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Evolution Problems with *m*-Accretive Operators and Perturbations

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Abstract: This paper is devoted to the study of perturbation evolution problems involving time-dependent m-accretive operators. We present for a specific class of m-accretive operators with convex weakly compact-valued perturbation, some results about the existence of absolutely continuous solutions and BRVC solutions. We finish by giving several applications to various domains such as relaxation results, second-order evolution inclusions, fractional-order equations coupled with m-accretive operators and Skorohod differential inclusions.

Keywords: evolution problems; *m*-accretive operator; perturbation; AC solution; BRVC solution; second-order evolution inclusion; Volterra integro-differential inclusion

MSC: 34A60; 34H05; 34K35; 47G20; 49J53; 28A25; 28B20; 28C20

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1. Introduction

In the present paper, we are mainly interested in the study of the perturbed evolution problem governed by a time-dependent m-accretive operator $A(t)$

$$-\frac{du}{dv}(t) \in A(t, u(t)) + f(t, u(t)),\ v\ a.e\ t \in [0, T]. \tag{1}$$

Here v is a given positive Radon measure on $[0, T]$, $u : [0, T] \to E$ is a right continuous function with bounded variation, du is its differential measure or Stieltjes measure, $\frac{du}{dv}$ is the density of du with respect to v, $t \mapsto A(t) : D(A(t)) \to 2^E$ is a time-dependent m-accretive operator, $f : [0, T] \times E \to E$ is a Caratheodory mapping. Multivalued perturbations $F : [0, T] \times E \to E$ are also considered.

When E is a separable Hilbert space and $A(t)$ is assumed to be a *Lipschitz in variation* or *bounded variation and continuous in variation* maximal monotone operator, the study of (1) is performed in [1] and includes the convex sweeping process (or Moreau's process), an area which enjoyed a great deal of intense activity with application to mechanics [2],

$$-\frac{du}{dr}(t) \in N_{C(t)}u(t) + f(t, u(t)),\ dr\ a.e\ t \in [0, T], \tag{2}$$

where $C(t)$ is a closed convex moving set, i.e., $d_H(C(t), C(s))) \le |r(t) - r(s)|$ for all $s, t \subset [0, T]$, $r : [0, T] \to [0, \infty[$ is a nondecreasing right continuous function with Stieltjes measure, dr, d_H denotes the Hausdorff distance on closed sets of E and $N_{C(t)}$ is the normal cone of the closed convex set $C(t)$. See [3–6] and the references therein.

Consequently, it is interesting to extend the theory outside of Hilbert space to a time-dependent m-accretive operator $A(t)$. To our knowledge, until now one cannot expect a positive answer to the existence problem of bounded variation and right continuous (BVRC) solution for (1) in the framework of a time-dependent m-accretive operator. The main difficulty is how to formulate the notion of solution to have convenient applications.

Therefore, it is important to know a few significant classes of m-accretive operators for which existence of absolutely continuous or bounded variation and right continuous (BVRC) solutions to (1) are proved.

In this regard, we present several new variants in the study of absolutely continuous and BVRC solutions for (1) with time-dependent m-accretive operator $A(t)$ and weakly compact-valued perturbation F. This leads to some remarkable applications such as periodic solutions, relaxation problems, second-order evolution driven with m-accretive operators with perturbation, fractional-order equation coupled with m-accretive operators, functional differential inclusion governed by m-accretive operators, sweeping process, and Skorohod differential inclusions.

Our techniques are essentially based on Moreau's catching-up algorithm [7] and deep results on the differential measures of vector functions of bounded variation [8,9]. We provide a new method for proofs that are simpler and that are independent in an essential way to the results in m-accretive theory. Very roughly, our method makes it possible to obtain, from sequences of partitions of the considered study interval, solutions as the limit of step approximations and provides an estimate of their velocity. Our results are studied from a theoretical point of view as well as in applications. They make it possible to obtain concrete solutions in various domains such as elastoplasticity, mechanics, traffic equilibria, and social and economic models.

2. Preliminaries and Background

We will use the following definitions and notations and summarize some basic results.

- Let E be a Banach space and E^* be its topological dual.
- $\overline{B}_E$ is the closed unit ball of E.
- $c(E)$, $cc(E)$, $ccwl(E)$, $cwk(E)$ is the collection of nonempty closed, closed convex, closed convex weakly locally compact which contain no line, weakly compact convex subsets of E respectively.
- If K is a subset of E, $\delta^*(.,K)$ is the support function of K. For any convex weakly compact subset K of E, $|K| := \sup_{x^*\in\overline{B}_{E^*}} |\delta^*(x^*,K)|$
- $\lambda := dt$ is the Lebesgue measure on $[0,T]$, $\mathcal{L}([0,T])$ is the σ-algebra of Lebesgue measurable subsets of $[0,T]$.
- $\mathcal{B}(E)$ is the Borel σ-algebra of E.
- A map u: $[0,T] \to E$ is absolutely continuous (shortly AC) if there exists an integrable mapping v such that $u(t) = u_0 + \int_0^t v(s)ds$; in this case $\dot{u} = v$ a.e. on I.
  A map u: $[0,T] \to E$ is BVRC if u is of bounded variation (shortly BV) and right continuous.
- $L^1_E([0,T],dt)$ (shortly $L^1_E([0,T])$) is the Banach space of Lebesgue–Bochner integrable functions $f : [0,T] \to E$.
- We denote by $\mathcal{W}^{1,\infty}_E([0,T])$ the set of all absolutely continuous mappings $v : [0,T] \to E$ such that $\dot{v} \in L^\infty_E([0,T])$.
- If X is a topological space, $\mathcal{C}_E(X)$ is the space of continuous mappings $u : X \to E$ equipped with the norm of uniform convergence.
- A set-valued mapping $F : [0,T] \rightrightarrows E$ is measurable if its graph belongs to $\mathcal{L}([0,T]) \otimes \mathcal{B}(E)$. A closed convex valued mapping $F : X \to cc(E)$ defined on a topological space X is scalarly upper semicontinuous if for every $y \in E^*$, the scalar function $\delta^*(y,F(.))$ is upper semicontinuous on X.
- Let E be a Banach space and E^* be its topological dual. Recall that operator A : $D(A) \subset E \to 2^E$ is accretive if $||x - \overline{x}|| \leq ||x - \overline{x} + \lambda(y - \overline{y})||$ for all $x, \overline{x} \in D(A), y \in$

$Ax, \overline{y} \in A\overline{x}$ and $\lambda > 0$ and A is m-accretive if, in addition, $R(I + \lambda A) = E$ for all $\lambda > 0$.

- If A is m-accretive, then,
  (i) for each $\lambda > 0$, the resolvent $J_\lambda = (I_E + \lambda A)^{-1}$ is single-valued and non-expensive, i.e.,
  $$||J_\lambda x - J_\lambda y|| \leq ||x - y||$$
  for each $(x, y) \in E$,
  (ii) the Yosida-approximation of A defined by
  $$A_\lambda := \frac{1}{\lambda}(I_E - J_\lambda)$$
  is single-valued, Lipschitz continuous with Lipschitz-constant $\frac{2}{\lambda}$,
  (iii) $A_\lambda x \in AJ_\lambda x$ for each $x \in E$,
  (iv) $||A_\lambda x|| \leq ||A^0 x||$ for each $x \in D(A)$ where $A^0 x$ is the element of minimum norm of Ax.

  Define $\langle x, y\rangle + = \lim_{t \to 0} \frac{1}{2t}[||x + ty||^2 - ||x||^2]$, for $x, y \in E$. Then A is accretive iff
  $$\langle x - \overline{x}, y - \overline{y}\rangle_+ \geq 0$$

for $x, \overline{x} \in D(A), y \in Ax, \overline{y} \in A\overline{x}$. The duality map $J : E \to 2^{E^*}$ is defined by $J(x) := \{x^* \in E^* : \langle x^*, x\rangle = ||x||^2 = ||x^*||^2\}, x \in E$. Then $\langle x, y\rangle_+ = \sup\{\langle x^*, y\rangle : x^* \in J(x)\} = \delta^*(y, J(x))$, for $x, y \in E$. We refer to [10–12] for the theory of accretive operators and evolution equations in Banach spaces.

3. Basic Hypotheses. Statement of Existence Theorems

We start this section by recalling some lemmas that are used in our proofs.
At first, the following is a discrete version of Gronwall's lemma.

Lemma 1. *Let (α_i), (β_i), (γ_i) and (a_i) be sequences of nonnegative real numbers such that $a_{i+1} \leq \alpha_i + \beta_i(a_0 + a_1 + ... + a_{i-1}) + (1 + \gamma_i)a_i$ for $i \in \mathbb{N}_0$. Then*
$$a_j \leq \left(a_0 + \sum_{k=0}^{j-1} \alpha_k\right) \cdot exp\left(\sum_{k=0}^{j-1} (k\beta_k + \gamma_k)\right) \text{ for } j \in \mathbb{N}_0.$$

The following version of Gronwall's lemma is crucial for your purpose.

Lemma 2. *Let μ be a positive Radon measure on $I = [0, T]$. Let $g \in L^1_{\mathbf{R}^+}(I, \mu)$ and $\beta \geq 0$ be such that , $\forall t \in I, 0 \leq \mu(\{t\})g(t) \leq \beta < 1$. Let $\varphi \in L^\infty_{\mathbf{R}^+}(I, \mu)$ satisfy*
$$\varphi(t) \leq \alpha + \int_{]0,t]} g(s)\varphi(s)\mu(ds), \ \forall t \in I,$$
where α is a positive constant. Then
$$\varphi(t) \leq \alpha \exp(\frac{1}{1 - \beta} \int_{]0,t]} g(s)\mu(ds)), \ \forall t \in I.$$

Proof. This lemma is due to M. Monteiro Marques. □

Lemma 3. *Let μ be a non-atomic positive Radon measure on the interval $I = [0, T]$. Let c, p be nonnegative real functions such that $c \in L^1([0, T], \mathbb{R}, \mu)$, $p \in L^\infty(I, \mathbb{R}; \mu)$, and let $\alpha \geq 0$. Assume that for $\mu - a.e.$ $t \in I$*
$$p(t) \leq \alpha + \int_0^t c(s)p(s)\mu(ds).$$

Then, for $\mu - a.e.$ $t \in [0, T]$,

$$p(t) \leq \alpha \cdot \exp\left(\int_0^t c(s)\mu(ds)\right).$$

The proof (see [1], Lemma 2.7) is not a consequence of the classical Gronwall lemma dealing with Lebesgue measure dt on I. It relies on a deep result of Moreau–Valadier on the derivation of (vector) functions of bounded variation [9].

In the present paper, E is a separable Banach space. We are mainly interested in the study of perturbed evolution problems governed by a time-dependent m-accretive operator $A(t)$

$$-\frac{du}{dv}(t) \in A(t, u(t)) + f(t, u(t)),\ \nu\ a.e.\ t \in [0, T], \tag{3}$$

where u: $[0, T] \to E$ is a right continuous function with bounded variation, du is its differential measure, ν is a positive measure on $[0, T]$, $\frac{du}{dv}$ is the density of the differential measure du with respect to the measure ν, $t \mapsto A(t)$: $D(A(t)) \to 2^E$ is a time-dependent m-accretive operator, f: $[0, T] \times E \to E$ is a Caratheodory mapping (multivalued perturbations F: $[0, T] \times E \to E$ are also considered). We also treat the case of m-accretive operator A: $D(A) \to 2^E$ with various perturbation

$$-\frac{du}{dt}(t) \in Au(t) + f(t, u(t)),\ dt\ a.e.\ t \in [0, T], \tag{4}$$

where u: $[0, T] \to E$ is absolutely continuous, f: $[0, T] \times E \to E$ is a Caratheodory mapping (multivalued perturbations F: $[0, T] \times E \to E$ are also considered). A special study of integro-differential Volterra inclusion is provided

$$-\frac{du}{dt}(t) \in Au(t) + \int_0^t f(t, s, u(s))ds,\ dt\ a.e.\ t \in [0, T], \tag{5}$$

where u: $[0, T] \to E$ is absolutely continuous, f: $[0, T] \times [0, T] \times E \to E$ is a Caratheodory mapping.

3.1. Existence Results for (3) in the Bounded Variation and Right Continuous Case

Our first result on the existence of the BVRC solution to a perturbed evolution problem (3) is stated with the following hypotheses: Let $t :\mapsto A(t) : D(A(t)) \to ccwl(E)$ be a time-dependent m-accretive operator satisfying:
($\mathcal{H}_1^A$) there exists a nonnegative real number c such that

$$\|A^0(t, x)\| \leq c(1 + \|x\|) \ \text{ for } \ t \in [0, T],\ x \in D(A(t));$$

($\mathcal{H}_2^A$) Γ: $t \mapsto D(A(t))$ has right closed graph $Gr(\Gamma)$, and for each $t \in [0, T]$, for each $k > 0$, the set $\{x \in D(A(t)) \colon ||x||| \leq k\}$ is relatively compact, shortly $D(A(t))$ is ball compact;
($\mathcal{H}_3^A$) $(t, x) \mapsto A(t, x)$: $Gr(\Gamma) \to ccwl(E)$ is scalar upper semicontinuous: for $t_n \downarrow t$, for $x_n \to x$ with $x_n \in D(A(t_n))$ and $x \in D(A(t))$,

$$\forall x^* \in E^*, \limsup_n \delta^*(x^*, A(t_n, x_n)) \leq \delta^*(x^*, A(t, x));$$

($\mathcal{H}_4^A$) There exists a nondecreasing and right continuous function r: $[0, T] \to [0, \infty[$ such that $r(T) < \infty$ with the Stieltjes measure dr such that for $t < \tau \subset [0, T]$, for $\lambda > 0$ and $x \in D(A(t))$

$$\|x - J_\lambda^{A(\tau)}(x)\| \leq (r(\tau) - r(t))(1 + ||A^0(t, x)||);$$

($\mathcal{H}^F$) Let F: $[0, T] \times E \to cwk(E)$ be a convex weakly compact-valued mapping such that

(i) F is scalarly $\mathcal{L}([0, T]) \otimes \mathcal{B}(E)$-measurable, i.e., for each $x^* \in E^*$, the scalar function $\delta^*(x^*, F(.,.))$ is $\mathcal{L}([0, T]) \otimes \mathcal{B}(E)$-measurable,

(ii) for each $t \in [0,T]$, $F(t,.)$ is scalarly upper semicontinuous, i.e., for each $x^* \in E^*$, the scalar function $\delta^*(x^*, F(t,.))$ is upper semicontinuous on E,

(iii) $F(t,x) \subset M(1+||x||)\overline{B}_E$ for all $(t,x) \in [0,T] \times E$ for some positive constant M.

