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Pre-Hausdorffness and Hausdorffness in Quantale-Valued Gauge Spaces

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Abstract: In this paper, we characterize each of T_0, T_1, Pre-Hausdorff and Hausdorff separation properties for the category **L-GS** of quantale-valued gauge spaces and $\mathcal{L}$-gauge morphisms. Moreover, we investigate how these concepts are related to each other in this category. We show that T_0, T_1 and T_2 are equivalent in the realm of Pre-Hausdorff quantale-valued gauge spaces. Finally, we compare our results with the ones in some other categories.

Keywords: $\mathcal{L}$-gauge space; topological category; separation; Hausdorffness

MSC: 54A05; 54B30; 54D10; 54A40; 18F60

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1. Introduction

In 1989, Lowen [1,2] introduced approach spaces as a common framework for both metric and topological spaces. More precisely, let X be a set and let $pqMet^{\infty}(X)$ be the set of all extended pseudo-quasi metrics (pseudo-reflexive property and triangular inequality) on X, $\mathfrak{D} \subseteq pqMet^{\infty}(X)$ and $d \in pqMet^{\infty}(X)$, then

(i) $\mathfrak{D}$ is called ideal if it is closed under the formation of finite suprema and if it is closed under the operation of taking smaller function.
(ii) $\mathfrak{D}$ dominates d if $\forall x \in X, \epsilon > 0$ and $\omega < \infty$ there exists $e \in \mathfrak{D}$ such that $d(x,.) \wedge \omega \leq e(x,.) + \epsilon$ and if $\mathfrak{D}$ dominates d, then $\mathfrak{D}$ is called saturated.

If $\mathfrak{D}$ is an ideal in $pqMet^{\infty}(X)$ and saturated, then $\mathfrak{D}$ is called gauge. The pair $(X, \mathfrak{D})$ is called a gauge-approach space [2]. Approach spaces can be defined by various distinct structures such as gauges, approach distances, approach systems or limit operators. Although these structures are conceptually different, they are equivalent, see [2].

Note that **Top**, the category of topological spaces and continuous maps, and **Met**, the category of metric spaces and non-expansive maps, can be embedded as a full and isomorphism-closed subcategory of **App**, the topological category of approach spaces and contractions. Therefore, metric and topological spaces are mostly studied in **App**.

Approach spaces are closely related to various disciplines and have several applications in practically all branches of mathematics, such as fixed point theory [3], convergence theory [4], domain theory [5] and probability theory [6]. Due to the widely recognized usefulness of approach spaces in research, several generalizations of approach spaces have emerged recently, including quantale-valued gauge spaces [7] and probabilistic approach spaces [8]. Quantale-valued bounded strong topological spaces and bounded interior spaces, which are frequently used by fuzzy mathematicians, have recently been used to characterize some quantale-valued approach spaces [9]. Although the classical approach structures (gauges, approach distances and approach systems) are equivalent, their arbitrary quantale generalizations are different, see Example 5.11 of [7,10].

Classical T_0 separation of topology has been extended to the topological category [11–13]. In 1991, Weck-Schwarz [14] and in 1995, Mehmet Baran and Hüseyin Altındiş [15] analyzed the relationship among these various generalizations of T_0 objects. T_0 objects are widely used to define and characterize various forms of Hausdorff [11] and sober [16] objects in topological categories.

Recall that a topological space (B, τ) is called a Pre-Hausdorff space if for each distinct pair $x, y \in B$, the subspace $(\{x, y\}, \tau_{\{x,y\}})$ is not indiscrete; then there exist disjoint neigbourhoods of x and y [11].

In 1994, Mielke [17] showed the important role of Pre-T_2 objects in general theory of geometric realization, their associated intervals and corresponding homotopic structures. In addition, in 1999, Mielke [18] used Pre-T_2 objects of topological categories to characterize decidable objects in Topos theory [19]. Another uses of Pre-Hausdorff objects is to define Hausdorff objects [11] in an arbitrary topological category. There is also a relationship between Pre-T_2 structures and partitions in some categories [20,21].

Note that there is no relationship between T_0 property and Pre-T_2 property. For example, let B be a set with at least two elements and τ be the indiscrete topology on B, then (B, τ) is Pre-T_2, but it is not T_0. If we take the cofinite topology τ_c on the set of real numbers $\mathbb{R}$, then $(\mathbb{R}, \tau_c)$ is T_0, but it is not Pre-T_2. However, if (B, τ) is a Pre-Hausdorff space, then by Theorem 3.5 of [22], all of T_0, T_1 and T_2 are equivalent.

The salient objectives of the paper are stated:

(i) To explicitly characterize each of T_0, $\overline{T_0}$ and T_1 separation properties in the category **L-GS** of quantale-valued gauge spaces and $\mathcal{L}$-gauge morphisms;
(ii) To give the characterization of each of Pre-$\overline{T_2}$, $\overline{T_2}$ and NT_2 in the category **L-GS**;
(iii) To examine the mutual relationship among all these separation axioms;
(iv) To compare our results with the ones in some other categories.

2. Preliminaries

In order theory, the *join* of a subset A of a partially ordered set $(L, \leq)$ where $\leq$ is any order on L, is the least upper bound (supremum) of A, denoted $\bigvee A$, and the *meet* of A is the greatest lower bound (infimum), denoted $\bigwedge A$. A *complete lattice* is a partially ordered set in which all subsets have both a join ($\bigvee$) and a meet ($\bigwedge$). For any complete lattice, the top and bottom elements are denoted by $\top$ and $\bot$, respectively. A complete lattice in which arbitrary joins distribute over arbitrary meets is said to be *completely distributive*.

Definition 1 ([23])**.** *A quantale $\mathcal{L} = (L, \leq, *)$ is a complete lattice $(L, \leq)$ endowed with a binary operation $*$ satisfying the following:*

(i) $(L, *)$ is a semi group;
(ii) $(\bigvee_{i \in I} a_i) * b = \bigvee_{i \in I} (a_i * b)$ and $b * (\bigvee_{i \in I} a_i) = \bigvee_{i \in I} (b * a_i)$ for all $a_i, b \in L$ and index-set I, i.e., $*$ is distributive over arbitrary joins.

Definition 2. *Let $(L, \leq)$ be a complete lattice, then the well-below relation $\lhd$ and the well-above relation $\prec$ are defined by*

(i) $a \lhd b$ if $\forall K \subseteq L$ such that $b \leq \bigvee K$ there exists $k \in K$ such that $a \leq k$;
(ii) $a \prec b$ if $\forall K \subseteq L$ such that $\bigwedge K \leq a$ there exists $k \in K$ such that $k \leq b$.

Definition 3 ([23])**.** *A quantale $\mathcal{L} = (L, \leq, *)$ is said to be*

(i) a commutative quantale if $(L, *)$ is a commutative semi-group;
(ii) an integral quantale if $a * \top = \top * a = a$ for all $a \in L$;
(iii) a value quantale if $\mathcal{L}$ is commutative and integral quantale with an underlying completely distributive lattice $(L, \leq)$ such that $\bot \lhd \top$ and $a \vee b \lhd \top$ for all $a, b \lhd \top$;
(iv) a linearly ordered quantale if either $a \leq b$ or $b \leq a$ for all $a, b \in L$.

Example 1. *(i)* *Lawvere's quantale, $L = [0, \infty]$ with the opposite order and addition as the quantale operation, where $c + \infty = \infty + c = \infty$ for all $c \in L$, is a linearly ordered value quantale [23,24].*

(ii) *Let $\mathcal{L} = ([0,1], \leq, *)$ be a triangular norm with a binary operation $*$ defined as $\forall a, b \in [0,1], a * b = a.b$ and named as a product triangular norm [25]. The triple $\mathcal{L} = ([0,1], \leq, .)$ is a commutative and integral quantale.*

(iii) *Let $\mathcal{L} = (\triangle^+, \leq, *)$ (a probabilistic quantale) where $\varphi * \Psi = \varphi.\Psi$ for all $\varphi, \Psi \in \triangle^+$, then $\mathcal{L}$ is not linearly ordered quantale [7].*

In this sequel, we consider only integral and commutative quantales $\mathcal{L}$ with underlying completely distributive lattices.

Definition 4 (cf. [7])**.** *Let X be a nonempty set. A map $m : X \times X \to \mathcal{L} = (L, \leq, *)$ is called an $\mathcal{L}$-metric on X if it satisfies for all $s \in X$, $m(s,s) = \top$, and for all $s, t, y \in X$, $m(s,t) * m(t,y) \leq m(s,y)$. The pair (X, m) is called an $\mathcal{L}$-metric space.*

A map $f : (X, m_X) \to (Y, m_Y)$ is called an $\mathcal{L}$-metric morphism if $m_X(s_1, s_2) \leq m_Y(f(s_1), f(s_2))$ for all $s_1, s_2 \in X$.

