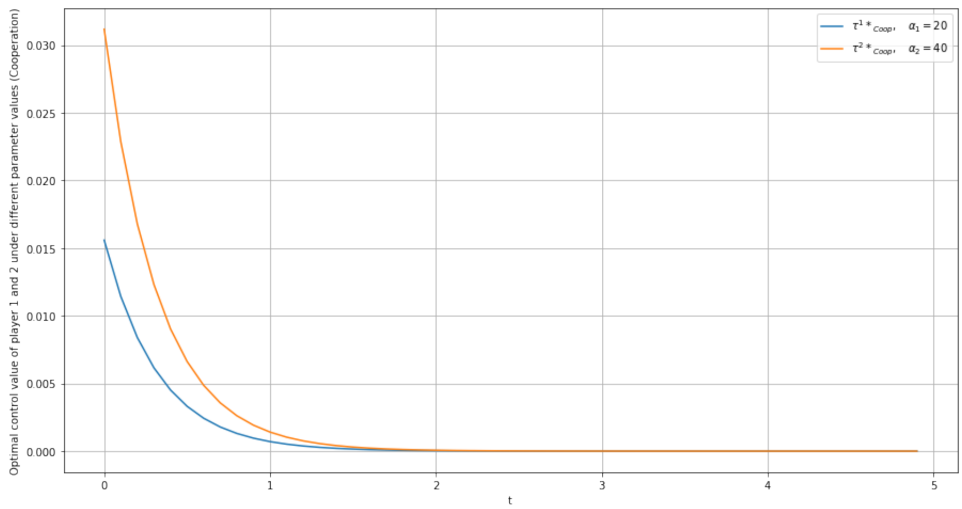

Figure 7. The optimal controls of two players under cooperative equilibrium and different ϕ.

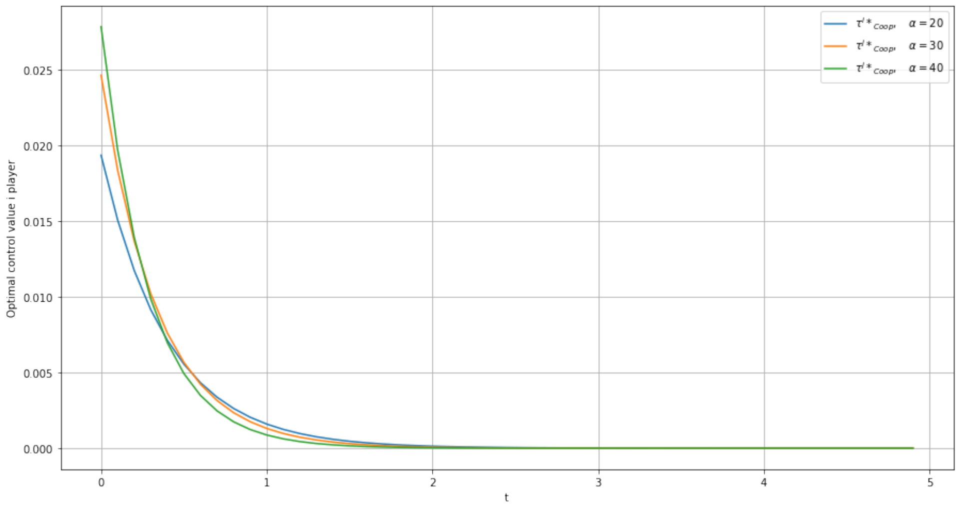

Figure 8. The optimal controls of player i under cooperative scenario and different α.

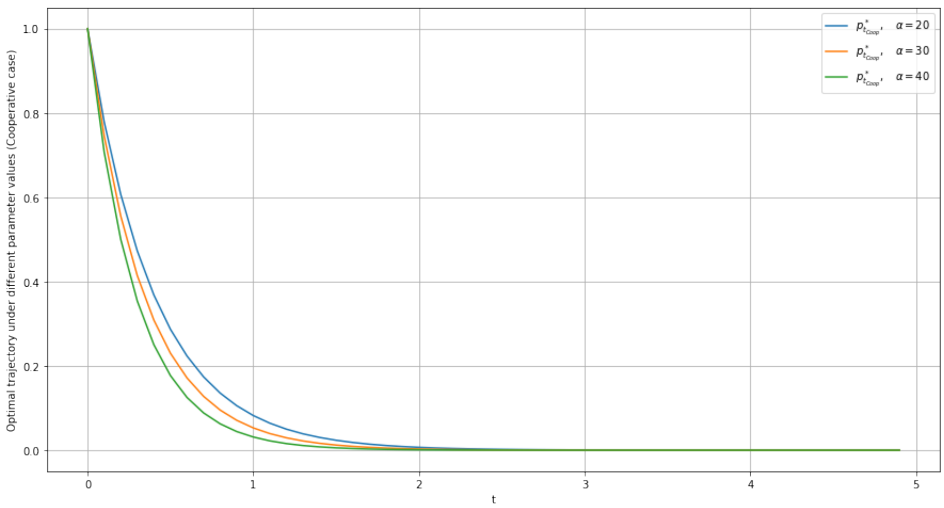

Figure 9. The optimal trajectory of pollution stock under cooperative scenario and different α.