We present at first our main existence result of BVRC solution.

Theorem 1. *Assume that E be a separable Banach space. Let $t \mapsto A(t)$: $D(A(t)) \to ccwl(E)$ be a time-dependent m-accretive operator satisfying $(\mathcal{H}_1^A)$, $(\mathcal{H}_2^A)$, $(\mathcal{H}_3^A)$, $(\mathcal{H}_4^A)$. Let F: $[0,T] \times E \to cwk(E)$ satisfying $(\mathcal{H}^F)$. Let $\nu = dr + \lambda$ and let $\frac{d\lambda}{d\nu}$ be the density of λ relatively to the measure ν. Then for all $u_0 \in D(A(0))$ the evolution problem*

$$-Du(t) \in A(t,u(t)) + F(t,u(t))$$

admits a BVRC solution u with $u(0) = u_0$, that is, there exists a BVRC mapping u: $[0,T] \to E$ and a Lebesgue-integrable mapping z: $[0,T] \to E$ such that

$$\begin{cases} u(0) = u_0 \in D(A(0)) \\ u(t) \in D(A(t)), \forall t \in [0,T] \\ \frac{du}{d\nu}(t) \in L^\infty_E([0,T],\nu) \\ z(t) \in F(t,u(t)),\ \lambda \text{ a.e} \\ -\frac{du}{d\nu}(t) \in A(t,u(t)) + z(t)\frac{d\lambda}{d\nu}(t), \nu \text{ a.e},\ t \in [0,T] \end{cases}$$

Proof. Let for each $(t,x) \in [0,T] \times E$, $f(t,x)$ be the element of minimal norm of $F(t,x)$, i.e., $f(t,x) = P_{F(t,x)}(0)$. For each $x \in E$, the map $t \mapsto f(t,x)$ is $\mathcal{L}([0,T])$-measurable by virtue of Theorem III-41(2) [13], and by $(\mathcal{H}_4^F)$

$$||f(t,x)|| \leq M(1+||x||),\ \forall (t,x) \in [0,T] \times E. \tag{6}$$

We choose a sequence $(\varepsilon_n)_n \subset [0,1]$ which decreases to 0 as $n \to \infty$ and a partition $0 = t_0^n < t_1^n < \ldots < t_{k_n}^n = T$ of I such that

$$t_{i+1}^n - t_i^n + dr(]t_i^n, t_{i+1}^n[) \leq \varepsilon_n \text{ for } i = 0, \ldots, k_n - 1. \tag{7}$$

Such a partition can be obtained by considering the measure $\nu = dr + \lambda$ using the constructions developed in Castaing-Marques [1].

For $i = 0, \ldots, k_n - 1$, let

$$\delta_{i+1}^n = dr(]t_i^n, t_{i+1}^n]) = r(t_{i+1}) - r(t_i^n);\ \eta_{i+1}^n = t_{i+1}^n - t_i^n;\ \beta_{i+1}^n = \nu(]t_i^n, t_{i+1}^n]). \tag{8}$$

We define a sequence of step-mappings u_n:$I \to E$ as follows:

$$u_n(t) = u_0^n = u_0 \in D(A(0)) \text{ for } t \in [0, t_1^n], \text{ and}$$

$$u_n(t) = u_i^n, t \in [t_i^n, t_{i+1}^n],\quad u_n(T) = u_{k_n}^n,$$

for $i = 0, 1, \ldots, k_n - 1$,

$$u_{i+1}^n = J_{\beta_{i+1}^n}^{A(t_{i+1}^n)}\left(u_i^n - \int_{t_i^n}^{t_{i+1}^n} f(s,u_i^n)d\lambda(s)\right) \text{ with } J_{\beta_{i+1}^n}^{A(t_{i+1}^n)}(x) = \left(I + \beta_{i+1}^n A(t_{i+1}^n)\right)^{-1}(x), \tag{9}$$

for $x \in E$. By construction $u_{i+1}^n \in D(A(t_{i+1}^n))$ and

$$-\frac{u_{i+1}^n - u_i^n}{\nu(]t_i^n, t_{i+1}^n])} \in A(t_{i+1}^n, u_{i+1}^n) + \frac{1}{\nu(]t_i^n, t_{i+1}^n])}\int_{t_i^n}^{t_{i+1}^n} f(s,u_i^n)d\lambda(s)$$

so that

$$-\frac{u_{i+1}^n - u_i^n + \int_{t_i^n}^{t_{i+1}^n} f(s,u_i^n)d\lambda(s)}{\nu(]t_i^n, t_{i+1}^n])} \in A(t_{i+1}^n, u_{i+1}^n). \tag{10}$$

We also define the bounded variation and right continuous mapping

$$v_n(t) = u_i^n + \frac{\nu(]t_i^n, t])}{\nu(]t_i^n, t_{i+1}^n])}(u_{i+1}^n - u_i^n + \int_{t_i^n}^{t_{i+1}^n} f(s, u_i^n) d\lambda(s)) - \int_{t_i^n}^{t} f(s, u_i^n) d\lambda(s)$$

on each interval $[t_i^n, t_{i+1}^n]$ so that v_n is bounded variation and right continuous on $[0, T]$.

Step 1. Estimates and convergence.

$$\|u_{i+1}^n - u_i^n\| \le \|J_{\beta_{i+1}^n}^{A(t_{i+1}^n)}(u_i^n - \int_{t_i^n}^{t_{i+1}^n} f(s, u_i^n) ds) - J_{\beta_{i+1}^n}^{A(t_{i+1}^n)}(u_i^n)\| + \|J_{\beta_{i+1}^n}^{A(t_{i+1}^n)} u_i^n - u_i^n\|$$

$$\le \int_{t_i^n}^{t_{i+1}^n} \|f(s, u_i^n)\| d\lambda(s) + r(t_{i+1}^n) - r(t_i^n)[1 + |A^0(t_i^n, u_i^n)|]$$

$$\le M(1 + ||u_i^n||)\beta_{i+1}^n + \beta_{i+1}^n[1 + c(1 + ||u_i^n||)]$$

(using (6), $(\mathcal{H}_1^A)$, $(\mathcal{H}_4^A)$). Whence we obtain

$$\|u_{i+1}^n - u_i^n\| \le (M + c)\beta_{i+1}^n \|u_i^n\| + (M + c + 1)\beta_{i+1}^n,$$

$$\|u_{i+1}^n\| \le (M + c + 1)\ \beta_{i+1}^n))\|u_i^n\| + (M + c + 1)\ \beta_{i+1}^n.$$

Then, by Gronwall discrete Lemma 1 it implies that for $n \in \mathbf{N}$, and $i = 0, \ldots, k_n$:

$$||u_i^n|| \le [||u_0|| + (M + c + 1) \sum_{i=0}^{k_n - 1} \beta_{i+1}^n] \cdot \exp\left[(M + c) \sum_{i=0}^{k_n - 1} \beta_{i+1}^n\right].$$

It follows that

$$||u_i^n|| \le \left[\ ||u_0|| + (M + c + 1)(T + r(T))\right] \cdot \exp\left[(M + c)(T + r(T))\right] =: K_1. \tag{11}$$

and then

$$||u_{i+1}^n - u_i^n|| \le K_2\ \beta_{i+1}^n. \tag{12}$$

Putting $K := max(K_1, K_2)$ we conclude that

$$\sup_{n \in \mathbb{N}} \|u_n\| \le K, \tag{13}$$

and

$$\sup_n var(u_n) = \sup_n \left(\sum_{i=0}^{k_n - 1} \|u_{i+1}^n - u_i^n\| \right) \le K(T + r(T)). \tag{14}$$

We note that u_n is uniformly bounded and bounded in variation (cf (13) and (14) and $\{u_n(t) \colon n \in \mathbf{N}\}$ is relatively compact for each t since $D(A(t))$ is ball compact according to $(\mathcal{H}_2^A)$ and the estimation (13) so that by Helly principle [14] we may assume u_n pointwise strongly converges to a BV mapping v. Now we will focus on the estimate for v_n. We will show that v_n has the density $\frac{dv_n}{dv}(.)$ with respect to ν with the estimation $||\frac{dv_n}{dv}(t)|| \le L,\ \nu$ a.e for some constant $L > 0$. Let us denote by $z_n(t) = f(t, u_i^n)$ for $t \in [t_i^n, t_{i+1}^n]$. We note that for any $t \in [0, T]$,

$$v_n(t) = v_n(0) + \int_{]0,t]} \phi_n(s) d\nu(s) - \int_0^t z_n(s) d\lambda(s),$$

where

$$\phi_n(t) := \sum_{i=0}^{k_n - 1} \frac{u_{i+1}^n - u_i^n + \int_{t_i^n}^{t_{i+1}^n} f(s, u_i^n) d\lambda(s)}{\nu(]t_i^n, t_{i+1}^n])} 1_{]t_i^n, t_{i+1}^n]}(t).$$

Since $\nu = dr + \lambda$, the Lebesgue measure λ is absolutely continuous with respect to the measure ν, it has a density $\frac{d\lambda}{d\nu} \in L^{\infty}_{\mathbf{R}}([0,T],\nu)$ relatively to ν, then the above expression of $v_n(t)$ is written as

$$v_n(t) = v_n(0) + \int_{]0,t]} [\phi_n(s) - z_n(s)\frac{d\lambda}{d\nu}(s)] d\nu(s).$$

Therefore dv_n has a density $\frac{dv_n}{d\nu}$ relatively to ν with $\frac{dv_n}{d\nu} \in L^{\infty}_{E}([0,T],\nu)$ and for ν a.e we have

$$\frac{dv_n}{d\nu}(t) = \phi_n(t) - z_n(t)\frac{d\lambda}{d\nu}(t),$$

$$\frac{dv_n}{d\nu}(t) + z_n(t)\frac{d\lambda}{d\nu}(t) = \phi_n(t) = \sum_{i=0}^{k_n-1} \frac{u^n_{i+1} - u^n_i + \int_{t^n_i}^{t^n_{i+1}} f(s,u^n_i)d\lambda(s)}{\nu(]t^n_i, t^n_{i+1}])} 1_{]t^n_i,t^n_{i+1}]}(t).$$

Please note that on any interval $[t^n_i, t^n_{i+1}]$, $||\phi_n(t)|| \le K_2 + M(1+K)$, using the estimate (12) of $||u^n_{i+1} - u^n_i||$, and

$$\int_{t^n_i}^{t^n_{i+1}} f(s,u^n_i)d\lambda \le M(1+K)\eta^n_{i+1},$$

using the above estimate of $||u^n_i||$. So

$$||\frac{dv_n}{d\nu}(t) + z_n(t)\frac{d\lambda}{d\nu}(s)|| \le K_2 + M(1+K), \tag{15}$$

as consequence

$$||\frac{dv_n}{d\nu}(t)|| \le K_2 + 2M(1+K) := L. \tag{16}$$

Hence, we have

$$v_n(t) = u_0 + \int_{]0,t]} \frac{dv_n}{d\nu}(t)d\nu(t), \forall t \in [0,T],$$

with the estimate (16). Define $\varphi_n, \theta_n : I \to I$ by $\varphi_n(t) = t^n_i$ and $\theta_n(t) = t^n_{i+1}$ for $t \in]t^n_i, t^n_{i+1}]$ and $\varphi_n(0) = \theta_n(0) = 0$, so that

$$v_n(\varphi_n(t)) = u_0 + \int_{]0,\varphi_n(t)]} \frac{dv_n}{d\nu}(t)\, d\nu(t), \forall t \in [0,T],$$

$$v_n(\theta_n(t)) = u_0 + \int_{]0,\theta_n(t)]} \frac{dv_n}{d\nu}(t)\, d\nu(t), \forall t \in [0,T],$$

with

$$-\frac{dv_n}{d\nu}(t) \in A(\theta_n(t))v_n(\theta_n(t) + f(t, v_n(\varphi_n(t))\frac{d\lambda}{d\nu}(t),\ d\nu\ a.e.$$

according to (10) and our notation. It is clear that $||v_n(\theta_n(t)) - v_n(t))|| \to 0$ when n goes to ∞ and so is the term $||v_n(\varphi_n(t)) - v_n(t))||$. As consequence, v_n is uniformly bounded, $||v_n(t)|| \le ||x_0|| + L\nu(]0,T]), \forall t \in [0,T], \forall n \in \mathbf{N}$, and *equi-right continuous with bounded variation*: $||v_n(t) - v_n(\tau)|| \le L\nu(]\tau,t]), \forall \tau < t \in [0,T]$. Hence the sequence v_n pointwise converges weakly to a BVRC mapping u,

$$u(t) = u_0 + \int_{]0,t]} \frac{du}{d\nu}(s)d\nu(s)$$

and we may assume that $\frac{dv_n}{d\nu}$ converge weakly in $L^1_E([0,T],\nu)$ to an integrable mapping $\frac{du}{d\nu}$ with $||\frac{du}{d\nu}|| \leq K_2 + 2M(1+K)$, ν a.e. By construction for every t we note that for all $t \in [t^n_i, t^n_{i+1}]$,

$$||v_n(t) - u_n(t)|| \leq ||v_n(t) - u^n_i|| \leq ||u^n_{i+1} - u^n_i|| + 2\int_{t^n_i}^{t^n_{i+1}} ||f(s, u^n_i)||ds$$

$$\leq ||u^n_{i+1} - u^n_i|| + 2M(1 + ||u^n_i||)(t^n_{i+1} - t^n_i) \leq [K + 2M(1+K)]\varepsilon_n.$$