The category whose objects are $\mathcal{L}$-metric spaces and morphisms are $\mathcal{L}$-metric morphisms is denoted by **L-MET**. Furthermore, we define **L-MET(X)** as the set of all $\mathcal{L}$-metrics on X.

Example 2. *(i)* *If $\mathcal{L} = (\{0,1\}, \leq, \wedge)$, then an $\mathcal{L}$-metric space is a preordered set.*

(ii) *If $\mathcal{L}$ is a Lawvere's quantale, then an $\mathcal{L}$-metric space is an extended pseudo-quasi metric space.*

(iii) *If $\mathcal{L} = (\triangle^+, \leq, *)$, then an $\mathcal{L}$-metric space is a probabilistic quasi metric space [23].*

Definition 5 (cf. [7])**.** *Let $\mathcal{H} \subseteq$ **L-MET(X)** and $m \in$ **L-MET(X)**.*

(i) *m is called locally supported by $\mathcal{H}$ if for all $s \in X$, $a \lhd \top$, $\perp \prec \omega$, there is $n \in \mathcal{H}$ such that $n(s,.) * a \leq m(s,.) \vee \omega$.*

(ii) *$\mathcal{H}$ is called locally directed if for all finite subsets $\mathcal{H}_0 \subseteq \mathcal{H}$, $\bigwedge_{m \in \mathcal{H}_0} m$ is locally supported by $\mathcal{H}$.*

(iii) *$\mathcal{H}$ is called locally saturated if for all $m \in$ **L-MET(X)** we have $m \in \mathcal{H}$ whenever m is locally supported by $\mathcal{H}$.*

(iv) *The set $\tilde{\mathcal{H}} = \{m \in$ **L-MET(X)** $: m$ is locally supported by $\mathcal{H}\}$ is called local saturation of $\mathcal{H}$.*

Definition 6 (cf. [7])**.** *Let X be a set. $\mathcal{G} \subseteq$ **L-MET(X)** is called an $\mathcal{L}$-gauge if $\mathcal{G}$ satisfies the following:*

(i) *$\mathcal{G} \neq \emptyset$.*

(ii) *$m \in \mathcal{G}$ and $m \leq n$ implies $n \in \mathcal{G}$.*

(iii) *$m, n \in \mathcal{G}$ implies $m \wedge n \in \mathcal{G}$.*

(iv) *$\mathcal{G}$ is locally saturated.*

The pair $(X, \mathcal{G})$ is called an $\mathcal{L}$-gauge space.

A map $f : (X, \mathcal{G}) \to (X', \mathcal{G}')$ is called an $\mathcal{L}$-gauge morphism if $m' \circ (f \times f) \in \mathcal{G}$ whenever $m' \in \mathcal{G}'$.

The category whose objects are $\mathcal{L}$-gauge spaces and morphisms are $\mathcal{L}$-gauge morphisms is denoted by **L-GS** (cf. [7]).

Definition 7 (cf. [7])**.** *Let $(X, \mathcal{G})$ be an $\mathcal{L}$-gauge space and let $\mathcal{H} \subseteq$ **L-MET(X)**. If $\tilde{\mathcal{H}} = \mathcal{G}$, then $\mathcal{H}$ is called a basis for the gauge $\mathcal{G}$.*

Proposition 1 (cf. [7])**.** *Let $\mathcal{L} = (L, \leq, *)$ be a value quantale. If $\emptyset \neq \mathcal{H} \subseteq$ **L-MET(X)** is locally directed, then $\mathcal{G} = \tilde{\mathcal{H}}$ is a gauge with $\mathcal{H}$ as a basis.*

Lemma 1 (pcf. [7]). *Let $\mathcal{L} = (L, \leq, *)$ be a value quantale, $(X_i, \mathfrak{B}_i)$ be the collection of $\mathcal{L}$-approach spaces and let $f_i : X \to (X_i, \mathfrak{B}_i)$ be a source. A basis for the initial $\mathcal{L}$-gauge on X is given by*

$$\mathcal{H} = \{ \bigwedge_{i \in K} m_i \circ (f_i \times f_i) : K \subseteq I\ finite, m_i \in \mathcal{G}_i, \forall i \in I\}$$

Lemma 2. *Let X be a nonempty set and $(X, \mathcal{G})$ be an $\mathcal{L}$-gauge space.*

*(i) The discrete $\mathcal{L}$-gauge structure on X is given by $\mathcal{G}_{dis} =$ **L-MET(X)** [26].*
(ii) The indiscrete $\mathcal{L}$-gauge structure on X is given by $\mathcal{G}_{ind} = \{\top\}$ [7].

Note that for a value quantale $\mathcal{L}$, the category **L-GS** is a topological category [27,28] over **Set** (the category of sets and functions) [7].

3. T_0 and T_1 Quantale-Valued Approach Spaces

Let X be a non-empty set and the wedge $X^2 \mathbb{V}_\triangle X^2$ be the pushout of the diagonal $\triangle : X \to X^2$ along itself [11].

A point (s,t) in $X^2 \mathbb{V}_\triangle X^2$ is denoted as $(s,t)_1$ if it lies in the first component and as $(s,t)_2$ if it lies in the second component. Note that $(s,t)_1 = (s,t)_2$ if $s = t$.

Definition 8 (cf. [11]). *$A : X^2 \mathbb{V}_\triangle X^2 \to X^3$, the principal axis map is defined by*

$$A(s,t)_i = \begin{cases} (s,t,s), & i = 1 \\ (s,s,t), & i = 2, \end{cases}$$

$S : X^2 \mathbb{V}_\triangle X^2 \to X^3$, the skewed axis map is defined by

$$S(s,t)_i = \begin{cases} (s,t,t), & i = 1 \\ (s,s,t), & i = 2, \end{cases}$$

and $\nabla : X^2 \mathbb{V}_\triangle X^2 \to X^2$, the fold map is defined by $\nabla(s,t)_i = (s,t)$ for $i = 1,2$.

Definition 9. *Let $U : \mathcal{E} \to$ **Set** be a topological functor and $X \in Ob(\mathcal{E})$ with $U(X) = B$.*

(i) X is $\overline{T_0}$ if the initial lift of the U-source $\{A : B^2 \mathbb{V}_\triangle B^2 \to U(X^3) = B^3$ and $\nabla : B^2 \mathbb{V}_\triangle B^2 \to UD(B^2) = B^2\}$ is discrete, where D is the discrete functor [11].
(ii) X is T_0 if X does not contain an indiscrete subspace with at least two points [13].
(iii) X is T_1 if the initial lift of the U-source $\{S : B^2 \mathbb{V}_\triangle B^2 \to U(X^3) = B^3$ and $\nabla : B^2 \mathbb{V}_\triangle B^2 \to UD(B^2) = B^2\}$ is discrete [11].

In **Top**, both $\overline{T_0}$ and T_0 are equivalent, and they reduce to the usual T_0 separation property [11,13]. Similarly, T_1 reduces to classical T_1 property [11].

Theorem 1. *An $\mathcal{L}$-gauge space $(X, \mathcal{G})$ is $\overline{T_0}$ if for all $s,t \in X$ with $s \neq t$, there exists $m \in \mathcal{G}$ such that $m(s,t) \wedge m(t,s) = \perp$.*

Proof. Suppose $(X, \mathcal{G})$ is $\overline{T_0}$ and $s,t \in X$ with $s \neq t$. Let $\mathcal{H}_{dis} = \{m_{dis}\}$ be a basis for the discrete $\mathcal{L}$-gauge where m_{dis} is the discrete $\mathcal{L}$-metric on $X^2 \mathbb{V}_\triangle X^2$. For $(s,t)_1, (s,t)_2 \in X^2 \mathbb{V}_\triangle X^2$ with $(s,t)_1 \neq (s,t)_2$. Note that

$$\begin{aligned} m_{dis}(\nabla(s,t)_1, \nabla(s,t)_2) &= m_{dis}((s,t),(s,t)) = \top \\ m(\pi_1 A(s,t)_1, \pi_1 A(s,t)_2) &= m(\pi_1(s,t,s), \pi_1(s,s,t)) = m(s,s) = \top \\ m(\pi_2 A(s,t)_1, \pi_2 A(s,t)_2) &= m(\pi_2(s,t,s), \pi_2(s,s,t)) = m(t,s) \\ m(\pi_3 A(s,t)_1, \pi_3 A(s,t)_2) &= m(\pi_3(s,t,s), \pi_3(s,s,t)) = m(s,t) \end{aligned}$$