As for the impact of abandonment fee ϕ, similar to the cooperative case, the player with higher environmental liability ϕ is asked to put more effort during the entire time horizon (see Figure 10). The same is true at the beginning of the game if we look at a representative player with different values for ϕ. However, in contrast with what we saw in the non-cooperative case (see Figure 11), here, higher ϕ does not result in a monotonic increase in reclamation effort rates at all points of time. The reason for this sharp difference between the impact of ϕ in these scenarios is that under cooperation, to minimize their joint optimal costs, both players will implement higher reclamation efforts when either player's environmental liabilities ϕ increases. However, under the non-cooperative scenario, if player i faces higher ϕ, player j may strategically lower their efforts with the knowledge that i will raise their efforts to avoid high abandonment fees. Hence, under cooperation pollution stock drops faster; consequently, the rate of cooperative reclamation efforts slow down, eventually.

$$\frac{\partial(\tau_t^z)^C}{\partial\phi^z} = \frac{2p_0\alpha^z r e^{\sigma^C t}}{(r-2d+\rho+\sqrt{D^C})^2}\Big[(1-((\alpha^i)^2+(\alpha^j)^2)(\phi^i+\phi^j)r\frac{1}{\sqrt{D^C}})(r-2d+\rho+\sqrt{D^C})- 2((\alpha^i)^2+(\alpha^j)^2)(\phi^i+\phi^j)r\frac{1}{\sqrt{D^C}}\Big]. \quad (27)$$

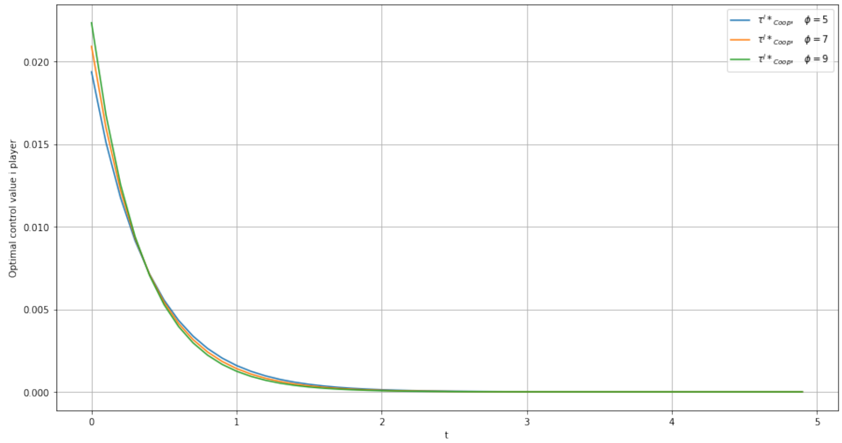

Figure 10. The optimal controls of player i under cooperative scenario and different ϕ.

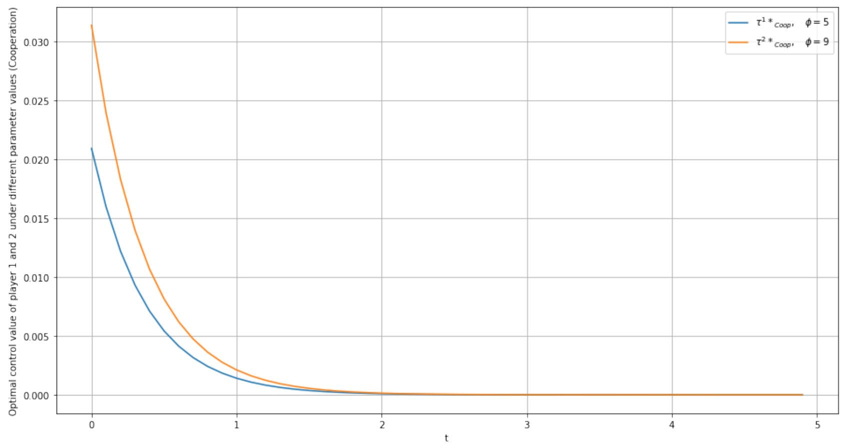

Figure 11. The optimal controls of two players under Nash equilibrium and different ϕ.