It implies that $v_n(t) - u_n(t) \to 0$ for all $t \in [0,T]$ and by identifying the limits, we have $u = v$. With our notations, recall that

$$-\frac{dv_n}{d\nu}(t) - f(t, v_n(\varphi_n(t)))\frac{d\lambda}{d\nu}(t) \in A(\theta_n(t), v_n(\theta_n(t)).$$

Now, we use Mazur's trick and $(\mathcal{H}^A_3)$ to finish the proof being ensured that $u(t) \in D(A(t))$ using the fact that $v_n(\theta_n(t)) \in D(A(\theta_n(t))\ \theta_n(t) \downarrow t$ and $v_n(\theta_n(t)) \to u(t)$. Let $\zeta_n(t) := f(t, v_n(\varphi_n(t))$ and $\zeta(t) := f(t, u(t))$. As $v_n(\varphi_n(t)) \to u(t)$, $\zeta_n(t) \to \zeta(t)$, so $\zeta_n \to \zeta$ weakly in $L^1_E([0,T],\lambda)$. Hence $\frac{dv_n}{d\nu}(.) + \zeta_n(.)\frac{d\lambda}{d\nu}(.)$ weakly converges in $L^1_E([0,T],\nu)$ to $\frac{du}{d\nu}(.) + z(.)\frac{d\lambda}{d\nu}(.)$. For convenient notation let

$$w_n(t) = -\frac{dv_n}{d\nu}(t) - \zeta_n(t)\frac{d\lambda}{d\nu}(t),$$

$$w(t) = -\frac{du}{d\nu}(t) - \zeta(t)\frac{d\lambda}{d\nu}(t).$$

Then $\{w_n\}$ weakly converges in $L^1_E([0,T],\nu)$ to w. We will show that

$$w(t) = -\frac{du}{d\nu}(t) - \zeta(t)\frac{d\lambda}{d\nu}(t) \in A(t, u(t)),\ \nu\ a.e.$$

By applying Mazur' s lemma, there exists a sequence $\{\hat{w}_n(.)\}$ which converges strongly in $L^1_E([0,T],\nu)$ to $w = -\frac{du}{d\nu}(.) - z(.)\frac{d\lambda}{d\nu}(.)$ with

$$\hat{w}_n(t) \in co\{w_k(t) : k \geq n\}.$$

Extracting a subsequence, we may ensure that $\hat{w}_n(t) \to w(t) = -\frac{du}{d\nu}(t) - z(t)\frac{d\lambda}{d\nu}(t)$ ν a.e. Consequently, for $t \notin N$ where N is a ν-negligible set , we have

$$w(t) = -\frac{du}{d\nu}(t) - z(t)\frac{d\lambda}{d\nu}(t) \in \bigcap_n \overline{co}\{\hat{w}_k(t) : k \geq n\}.$$

It follows that for $t \notin N$ and for any $x^* \in E^*$,

$$\langle x^*, w\rangle \leq \inf_n \sup_{k\geq n} \langle x^*, \hat{w}_k\rangle.$$

So, by the above fact we obtain

$$\langle x^*, w(t)\rangle \leq \limsup \delta^*(x^*, A(\theta_n(t), v_n(\theta_n(t)) \leq \delta^*(x^*, A(t, u(t)),$$

because $\theta_n(t) \downarrow t$, $v_n(\theta_n(t) \in D(A(\theta_n(t))$ and $v_n(\theta_n(t) \to u(t)$ with $u(t) \in D(A(t))$. As consequence by ([13], Prop. III.35) we obtain

$$w(t) = -\frac{du}{d\nu}(.) - z(.)\frac{d\lambda}{d\nu}(.) \in A(t, u(t)).$$

It remains to check that $z(t) \in F(t, u(t)), \lambda$ a.e. However, this fact is clearly true thanks to the property F. Indeed from $z_n(t) \in F(t, v_n(\varphi_n(t))$ we have for any $x^* \in E^*$,

$$\langle x^*, z_n(t)\rangle \leq \delta^*(x^*, F(t, v_n(\varphi_n(t))).$$

Then by integrating on any λ-measurable set $Q \subset [0, T]$ and by noting that z_n, $t \mapsto \delta^*(x^*, F(t, v_n(\varphi_n(t))$ and $t \to \delta^*(x^*, F(t, u(t))$ are Borel, we obtain

$$\int_Q \langle x^*, z_n(t)\rangle dt \leq \int_Q \delta^*(x^*, F(t, v_n(\varphi_n(t))dt.$$

Passing to the limit yields

$$\int_Q \langle x^*, z(t)\rangle dt \leq \limsup_n \int_Q \delta^*(x^*, F(t, v_n(\varphi_n(t)))dt$$

$$\leq \int_Q \limsup_n \delta^*(x^*, F(t, v_n(\varphi_n(t)))dt \leq \int_Q \delta^*(x^*, F(t, u(t)))dt,$$

so that

$$\langle x^*, z(t)\rangle \leq \delta^*(x^*, F(t, u(t))\ \lambda\ a.e\ .$$

Since E is separable, and $t \mapsto F(t, u(t))$ is measurable, by ([13], Proposition III-35), we conclude that $z(t) \in F(t, u(t))$, λ *a.e.* $\square$

We provide some direct corollaries of Theorem 1.

Corollary 1. *Assume that E separable Banach space. Let $t \mapsto A(t) : D(A(t)) \to ccwl(E)$ be a time-dependent m-accretive operator satisfying $(\mathcal{H}_1^A)$, $(\mathcal{H}_2^A)$, $(\mathcal{H}_3^A)$, $(\mathcal{H}_4^A)$. Let $f : [0, T] \times E \to E$ such that*

(i) $f(., x)$ *is $\mathcal{L}([0, T])$-measurable on $[0, T]$ for all $x \in E$,*
(ii) $f(t, x)$ *is continuous on E for all $t \in [0, T]$,*
(iii) $||f(t, x)|| \leq M(1 + ||x||)$ *for all $(t, x) \in [0, T] \times E$.*

Let $\nu = dr + \lambda$ and let $\frac{d\lambda}{d\nu}(.)$ be the density of λ with respect to the measure ν. Then for all $u_0 \in D(A(0))$ the evolution problem

$$-Du(t) \in A(t, u(t)) + f(t, u(t))$$

admits at least a BVRC solution u with $u(0) = u_0$, i.e., there exists a BVRC function $u : [0, T] \to E$ such that

$$\begin{cases} u(0) = u_0 \in D(A(0) \\ u(t) \in D(A(t)), \forall t \in [0, T] \\ \frac{du}{d\nu}(t) \in L_E^\infty([0, T], \nu) \\ -\frac{du}{d\nu}(t) \in A(t, u(t)) + f(t, u(t))\frac{d\lambda}{d\nu}(t),\ \nu\ a.e. \end{cases}$$

Corollary 2. *Assume that E is a separable Hilbert space. Let $t \mapsto A(t) : D(A(t)) \to ccwl(E)$ be a time-dependent m-accretive operator satisfying $(\mathcal{H}_1^A)$, $(\mathcal{H}_2^A)$, $(\mathcal{H}_3^A)$, $(\mathcal{H}_4^A)$. Let $f : [0, T] \times E \to E$ such that*

(i) $f(., x)$ *is $\mathcal{L}([0, T])$-measurable on $[0, T]$ for all $x \in E$,*
(ii) $||f(t, x) - f(t, y)|| \leq M||x - y||$ *for all $t, x, y \in [0, T] \times E \times E$,*
(iii) $||f(t, x)|| \leq M(1 + ||x||)$ *for all $(t, x) \in [0, T] \times E$,*

for some constant $M > 0$. Let $\nu = dr + \lambda$ and let $\frac{d\lambda}{d\nu}(.)$ be the density of λ relatively to the measure ν. Assume further that there is $\beta \in]0, 1[$ such that $\forall t \in [0, T]$, $0 \leq 2M\frac{d\lambda}{d\nu}(t)d\nu(\{t\}) \leq \beta < 1$. Then for all $u_0 \in D(A(0))$ the evolution problem

$$-Du(t) \in A(t, u(t)) + f(t, u(t))$$

admits a unique BVRC solution u with $u(0) = u_0$, i.e., there exists a unique BVRC function u: $[0, T] \to E$ such that

$$\begin{cases} u(0) = u_0 \in D(A(0)) \\ u(t) \in D(A(t)), \forall t \in [0, T] \\ \frac{du}{d\nu}(t) \in L^\infty_E([0, T], d\nu) \\ -\frac{du}{d\nu}(t) \in A(t, u(t)) + f(t, u(t))\frac{d\lambda}{d\nu}(t),\ \nu \text{ a.e.} \end{cases}$$

Proof. We need only to prove the uniqueness, suppose that there are two BVRC solutions u and v,

$$-\frac{du}{d\nu}(t) - f(t, u(t))\frac{d\lambda}{d\nu}(t) \in A(t, u(t)),$$
$$-\frac{dv}{d\nu}(t) - f(t, v(t))\frac{d\lambda}{d\nu}(t) \in A(t, v(t)).$$

By the monotonicity of $A(t)$ we obtain

$$\left\langle \frac{dv}{d\nu}(t) - \frac{du}{d\nu}(t) + \frac{d\lambda}{d\nu}(t)f(t, v(t)) - \frac{d\lambda}{d\nu}(t)f(t, u(t)), v(t) - u(t) \right\rangle \leq 0$$

by the Lipschitz condition on $f(t, \cdot)$,

$$\left\langle \frac{dv}{d\nu}(t) - \frac{du}{d\nu}(t), v(t) - u(t) \right\rangle \leq \left\langle \frac{d\lambda}{d\nu}(t)f(t, u(t)) - \frac{d\lambda}{d\nu}(t)f(t, v(t)), v(t) - u(t) \right\rangle$$
$$\leq M\frac{d\lambda}{d\nu}(t)\|v(t) - u(t)\|^2.$$

Then, u and v are of bounded variation and right continuous and have the density $\frac{du}{d\nu}$ and $\frac{dv}{d\nu}$ relatively to $d\nu$, by a result of Moreau concerning the differential measure [8], $||v - u||^2$ is BVRC and we have

$$d||v - u||^2 \leq 2\langle v(.) - u(.), \frac{dv}{d\nu}(.) - \frac{du}{d\nu}(.)\rangle d\nu$$

so that by integrating on $]0, t]$ and using the above estimate we obtain

$$||v(t) - u(t)||^2 = \int_{]0,t]} d||u - v||^2 d\nu(t) \leq \int_{]0,t]} 2\langle v(.) - u(.), \frac{dv}{d\nu}(.) - \frac{du}{d\nu}(.)\rangle d\nu(t)$$
$$\leq \int_{]0,t]} 2M\frac{d\lambda}{d\nu}(t)\|v(t) - u(t)\|^2 d\nu(t).$$

According to the assumption $0 \leq 2M\frac{d\lambda}{d\nu}(t)d\nu(\{t\}) \leq \beta < 1$ and using Grownwall's Lemma (Lemma 2), we deduce from the last inequality that $u = v$ in $[0, T]$. This completes the proof. □

A concrete application is given by the convex sweeping process in a separable Hilbert space E. If $A(t, x) = N_{C(t)}x$, where C: $[0, T] \to E$ is a closed convex valued mapping and $N_{C(t)}x$ is the normal cone of $C(t)$ at the point $x \in C(t)$, one deduce the existence of BVRC solutions of a closed convex and nonconvex sweeping process. See e.g., [15]. For more information on the existence BVRC of solutions to the sweeping process we refer to [2,7,16]. The above results shed new light on the problem of the existence of BVRC solutions for a class of time-dependent m-accretive operators with convex weakly compact perturbation. At this point, compare with some related results in the literature [17–19] dealing with mild solutions for evolution inclusion driven by fixed m-accretive operator A with convex compact perturbation. Here our result is strong and new. Further applications will be provided.

3.2. Existence Results of Absolutely Continuous Solutions

We begin this section by recalling at first an important result in ([20], Theorem 4.6) dealing with the existence of AC solution for problem (1).

Theorem 2. *Assume that E is a separable reflexive uniformly convex space along with the dual E^*. Let A: $D(A) \to 2^E$ is an m-accretive operator satisfying*
$(\mathcal{H}_1)$ $||A^0x|| \le c(1+||x||)$ for all $x \in D(A)$ where c is a positive constant,
$(\mathcal{H}_2)$ $D(A)$ closed and for each $k > 0$, the set $\{x \in D(A) : ||x|| \le k\}$ is compact.
Let $f : [0,T] \times [0,T] \times E \to E$ satisfying to the conditions
$(\mathcal{H}_3)(t,s) \to f(t,s,x)$ is Lebesgue measurable on $[0,T] \times [0,T]$, $\forall(t,s,x) \in [0,T] \times [0,T] \times E$,
$(\mathcal{H}_4)$ $\|f(t,s,x) - f(t,s,y) \le M||x-y||$, $\forall t,s \in [0,T], \forall x,y \in E$,
$(\mathcal{H}_5)||f(t,s,x)||| \le M(1+||x||)$, $\forall t,s,x \in [0,T] \times [0,T] \times E$,
where M is positive constant.
Then, for every $u_0 \in D(A)$, there exists a unique $\mathcal{W}_E^{1,\infty}([0,T])$-mapping $u : [0,T] \to E$ such that

$$\begin{cases} u(0) = u_0, \\ u(t) \in D(A), \forall t \in [0,T], \\ -\dot{u}(t) \in Au(t) + \int_0^t f(t,\tau,u(\tau))d\tau \, a.e. \end{cases}$$