Since $(s,t)_1 \neq (s,t)_2$ and $(X,\mathcal{G})$ is $\overline{T_0}$, by Lemma 1 and Definition 9 (i),

$$\begin{aligned}\bot &= \bigwedge\{m_{dis}(\nabla(s,t)_1,\nabla(s,t)_2), m(\pi_k A(s,t)_1,\pi_k A(s,t)_2)(k=1,2,3)\}\\ &= \bigwedge\{\top, m(s,t), m(t,s)\}\\ &= m(s,t)\wedge m(t,s)\end{aligned}$$

Conversely, let $\overline{\mathcal{H}}$ be the initial $\mathcal{L}$-gauge basis on $X^2\mathbb{V}_\triangle X^2$ induced by $A: X^2\mathbb{V}_\triangle X^2 \to U(X^3,\mathcal{G}^3) = X^3$ and $\nabla : X^2\mathbb{V}_\triangle X^2 \to U(X^2,\mathcal{G}_{dis}) = X^2$, where, by Lemma 2 (i), $\mathcal{G}_{dis} =$ **L-MET(X)** is the discrete $\mathcal{L}$-gauge on X^2, and $\mathcal{G}^3$ is the product structure on X^3 induced by the projection maps $\pi_k : X^3 \to X$ for $k = 1,2,3$.

Suppose for all $s,t \in X$ with $s \neq t$, there exists $m \in \mathcal{G}$ such that $m(s,t)\wedge m(t,s) = \bot$. Let $\overline{m} \in \overline{\mathcal{H}}$ and $u,v \in X^2\mathbb{V}_\triangle X^2$.

Case I: If $u = v$, then $\overline{m}(u,v) = \overline{m}(u,u) = \top$

Case II: If $u \neq v$ and $\nabla u \neq \nabla v$, then $m_{dis}(\nabla u,\nabla v) = \bot$ since m_{dis} is discrete. By Lemma 1,

$$\begin{aligned}\overline{m}(u,v) &= \bigwedge\{m_{dis}(\nabla u,\nabla v), m(\pi_k Au,\pi_k Av)(k=1,2,3)\}\\ &= \bigwedge\{\bot, m(\pi_1 Au,\pi_1 Av), m(\pi_2 Au,\pi_2 Av), m(\pi_3 Au,\pi_3 Av)\} = \bot\end{aligned}$$

Case III: Suppose $u \neq v$ and $\nabla u = \nabla v$. If $\nabla u = (s,t) = \nabla v$ for some $s,t \in X$ with $s \neq t$, then $u = (s,t)_1$ and $v = (s,t)_2$ or $u = (s,t)_2$ and $v = (s,t)_1$ since $u \neq v$.

If $u = (s,t)_1$ and $v = (s,t)_2$, then

$$\begin{aligned}m_{dis}(\nabla u,\nabla v) &= m_{dis}(\nabla(s,t)_1,\nabla(s,t)_2)\\ &= m_{dis}((s,t),(s,t)) = \top\\ m(\pi_1 Au,\pi_1 Av) &= m(\pi_1 A(s,t)_1,\pi_1 A(s,t)_2)\\ &= m(s,s) = \top\\ m(\pi_2 Au,\pi_2 Av) &= m(\pi_2 A(s,t)_1,\pi_2 A(s,t)_2) = m(t,s)\\ m(\pi_3 Au,\pi_3 Av) &= m(\pi_3 A(s,t)_1,\pi_3 A(s,t)_2) = m(s,t)\end{aligned}$$

It follows that

$$\begin{aligned}\overline{m}(u,v) &= \overline{m}((s,t)_1,(s,t)_2)\\ &= \bigwedge\{m_{dis}(\nabla(s,t)_1,\nabla(s,t)_2), m(\pi_k A(s,t)_1,\pi_k A(s,t)_2)(k=1,2,3)\}\\ &= \bigwedge\{\top, m(s,t), m(t,s)\}\\ &= m(s,t)\wedge m(t,s)\end{aligned}$$

By the assumption $m(s,t)\wedge m(t,s) = \bot$, and we have $\overline{m}(u,v) = \bot$.
Similarly, if $u = (s,t)_2$ and $v = (s,t)_1$, then $\overline{m}(u,v) = \bot$.
Therefore, for all $u,v \in X^2\mathbb{V}_\triangle X^2$, we have

$$\overline{m}(u,v) = \begin{cases}\top, & u = v\\ \bot, & u \neq v\end{cases}$$

and by Lemma 2 (i), $\overline{m}$ is the discrete $\mathcal{L}$-metric on $X^2\mathbb{V}_\triangle X^2$. Hence, by Definition 9 (i), $(X,\mathcal{G})$ is $\overline{T_0}$. □

Note that in a quantale $(L,\leq,*)$, if $p \in L$ and $p \neq \top$, then p is called a prime element if $a \wedge b \leq p$ implies $a \leq p$ or $b \leq p$ for all $a,b \in L$.

Corollary 1. *Let $(X,\mathcal{G})$ be an $\mathcal{L}$-gauge space where $\mathcal{L}$ has a prime bottom element. $(X,\mathcal{G})$ is $\overline{T_0}$ if for all $s,t \in X$ with $s \neq t$, there exists $m \in \mathcal{G}$ such that $m(s,t) = \bot$ or $m(t,s) = \bot$.*

Proof. It follows from the definition of the prime bottom element and Theorem 1. □

Theorem 2. *An $\mathcal{L}$-gauge space $(X, \mathcal{G})$ is T_0 if for all $s, t \in X$ with $s \neq t$, there exists $m \in \mathcal{G}$ such that $m(s,t) < \top$ or $m(t,s) < \top$.*

Proof. Let $(X, \mathcal{G})$ be T_0, $B = \{s, t\} \subset X$ and $\mathcal{H}_B$ be the initial $\mathcal{L}$-gauge basis induced by $i : B \to (X, \mathcal{L})$ and $m_B \in \mathcal{H}_B$. For all $s, t \in X$ with $s \neq t$, $m_B(s,t) = m(i(s), i(t)) = m(s,t)$ or $m_B(t,s) = m(i(t), i(s)) = m(t,s)$. It follows that $m(s,t) < \top$ or $m(t,s) < \top$; otherwise $m(s,t) = \top = m(t,s)$, and X contains an indiscrete subspace with at least two elements.

Conversely, suppose the condition holds. Let B be an indiscrete subspace of X with at least two elements $s, t \in B$ with $s \neq t$. Let $\mathcal{H}_B$ be the initial $\mathcal{L}$-gauge basis induced by $i : B \to (X, \mathcal{L})$ and $m_B \in \mathcal{H}_B$. It follows that $T = m_B(s,t) = m(i(s), i(t)) = m(s,t)$ and $T = m_B(t,s) = m(i(t), i(s)) = m(t,s)$ and consequently, $m(s,t) = \top = m(t,s)$, a contradiction to our assumption. Therefore, X does not contain an indiscrete subspace with at least two elements. Hence, by Definition 9 (ii), $(X, \mathcal{G})$ is T_0. □

Theorem 3. *An $\mathcal{L}$-gauge space $(X, \mathcal{G})$ is T_1 if for all $s, t \in X$ with $s \neq t$, there exists $m \in \mathcal{G}$ such that $m(s,t) = \bot = m(t,s)$.*

Proof. Suppose that $(X, \mathcal{G})$ is T_1 and $s, t \in X$ with $s \neq t$. Let $u = (s,t)_1$, $v = (s,t)_2 \in X^2 \mathbb{V}_{\triangle} X^2$. Note that

$$\begin{aligned} m_{dis}(\nabla u, \nabla v) &= m_{dis}((s,t),(s,t)) = \top \\ m(\pi_1 Su, \pi_1 Sv) &= m(\pi_1(s,t,t), \pi_1(s,s,t)) = m(s,s) = \top \\ m(\pi_2 Su, \pi_2 Sv) &= m(\pi_2(s,t,t), \pi_2(s,s,t)) = m(t,s) \\ m(\pi_3 Su, \pi_3 Sv) &= m(\pi_3(s,t,t), \pi_3(s,s,t)) = m(t,t) = \top \end{aligned}$$

where m_{dis} is the discrete $\mathcal{L}$-metric on $X^2 \mathbb{V}_{\triangle} X^2$ and $\pi_k : X^3 \to X$ are the projection maps for $k = 1,2,3$. Since $u \neq v$ and $(X, \mathcal{G})$ is T_1, by Lemma 1 and Definition 9 (iii),

$$\bot = \bigwedge \{m_{dis}(\nabla u, \nabla v), m(\pi_k Su, \pi_k Sv)(k = 1,2,3)\} = \bigwedge \{\top, m(t,s)\} = m(t,s)$$

Similarly, if $u = (s,t)_2$, $v = (s,t)_1 \in X^2 \mathbb{V}_{\triangle} X^2$, then

$$\bot = \bigwedge \{m_{dis}(\nabla u, \nabla v), m(\pi_k Su, \pi_k Sv)(k = 1,2,3)\} = \bigwedge \{\top, m(s,t)\} = m(s,t)$$

Conversely, let $\overline{\mathcal{H}}$ be the initial $\mathcal{L}$-gauge basis on $X^2 \mathbb{V}_{\triangle} X^2$ induced by $S : X^2 \mathbb{V}_{\triangle} X^2 \to U(X^3, \mathcal{G}^3) = X^3$ and $\nabla : X^2 \mathbb{V}_{\triangle} X^2 \to U(X^2, \mathcal{G}_{dis}) = X^2$ where, by Proposition 2, $\mathcal{G}_{dis} =$ **L-MET(X)** is the discrete $\mathcal{L}$-gauge on X^2 and $\mathcal{G}^3$ is the product structure on X^3 induced by the projection maps $\pi_k : X^3 \to X$ for $k = 1,2,3$.