6. Normalized Value of Cooperation

In this section, we present a novel method to compare the non-cooperative and cooperative solutions based on their total costs by introducing the concept of normalized value of cooperation (NVC). A similar approach was discussed in [17]; however, in [17] the value of cooperation was computed for individual players for the entire period of the game and hence, it did not depend on a current time instant. In this paper, NVC_t is computed by taking the difference between the total cost of the two players in the Nash equilibrium and the cooperative case then dividing this value by the total cost of all players under the Nash equilibrium in the subgame of the game beginning at the time t. Note, that in this case, we consider only the subgames that start from the optimal trajectory. Hence, NVC_t takes values between 0 and 1, where 0 means that both costs coincide, i.e., there will be no difference between playing the game cooperatively or non-cooperatively from time t on, cost-wise. A value close to 1 means that the total sum of the costs corresponding to the Nash are much larger than the costs for the cooperative case. Intuitively, NVC value reveals how big the gap between cooperation and non-cooperation costs will be if players continue to play the game non-cooperatively.

$$NVC_t = \frac{\sum_{z=i,j}(C_t^z)^N - \sum_{z=i,j}(C_t^z)^C}{\sum_{z=i,j}(C_t^z)^N}. \tag{28}$$

Below, we illustrate the computation of the normalized value of cooperation. Consider a subgame starting at some point θ, then the life-time cost of a player z is:

$$C_{\theta}{}^{z} = E\left[\int_{\theta}^{\infty} \frac{(\tau_t^z)^2}{2} e^{-\rho t} dt\right] + E\left[\phi^z \frac{p_{T^z}^2}{2} e^{-\rho T^z}\right] =$$
$$\frac{1}{1-F(\theta)} \int_{\theta}^{\infty} \frac{(\tau_t^z)^2}{2} e^{-\rho t}(1-F(t))dt + \frac{1}{1-F(\theta)} \int_{\theta}^{\infty} \phi^z \frac{p_t^2}{2} e^{-\rho t} f(t) dt =$$
$$\frac{1}{1-F(\theta)} \int_{\theta}^{\infty} \left[\frac{(\tau_t^z)^2}{2} e^{-\rho t}(1-F(t)) + \phi^z \frac{p_t^2}{2} e^{-\rho t} f(t)\right] dt = e^{r\theta} \int_{\theta}^{\infty} \left[\frac{(\tau_t^z)^2}{2} e^{-(\rho+r)t} + \phi^z r \frac{p_t^2}{2} e^{-(\rho+r)t}\right] dt =$$
$$e^{r\theta} \int_{\theta}^{\infty} \left[\frac{(\tau_t^z)^2}{2} + \phi^z r \frac{p_t^2}{2}\right] e^{-(\rho+r)t} dt, \quad z = \overline{i,j}. \tag{29}$$

Thus, in cooperative case, the joint cost of the two players is:

$$C_\theta^i + C_\theta^j = e^{r\theta} \int_{\theta}^{\infty} (-1) \left[\frac{(\tau_t^i)^2}{2} + \frac{\left(\tau_t^j\right)^2}{2} + (\phi^i + \phi^j) r \frac{p_t^2}{2}\right] e^{-(\rho+r)t} dt =$$
$$(-1)\frac{1}{2} e^{r\theta} \int_{\theta}^{\infty} \left[\frac{4((\alpha^i)^2 + (\alpha^j)^2)(\phi^i+\phi^j)^2 r^2}{(r-2d+\rho+\sqrt{D^C})^2} p_0^2 e^{2\sigma^C t} + (\phi^i+\phi^j) r p_0^2 e^{2\sigma^C t}\right] e^{-(\rho+r)t} dt =$$
$$(-1) e^{r\theta} \frac{(\phi^i+\phi^j) r p_0^2}{2} \int_{\theta}^{\infty} \left[\frac{4((\alpha^i)^2 + (\alpha^j)^2)(\phi^i+\phi^j) r}{(r-2d+\rho+\sqrt{D^C})^2} + 1\right] e^{-\sqrt{D^C} t} dt =$$
$$(-1) e^{r\theta} \frac{(\phi^i+\phi^j) r p_0^2}{(-2)\sqrt{D^C}} \left[\frac{4((\alpha^i)^2 + (\alpha^j)^2)(\phi^i+\phi^j) r}{(r-2d+\rho+\sqrt{D^C})^2} + 1\right] (0 - e^{-\sqrt{D^C}\theta}) =$$
$$(-1) \frac{(\phi^i+\phi^j) r p_0^2}{2\sqrt{D^C}} \left[\frac{4((\alpha^i)^2 + (\alpha^j)^2)(\phi^i+\phi^j) r}{(r-2d+\rho+\sqrt{D^C})^2} + 1\right] e^{r-\sqrt{D^C}\theta}. \tag{30}$$