Lemma 4. *Assume that E and E^* are uniformly convex reflexive separable and $A(t) : D(A(t)) \to cc(E)$ is a time-dependent m-accretive operator satisfying:*
$(\mathcal{A}_1)$: $t \to A_\lambda(t,x)$ is $\mathcal{L}([0,T])$-measurable for all $\lambda > 0$ and for all $x \in E$,
$(\mathcal{A}_2)$: $|A_\lambda(t,x)| \le |A^0(t,x)| \le c(1+||x||)$ for all $\lambda > 0$ and for all fixed $x \in D(A(t))$ where c is a positive constant.
Then the operator $\mathcal{A} : D(\mathcal{A}) \subset L_E^2([0,T],dt) \to 2^{L_E^2([0,T],dt)}$,

$$\mathcal{A}u = \{v \in L_E^2([0,T],dt) : v(t) \in A(t,u(t)), \ a.e \quad t \in [0,T]\},$$

for each $u \in D(\mathcal{A})$ where $D(\mathcal{A})$ is defined by
$D(\mathcal{A}) := \{u \in L_E^2([0,T],dt) : u(t) \in D(A(t))$ a.e. $t \in [0,T]$: exist *$v \in L_E^2([0,T],dt)$: $v(t) \in A(t,u(t))$, a.e. $t \in [0,T]\}$ is m-accretive. As consequence, the graph of $\mathcal{A}$ is strongly weakly closed.*

Proof. It is easy to see that $\mathcal{A}$ is accretive in $L_E^2([0,T],dt)$, namely

$$||f-\overline{f}||_{L_E^2([0,T],dt)} \le ||(f-\overline{f}) + \lambda(g-\overline{g})||_{L_E^2([0,T],dt)}, \forall g \in \mathcal{A}(f), \overline{g} \in \mathcal{A}(\overline{f}).$$

We need to check that $R(I_{L_E^2([0,T],dt)} + \lambda\mathcal{A}) = L_E^2([0,T],dt)$ for each $\lambda > 0$. Let $g \in L_E^2([0,T],dt)$. Then $t \mapsto v_\lambda(t) = [I_E + \lambda A(t)]^{-1}g(t) = J_\lambda(t,g(t)) = g(t) - \lambda A_\lambda(t,g(t))$. We note that $A_\lambda(t,.)$ is $\frac{2}{\lambda}$-Lipschitz map in E and $A_\lambda(.,x)$ is $\mathcal{L}([0,T])$-measurable for all $\lambda > 0$ and for all $x \in E$. Set $h(t) = A_\lambda(t,g(t)) = (A_\lambda(t,g(t)) - A_\lambda(t,u(t)) + A_\lambda(t,u(t))$. Then h is measurable with $||h(t)|| \le \frac{2}{\lambda}||g(t) - u(t)|| + ||A_\lambda(t,u(t)||$ and so we deduce that $h \in L_E^2([0,T],dt)$ because u and g belong to $L_E^2([0,T],dt)$ and $t \mapsto A_\lambda(t,u(t))$ is $\mathcal{L}([0,T])$-measurable and belongs to $L_E^2([0,T],dt)$ because $||A_\lambda(t,u(t))|| \le ||A^0(t,u(t))|| \le c(1+||u(t)||)$ for all $t \in [0,T]$. This proves that $v_\lambda \in L_E^2([0,T],dt)$ and $g(t) \in v_\lambda(t) + \lambda A v_\lambda(t))$ so that $R(I_{L_E^2([0,T],dt)} + \lambda\mathcal{A}) = L_E^2([0,T],dt)$. So, we conclude that the m-accretiveness of $\mathcal{A}$. □

Remark 1. *This lemma has some importance in further application. If $A : D(A) \to cc(E)$ is a fixed m-accretive operator, the result is obvious. See e.g., [12].*

Our second result on the existence of AC solution to a perturbed evolution problem (3) with time-dependent m-accretive operator is stated as follows.

Theorem 3. *Let E be a separable Banach space. Let $t \mapsto A(t) : D(A(t)) \to ccwl(E)$ be a time-dependent m-accretive operator satisfying*
$(\mathcal{H}_1^A)$ *there exists a nonnegative real number c such that*

$$||A^0(t,x)|| \leq c(1 + \|x\|) \ \textit{for} \ \ t \in [0,T], \ x \in D(A(t)),$$

$(\mathcal{H}_2^A)$ *$t \mapsto D(A(t))$ has closed graph, $Gr(D(A))$, and $\bigcup_{t\in[0,T]} D(A(t))$ is ball compact,*
$(\mathcal{H}_3^A)$ *$(t,x) \mapsto A(t,x) : Gr(D(A)) \to ccwl(E)$ is scalar upper semicontinuous: for $t_n \downarrow t$, for $x_n \to x$ with $x_n \in D(A(t_n))$ and $x \in D(A(t))$,*

$$\forall x^* \in E^*, \limsup_n \delta^*(x^*, A(t_n, x_n)) \leq \delta^*(x^*, A(t,x)),$$

$(\mathcal{H}_4^A)'$ *There exists a nondecreasing and absolutely continuous function $\beta : [0,T] \to [0,\infty[$ with $\dot\beta \in L^2$, such that for $t < \tau \subset [0,T]$, for $\lambda > 0$ and $x \in D(A(t))$,*

$$\|x - J_\lambda^{A(\tau)}(x)\| \leq (\beta(\tau) - \beta(t))(1 + ||A^0(t,x)||),$$

$(\mathcal{H}^F)$ *Let $F : [0,T] \times E \to ck(E)$ be a convex compact-valued mapping such that*

(i) *F is scalarly $\mathcal{L}([0,T]) \otimes \mathcal{B}(E)$-measurable, i.e., for each $x^* \in E^*$, the scalar function $\delta^*(x^*, F(.,.))$ is $\mathcal{L}([0,T]) \otimes \mathcal{B}(E)$-measurable,*
(ii) *for each $t \in [0,T]$, $F(t,.)$ is scalarly upper semicontinuous, i.e., for each $x^* \in E^*$, the scalar function $\delta^*(x^*, F(t,.))$ is upper semicontinuous on E,*
(iii) *$F(t,x) \subset M(1 + ||x||)\overline{B}_E$ for all $(t,x) \in [0,T] \times E$ for some positive constant M.*

Then for all $u_0 \in D(A(0))$ the evolution problem

$$-Du(t) \in A(t, u(t)) + F(t, u(t))$$

admits an absolutely continuous solution u with $u(0) = u_0$, that is, there exists an absolutely continuous mapping $u : [0,T] \to E$ such that

$$\begin{cases} u(0) = u_0 \in D(A(0)) \\ u(t) \in D(A(t)), \forall t \in [0,T] \\ \frac{du}{dt}(t) \in L^2_E([0,T], dt) \\ z(t) \in F(t, u(t)), \ dt \ a.e \\ -\frac{du}{dt}(t) \in A(t, u(t)) + z(t), \ dt \, a.e., \ t \in [0,T]. \end{cases}$$

Proof. We will use the ideas and techniques of Theorem 1 above and Theorem 1 in [21]. For the sake of completeness, we give here the proof. Let for each $(t,x) \in [0,T] \times E$, $f(t,x)$ the element of minimal norm of $F(t,x)$, i.e., $f(t,x) = P_{F(t,x)}(0)$. For each $x \in E$, the map $t \mapsto f(t,x)$ is $\mathcal{L}([0,T])$-measurable by virtue of Theorem III-41(2) [13], and by $(\mathcal{H}_4^F)$

$$||f(t,x)|| \leq M(1 + ||x||), \ \forall (t,x) \in [0,T] \times E. \tag{17}$$

We choose any sequence $(\varepsilon_n)_n \subset I$ which decreases to 0 as $n \to \infty$ and any sequence of partition $0 = t_0^n < t_1^n < \ldots < t_k^n = T$ of $I = [0,T]$ such that $(t_{i+1}^n - t_i^n) + \int_{t_i^n}^{t_{i+1}^n} \dot\beta(\tau) d\tau < \varepsilon_n$ for $i = 0, \ldots, k_n - 1$, which is allowed since $t \to \beta(t) + t$ is absolutely continuous. Without loss of generality, we may assume that $\beta(0) = 0$. Let for $i = 0, \ldots, k_n - 1$, $\delta_{i+1}^n = (t_{i+1}^n - t_i^n)$, $\beta_{i+1}^n = \int_{t_i^n}^{t_{i+1}^n} \dot\beta(\tau) d\tau$ so that $\beta(t_{i+1}^n) - \beta(t_i^n) = \beta_{i+1}^n$, $\eta_{i+1}^n := \delta_{i+1}^n + \beta_{i+1}^n \leq \varepsilon_n$. We define the mapping $v_n : I \to E$ by

$$v_n(t) = u_i^n + \frac{t - t_i^n}{t_{i+1}^n - t_i^n}(u_{i+1}^n - u_i^n + \int_{t_i^n}^{t_{i+1}^n} f(\tau, u_i^n) d\tau) - \int_{t_i^n}^{t} f(\tau, u_i^n) d\tau,$$

for $t \in [t_i^n, t_{i+1}^n]$ where for $i = 0, 1, \ldots, k_n - 1$,

$$u_{i+1}^n = J_{\delta_{i+1}^n}^{A(t_{i+1}^n)}(u_i^n - \int_{t_i^n}^{t_{i+1}^n} f(\tau, u_i^n)d\tau).$$

By construction

$$u_{i+1}^n \in D(A(t_{i+1}^n),$$

$$-\frac{1}{\delta_{i+1}^n}(u_{i+1}^n - u_i^n + \int_{t_i^n}^{t_{i+1}^n} f(\tau, u_i^n)d\tau) \in A(t_{i+1}^n, u_{i+1}^n).$$

Let us define $\theta_n(t) = t_{i+1}^n, \varphi_n(t) = t_i^n$ for $t \in [t_i^n, t_{i+1}^n], i = 0, 1, \ldots, k_n - 1$ and $\theta_n(0) = \varphi_n(0)$. so that

$$-\dot{v}_n(t) \in A(\theta_n(t), v_n(\theta_n(t)) + f(\tau, v_n(\varphi_n(\tau))$$

a.e. with $|\theta_n(t) - t| \to 0$ and $|\varphi_n(t) - t| \to 0$ as $n \to \infty$.

Step 1. Estimates and convergence.

$$||u_{i+1}^n - u_i^n|| = ||J_{\delta_{i+1}^n}^{A(t_{i+1}^n)}(u_i^n - \int_{t_i^n}^{t_{i+1}^n} f(\tau, u_i^n)d\tau) - u_i^n||$$

$$\le ||J_{\delta_{i+1}^n}^{A(t_{i+1}^n)}(u_i^n - \int_{t_i^n}^{t_{i+1}^n} f(\tau, u_i^n)d\tau) - J_{\delta_{i+1}^n}^{A(t_{i+1}^n)}(u_i^n)||$$

$$+||J_{\delta_{i+1}^n}^{A(t_{i+1}^n)}(u_i^n) - u_i^n||$$

$$\le M(1 + ||u_i^n||)\delta_{i+1}^n + \beta_{i+1}^n(1 + |A^0(t_i^n, u_i^n)|)$$

$$\le M(1 + ||u_i^n||)\delta_{i+1}^n + \beta_{i+1}^n(1 + c(1 + ||u_i^n||))$$

$$\le [(M + c)||u_i^n|| + M + c + 1]\eta_{i+1}^n.$$

Set $M_1 = M + c, M_2 = M + c + 1$, it comes that

$$||u_{i+1}^n|| \le (1 + M_1\eta_{i+1}^n)||u_i^n|| + M_2\eta_{i+1}^n.$$

Then by Lemma 1 we obtain

$$||u_i^n|| \le (||u_0|| + M_2(T + \beta(T)))exp.M_1(T + \beta(T)) := c_1$$

and

$$||u_{i+1}^n - u_i^n|| \le (M_1c_1 + M_2)\eta_{i+1}^n := c_2\eta_{i+1}^n.$$

For all $t \in [t_i^n, t_{i+1}^n]$ we deduce that

$$||v_n(t)|| \le ||u_{i+1}^n|| + 2||u_i^n|| + 2\int_{t_i^n}^{t_{i+1}^n} f(\tau, u_i^n)d\tau$$

$$\le 3c_1 + 2M(1 + c_1)(t_{i+1}^n - t_i^n) \le 3c_1 + 2M(1 + c_1)T =: l_1$$

with

$$||v_n(t_{i+1}^n) - v_n(t_i^n) = ||u_{i+1}^n - u_i^n|| \le c_2\eta_{i+1}^n.$$

Set $K = \max(l_1, c_2)$ we obtain

$$\sup_{n\in\mathbf{N}} ||v_n|| \le K, \ \sup_{n\in\mathbf{N}} var(v_n) = \sup_{n\in\mathbf{N}} \sum_{i=0}^{k_n-1} ||u_{i+1}^n - u_i^n|| \le K(T + \beta(T)).$$

Estimate of $\dot{v}_n$. For all $t \in]t_i^n, t_{i+1}^n[$ we have

$$||\dot{v}_n(t)|| \leq \frac{||u_{i+1}^n - u_i^n||}{t_{i+1}^n - t_i^n} + 2M(1 + c_1).$$

Set for all $t \in I$, $\gamma(t) = c_2(1 + \beta(t))$ so that

$$||u_{i+1}^n - u_i^n|| \leq \int_{t_i^n}^{t_{i+1}^n} \gamma(\tau) d\tau \leq (t_{i+1}^n - t_i^n)^{\frac{1}{2}} (\int_{t_i^n}^{t_{i+1}^n} \gamma(\tau)^2 d\tau)^{\frac{1}{2}}.$$

Whence

$$||\dot{v}_n||_{L_E^2} = \int_0^T ||\dot{v}_n(\tau)||^2 d\tau = \sum_{i=0}^{k_n-1} \int_{t_i^n}^{t_{i+1}^n} ||\dot{v}_n(\tau)||^2 d\tau$$

$$\leq 2 \sum_{i=0}^{k_n-1} (\int_{t_i^n}^{t_{i+1}^n} ([\frac{||u_{i+1}^n - u_i^n||}{t_{i+1}^n - t_i^n}]^2 + \big[2M(1 + c_1)\big]^2) d\tau$$

$$= 2 \sum_{i=0}^{k_n-1} ([\frac{||u_{i+1}^n - u_i^n||}{t_{i+1}^n - t_i^n}]^2 + \big[2M(1 + c_1)\big]^2)(t_{i+1}^n - t_i^n)$$

$$\leq 2 \sum_{i=0}^{k_n-1} (\int_{t_i^n}^{t_{i+1}^n} \gamma(\tau)^2 d\tau + \big[2M(1 + c_1)\big]^2)(t_{i+1}^n - t_i^n))$$

$$\leq 2||\gamma||_{L^2}^2 + 2T\big[2M(1 + c_1)\big]^2 = c_3.$$

As consequence

$$||v_n(t) - v_n(s)|| \leq \int_s^t |\dot{v}_n(\tau) d\tau \leq c_3(t - s)|^{\frac{1}{2}}, \ \forall (t, s \in [0, T].$$