Suppose for all $s, t \in X$ with $s \neq t$, there exists $m \in \mathcal{G}$ such that $m(s,t) = \bot = m(t,s)$. Let $\overline{m} \in \overline{\mathcal{H}}$ and $u, v \in X^2 \mathbb{V}_{\triangle} X^2$.

Case I: If $u = v$, then $\overline{m}(u,v) = \overline{m}(u,u) = \top$

Case II: If $u \neq v$ and $\nabla u \neq \nabla v$, then $m_{dis}(\nabla u, \nabla v) = \bot$ since m_{dis} is a discrete structure on X^2. By Lemma 1,

$$\begin{aligned} \overline{m}(u,v) &= \bigwedge \{m_{dis}(\nabla u, \nabla v), m(\pi_k Su, \pi_k Sv)(k = 1,2,3)\} \\ &= \bigwedge \{\bot, m(\pi_1 Su, \pi_1 Sv), m(\pi_2 Su, \pi_2 Sv), m(\pi_3 Su, \pi_3 Sv)\} = \bot \end{aligned}$$

Case III: Suppose $u \neq v$ and $\nabla u = \nabla v$. If $\nabla u = (s,t) = \nabla v$ for some $s, t \in X$ with $s \neq t$, then $u = (s,t)_1$ and $v = (s,t)_2$ or $u = (s,t)_2$ and $v = (s,t)_1$ since $u \neq v$.

If $u = (s,t)_1$ and $v = (s,t)_2$, then by Lemma 1,

$$\begin{aligned}\overline{m}(u,v) &= \overline{m}((s,t)_1,(s,t)_2)\\ &= \bigwedge\{m_{dis}(\nabla(s,t)_1,\nabla(s,t)_2), m(\pi_k S(s,t)_1,\pi_k S(s,t)_2)(k=1,2,3)\\ &= \bigwedge\{\top, m(t,s)\} = m(t,s) = \bot\end{aligned}$$

since $s \neq t$ and $m(t,s) = \bot$.

Similarly, if $u = (s,t)_2$ and $v = (s,t)_1$, then $m(s,t) = \bot$.

Hence, for all $u, v \in X^2\mathbb{V}_\triangle X^2$, we obtain

$$\overline{m}(u,v) = \begin{cases}\top, & u = v\\ \bot, & u \neq v\end{cases}$$

and it follows that $\overline{m}$ is the discrete $\mathcal{L}$-metric on $X^2\mathbb{V}_\triangle X^2$. By Definition 9 (iii), $(X,\mathcal{G})$ is T_1. □

4. (Pre-)Hausdorff $\mathcal{L}$-Gauge Spaces

Definition 10. *Let $U : \mathcal{E} \to$ **Set** be a topological functor and $X \in Ob(\mathcal{E})$ with $U(X) = B$.*

(i) X is Pre-$\overline{T_2}$ if the initial lifts of U-sources $\{A : B^2\mathbb{V}_\triangle B^2 \to U(X^3) = B^3$ and $\{S : B^2\mathbb{V}_\triangle B^2 \to U(X^3) = B^3$ coincide [11].

(ii) X is $\overline{T_2}$ if X is $\overline{T_0}$ and Pre-$\overline{T_2}$ [11].

(iii) X is NT_2 if X is T_0 and Pre-$\overline{T_2}$ [29].

In **Top**, both $\overline{T_2}$ and NT_2 are equivalent, and they reduce to the usual T_2 [11,13]. By Theorem 2.1 of [30], a topological space (B,τ) is Pre-Hausdorff if the initial topologies on $B^2\mathbb{V}_\triangle B^2$ induced by the maps A and S agree.

Theorem 4. *An $\mathcal{L}$-gauge space $(X,\mathcal{G})$ is Pre-$\overline{T_2}$ if there exists $m \in \mathcal{G}$ such that the following conditions are satisfied.*

(I) For all $s,t \in X$ with $s \neq t$, $m(s,t) \wedge m(t,s) = m(s,t) = m(t,s)$.

(II) For any three distinct points $s,t,y \in X$, $m(t,s) \wedge m(y,s) \wedge m(t,y) = m(t,s) \wedge m(y,s) = m(s,t) \wedge m(y,t) = m(y,s) \wedge m(t,y)$.

(III) For any four distinct points $s,t,y,z \in X$, $m(s,y) \wedge m(t,y) \wedge m(t,z) = m(s,y) \wedge m(t,y) \wedge m(s,z) = m(s,z) \wedge m(t,y) \wedge m(t,z) = m(s,y) \wedge m(t,z) \wedge m(s,z)$.

Proof. Suppose that $(X,\mathcal{G})$ is Pre-$\overline{T_2}$ and $s,t \in X$ with $s \neq t$. Let $\pi_k : X^3 \to X$, $k = 1,2,3$ be the projection maps.

Suppose $u = (s,t)_1, v = (s,t)_2 \in X^2\mathbb{V}_\triangle X^2$. Note that

$$\begin{aligned}m(\pi_1 Au,\pi_1 Av) &= m(\pi_1(s,t,s),\pi_1(s,s,t)) = m(s,s) = \top\\ m(\pi_2 Au,\pi_2 Av) &= m(\pi_2(s,t,s),\pi_2(s,s,t)) = m(t,s)\\ m(\pi_3 Au,\pi_3 Av) &= m(\pi_3(s,t,s),\pi_3(s,s,t)) = m(s,t)\end{aligned}$$

and

$$\begin{aligned}m(\pi_1 Su,\pi_1 Sv) &= m(\pi_1(s,t,t),\pi_1(s,s,t)) = m(s,s) = \top\\ m(\pi_2 Su,\pi_2 Sv) &= m(\pi_2(s,t,t),\pi_2(s,s,t)) = m(t,s)\\ m(\pi_3 Su,\pi_3 Sv) &= m(\pi_3(s,t,t),\pi_3(s,s,t)) = m(t,t) = \top\end{aligned}$$

Since $(X, \mathcal{G})$ is Pre-$\overline{T_2}$ and by Definition 10 (i), we have

$$\begin{aligned} \bigwedge\{m(\pi_k Au, \pi_k Av) : k = 1,2,3\} &= \bigwedge\{m(\pi_k Su, \pi_k Sv) : k = 1,2,3\} \\ \bigwedge\{\top, m(s,t), m(t,s)\} &= \bigwedge\{\top, m(t,s)\} \\ m(s,t) \wedge m(t,s) &= m(t,s) \end{aligned}$$

Let $u = (s,t)_2, v = (s,t)_1 \in X^2\mathbb{V}_\triangle X^2$. Similarly, since $(X, \mathcal{G})$ is Pre-$\overline{T_2}$ and by Definition 10 (i), we have $m(s,t) \wedge m(t,s) = m(s,t)$, and consequently $m(s,t) \wedge m(t,s) = m(s,t) = m(t,s)$.