Finally, the total cost of the two players for a subgame beginning from time t under cooperation is:

$$\sum_{z=i,j} (C_t^z)^C = (-1) \frac{(\phi^i+\phi^j) r p_0^2}{2\sqrt{D^C}} \left[\frac{4((\alpha^i)^2 + (\alpha^j)^2)(\phi^i+\phi^j) r}{(r-2d+\rho+\sqrt{D^C})^2} + 1\right] e^{r-\sqrt{D^C} t}. \tag{31}$$

Now, we turn to calculating the sum of the total cost of the two players for the subgame in case of Nash Equilibrium:

$$C_\theta^i + C_\theta^j = e^{r\theta} \int_{\theta}^{\infty} (-1) \left[\frac{(\tau_t^i)^2}{2} + \frac{\left(\tau_t^j\right)^2}{2} + (\phi^i + \phi^j) r \frac{p_t^2}{2}\right] e^{-(\rho+r)t} dt =$$
$$(-1)\frac{1}{2} e^{r\theta} \int_{\theta}^{\infty} \left[\frac{4((\alpha^i\phi^i)^2 + (\alpha^j\phi^j)^2) r^2}{(r-2d+\rho+\sqrt{D^N})^2} p_0^2 e^{2\sigma^N t} + (\phi^i+\phi^j) r p_0^2 e^{2\sigma^N t}\right] e^{-(\rho+r)t} dt =$$
$$(-1) e^{r\theta} \frac{r p_0^2}{2} \int_{\theta}^{\infty} \left[\frac{4((\alpha^i\phi^i)^2 + (\alpha^j\phi^j)^2) r}{(r-2d+\rho+\sqrt{D^N})^2} + (\phi^i+\phi^j)\right] e^{-\sqrt{D^N} t} dt =$$
$$(-1) e^{r\theta} \frac{r p_0^2}{(-2)\sqrt{D^N}} \left[\frac{4((\alpha^i\phi^i)^2 + (\alpha^j\phi^j)^2) r}{(r-2d+\rho+\sqrt{D^N})^2} + (\phi^i+\phi^j)\right] (0 - e^{-\sqrt{D^N}\theta}) =$$
$$(-1) \frac{r p_0^2}{2\sqrt{D^N}} \left[\frac{4((\alpha^i\phi^i)^2 + (\alpha^j\phi^j)^2) r}{(r-2d+\rho+\sqrt{D^N})^2} + (\phi^i+\phi^j)\right] e^{r-\sqrt{D^N}\theta}. \tag{32}$$

Finally, the sum of the total cost of the players for a subgame commencing at the time t when they play Nash is:

$$\sum_{z=i,j}(C_t^z)^N = -1)\frac{rp_0^2}{2\sqrt{D^N}}\left[\frac{4((\alpha^i\phi^i)^2+(\alpha^j\phi^j)^2)r}{(r-2d+\rho+\sqrt{D^N})^2}+(\phi^i+\phi^j)\right]e^{r-\sqrt{D^N}t}. \quad (33)$$

In Figure 12, the normalized value of cooperation is demonstrated for our baseline parameter values. The notion of NVC can be used in many insightful ways. For example, suppose players would only continue to play non-cooperatively up to a point where the losses of continuing in that manner become too high in comparison to the cooperation. That is, when they reach such a threshold, they may find it too costly to continue to not cooperate and choose to switch to cooperation. Figure 12 presents a threshold level of 40%. So, we can find the time instant when the losses associated with Nash exceed the losses associated with cooperative case by 40%.

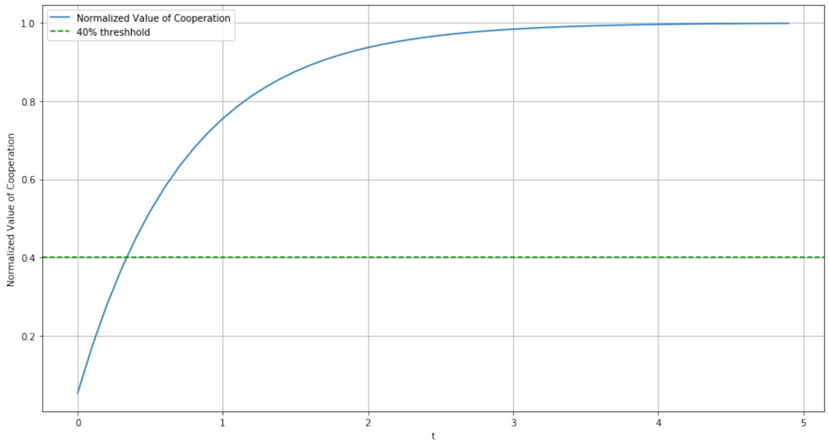

Figure 12. Normalized Value of Cooperation.