Since $v_n(\theta_n(t)) = u_n(\theta_n(t))$ and $\bigcup_{t \in [0,T]} D(A(t))$ is ball compact, $\{v_n(\theta_n(t)\}$ is relatively compact and so is $\{v_n(t)\}$. By Ascoli theorem, $\{v_n\}$ converges uniformly to an absolutely continuous mapping v and we may assure that $\{\dot{v}_n\}$ weakly converge in $L_E^2([0, T), dt)$ to $\dot{v}$. With our notations, recall that

$$-\frac{dv_n}{dt}(t) - f(t, v_n(\varphi_n(t)) \in A(\theta_n(t), v_n(\theta_n(t)).$$

Now, we use the Mazur's trick and $(\mathcal{H}_3^A)$ to finish the proof being ensured that $v(t) \in D(A(t))$ using the fact that $v_n(\theta_n(t)) \in D(A(\theta_n(t))$, $\theta_n(t) \downarrow t$ and $v_n(\theta_n(t)) \to v(t)$. Let $\zeta_n(t) := f(t, v_n(\varphi_n(t))$ and $\zeta(t) := f(t, v(t))$. As $v_n(\varphi_n(t)) \to v(t)$, $\zeta_n(t) \to \zeta(t)$, so $\zeta_n \to \zeta$ weakly in $L_E^2([0, T], dt)$. Hence $\frac{dv_n}{dt}(.) + \zeta_n(.)$ weakly converges in $L_E^2([0, T], dt)$ to $\frac{du}{dt}(.) + z(.)$. For convenient notation let

$$w_n(t) = -\frac{dv_n}{dt}(t) - \zeta_n(t),$$

$$w(t) = -\frac{dv}{dt}(t) - \zeta(t).$$

Then $\{w_n\}$ weakly converges in $L_E^2([0, T], dt)$ to w. We will show that

$$w(t) = -\frac{dv}{dt}(t) - \zeta(t) \in A(t, v(t)), \ dt \ a.e.$$

By applying Mazur' s lemma, there exists a sequence $\{\hat{w}_n(.)\}$ which converges strongly in $L_E^2([0, T], dt)$ to $w = -\frac{dv}{dt}(.) - z(.)$ with

$$\hat{w}_n(t) \in co\{w_k(t) : k \geq n\}.$$

Extracting a subsequence, we may ensure that $\hat{w}_n(t) \to w(t) = -\frac{dv}{dt}(t) - z(t)$ λ a.e. Consequently, for $t \notin N$ where N is a λ-negligible set , we have

$$w(t) = -\frac{dv}{dt}(t) - z(t) \in \bigcap_n \overline{co}\{\hat{w}_k(t) : k \geq n\}.$$

It follows that for $t \notin N$ and for any $x^* \in E^*$,

$$\langle x^*, w\rangle \leq \inf_n \sup_{k\geq n} \langle x^*, \hat{w}_k\rangle.$$

So, by the above fact we obtain

$$\langle x^*, w(t)\rangle \leq \limsup_n \delta^*(x^*, A(\theta_n(t), v_n(\theta_n(t)) \leq \delta^*(x^*, A(t, v(t)),$$

because $\theta_n(t) \downarrow t$, $v_n(\theta_n(t) \in D(A(\theta_n(t))$ and $v_n(\theta_n(t) \to v(t)$ with $v(t) \in D(A(t))$. As consequence by ([13], Prop. III.35) we obtain

$$w(t) = -\frac{dv}{dt}(.) - z(.) \in A(t, v(t)).$$

It remains to check that $z(t) \in F(t, u(t)), \lambda$ a.e. However, this fact is clearly true thanks to the property of F. Indeed from $z_n(t) \in F(t, v_n(\varphi_n(t))$ we have for any $x^* \in E^*$,

$$\langle x^*, z_n(t)\rangle \leq \delta^*(x^*, F(t, v_n(\varphi_n(t)).$$

Then by integrating on any λ-measurable set $Q \subset [0, T]$ and by noting that z_n, $t \mapsto \delta^*(x^*, F(t, v_n(\varphi_n(t)))$ and $t \to \delta^*(x^*, F(t, v(t)))$ are Borel, we obtain

$$\int_Q \langle x^*, z_n(t)\rangle dt \leq \int_Q \delta^*(x^*, F(t, v_n(\varphi_n(t)))dt.$$

Passing to the limit yields

$$\int_Q \langle x^*, z(t)\rangle dt \leq \limsup_n \int_Q \delta^*(x^*, F(t, v_n(\varphi_n(t)))dt$$

$$\leq \int_Q \limsup_n \delta^*(x^*, F(t, v_n(\varphi_n(t))dt \leq \int_Q \delta^*(x^*, F(t, u(t)))dt$$

so that

$$\langle x^*, z(t)\rangle \leq \delta^*(x^*, F(t, u(t)))\ \lambda\ a.e.$$

Since E is separable, and $t \mapsto F(t, v(t))$ is measurable, by ([13], Proposition III-35), we conclude that $z(t) \in F(t, u(t)),\ dt\ a.e.$ $\square$

4. Applications

Our first application uses the results of Theorem 1 on the existence of BVRC solutions in the framework of time-dependent m-accretive operators in a separable Hilbert space.

4.1. Second-Order Evolution Inclusion Driven by a Time-Dependent m-Accretive Operator. The BVRC Case

Theorem 4. *Assume that E is a separable Hilbert space. Let $t \mapsto A(t) : D(A(t)) \to cc(E)$ be a time-dependent m-accretive operator satisfying $(\mathcal{H}_1^A)$, $(\mathcal{H}_2^A)$, $(\mathcal{H}_3^A)$, $(\mathcal{H}_4^A)$. Let $f : [0, T] \times E \times E \to E$ be a continuous mapping satisfying*

(i) $||f(t, x, y)|| \leq M(1 + ||x||),\ \forall t, x, y \in [0, T] \times E \times E.$
(ii) $||f(t, x, z) - f(t, y, z)|| \leq M||x - y||,\ \forall t, x, y, z \in [0, T] \times E \times E \times E.$

Let $\nu = dr + \lambda$ and let $\frac{d\lambda}{d\nu}$ the density of λ with respect to the measure $d\nu$. Assume further that there is $\beta \in]0,1[$ such that $\forall t \in I$, $0 \le 2M\frac{d\lambda}{d\nu}(t)d\nu(\{t\}) \le \beta < 1$. Then for $u_0 \in D(A(0)), x_0 \in E$, there are a BVRC mapping $u : [0,T] \to E$ with density $\frac{du}{d\nu}$ relatively to $d\nu$, and an AC mapping $w : [0,T] \to E$ satisfying

$$\begin{cases} w(t) = x_0 + \int_0^t u(s)ds, \ \ t \in [0,T] \\ u(0) = u_0 \in D(A(0)) \\ u(t) \in D(A(t)), \forall t \in [0,T] \\ \frac{du}{d\nu}(t) \in L_E^\infty([0,T], d\nu) \\ -\frac{du}{d\nu}(t) \in A(t, u(t)) + f(t, u(t), w(t))\frac{d\lambda}{d\nu}(t), \ \nu \ a.e. \end{cases}$$

Proof. For any continuous mapping $h : [0,T] \to E$, the mapping $f_h(t,x) := f(t,x,h(t))$ is measurable on $[0,T]$ for any $x \in E$ and satisfies $||f_h(t,x)|| \le M(1+||x||), \ \forall t, x \in [0,T] \times E$ and $||f_h(t,x) - f_h(t,y))|| \le M||x-y||, \ \forall t,x,y \in [0,T] \times E \times E$, so by Theorem 1 and Corollary 2 there is a unique BVRC solution v_h to the inclusion

$$\begin{cases} v_h(0) = u_0 \in D(A(0)) \\ v_h(t) \in D(A(t)), \forall t \in [0,T] \\ -\frac{dv_h}{d\nu}(t) \in A(t, v_h(t)) + f(t, v_h(t), h(t))\frac{d\lambda}{d\nu} \ d\nu\text{-a.e} \end{cases}$$

with $\frac{dv_h}{d\nu} \in K\overline{B}_E$, where K is a positive generic constant so that $||v_h(t)|| \le K\nu([0,T]) := L, \ t \in [0,T]$. Let us consider the closed convex subset $\mathcal{X}$ in the Banach space $\mathcal{C}_E([0,T])$ defined by

$$\mathcal{X} := \{u_f : [0,T] \to E : u_f(t) = u_0 + \int_0^t f(s)ds, \ f \in S^1_{L\overline{B}_E}, \ t \in [0,T]\},$$

where $S^1_{L\overline{B}_E}$ denotes the set of all dt-integrable selections of the convex weakly compact-valued constant multifunction $L\overline{B}_E$. Now for each $h \in \mathcal{X}$ let us consider the mapping

$$\Phi(h)(t) := u_0 + \int_0^t v_h(s)ds, \ t \in [0,T].$$

Then it is clear that $\Phi(h) \in \mathcal{X}$. Further we have $||v_h(t)|| \le L$ for all $t \in [0,T]$ so that the set $\Gamma(t) := \{x \in D(A(t)) : ||x|| \le L\}$ is compact by $(\mathcal{H}_2^A)$ and nonempty because $v_h(t) \in D(A(t))$, as consequence $\Phi(h)(t) \in u_0 + \int_0^t \overline{co}[\Gamma(s)]ds$. Since $s \mapsto \overline{co}[\Gamma(s)]$ is a convex compact-valued and integrably bounded multifunction, the second member is convex compact-valued [22]. Hence $\Phi(\mathcal{X})$ is equicontinuous and relatively compact in the Banach space $\mathcal{C}_E([0,T])$. Now we check that Φ is continuous. It is sufficient to show that if (h_n) converges uniformly to h in $\mathcal{X}$, then the BVRC solution v_{h_n} associated with h_n

$$\begin{cases} v_{h_n}(0) = u_0 \in D(A(0)) \\ v_{h_n}(t) \in D(A(t)), \forall t \in [0,T] \\ -\frac{dv_{h_n}}{d\nu}(t) \in A(t, v_{h_n}(t)) + f(t, v_{h_n}(t), h_n(t))\frac{d\lambda}{d\nu}(t) \ \nu\text{-a.e.} \end{cases}$$

pointwise converges to the BVRC solution v_h associated with h

$$\begin{cases} v_h(0) = u_0 \in D(A(0)) \\ v_h(t) \in D(A(t)), \forall t \in [0,T] \\ -\frac{dv_h}{d\nu}(t) \in A(t, v_h(t)) + f(t, v_h(t), h(t))\frac{d\lambda}{d\nu}(t) \ d\nu\text{-a.e.} \end{cases}$$

As $(v_{h_n}(t))$ is relatively compact, for each $t \in [0,T]$ and (v_{h_n}) is uniformly bounded and bounded in variation since $||v_{h_n}(t) - v_{h_n}(\tau)|| \leq K(\nu(]\tau,t]))$, $\tau \leq t \in [0,T]$, by the Helly principle [14] we may assume that (v_{h_n}) pointwise converges to a BV mapping u. As $v_{h_n} = v_0 + \int_{]0,t]} \frac{dv_{h_n}}{d\nu} d\nu$, $t \in [0,T]$ and $\frac{dv_{h_n}}{dr}(s) \in K\overline{B}_H$, $s \in [0,T]$, we may assume that $(\frac{dv_{h_n}}{d\nu})$ converges weakly in $L^2_E([0,T], d\nu)$ to $w \in L^2_E([0,T], d\nu)$ with $w(t) \in K\overline{B}_E$, $t \in [0,T]$ so that

$$\text{weak-}\lim_n v_{h_n} = u_0 + \int_0^t w(s)d\nu(s) := z(t),\ t \in [0,T].$$

By identifying the limits, we obtain

$$u(t) = z(t) = u_0 + \int_0^t w(s)d\nu(s)$$

with $\frac{du}{d\nu} = w$ so that $\lim_n f(t, v_{h_n}(t), h_n(t)) = f(t, u(t), h(t))$, $t \in [0,T]$. As consequence $(\frac{dv_{h_n}}{dr} + f(., v_{h_n}(.), h_n(.))\frac{d\lambda}{d\nu}(.))$ weakly converges to $\frac{dv}{d\nu} - f(., u(.), h(.))\frac{d\lambda}{d\nu}(.)$ in $L^2_E([0,T], E, d\nu)$. From the inclusion

$$-\frac{dv_{h_n}}{d\nu}(t) - f(t, v_{h_n}(t), h_n(t))\frac{d\lambda}{d\nu}(t)) \in A(t, v_{h_n}(t))\ \nu\ a.e$$

we show, using the m-accretive extension $\mathcal{A}$ in $L^2_E([0,T], d\nu)$ defined by Lemma 4, the inclusion $-\frac{du}{d\nu}(t) - f(t, u(t), h(t))\frac{d\lambda}{d\nu}(t) \in A(t, u(t))\ \nu\ a.e$. Indeed, as $-\dot{v}_n - z_n \in \mathcal{A}(v_{h_n})$ where $-\dot{u}_n - z_n \in L^2_E(0,T], d\nu)$ with $-\dot{u}_n - z_n$ weakly converging in $L^2_E(0,T], dt)$ to $-\dot{u} - z$ and v_{h_n} strongly convergent $L^2_E(0,T], d\nu)$ to u, and the the graph of $\mathcal{A}$ is strongly weakly sequentially closed, we deduce the required inclusion

$$-\frac{du}{d\nu}(t) - f(t, u(t), h(t))\frac{d\lambda}{d\nu}(t) \in A(t, u(t))\ \nu\ a.e$$

with $u(0) = u_0 \in D(A(0))$ and $u(t) \in D(A(t))$ so that by uniqueness $u = v_h$. Now let us check that $\Phi : \mathcal{X} \to \mathcal{X}$ is continuous. Let $h_n \to h$. We have

$$\Phi(h_n)(t) - \Phi(h)(t) = \int_0^t v_{h_n}(s)ds - \int_0^t v_h(s)ds = \int_0^t [v_{h_n}(s) - v_h(s)]ds.$$

As $||v_{h_n}(.) - v_h(.)|| \to 0$ pointwisely and is uniformly bounded : $||v_{h_n}(.) - v_h(.)|| \leq 2L$, we conclude that

$$\sup_{t\in[0,T]} ||\Phi(h_n)(t) - \Phi(h)(t)|| \leq \sup_{t\in[0,T]} \int_0^t ||v_{h_n}(.) - v_h(.)||ds \to 0$$

so that $\Phi(h_n) - \Phi(h) \to 0$ in $\mathcal{C}_H([0,T])$. Since $\Phi : \mathcal{X} \to \mathcal{X}$ is continuous and $\Phi(\mathcal{X})$ is relatively compact in $\mathcal{C}_E([0,T])$, by Schauder theorem has a fixed point, say $h = \Phi(h) \in \mathcal{X}$, which means

$$h(t) = \Phi(h)(t) = u_0 + \int_0^t v_h(s)ds,\ t \in [0,T]$$

$$\begin{cases} v_h(0) = u_0 \in D(A(0)) \\ -\frac{dv_h}{d\nu}(t) \in A(t, v_h(t)) + f(t, v_h(t), h(t))\ \nu\text{-a.e} \end{cases}$$

□

4.2. Second-Order Evolution Inclusion Driven by m-Accretive Operator. The AC Case

In the same spirit we present a new second-order evolution involving an integro-differential Volterra equation with an m-accretive operator in a reflexive separable uniformly convex space.