Let s, t, y be any three distinct points of X. Since $(X, \mathcal{G})$ is Pre-$\overline{T_2}$ and by Definition 10 (i), we have

$$\begin{aligned} \bigwedge\{m(\pi_k A(t,y)_1, \pi_k A(s,y)_2) : k = 1,2,3\} &= \bigwedge\{m(\pi_k S(t,y)_1, \pi_k S(s,y)_2) : k = 1,2,3\} \\ \bigwedge\{m(t,s), m(y,s), m(t,y)\} &= \bigwedge\{\top, m(t,s), m(y,s)\}, \\ \bigwedge\{m(\pi_k A(s,y)_1, \pi_k A(t,y)_2) : k = 1,2,3\} &= \bigwedge\{m(\pi_k S(s,y)_1, \pi_k S(t,y)_2) : k = 1,2,3\} \\ \bigwedge\{m(s,t), m(y,t), m(s,y)\} &= \bigwedge\{\top, m(s,t), m(y,t)\}, \end{aligned}$$

and

$$\begin{aligned} \bigwedge\{m(\pi_k A(s,t)_1, \pi_k A(y,t)_2) : k = 1,2,3\} &= \bigwedge\{m(\pi_k S(s,t)_1, \pi_k S(y,t)_2) : k = 1,2,3\} \\ \bigwedge\{m(s,y), m(t,y), m(s,t)\} &= \bigwedge\{\top, m(s,y), m(t,y)\}. \end{aligned}$$

By condition (I), we have $m(t,s) \wedge m(y,s) \wedge m(t,y) = m(t,s) \wedge m(y,s) = m(s,t) \wedge m(y,t) = m(y,s) \wedge m(t,y)$.

Let s, t, y, z be any four distinct points of X. Since $(X, \mathcal{G})$ is Pre-$\overline{T_2}$ and by Definition 10 (i), we have

$$\begin{aligned} \bigwedge\{m(\pi_k A(s,t)_1, \pi_k A(y,z)_2) : k = 1,2,3\} &= \bigwedge\{m(\pi_k S(s,t)_1, \pi_k S(y,z)_2) : k = 1,2,3\} \\ \bigwedge\{m(s,y), m(t,y), m(s,z)\} &= \bigwedge\{m(s,y), m(t,y), m(t,z)\}, \\ \bigwedge\{m(\pi_k A(s,t)_1, \pi_k A(z,y)_2) : k = 1,2,3\} &= \bigwedge\{m(\pi_k S(s,t)_1, \pi_k S(z,y)_2) : k = 1,2,3\} \\ \bigwedge\{m(s,z), m(t,z), m(s,y)\} &= \bigwedge\{m(s,z), m(t,z), m(t,y)\}, \end{aligned}$$

and

$$\begin{aligned} \bigwedge\{m(\pi_k A(z,y)_1, \pi_k A(t,s)_2) : k = 1,2,3\} &= \bigwedge\{m(\pi_k S(z,y)_1, \pi_k S(t,s)_2) : k = 1,2,3\} \\ \bigwedge\{m(z,t), m(y,t), m(z,s)\} &= \bigwedge\{m(z,t), m(y,t), m(y,s)\}. \end{aligned}$$

By condition (I), we have $m(s,y) \wedge m(t,y) \wedge m(t,z) = m(s,y) \wedge m(t,y) \wedge m(s,z) = m(s,z) \wedge m(t,y) \wedge m(t,z) = m(s,y) \wedge m(t,z) \wedge m(s,z)$.

Conversely, suppose that the conditions hold. Then, we will show that $(X, \mathcal{G})$ is Pre-$\overline{T_2}$. Let $\overline{\mathcal{H}}$ and $\mathcal{H}'$ be two initial $\mathcal{L}$-gauge bases on $X^2\mathbb{V}_\triangle X^2$ induced by $A : X^2\mathbb{V}_\triangle X^2 \to U(X^3, \mathcal{G}^3) = X^3$ and $S : X^2\mathbb{V}_\triangle X^2 \to U(X^3, \mathcal{G}^3) = X^3$, respectively, and $\mathcal{G}^3$ be the product structure on X^3 induced by $\pi_k : X^3 \to X$ the projection map for $k = 1,2,3$. Let $\overline{m}$ and m' be any two $\mathcal{L}$-metrics in $\overline{\mathcal{H}}$ and $\mathcal{H}'$, respectively. We need to show that $\overline{m} = m'$.

First, note that $\overline{m}$ and m' are symmetric by assumption (I), $m(s,t) \wedge m(t,s) = m(s,t) = m(t,s)$.

Suppose u and v are any two points in $X^2\mathbb{V}_\triangle X^2$.

If $u = v$, then $\overline{m}(u,v) = \overline{m}(u,u) = \top = m'(u,u) = m'(u,v)$.

If $u \neq v$, and they are in the same component of $X^2\mathbb{V}_\triangle X^2$, i.e., $u = (s,t)_i$ and $v = (y,z)_i$ for $i = 1,2$, then

$$\begin{aligned}\overline{m}(u,v) &= \bigwedge\{m(\pi_k Au, \pi_k Av) : k = 1,2,3\} \\ &= \bigwedge\{m(s,y), m(t,z)\} \\ &= \bigwedge\{m(\pi_k Su, \pi_k Sv) : k = 1,2,3\} \\ &= m'(u,v)\end{aligned}$$

Suppose $u \neq v$, and they are in the different component of $X^2\mathbb{V}_\triangle X^2$. We have:
Case I: $u = (s,t)_1$ or $(t,s)_1$ and $v = (s,t)_2$ or $(t,s)_2$ for $s \neq t$.
If $u = (s,t)_1$ and $v = (s,t)_2$ (resp. $v = (t,s)_2$), then

$$\overline{m}(u,v) = \bigwedge\{m(\pi_k Au, \pi_k Av) : k = 1,2,3\} = m(s,t) \wedge m(t,s) \;\; (resp.\; m(s,t)),$$

$$m'(u,v) = \bigwedge\{m(\pi_k Su, \pi_k Sv) : k = 1,2,3\} = m(t,s) \;\; (resp.\; m(s,t) \wedge m(t,s))$$

Consequently, we have $\overline{m}(u,v) = m'(u,v)$ by assumption (I).
Similarly, if $u = (t,s)_1$ and $v = (s,t)_2$ (resp. $v = (t,s)_2$), then $\overline{m}(u,v) = m'(u,v)$.
Case II: $u = (s,t)_1$, $(s,y)_1$, $(t,y)_1$, $(t,s)_1$, $(y,s)_1$ or $(y,t)_1$ and $v = (s,t)_2$, $(s,y)_2$, $(t,y)_2$, $(t,s)_2$, $(y,s)_2$ or $(y,t)_2$ for three distinct points s,t,y of X.

If $u = (s,t)_1$ or $(t,s)_1$ and $v = (s,t)_2$ or $(t,s)_2$, $u = (s,y)_1$ or $(y,s)_1$ and $v = (s,y)_2$ or $(y,s)_2$, $u = (t,y)_1$ or $(y,t)_1$ and $v = (t,y)_2$ or $(y,t)_2$, then by case I, we have $\overline{m}(u,v) = m'(u,v)$.

If $u = (s,t)_1$ and $v = (s,y)_2$ or $(t,y)_2$ (resp. $u = (t,s)_1$ and $v = (s,y)_2$ or $(t,y)_2$), then by assumption (I),

$$\overline{m}(u,v) = \bigwedge\{m(\pi_k Au, \pi_k Av) : k = 1,2,3\} = m(t,s) \wedge m(s,y) \;\; (resp.\; m(t,s) \wedge m(t,y)),$$

$$m'(u,v) = \bigwedge\{m(\pi_k Su, \pi_k Sv) : k = 1,2,3\} = m(t,s) \wedge m(t,y) \;\; (resp.\; m(t,s) \wedge m(s,y)),$$

and by assumption (II), we have $\overline{m}(u,v) = m'(u,v)$.

If $u = (s,t)_1$ and $v = (y,s)_2$ or $u = (t,s)_1$ and $v = (y,t)_2$ (resp. $u = (s,t)_1$ and $v = (y,t)_2$ or $u = (t,s)_1$ and $v = (y,s)_2$), then by assumption (I),

$$\overline{m}(u,v) = \bigwedge\{m(\pi_k Au, \pi_k Av) : k = 1,2,3\} = m(s,y) \wedge m(t,y) \;\; (resp.\; m(s,y) \wedge m(t,y) \wedge m(s,t)),$$

$$m'(u,v) = \bigwedge\{m(\pi_k Su, \pi_k Sv) : k = 1,2,3\} = m(s,y) \wedge m(t,y) \wedge m(s,t) \;\; (resp.\; m(s,y) \wedge m(t,y)),$$

and by assumption (II), we have $\overline{m}(u,v) = m'(u,v)$.