7. Conclusions

In this paper, we consider a two-player differential game of reclamation of an extraction site, where each firm's planning horizon presents the period that their extraction of the resources from each site is economically viable. Hence, in our model, the planning horizon is defined by a random duration determined on the infinite time horizon. We compute the cooperative and the Nash equilibrium solutions for the discounted optimal control problem defined on an infinite interval. The corresponding optimal solutions and the respective payoff functions are computed explicitly.

We use numerical analysis to provide insights into how the cooperative and the Nash solutions compare, and also how our key parameters affect the players' choices and the pollution stock under different scenarios. We also define the concept of "normalized value of cooperation" and explain how this concept could help us to better characterize the losses the players will face if they continue to refrain from cooperation.

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References

1. Marsiglio, S.; Masoudi N. Reclamation of a Resource Extraction Site: A Dynamic Game Approach. *Metroeconomica* **2022**, *73*, 770–802. [CrossRef]
2. Mitchell, A.L.; Casman, E.A. Economic incentives and regulatory framework for shale gas well site reclamation in Pennsylvania. *Environ. Sci. Technol.* **2011**, *45*, 9506–9514. [CrossRef] [PubMed]
3. Lappi, P. Optimal clean-up of polluted sites. *Resour. Energy Econ.* **2018**, *54*, 53–68. [CrossRef]
4. Lappi, P. A model of optimal extraction and site reclamation. *Resour. Energy Econ.* **2020**, *29*, 101–126. [CrossRef]
5. Yang, P.; Davis, G. Non-renewable resource extraction under financial incentives to reduce and reverse stock pollution. *J. Environ. Econ. Manag.* **2018**, *92*, 282–299. [CrossRef]
6. Basar, T.; Olsder G. *Dynamic Noncooperative Game Theory*; SIAM: New York, NY, USA, 1999.
7. Dockner, E.J.; Jørgensen, S.; Long, N.V.; Sorger, G. Differential Games in Economics and Management Science. In *Cambridge Books*; Cambridge University Press: Cambridge, UK, 2000.
8. Jørgensen, S.; Zaccour, G. Developments in differential game theory and numerical methods: Economic and management applications. *Comput. Manag. Sci.* **2007**, *4*, 159–181. [CrossRef]
9. Yeung, D.; Petrosyan, L. *Cooperative Stochastic Differential Games*; Springer Science, Business Media: Berlin/Heidelberg, Germany, 2006.
10. Yaari, M. Uncertain lifetime, life insurance, and the theory of the consumer. *Rev. Econ. Stud.* **1965**, *32*, 137–150. [CrossRef]
11. Petrosyan, L.; Shevkoplyas, E. Cooperative differential games with stochastic time. *Vestn. Petersburg Univ. Math.* **2000**, *33*, 18–23.
12. Gromova, E.; Malakhova, A.; Palestini, A. Payoff Distribution in a Multi-Company Extraction Game with Uncertain Duration. *Mathematics* **2018**, *6*, 165. [CrossRef]
13. Marin-Solano, J.; Shevkoplyas, E. Non-constant discounting and differential games with random time horizon. *Automatica* **2011**, *47*, 2626–2638. [CrossRef]
14. Kostyunin, S.; Palestini, A.; Shevkoplyas, E. On a nonrenewable resource extraction game played by asymmetric firms. *J. Optim. Theory Appl.* **2014**, *163*, 660–673. [CrossRef]
15. Pontryagin, L.; Boltyanskii, V.; Gamkrelidze, R.; Mishchenko, E. *The Mathematical Theory of Optimal Processes*; Interscience: New York, NY, USA, 1962.
16. Grass, D.; Caulkins, J.P.; Feichtinger, G.; Tragler, G.; Behrens, D.A. *Optimal Control of Nonlinear Processes: With Applications in Drugs, Corruption, and Terror*; Springer: Berlin/Heidelberg, Germany, 2008.
17. Chebotareva, A.; Su, S.; Voronina, E.; Gromova, E. Value of Cooperation in a Differential Game of Pollution Control. In *International Conference on Mathematical Optimization Theory and Operations Research*; Springer: Cham, Switzerland, 2022; pp. 221–234.