Theorem 5. *Assume that E is a separable reflexive uniformly convex space along with the dual E^*. Let $A : D(A) \to 2^E$, is an m-accretive operator satisfying (H_1) and (H_2).*

Let $f : [0,T] \times [0,T] \times E \times E \to E$ be a Caratheodory mapping satisfying

(i) $||f(t,s,x,y)|| \leq M(1 + ||x||),\ \forall (t,s,x,y) \in [0,T \times [0,T]] \times E \times E.$

(ii) $||f(t,s,x,z) - f(t,s,y,z)|| \leq M||x-y||,\ \forall (t,s,x,y,z) \in [0,T] \times [0,T] \times E \times E \times E.$

Then for $u_0 \in D(A), x_0 \in E$, there is an AC mapping u: $[0,T] \to E$, and an AC mapping w: $[0,T] \to H$ satisfying

$$\begin{cases} w(t) = x_0 + \int_0^t u(s)ds, \quad t \in [0,T] \\ u(0) = u_0 \in D(A) \\ u(t) \in D(A), \forall t \in [0,T] \\ \frac{du}{dt}(t) \in L_E^\infty([0,T], dt) \\ -\frac{du}{dt}(t) \in Au(t) + \int_0^t f(t,s,u(s),w(s))ds,\ dt \text{ a.e.} \end{cases}$$

Proof. For any continuous h: $[0,T] \to E$, the mapping $f_h(t,s,x) := f(t,s,x,h(t))$ is measurable on $[0,T] \times [0,T]$, $\forall x \in E$, continuous on E, $\forall (t,s) \in [0,T] \times [0,T]$ and satisfies $||f_h(t,s,x)|| \leq M(1 + ||x||)$, $\forall (t,s,x) \in [0,T] \times [0,T] \times E$ and $||f_h(t,s,x) - f_h(t,s,y))|| \leq M||x-y||$, $\forall (t,s,x,y) \in [0,T] \times [0,T] \times E \times E$, so by Theorem 2 or ([20], Theorem 4.6) there is a unique AC solution v_h to the inclusion

$$\begin{cases} v_h(0) = u_0 \in D(A) \\ v_h(t) \in D(A), \forall t \in [0,T] \\ -\frac{dv_h}{dt}(t) \in Av_h(t) + \int_0^t f(t,s,v_h(s),h(s))ds\ dt\text{-a.e} \end{cases}$$

with v_h uniformly bounded and equi-absolutely continuous: $\frac{dv_h}{dt} \in K\overline{B}_E$, where K is a positive generic constant so that $||v_h(t)|| \leq L,\ t \in [0,T]$. Let consider the closed convex subset $\mathcal{X}$ in the Banach space $\mathcal{C}_E([0,T])$ defined by

$$\mathcal{X} := \{u_f : [0,T] \to E : u_f(t) = u_0 + \int_0^t g(s)ds,\ g \in S^1_{L\overline{B}_E},\ t \in [0,T]\},$$

where $S^1_{L\overline{B}_E}$ denotes the set of all dt-integrable selections of the convex weakly compact-valued constant multifunction $L\overline{B}_E$. Now for each $h \in \mathcal{X}$ let us consider the mapping

$$\Phi(h)(t) := u_0 + \int_0^t v_h(s)ds,\ t \in [0,T].$$

Then it is clear that $\Phi(h) \in \mathcal{X}$. We have $||v_h(t)|| \leq L$ for all $t \in [0,T]$ so that the set $\Gamma := \{x \in D(A) : ||x|| \leq L\}$ is compact by (H_2) and nonempty because $v_h(t) \in \Gamma$. As consequence for any $h \in \mathcal{X}$ and for any $t \in [0,T]$ the inclusion holds $\Phi(h)(t) \in u_0 + \int_0^t \overline{co}[\Gamma]ds$ and since $\overline{co}[\Gamma]$ is a convex compact-valued and integrably bounded multifunction, the second member is convex compact-valued [22]. Hence $\Phi(\mathcal{X})$ is equicontinuous and relatively compact in the Banach space $\mathcal{C}_E([0,T])$. Now we check that Φ is continuous. It is sufficient to show that if (h_n) converges uniformly to h in $\mathcal{X}$, then the AC solution v_{h_n} associated with h_n

$$\begin{cases} v_{h_n}(0) = u_0 \in D(A) \\ v_{h_n}(t) \in D(A), \forall t \in [0,T] \\ -\frac{dv_{h_n}}{dv}(t) \in Av_{h_n}(t) + \int_0^t f(t,s,v_{h_n}(s),h_n(s)ds\ dt\text{-a.e.} \end{cases}$$

pointwise converges to the AC solution v_h associated with h

$$\begin{cases} v_h(0) = u_0 \in D(A) \\ v_h(t) \in D(A), \forall t \in [0,T] \\ -\dfrac{dv_h}{dv}(t) \in Av_h(t) + \displaystyle\int_0^t f(t,s,v_h(s),h(s))ds\ dt\text{-a.e.} \end{cases}$$

As $(v_{h_n}(t))$ is relatively compact and v_{h_n} is equi-absolutely continuous, we may assume that (v_{h_n}) converge uniformly to a continuous mapping u. As $v_{h_n} = v_0 + \int_{]0,t]} \frac{dv_{h_n}}{dt} dt,\ t \in [0,T]$ and $\frac{dv_{h_n}}{dt}(s) \in K\overline{B}_E,\ s \in [0,T]$, we may assume that $(\frac{dv_{h_n}}{dt})$ converges weakly in $L^2_E([0,T],dt)$ to $w \in L^2_E([0,T],dt)$ with $w(t) \in K\overline{B}_E,\ t \in [0,T]$ so that

$$\text{weak-}\lim_n v_{h_n} = u_0 + \int_0^t w(s)dv(s) := z(t),\ t \in [0,T].$$

By identifying the limits, we obtain

$$u(t) = z(t) = u_0 + \int_0^t w(s)d(s),$$

with $\frac{du}{dt} = w$ so that $\lim_n \int_0^t f(t,s,v_{h_n}(s),h_n(s))ds = \int_0^t f(t,s,u(s),h(s))ds,\ t \in [0,T]$. As consequence $z_n(t) := \int_0^t f(t,s,v_{h_n}(s),h_n(s))ds \to z(t) := \int_0^t f(t,s,u(s),h(s))ds$ weakly in $L^2_E([0,T],dt)$. From the inclusion

$$-\frac{dv_{h_n}}{dt}(t) - z_n(t) \in Av_{h_n}(t)\ dt\ a.e\,,$$

we show, using the m-accretive $\mathcal{A}$ in $L^2_E([0,T],dt)$ defined by Lemma 4 (or [12], Lemma 1.4.2)), the inclusion

$$-\frac{du}{dt}(t) - \int_0^t f(t,s,u(s),h(s))ds \in Au(t)\ dt\ a.e$$

with $u(0) = u_0 \in D(A)$ and $u(t) \in D(A)$ so that by uniqueness $u = v_h$. Now let us check that Φ: $\mathcal{X} \to \mathcal{X}$ is continuous. Let $h_n \to h$. We have

$$\Phi(h_n)(t) - \Phi(h)(t) = \int_0^t v_{h_n}(s)dv(s) - \int_0^t v_h(s)ds = \int_0^t [v_{h_n}(s) - v_h(s)]ds.$$

As $||v_{h_n}(.) - v_h(.)|| \to 0$ pointwise and is uniformly bounded: $||v_{h_n}(.) - v_h(.)|| \le 2L$, we conclude that

$$\sup_{t\in[0,T]} ||\Phi(h_n)(t) - \Phi(h)(t)|| \le \int_0^T ||v_{h_n}(.) - v_h(.)||ds \to 0.$$

So that $\Phi(h_n) - \Phi(h) \to 0$ in $\mathcal{C}_H([0,T])$. Since Φ: $\mathcal{X} \to \mathcal{X}$ is continuous and $\Phi(\mathcal{X})$ is relatively compact in $\mathcal{C}_E([0,T])$, by Schauder theorem Φ has a fixed point, say $h = \Phi(h) \in \mathcal{X}$, which means

$$h(t) = \Phi(h)(t) = u_0 + \int_0^t v_h(s)ds,\ t \in [0,T]$$

$$\begin{cases} v_h(0) = u_0 \in D(A) \\ -\dfrac{dv_h}{dt}(t) \in Av_h(t) + \displaystyle\int_0^t f(t,s,v_h(s),h(s))ds\ dt\text{-a.e.} \end{cases}$$

□

4.3. Optimal Control Problem Governed by an Integro-Differential Volterra Accretive Operator

Let E be a separable reflexive uniformly convex space along with the dual E^*. Let us consider a convex weakly compact-valued mapping $C : I \rightrightarrows \overline{B}_E$ with bounded right continuous retraction in the sense, there is a bounded and right continuous function $\rho : I \to \mathbf{R}^+$ such that $d_H(C(t), C(\tau)) \leq \rho(\tau) - \rho(t), \forall t \leq \tau \in I$ and such that its graph is Borel, i.e., $Graph(C) \in \mathcal{B}(I) \otimes \mathcal{B}(E)$. We consider the control sets given by

$$S_C^{BVRC} := \{u : I \to E, u \text{ is BVRC},\ u(t) \in C(t),\ \forall t \in I\},$$

$$S_C^{\infty} := \{u \in L^{\infty}(I, E, d\lambda),\ u(t) \in C(t),\ \ \forall t \in I\}.$$

By J.J. Moreau ([23], Prop.5 d, p. 198) and Valadier [24] these sets are nonempty and $clS_C^{BVRC} = S_C^{\infty}$, where cl denotes the closure with respect to the $\sigma(L_E^{\infty}, L_{E^*}^1)$-topology. Shortly S_C^{BVRC} is dense in S_C^{∞} with respect to this topology. Then we have the following relaxation results in a control problem governed by an integro-differential Volterra m-accretive operator given above.

Theorem 6. *Assume that E and E^* are reflexive separable and uniformly convex spaces. Let $I = [0, T]$ and $A : D(A) \to 2^E$, is an m-accretive operator satisfying (H_1) and (H_2).*

Let $a : I \times I \times E \to \mathbf{R}$ be a Caratheodory mapping

(i) $|a(t, s, x)| \leq M, \forall (t, s, x \in I \times I \times E,$

(ii) $|a(t, s, x) - a(t, s, y)| \leq M||x - y||, \forall t, s \in I \times I, \forall x, y \in E \times E.$

Then the following hold:

(a) *the AC solution set $\mathcal{S}_{\mathcal{V}_C^{\infty}}$ to the inclusion*

$$\begin{cases} u(0) = u_0 \in D(A) \\ u(t) \in D(A), \forall t \in I \\ \frac{du}{dt}(t) \in L_E^{\infty}(I, \lambda) \\ -\frac{du}{dt}(t) \in Au(t) + \int_0^t a(t, s, u(s))h(s)ds,\ \lambda \text{ a.e. } t \in [0, T], h \in \mathcal{V}_C^{\infty} \end{cases}$$

is nonempty and compact in $\mathcal{C}_E(I)$.

(b) *The AC solution set $\mathcal{S}_{\mathcal{V}_C}$ to the inclusion*

$$\begin{cases} u(0) = u_0 \in D(A) \\ u(t) \in D(A), \forall t \in I \\ \frac{du}{dt}(t) \in L_E^{\infty}(I, \lambda) \\ -\frac{du}{dt}(t) \in Au(t) + \int_0^t a(t, s, u(s))h(s)ds,\ \lambda \text{ a.e. } t \in I, h \in \mathcal{V}_C \end{cases}$$

is nonempty and is dense in the compact set $\mathcal{S}_{\mathcal{V}_C^{\infty}}$.