If $u = (s,y)_1$ and $v = (s,t)_2$ or $(y,t)_2$ (resp. $u = (y,s)_1$ and $v = (s,t)_2$ or $(y,t)_2$), then by assumption (I),

$$\overline{m}(u,v) = \bigwedge\{m(\pi_k Au, \pi_k Av) : k = 1,2,3\} = m(y,s) \wedge m(s,t) \;\; (resp.\; m(y,s) \wedge m(y,t)),$$

$$m'(u,v) = \bigwedge\{m(\pi_k Su, \pi_k Sv) : k = 1,2,3\} = m(y,s) \wedge m(y,t) \;\; (resp.\; m(y,s) \wedge m(s,t)),$$

and by assumption (II), we have $\overline{m}(u,v) = m'(u,v)$.

If $u = (s,y)_1$ and $v = (t,y)_2$ or $u = (y,s)_1$ and $v = (t,s)_2$ (resp. $u = (s,y)_1$ and $v = (t,s)_2$ or $u = (y,s)_1$ and $v = (t,y)_2$), then by assumption (I),

$$\overline{m}(u,v) = \bigwedge\{m(\pi_k Au, \pi_k Av) : k = 1,2,3\} = m(s,t) \wedge m(y,t) \wedge m(s,y) \;\; (resp.\; m(s,t) \wedge m(y,t)),$$

$$m'(u,v) = \bigwedge\{m(\pi_k Su, \pi_k Sv) : k = 1,2,3\} = m(s,t) \wedge m(y,t) \;\; (resp.\; m(s,t) \wedge m(y,t) \wedge m(s,y)),$$

and by assumption (II), we have $\overline{m}(u,v) = m'(u,v)$.

If $u = (t,y)_1$ and $v = (s,y)_2$ or $u = (y,t)_1$ and $v = (s,t)_2$ (resp. $u = (t,y)_1$ and $v = (s,t)_2$ or $u = (y,t)_1$ and $v = (s,y)_2$), then by assumption (I),

$$\overline{m}(u,v) = \bigwedge\{m(\pi_k Au, \pi_k Av) : k = 1,2,3\} = m(t,s) \wedge m(y,s) \wedge m(t,y) \ \ (resp.\ m(t,s) \wedge m(y,s)),$$

$$m'(u,v) = \bigwedge\{m(\pi_k Su, \pi_k Sv) : k = 1,2,3\} = m(t,s) \wedge m(y,s) \ \ (resp.\ m(t,s) \wedge m(y,s) \wedge m(t,y)),$$

and by assumption (II), we have $\overline{m}(u,v) = m'(u,v)$.

If $u = (t,y)_1$ and $v = (t,s)_2$ or $(y,s)_2$ (resp. $u = (y,t)_1$ and $v = (t,s)_2$ or $(y,s)_2$), then by assumption (I),

$$\overline{m}(u,v) = \bigwedge\{m(\pi_k Au, \pi_k Av) : k = 1,2,3\} = m(y,t) \wedge m(t,s) \ \ (resp.\ m(y,t) \wedge m(y,s)),$$

$$m'(u,v) = \bigwedge\{m(\pi_k Su, \pi_k Sv) : k = 1,2,3\} = m(y,t) \wedge m(y,s) \ \ (resp.\ m(y,t) \wedge m(t,s)),$$

and by assumption (II), we have $\overline{m}(u,v) = m'(u,v)$.

Case III: Let s,t,y,z be four distinct points of X.

If $u = (s,t)_1$ and $v = (y,z)_2$ (resp. $u = (y,z)_1$ and $v = (s,t)_2$), then by assumption (I),

$$\begin{aligned}
\overline{m}(u,v) &= \bigwedge\{m(\pi_k Au, \pi_k Av) : k = 1,2,3\} \\
&= m(s,y) \wedge m(t,y) \wedge m(s,z) \\
m'(u,v) &= \bigwedge\{m(\pi_k Su, \pi_k Sv) : k = 1,2,3\} \\
&= m(s,y) \wedge m(t,y) \wedge m(t,z) \\
&= (resp.\ m(s,y) \wedge m(t,z) \wedge m(s,z))
\end{aligned}$$

and by assumption (III), we have $\overline{m}(u,v) = m'(u,v)$.

If $u = (s,t)_1$ and $v = (z,y)_2$ (resp. $u = (z,y)_1$ and $v = (s,t)_2$), then by assumption (I),

$$\begin{aligned}
\overline{m}(u,v) &= \bigwedge\{m(\pi_k Au, \pi_k Av) : k = 1,2,3\} \\
&= m(s,y) \wedge m(t,z) \wedge m(s,z) \\
m'(u,v) &= \bigwedge\{m(\pi_k Su, \pi_k Sv) : k = 1,2,3\} \\
&= m(s,z) \wedge m(t,y) \wedge m(t,z) \\
&= (resp.\ m(s,y) \wedge m(t,y) \wedge m(s,z))
\end{aligned}$$

and by assumption (III), we have $\overline{m}(u,v) = m'(u,v)$.

If $u = (t,s)_1$ and $v = (y,z)_2$ (resp. $u = (y,z)_1$ and $v = (t,s)_2$), then by assumption (I),

$$\begin{aligned}
\overline{m}(u,v) &= \bigwedge\{m(\pi_k Au, \pi_k Av) : k = 1,2,3\} \\
&= m(s,y) \wedge m(t,y) \wedge m(t,z) \\
m'(u,v) &= \bigwedge\{m(\pi_k Su, \pi_k Sv) : k = 1,2,3\} \\
&= m(s,y) \wedge m(t,y) \wedge m(s,z) \\
&= (resp.\ m(s,z) \wedge m(t,y) \wedge m(t,z))
\end{aligned}$$

and by assumption (III), we have $\overline{m}(u,v) = m'(u,v)$.

If $u = (t,s)_1$ and $v = (z,y)_2$ (resp. $u = (z,y)_1$ and $v = (t,s)_2$), then by assumption (I),

$$\begin{aligned}
\overline{m}(u,v) &= \bigwedge\{m(\pi_k Au, \pi_k Av) : k = 1,2,3\} \\
&= m(s,z) \wedge m(t,y) \wedge m(t,z)
\end{aligned}$$

$$\begin{aligned} m'(u,v) &= \bigwedge\{m(\pi_k Su, \pi_k Sv) : k = 1,2,3\} \\ &= m(s,y) \wedge m(t,z) \wedge m(s,z) \\ &= (resp.\ m(s,y) \wedge m(t,y) \wedge m(t,z)) \end{aligned}$$

and by assumption (III), we have $\overline{m}(u,v) = m'(u,v)$.

Similarly, if $u = (s,y)_1$ and $v = (t,z)_2$, $u = (t,z)_1$ and $v = (s,y)_2$, $u = (s,y)_1$ and $v = (z,t)_2$, $u = (z,t)_1$ and $v = (s,y)_2$, $u = (y,s)_1$ and $v = (t,z)_2$, $u = (t,z)_1$ and $v = (y,s)_2$, $u = (y,s)_1$ and $v = (z,t)_2$, $u = (z,t)_1$ and $v = (y,s)_2$, and if $u = (s,z)_1$ and $v = (t,y)_2$, $u = (t,y)_1$ and $v = (s,z)_2$, $u = (s,z)_1$ and $v = (y,t)_2$, $u = (y,t)_1$ and $v = (s,z)_2$, $u = (z,s)_1$ and $v = (t,y)_2$, $u = (t,y)_1$ and $v = (z,s)_2$, $u = (z,s)_1$ and $v = (y,t)_2$, $u = (y,t)_1$ and $v = (z,s)_2$, then by assumption (III), we have $\overline{m}(u,v) = m'(u,v)$.

Hence, for all points $u,v \in X^2 \mathbb{V}_{\triangle} X^2$, we obtain $\overline{m}(u,v) = m'(u,v)$, and by Lemma 1 and Definition 10 (i), $(X,\mathcal{G})$ is Pre-$\overline{T_2}$. □

Corollary 2. *Let $(X,\mathcal{G})$ be an $\mathcal{L}$-gauge space, where $\mathcal{L}$ is a linearly ordered quantale. $(X,\mathcal{G})$ is Pre-$\overline{T_2}$ if there exists $m \in \mathcal{G}$ such that for any distinct points $s,t,y,z \in X$, the following conditions are satisfied.*

(I) $m(s,t) = m(t,s)$.
(II) $m(s,t) = m(s,y) \leq m(t,y)$ *or* $m(s,t) = m(t,y) \leq m(s,y)$ *or* $m(s,y) = m(t,y) \leq m(s,t)$.
(III) $m(s,y) = m(t,y) \leq m(s,z), m(t,z)$ *or* $m(s,y) = m(s,z) \leq m(t,y), m(t,z)$ *or* $m(s,y) = m(t,z) \leq m(t,y), m(s,z)$ *or* $m(t,y) = m(s,z) \leq m(s,y), m(t,z)$ *or* $m(t,y) = m(t,z) \leq m(s,y), m(s,z)$ *or* $m(s,z) = m(t,z) \leq m(s,y), m(t,y)$.

Theorem 5. *An $\mathcal{L}$-gauge space $(X,\mathcal{G})$ is $\overline{T_2}$ if $(X,\mathcal{G})$ is discrete.*

Proof. By Definition 10 (ii), Theorems 1 and 4, the condition $m(s,t) = m(t,s) = \bot$ for all $s \neq t$ implies that m is the discrete $\mathcal{L}$-metric and if such a $m \in \mathcal{G}$ exists, then $\mathcal{G}$ contains all $\mathcal{L}$-metrics on X, i.e., $\mathcal{G}_{dis} = \{d \in \textbf{L-MET(X)} : d \geq m\}$, and consequently, $(X,\mathcal{G})$ is discrete. □

Theorem 6. *An $\mathcal{L}$-gauge space $(X,\mathcal{G})$ is NT_2 if there exists $m \in \mathcal{G}$ such that the following conditions are satisfied.*

(I) *For all $s,t \in X$ with $s \neq t$, $m(s,t) \wedge m(t,s) = m(s,t) = m(t,s) < \top$.*
(II) *For any three distinct points $s,t,y \in X$, $m(t,s) \wedge m(y,s) \wedge m(t,y) = m(t,s) \wedge m(y,s) = m(s,t) \wedge m(y,t) = m(y,s) \wedge m(t,y)$.*
(III) *For any four distinct points $s,t,y,z \in X$, $m(s,y) \wedge m(t,y) \wedge m(t,z) = m(s,y) \wedge m(t,y) \wedge m(s,z) = m(s,z) \wedge m(t,y) \wedge m(t,z) = m(s,y) \wedge m(t,z) \wedge m(s,z)$.*

Proof. It follows from Definition 10 (iii), Theorems 2 and 4. □

Corollary 3. *A $(X,\mathcal{G})$, where $\mathcal{L}$ is a linearly ordered quantale, is NT_2 if there exists $m \in \mathcal{G}$ such that for any distinct points $s,t,y,z \in X$, the following conditions are satisfied.*

(I) $m(s,t) = m(t,s) < \top$.
(II) $m(s,t) = m(s,y) \leq m(t,y)$ *or* $m(s,t) = m(t,y) \leq m(s,y)$ *or* $m(s,y) = m(t,y) \leq m(s,t)$.
(III) $m(s,y) = m(t,y) \leq m(s,z), m(t,z)$ *or* $m(s,y) = m(s,z) \leq m(t,y), m(t,z)$ *or* $m(s,y) = m(t,z) \leq m(t,y), m(s,z)$ *or* $m(t,y) = m(s,z) \leq m(s,y), m(t,z)$ *or* $m(t,y) = m(t,z) \leq m(s,y), m(s,z)$ *or* $m(s,z) = m(t,z) \leq m(s,y), m(t,y)$.

Example 3. *Let X be a set with at least two points and $(X,\mathcal{G})$ be an indiscrete $\mathcal{L}$-gauge space. Then, by Theorem 3.3 of [22], $(X,\mathcal{G})$ is Pre-$\overline{T_2}$, but by Theorems 1, 3 and 5, $(X,\mathcal{G})$ is neither T_0, $\overline{T_0}$, T_1, $\overline{T_2}$ nor NT_2.*

Theorem 7. *Let $(X, \mathcal{G})$ be a Pre-$\overline{T_2}$ $\mathcal{L}$-gauge space, then the following are equivalent.*

1. $(X, \mathcal{G})$ *is* $\overline{T_2}$.
2. $(X, \mathcal{G})$ *is* T_1.
3. $(X, \mathcal{G})$ *is* $\overline{T_0}$.

Proof. Combine Theorems 1 and 3–5. □

5. Comparative Evaluation

In this section, we compare our results with the ones in some other categories.

Let $\mathcal{E}$ be a topological category, and let $\mathbf{T}(\mathcal{E})$ be the full subcategory of $\mathcal{E}$ consisting of all $\mathbf{T}$ objects where $\mathbf{T}$ is $\overline{T_0}$, T_1, Pre-$\overline{T_2}$ or $\overline{T_2}$.

By Theorem 3.4 of [22], the full subcategory $\mathbf{Pre\text{-}T_2}(\mathcal{E})$ of $\mathcal{E}$ consisting of all Pre-$\overline{T_2}$ objects in $\mathcal{E}$ is a topological category.

Theorem 8. *The following categories are isomorphic.*

1. $\overline{T_0}$ $(\mathbf{Pre\text{-}T_2}(\boldsymbol{L\text{-}GS}))$.
2. T_1 $(\mathbf{Pre\text{-}T_2}(\boldsymbol{L\text{-}GS}))$.
3. $\overline{T_2}$ $(\mathbf{Pre\text{-}T_2}(\boldsymbol{L\text{-}GS}))$.
4. T_1 $(\boldsymbol{L\text{-}GS})$.
5. $\overline{T_2}$ $(\boldsymbol{L\text{-}GS})$.

Proof. It follows from Theorem 3.5 of [22] and Theorems 3, 5 and 7. □

We can infer the following:

(1) In **L-GS**,
   (a) By Theorems 1–3 and 5, $\overline{T_2} = T_1 \implies \overline{T_0} \implies T_0$.
   (b) By Theorems 4–6, if an $\mathcal{L}$-gauge space $(X, \mathcal{G})$ is $\overline{T_2}$, then $(X, \mathcal{G})$ is both NT_2 and Pre-$\overline{T_2}$.
   (c) By Theorem 7, $(X, \mathcal{G})$ is a Pre-Hausdorff $\mathcal{L}$-gauge space, then $\overline{T_0}$, T_1 and $\overline{T_2}$ are equivalent.

(2) In the category **App** of approach spaces and contraction maps, T_0, $\overline{T_0}$ and T_1 separation axioms, given in [2,31] are the special forms of our results. For example, if we take Lawvere's quantale [23,24], then Theorems 1 and 3 reduce to Theorems 3.1.3 and 3.2.3 of [31], respectively.

(3) For the category **Top**, $\overline{T_2} = NT_2 \implies T_1 \implies \overline{T_0} = T_0$ and $\overline{T_2} = NT_2 \implies$ Pre-$\overline{T_2}$ [13,29,30]. Moreover, in the realm of Pre-T_2 property, by Theorem 3.5 of [22], all of T_0, T_1 and T_2 are equivalent.

(4) (a) In category **Prox** of proximity spaces and proximity maps, if a proximity space (X, z) is $\overline{T_0}$ or T_1 or $\overline{T_2}$, then (X, z) is Pre-$\overline{T_2}$ [32]. Similarly, in category **CHY** of Cauchy spaces and Cauchy continuous maps, $T_0 = \overline{T_0} = T_1 = \overline{T_2} \implies$ Pre-$\overline{T_2}$ [33].
   (b) In category **Born** of bornological spaces and bounded maps, if a bornological space is T_0, then it is $\overline{T_0}$ or T_1 or $\overline{T_2}$ or Pre-$\overline{T_2}$ [14,15,29]. However, in category **Lim** of limit spaces and filter convergence maps, $T_1 \implies T_0 = \overline{T_0}$ [15].
   (c) In **ConFCO** (the category of constant filter convergence spaces and continuous maps), $T_0 = \overline{T_0} = T_1$ and $\overline{T_2} = NT_2 \implies$ Pre-$\overline{T_2}$ [34]. In the realm of Pre-$\overline{T_2}$ property, $T_0 = \overline{T_0} = T_1 = \overline{T_2} = NT_2$ [22,34]. In **ConLFCO** (the category of constant local filter convergence spaces and continuous maps), $T_0 \implies \overline{T_0} = T_1$ and $T_0 = NT_2 \implies \overline{T_2} \implies$ Pre-$\overline{T_2}$ [34]. In the realm of Pre-$\overline{T_2}$ property, $T_0 = NT_2 \implies \overline{T_2} = \overline{T_0} = T_1$ [22,34].
   (d) In the category ∞**pqsMet** of extended pseudo-quasi-semi-metric spaces and contraction maps, $T_1 = \overline{T_2} \implies \overline{T_0} \implies T_0$ and $\overline{T_2} \implies NT_2 \implies$ Pre-

$\overline{T_2}$ [20,35]. Furthermore, in the realm of Pre-$\overline{T_2}$ property, $\overline{T_0} = T_1 = \overline{T_2}$ and $NT_2 = T_0$ [20,35].