Proof. We first note that for each Borel measurable selection h of C, the function $f_h(t, s, x) := a(t, s, x)h(s)$ satisfies to the conditions: $||f_h(t, s, x)|| \leq M$, $||f_h(t, s, x) - f_h(t, s, y)|| \leq M(||x - y||, \forall (t, s \in I \times I, \forall x, y \in E \times E$, and $(t, s) \to f_h(t, s, x)$ is Lebesgue measurable on $I \times I$, in particular if h is a BVRC selection of C and if $v : I \to E$ is AC , then $(t, s) \mapsto a(t, s, v(s))h(s)$ is Lebesgue measurable and bounded. By Theorem 2 or (Theorem 4.6, [20]) for each Borel measurable selection h of C, there is a unique AC solution v_h to the inclusion

$$\begin{cases} v_h(0) = u_0 \in D(A) \\ v_h(t) \in D(A), \forall t \in I \\ \frac{dv_h}{dt}(t) \in L_E^{\infty}(I, \lambda) \\ -\frac{dv_h}{dt}(t) \in Av_h(t) + \int_0^t a(t, s, v_h(s))h(s)ds,\ \lambda \text{ a.e.} \end{cases}$$

So, the set of solutions are given by: $\mathcal{S}_{\mathcal{V}_C^\infty} = \{v_h : h \in \mathcal{V}_C^\infty\}$ and $\mathcal{S}_{\mathcal{V}_C} = \{v_h : h \in \mathcal{V}_C\}$. Let $\{h_n \in \mathcal{V}_C^\infty\}$ $\sigma(L_E^\infty(I,\lambda), L_{E^*}^1(I,\lambda))$ converging to $h \in \mathcal{V}_C^\infty$. As shown in the proof of Theorem 4.6 in [20], the sequence of AC solution (v_{h_n}) is equi-absolutely continuous with $\{v_{h_n}(t)\}$ relatively compact. Namely

$$(v_{h_n}) \subset \mathcal{X} := \{v : I \to E : v(t) = u_0 + \int_0^t \frac{dv}{ds}(s)ds, t \in I, ||\frac{dv}{ds}(s)|| \leq K\},$$

where K is a positive generic constant which depends only on u_0, A and M. Since $\{v_{h_n}(t)\}$ relatively compact, we may assume that v_{h_n} converges uniformly to an AC mapping $v : I \to E$ with $\frac{dv_{h_n}}{dt} \to \frac{dv}{dt}$ weakly in $L_E^2(I,\lambda)$. Further it is clear that $a(t,s,v_{h_n}(s)) \to a(t,s,v(s))$ pointwise. Let $z_n(t) := \int_0^t a(t,s,v_{h_n}(s))h_n(s)ds, z(t) := \int_0^t a(t,s,v_h(s))h(s)ds$. We assert **the main fact** : $z_n \to z$ weakly in $L_E^2(I,\lambda)$. It is clear that z_n and z are Lebesgue measurable by Fubini–Lebesgue integral and the separability of the space E. A crucial fact is $a(t,.,v_{h_n}(.)) - (a(t,.,v_h(.))$ converge to 0, it converges to 0 uniformly on uniformly integrable sets, alias Mackey converges to 0. As consequence, the assertion follows. Indeed, let $g \in L_E^2(I)$. Then we have by integration

$$\lim_n \int_0^T \langle g(t), z_n(t)\rangle dt = \lim_n \int_0^T \langle g(t), \int_0^t a(t,s,v_{h_n}(s))h_n(s)ds\rangle dt$$

$$= \lim_n \int_0^T [\int_0^t a(t,s,v_{h_n}(s)), \langle g(t), h_n(s)\rangle ds]dt$$

$$= \int_0^T \lim_n [\int_0^t a(t,s,v_{h_n}(s)), \langle g(t), h_n(s)\rangle ds]dt = \int_0^T [\int_0^t a(t,s,v_h(s)), \langle g(t), h(s)\rangle ds]dt$$

$$= \int_0^T \langle g(t), z(t)\rangle dt.$$

From $\frac{dv_{h_n}}{dt} + z_n \to \frac{dv}{dt} + z$ weakly $L_E^2(I,\lambda)$ and the inclusion

$$-\frac{dv_{h_n}}{dt}(t) - \int_0^t a(t,s,v_h(s))h(s)ds \in Av_{h_n}(t) \ \lambda \ a.e\,,$$

we deduce

$$-\frac{dv}{dt}(t) - \int_0^t a(t,s,v_h(s))h(s)ds \in Av(t) \ \lambda \ a.e\,,$$

by repeating the convergence limit involving the accretive argument given in the proof of ([20], Theorem 4.6) via Lemma 4 (or [12], Lemma 1.4.2). By uniqueness we have $v = v_h$. We conclude that the mapping $\phi : h \mapsto v_h$ from the compact metrizable set $\mathcal{V}_C^\infty \subset L_E^\infty(I,\lambda)$ to $\mathcal{C}_E(I)$ is continuous. Hence $\{v_h : h \in \mathcal{V}_C\}$ is compact in $\mathcal{C}_E(I)$, since $\mathcal{V}_C$ is dense in $\mathcal{V}_C^\infty$, the latter $\{v_h : h \in \mathcal{V}_C\}$ is dense in the first $\{v_h : h \in \mathcal{V}_C^\infty\}$. □

Theorem 7. *Assume that E and E^* are reflexive separable and uniformly convex spaces. Let $A : D(A) \to 2^E$, is an m-accretive operator satisfying (H_1) and (H_2).*
Let $\overline{B}_E$ be the closed unit ball in E and let $Ext(\overline{B}_E)$ the set of extreme points of $\overline{B}_E$ and

$$\mathcal{M}_{\overline{B}_E} := \{u \in L_E^\infty([0,T],\lambda),\ u(t) \in \overline{B}_E,\ \ \forall t \in [0,T]\},$$

$$\mathcal{M}_{Ext(\overline{B}_E)} := \{u \in L_E^\infty([0,T],\lambda),\ u(t) \in Ext(\overline{B}_E),\ \ \forall t \in [0,T]\}.$$

Then the following hold:

(a) *the AC solution set* $\mathcal{S}_{\mathcal{M}_{\overline{B}_E}}$ *to the inclusion*

$$\begin{cases} u(0) = u_0 \in D(A) \\ u(t) \in D(A), \forall t \in [0,T] \\ \frac{du}{dt}(t) \in L^\infty_E([0,T],\lambda) \\ -\frac{du}{dt}(t) \in Au(t) + \int_0^t a(t,s,u(s))h(s)ds,\ \lambda\ ,\ \lambda\ a.e.\ t \in [0,T], h \in \mathcal{M}_{\overline{B}_E} \end{cases}$$

is nonempty and compact in $\mathcal{C}(I,E)$.

(b) *The AC solution set* $\mathcal{S}_{\mathcal{M}_{Ext(\overline{B}_E)}}$ *to the inclusion*

$$\begin{cases} u(0) = u_0 \in D(A) \\ u(t) \in D(A), \forall t \in [0,T] \\ \frac{du}{dt}(t) \in L^\infty([0,T],E;\lambda) \\ -\frac{du}{dt}(t) \in Au(t) + \int_0^t a(t,s,u(s))h(s)ds,\ \lambda\ a.e.\ t \in [0,T], h \in \mathcal{M}_{Ext(\overline{B}_E)} \end{cases}$$

is nonempty and is dense in the compact set $\mathcal{S}_{\mathcal{M}_{\overline{B}_E}}$.

Proof. We use the same tool as in the proof of Theorem 5 by noting that $\mathcal{M}_{Ext(\overline{B}_E)}$ is dense in $\mathcal{M}_{\overline{B}_E}$ with respect to the $\sigma(L^\infty_E([0,T],\lambda), L^1_{E^*}([0,T],\lambda))$ by virtue of Ljapunov theorem. □

Theorems 5–7 are new applications of the above results and tools. There is a sharp similarity with the inclusion driven with an m-accretive operator A of the form $m(t) \in \dot{u}(t) + Au(t)$ where $m \in W^{1,1}([0,T],E)$ and E is a reflexive Banach space, see Barbu ([10], Theorem 2.2, p. 131) and the inclusion $f(t) \in \dot{u}(t) + Au(t)$ where f is function of bounded variation, see [25], Corollary 1 of Proposition 6). Taking account of these facts, we develop in this spirit some related results dealing with mild solution. It is well-known that given an m-accretive operator $A : D(A) \rightrightarrows E$, for each $x_0 \in D(A)$ and $f \in L^1_E[0,T],\lambda)$ there exists a unique mild solution to the inclusion $-\frac{du}{dt}(t) \in Au(t) + f(t), u(0) = x_0 \in D(A)$. That is a celebrated result due to Benilan–Crandall–Evans–Kobayashi ([12], Theorem 1.7.4). Further let u_f, u_g be two mild solutions with $u_f(0) = u_g(0) = x_0 \in D(A)$ corresponding to $f, g \in L^1_E[0,T],\lambda)$, then we have the estimation ([12], Theorem 1.7.5)

$$||u_f(t) - u_g(t)|| \le 2\int_0^t \langle u_f(s) - u_g(s), f(s) - g(s)\rangle_+ ds = 2\int_0^t \langle j(u_f(s) - u_g(s)), f(s) - g(s)\rangle ds$$

where $j : E \to E^*$ is the single-valued duality mapping, taking account of the dual space E^* is uniformly convex and reflexive. Let $\mathcal{H}$ be a weakly compact subset in $L^1_E[0,T],\lambda)$. We are concerned with the solution set $\{u_f : f \in \mathcal{H}\}$ and the property $f \to u_f$ from $\mathcal{H}$ to $C_E([0,T]$ related to the above inclusion. For this purpose, we produce a fairly useful lemma.

Lemma 5. *Let E be a reflexive separable space such that its dual is uniformly convex. Let* (u_n), (v_n) *be two sequences in* $C_E([0,T])$ *and* $(f_n), (g_n)$ *two sequences in* $L^1_E([0,T],dt)$. *If* $\lim_n u_n = u, \lim_n v_n = v$ *strongly in* $C_E([0,T])$ *and* $\lim_n f_n = f, \lim_n g_n = g$ *weakly in* $L^1_E([0,T],dt)$, *then*

$$\lim_n \int_0^T \langle u_n(s) - v_n(s), f_n(s) - g_n(s)\rangle_+ ds = \int_0^T \langle u(s) - v(s), f(s) - g(s)\rangle_+ ds.$$

Proof. We have

$$|\int_0^T \langle u_n(s) - v_n(s), f_n(s) - g_n(s)\rangle_+ ds - \int_0^T \langle u(s) - v(s), f(s) - g(s)\rangle_+ ds|$$

$$= |\int_0^T j(u_n(s) - v_n(s)), f_n(s) - g_n(s)\rangle ds - \int_0^T j(u(s) - v(s)), f(s) - g(s)\rangle ds|$$

$$\leq |\int_0^T j(u_n(s)-v_n(s)), f_n(s)-g_n(s)\rangle ds - \int_0^T j(u(s)-v(s)), f_n(s)-g_n(s)\rangle ds|$$

$$+|\int_0^T j(u(s)-v(s)), f_n(s)-g_n(s)\rangle ds - \int_0^T j(u(s)-v(s)), f(s)-g(s)\rangle ds|.$$

As $j(u(.)-v(.)) \in L^\infty_{E^*}([0,T],dt)$, and $f_n - g_n \to 0$ weakly in $L^1_E([0,T],dt)$, it is obvious that $\int_0^T j(u(s)-v(s)), f_n(s)-g_n(s)\rangle ds - \int_0^T j(u(s)-v(s)), f(s)-g(s)\rangle ds \to 0$. As $j(u_n(.)-v_n(.)-j(u(.)-v(.))$ is uniformly bounded and pointwise converges in measure to 0, it converges to 0 uniformly on uniformly integrable sets of $L^1_E([0,T],dt)$, In other terms its converges to 0 with respect to the Mackey topology $\tau(L^\infty_{E^*}([0,T],dt), L^1_E([0,T],dt))$ (If $E = \mathbf{R}^d$, one may invoke a classical fact that on bounded subsets of $L^\infty([0,T],E;dt)$ the topology of convergence in measure coincides with the topology of uniform convergence on uniformly integrable sets, i.e., on relatively weakly compact subsets, alias the Mackey topology. This is a lemma due to Grothendieck [26] [Ch.5 §4 no 1 Prop. 1 and exercise]), therefore

$$\lim_n \int_0^T \langle j(u_n(s)-v_n(s)) - j(u(s)-v(s)), f_n(s)\rangle ds = 0$$

and so is

$$\lim_n \int_0^T \langle j(u_n(s)-v_n(s)) - j(u(s)-v(s)), g_n(s)\rangle ds = 0$$

because$\{f_n\}$ and $\{g_n\}$ is uniformly integrable in $L^1_E([0,T],dt)$. □

The preceding lemma with its tool occurs in several applications when dealing with mild solutions for m-accretive operators. See e.g., Crandall and Nohel [25,27], Bothe [17,18], Tolstonogov [19] and Wrabie [12]. However, we do not go to this direction that is out of the scope of the work.

4.4. An Application to Fractional Equation Coupled with a Volterra Integro-Differential Evolution

We are interested in the following fractional-order boundary problem involving an evolution governed by an m-accretive operator A: $D(A) \to E$ with perturbation.

$$D^\alpha h(t) + \lambda D^{\alpha-1}h(t) = u(t),\ t \in [0,1], \tag{18}$$

$$I^\beta_{0^+}h(t)|_{t=0} := \lim_{t\to 0}\int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)}h(s)ds = 0, \tag{19}$$

$$h(1) = I^\gamma_{0^+}h(1) = \int_0^1 \frac{(1-s)^{\gamma-1}}{\Gamma(\gamma)}h(s)ds, \tag{20}$$

$$-\frac{du}{dt}(t) \in Au(t) + f(t,h(t),u(t)) \text{ a.e. } t \in I,$$

where $\alpha \in]1,2]$, $\beta \in [0, 2-\alpha]$, $\lambda \geq 0$, $\gamma > 0$ are given constants, D^α is the standard Riemann–Liouville fractional derivative, Γ is the gamma function and $f : [0,1] \times E \times E \to E$ is a single-valued mapping.

Definition 1 (Fractional Bochner integral)**.** *Let $f : [0,1] \to H$. The fractional Bochner integral of order $\alpha > 0$ of the function f is defined by*

$$I^\alpha_{a^+}f(t) := \int_a^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}f(s)ds,\ t > a.$$

We refer to [28–30], for the general theory of Fractional Calculus and Fractional Differential Equations.

We denote by $W^{\alpha,1}_{B,E}([0,1])$ the space of all continuous functions in $\mathcal{C}_E([0,1])$ such that their Riemann–Liouville fractional derivative of order $\alpha-1$ are continuous and their Riemann–Liouville fractional derivative of order α are Bochner integrable.