(e) In category **CP** of pair spaces and pair preserving maps, all pair spaces are $\overline{T_0}$, T_1, $\overline{T_2}$ and Pre-$\overline{T_2}$ [16]. Moreover, $T_0 = NT_2 \implies \overline{T_2} = \overline{T_0} = T_1$ =Pre-$\overline{T_2}$ [16].

(5) (a) For any arbitrary topological category, there is no relationship between $\overline{T_0}$ and T_0 [15]. In addition, it is shown in [29] that the notions of $\overline{T_2}$ and NT_2 are independent of each other, in general. However, in the realm of Pre-T_2 property, by Theorem 3.5 of [22], all of $\overline{T_0}$, T_1 and $\overline{T_2}$ are equivalent.

(b) By Corollary 2.7 of [36], if $U : \mathcal{E} \to$ **Set** is normalized (i.e., U is topological and there is only one structure on a one-point set and $\emptyset$, the empty set), then $\overline{T_0}$, T_1, Pre-$\overline{T_2}$ and $\overline{T_2}$ imply $\overline{T_0}$ at p, T_1 at p, Pre-$\overline{T_2}$ at p and $\overline{T_2}$ at p, respectively. In **L-GS**, by Theorems 3.1–3.4 of [26], if an $\mathcal{L}$-gauge space $(X, \mathcal{G})$ is $\overline{T_0}$ (or T_1), then $(X, \mathcal{G})$ is $\overline{T_0}$ at p (or T_1 at p).

6. Conclusions

Firstly, we characterized T_0, $\overline{T_0}$, T_1, Pre-$\overline{T_2}$, $\overline{T_2}$ and NT_2 $\mathcal{L}$-gauge spaces and showed that $\overline{T_2} = T_1 \implies \overline{T_0} \implies T_0$. Moreover, we obtained that an $\mathcal{L}$-gauge space $(X, \mathcal{G})$ is $\overline{T_2}$, then $(X, \mathcal{G})$ is both NT_2 and Pre-$\overline{T_2}$, and in the realm of Pre-Hausdorff quantale-valued gauge spaces, $\overline{T_0}$, T_1 and $\overline{T_2}$ are equivalent. Finally, we compared our results with the ones in some other categories. Considering these results, the following can be treated as open research problems:

(i) Can one characterize each of T_3, T_4, irreducible, compact, connected, sober and zero-dimensional quantale-valued gauge spaces?

(ii) Can one present the Urysohn's Lemma, the Tietze Extension Theorem and the Tychonoff Theorem for the category **L-GS**?

(iii) How can one characterize T_0, $\overline{T_0}$, T_1, Pre-$\overline{T_2}$, $\overline{T_2}$ and NT_2 separation axioms for quantale generalization of other approach structures such as approach distances and approach systems, and what would be their relation to each other?

(iv) In the category **App** of approach spaces and contraction maps, what would be the characterization of Pre-$\overline{T_2}$, $\overline{T_2}$ and NT_2 properties?

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References

1. Lowen, R. Approach spaces A common Supercategory of Top and Met. *Math. Nachr.* **1989**, *141*, 183–226. [CrossRef]
2. Lowen, R. *Approach Spaces: The Missing Link in the Topology-Uniformity-Metric Triad*; Oxford University Press: New York, NY, USA, 1997.
3. Colebunders, E.; De Wachter, S.; Lowen, R. Fixed points of contractive maps on dcpo's. *Math. Struct. Comput. Sci.* **2014**, *24*, 18. [CrossRef]
4. Brock, P.; Kent, D. On convergence approach spaces. *Appl. Categ. Struct.* **1998**, *6*, 117–125. [CrossRef]
5. Colebunders, E.; De Wachter, S.; Lowen, R. Intrinsic approach spaces on domains. *Topol. Appl.* **2011**, *158*, 2343–2355. [CrossRef]
6. Berckmoes, B.; Lowen, R.; Van Casteren, J. Approach theory meets probability theory. *Topol. Appl.* **2011**, *158*, 836–852. [CrossRef]
7. Jäger, G.; Yao, W. Quantale-valued gauge spaces. *Iran. J. Fuzzy Syst.* **2018**, *15*, 103–122.
8. Jäger, G. Probabilistic approach spaces. *Math. Bohem.* **2017**, *142*, 277–298. [CrossRef]

9. Jäger, G. Quantale-valued generalizations of approach spaces and quantale-valued topological spaces. *Quaest. Math.* **2019**, *42*, 1313–1333. [CrossRef]
10. Jäger, G. Quantale-valued generalization of approach spaces: L-Approach Systems. *Topol. Proc.* **2018**, *51*, 253–276.
11. Baran, M. Separation properties. *Indian J. Pure Appl. Math.* **1991**, *23*, 333–341.
12. Brümmer, G.C.L. A Categorical Study of Initiality in Uniform Topology. Ph.D. Thesis, University of Cape Town, Cape Town, South Africa, 1971.
13. Marny, T. Rechts-Bikategoriestrukturen in Topologischen Kategorien. Ph.D. Thesis, Freie Universität, Berlin, Germany, 1973.
14. Weck-Schwarz, S. T_0-objects and separated objects in topological categories. *Quaest. Math.* **1991**, *14*, 315–325. [CrossRef]
15. Baran, M.; Altındiş, H. T_0 objects in topological categories. *J. Univ. Kuwait Sci.* **1995**, *22*, 123–127.
16. Baran, M.; Abughalwa, H. Sober Spaces, *Turk. J. Math.* **2022**, *46*, 299–310.
17. Mielke, M.V. Separation Axioms and Geometric Realizations. *Indian J. Pure Appl. Math.* **1994**, *25*, 711–722.
18. Mielke, M.V. Hausdorff separations and decidability. In *Symposium on Categorical Topology*; University of Cape Town: Cape Town, South Africa, 1999; pp. 155–160.
19. Johnstone, P.T.; Stone Spaces. L.M.S. *Mathematics Monograph: No. 10*; Academic Press: New York, NY, USA, 1977.
20. Baran, T.M.; Kula, M. Separation Axioms, Urysohn's Lemma and Tietze Extention Theorem for Extended Pseudo-Quasi-Semi Metric Spaces. *Filomat* **2022**, *36*, 703–713. [CrossRef]
21. Stine, J. Initial hulls and zero dimensional objects. *Publ. Math. Debr.* **2013**, *82*, 359–371. [CrossRef]
22. Baran, M. PreT_2-objects in topological categories. *Appl. Categ. Struct.* **2009**, *17*, 591–602. [CrossRef]
23. Flagg, R.C. Quantales and continuity spaces. *Algebra Universalis* **1997**, *37*, 257–276. [CrossRef]
24. Lawvere, F.W. Metric spaces, generalized logic, and closed categories. *Rend. Sem. Mat. Fis. Milano* **1973**, *43*, 135–166. [CrossRef]
25. Klement, E.P.; Mesiar, R.; Pap, E. *Triangular Norms*; Springer: Dordrecht, The Netherlands, 2000.
26. Qasim, M.; Özkan, S. The notion of Closedness and D-connectedness in quantale-valued approach space. *Categories Gen. Algebraic Struct. Appl.* **2020**, *12*, 149–173. [CrossRef]
27. Adámek, J.; Herrlich, H.; Strecker, G.E. *Abstract and Concrete Categories: The Joy of Cats*; John Wiley & Sons: New York, NY, USA, 1990.
28. Preuss, G. *Theory of Topological Structures: An Approach to Categorical Topology*; D. Reidel Publishing Company: Dordrecht, The Netherlands, 1988.
29. Baran, M.; Altındiş, H. T_2 objects in topological categories. *Acta Math. Hung.* **1996**, *22*, 41–48. [CrossRef]
30. Baran, M. Separation Properties in Topological Categories. *Math. Balk.* **1996**, *10*, 39–48.
31. Qasim, M. T_0 and T_1 Approach Spaces. Ph.D. Thesis, Erciyes University, Kayseri, Turkey, 2018.
32. Kula, M.; Özkan, S.; Maraşlı, T. Pre-Hausdorff and Hausdorff proximity spaces. *Filomat* **2017**, *31*, 3837–3846. [CrossRef]
33. Kula, M. A note on Cauchy spaces. *Acta Math. Hung.* **2011**, *133*, 14–32. [CrossRef]
34. Baran, M. Separation Properties in Categories of Constant Convergence Spaces. *Turk. J. Math.* **1994**, *18*, 238–248.
35. Baran, T.M.; Kula, M. T_1 Extended Pseudo-Quasi-Semi Metric Spaces. *Math. Sci. Appl. E-Notes* **2017**, *5*, 40–45.
36. Baran, M. Generalized local separation properties. *Indian J. Pure Appl. Math.* **1994**, *25*, 615–620.