For the proof of our theorems, we will need some elementary lemmas and theorems taken from reference [31].

Green function and its properties. Let $\alpha\in]1,2]$, $\beta\in[0,2-\alpha]$, $\lambda\geq 0$, $\gamma>0$ and $G:[0,1]\times[0,1]\to\mathbf{R}$ be a function defined by

$$G(t,s)=\varphi(s)I^{\alpha-1}_{0^+}(\exp(-\lambda t))+\begin{cases}\exp(\lambda s)I^{\alpha-1}_{s^+}(\exp(-\lambda t)), & 0\leq s\leq t\leq 1,\\ 0, & 0\leq t\leq s\leq 1,\end{cases}$$

where

$$\varphi(s)=\frac{\exp(\lambda s)}{\mu_0}\Big[\Big(I^{\alpha-1+\gamma}_{s^+}(\exp(-\lambda t))\Big)(1)-\Big(I^{\alpha-1}_{s^+}(\exp(-\lambda t))\Big)(1)\Big]$$

with

$$\mu_0=\Big(I^{\alpha-1}_{0^+}(\exp(-\lambda t))\Big)(1)-\Big(I^{\alpha-1+\gamma}_{0^+}(\exp(-\lambda t))\Big)(1).$$

Lemma 6. *Let G be the function defined above.*

(i) $G(\cdot,\cdot)$ *satisfies the following estimate*

$$|G(t,s)|\leq\frac{1}{\Gamma(\alpha)}\left(\frac{1+\Gamma(\gamma+1)}{|\mu_0|\Gamma(\alpha)\Gamma(\gamma+1)}+1\right)=M_G.$$

(ii) *If $u\in W^{\alpha,1}_{B,E}([0,1])$ satisfies boundary conditions equations*(18), (19), (20), *then*

$$u(t)=\int_0^1 G(t,s)\Big(D^\alpha u(s)+\lambda D^{\alpha-1}u(s)\Big)ds\quad\textit{for every } t\in[0,1].$$

(iii) *Let $f\in L^1_E([0,1])$ and let $u_f:[0,1]\to H$ be the function defined by*

$$u_f(t):=\int_0^1 G(t,s)f(s)ds\quad\textit{for}\quad t\in[0,1].$$

Then

$$I^\beta_{0^+}u_f(t)|_{t=0}=0\quad\textit{and}\quad u_f(1)=\Big(I^\gamma_{0^+}u_f\Big)(1).$$

Moreover, $u_f\in W^{\alpha,1}_{B,E}([0,1])$ and we have

$$\Big(D^\alpha u_f\Big)(t)+\lambda\Big(D^{\alpha-1}u_f\Big)(t)=f(t)\quad\textit{for all}\quad t\in[0,1].$$

The following theorem characterizes the topological structure of the solutions set.

Theorem 8. *Let $I=[0,1]$ and $X:I\rightrightarrows E$ be a convex compact-valued measurable set-valued map such that $X(t)\subset\gamma\overline{B}_H$ for all $t\in I$, where γ is a positive constant and S^1_X be the set of all measurable selections of X. Then the $W^{\alpha,1}_{B,E}(I)$-solutions set of problem*

$$\begin{cases}D^\alpha u(t)+\lambda D^{\alpha-1}u(t)=f(t), f\in S^1_X,\ \text{a.e. } t\in I\\ I^\beta_{0^+}u(t)|_{t=0}=0,\quad u(1)=I^\gamma_{0^+}u(1)\end{cases}\tag{21}$$

is a convex compact subset in $\mathcal{C}_E(I)$.

The following extends Theorem 5 in [32] into the m-accretive setting.

Theorem 9. *Let $I = [0, 1]$. Assume that E and E^* are reflexive separable and uniformly convex. A: $D(A) \to 2^E$, is an m-accretive operator satisfying $(\mathcal{H}_1)$ and $(\mathcal{H}_2)$.*

Let $f : I \times I \times H \times H \to H$ such that

(i) $f(.,.,x,y)$ is Lebesgue measurable on $I \times I$ for all $(x,y) \in H \times H$
(ii) $f(t,\tau,.,.)$ is continuous on $H \times H$ for all $(t,\tau) \in I \times I$.
(iii) $||f(t,\tau,x,y)|| \leq M(1 + ||y||)$ for all $(t,\tau,x,y) \in I \times I \times H \times H$,
(iv) $||f(t,\tau,x,y) - f(t,\tau,x,z)|| \leq M||y - z||$, for all $(t,\tau,x,y,z) \in I \times I \times E \times E \times E$ for some positive constant M.

Then there is a $W^{\alpha,1}_{B,E}(I)$ mapping $x : I \to E$ and an absolutely continuous mapping $v : I \to H$ satisfying

$$\begin{cases} D^\alpha x(t) + \lambda D^{\alpha-1}x(t) = v(t), \ t \in I \\ I^\beta_{0^+} x(t)|_{t=0} = 0, \quad x(1) = I^\gamma_{0^+} x(1) \\ v(t) \in D(A), \ t \in I \\ -\frac{dv}{dt}(t) \in Av(t) + \int_0^t f(t,\tau,x(\tau),v(\tau))d\tau \quad \text{a.e. } t \in I. \end{cases}$$

Proof. For any continuous $h : I \to E$, the mapping $f_h(t,s,x) := f(t,s,x,h(t))$ is measurable on $I \times I$, $\forall x \in E$, continuous on E, $\forall(t,s) \in I \times I$ and satisfies $||f_h(t,s,x)|| \leq M(1 + ||x||)$, $\forall(t,s,x) \in I \times I \times E$ and $||f_h(t,s,x) - f_h(t,s,y))|| \leq M||x - y||$, $\forall(t,s,x,y) \in I \times I \times E \times E$, so by Theorem 2 or (Theorem 4.6, [20]) there is a unique AC solution v_h to the inclusion

$$\begin{cases} v_h(0) = u_0 \in D(A) \\ v_h(t) \in D(A), \forall t \in I \\ -\frac{dv_h}{dt}(t) \in Av_h(t) + \int_0^t f(t,s,v_h(s),h(s))ds \ dt\text{-a.e} \end{cases}$$

with v_h uniformly bounded and equi-absolutely continuous: $\frac{dv_h}{dt} \in K\overline{B}_E$, where K is a positive generic constant so that $||v_h(t)|| \leq L$, $t \in I$. Let us consider the multivalued mapping defined by

$$\mathcal{X} := \{u_f : [0,1] \to E : u_f(t) = \int_0^1 G(t,s)f(s)ds, \ f \in S^1_{L\overline{B}_E}, \ t \in I\}.$$

We note that $\mathcal{X}$ closed convex and equi-Lipschitz ([32], Theorem 4.1). Now for each $h \in \mathcal{X}$, let us consider the unique absolutely continuous solution u_h to

$$\begin{cases} -\dot{u}_h(t) \in Au_h(t) + \int_0^t f(t,\tau,h(\tau),u_h(\tau))d\tau \quad \text{a.e. } t \in I \\ u_h(t) \in D(A), \forall t \in I \\ u_h(0) = u_0 \in D(A). \end{cases}$$

For each $h \in \mathcal{X}$ let us set

$$\Phi(h)(t) = \int_0^1 G(t,s)u_h(s)ds, \ t \in I.$$

Then it is clear that $\Phi(h) \in \mathcal{X}$. We have $||u_h(t)|| \leq L$ for all $t \in I$ so that the set $\Gamma := \{x \in D(A) : ||x|| \leq L\}$ is compact by $(\mathcal{H}_2)$ and nonempty because $v_h(t) \in \Gamma$. As consequence for any $h \in \mathcal{X}$ and for any $t \in I$, the inclusion holds

$$\Phi(h) \in \mathcal{Y} := \{u_f : I \to E : u_f(t) = \int_0^1 G(t,s)f(t)ds, \ f \in S^1_{\overline{co}[\Gamma]}, \ t \in I\}.$$

By ([32], Theorem 4.1) $\mathcal{Y}$ convex compact and equi-Lipschitz. Hence $\Phi(\mathcal{X})$ is equicontinuous and relatively compact in the Banach space $\mathcal{C}_E([0,T])$ because $\Phi(\mathcal{X}) \subset \mathcal{Y}$. Now we check that Φ is continuous. It is sufficient to show that if (h_n) converges uniformly to h in $\mathcal{X}$, then the absolutely continuous solution u_{h_n} associated with h_n

$$\begin{cases} u_{h_n}(0) = u_0 \in D(A) \\ u_{h_n}(t) \in D(A), \forall t \in I \\ -\dot{u}_{h_n}(t) \in Au_{h_n}(t) + \int_0^t f(t,\tau,h_n(\tau),u_{h_n}(\tau))d\tau \quad \text{a.e. } t \in I \end{cases}$$

uniformly converges to the absolutely solution u_h associated with h

$$\begin{cases} u_h(0) = u_0 \in D(A) \\ u_h(t) \in D(A), \forall t \in I \\ -\dot{u}_h(t) \in Au_h(t) + \int_0^t f(t,\tau,h(\tau),u_h(\tau))d\tau \quad \text{a.e. } t \in I. \end{cases}$$

This needs a careful look. We note that u_{h_n} is equicontinuous with $||\dot{u}_{h_n}(t)|| \le K$ for almost all $t \in I$ and for all $n \in N$ and relatively compact. So, by extracting subsequence, we may assume that $u_{h_n}(t) \to v(t) = v(0) + \int_0^t \dot{v}(s)ds$ uniformly with $\dot{u}_{h_n}$ weakly converging in $L^2_H(I)$ to $\dot{v}$ with $||\dot{v}(t)|| \le K$ for a.e $t \in I$. Please note that $f(t,\tau,h_n(\tau),u_{h_n}(\tau)) \to f(t,\tau,h_n(\tau),u_{h_n}(\tau))$ for all $t,\tau \in I \times I$. For simplicity, note

$$z_n(t) = \int_0^t f(t,\tau,h_n(\tau),u_{h_n}(\tau)), \forall t \in I,$$

$$z(t) = \int_0^t f(t,\tau,h(\tau),u_h(\tau)), \forall t \in I.$$

We mention at first that these mappings are Lebesgue measurable by the Fubini-Bochner property and the separability of the space E. Second, by the growth condition and the boundedness of u_{h_n} and u_h, z_n and z are uniformly bounded, say $||z_n(t)|| \le \kappa$ $||z(t)|| \le \kappa, \forall n \in \mathbf{N}, \forall t \in I$. As consequence z_n is uniformly bounded measurable and pointwise converge to the measurable mapping z. Hence $\dot{u}_{h_n}(t) + z_n(t) \to \dot{v}(t) + z(t)$ weakly in $L^2_E(I)$. Applying the accretive extension of $\mathcal{A}$ (cf Lemma 4) gives $-\frac{dv}{dt}(t) \in Av(t) + z(t)$ a.e. with $v(t) \in D(A)$ for all $t \in I$ so that by uniqueness $v = u_h$. Since $h_n \to h$, we have

$$\Phi(h_n)(t) - \Phi(h)(t) = \int_0^1 G(t,s)u_{h_n}(s)ds - \int_0^1 G(t,s)u_h(s)ds$$

$$= \int_0^1 G(t,s)[u_{h_n}(s) - u_h(s)]ds$$

$$\le \int_0^1 M_G||u_{h_n}(s) - u_h(s)||ds.$$

As $||u_{h_n}(\cdot) - u_h(\cdot)|| \to 0$ uniformly we conclude that

$$\sup_{t \in I} ||\Phi(h_n)(t) - \Phi(h)(t)|| \le \int_0^1 M_G||u_{h_n}(\cdot) - u_h(\cdot)||ds \to 0$$

so that $\Phi(h_n) \to \Phi(h)$ in $\mathcal{C}_E(I)$. Since $\Phi : \mathcal{X} \to \mathcal{X}$ is continuous with $\Phi(\mathcal{X})$ relatively compact in $\mathcal{C}_E([0,T])$ by Schauder theorem Φ has a fixed point, say $h = \Phi(h) \in \mathcal{X}$. This means that

$$h(t) = \Phi(h)(t) = \int_0^1 G(t,s)u_h(s)ds,$$

with
$$\begin{cases} u_h(0) \in D(A) \\ u_h(t) \in D(A), \forall t \in I \\ -\dot{u}_h(t) \in Au_h(t) + \int_0^t f(t,\tau,h(\tau),u_h(\tau))d\tau \quad \text{a.e. } t \in I. \end{cases}$$

This means that we have just shown that there exists a mapping $h \in W^{\alpha,\infty}_{B,E}(I)$ satisfying

$$\begin{cases} D^\alpha h(t) + \lambda D^{\alpha-1}h(t) = u_h(t), \\ I^\beta_{0^+}h(t)|_{t=0} = 0, \quad h(1) = I^\gamma_{0^+}h(1) \\ u_h(0) \in D(A) \\ u_h(t) \in D(A), \forall t \in I \\ -\dot{u}_h(t) \in Au_h(t) + \int_0^t f(t,\tau,h(\tau),u_h(\tau))d\tau \quad \text{a.e. } t \in I. \end{cases}$$

□

Several variants are available by considering time-dependent m-accretive operator e.g., Theorems 1–3 and other type of fractional equations, e.g., the Caputo fractional equation with Caputo fractional derivatives.

4.5. Skorohod Problem Driven by Operator

To finish the paper, we provide some new versions of Skorohod problem for an evolution inclusion driven by time dependent operator A_t in the vein of Castaing et al. [33], Falkowski and Słominski [34], Rascanu [35], and Maticiuc, Rascanu, Slominski and Topolewski [36].

We begin by recalling some notations which are used in next proofs (See [37] Definition 1.5).

Let $\mathcal{D}([s,t])$ be the set of all dissections $D = \{s = t_0 < t_1 < \ldots < t_n = t\}$ of $[s,t]$ of $[0,T]$.

If $z : [0,T] \to E$, for $0 \le s \le t \le T$, the 1-variation of z on $[s,t]$ is defined as

$$|z|_{1-var:[s,t]} = \sup_{(t_i)\in\mathcal{D}([s,t])} \Sigma\|z(t_{i+1}) - z(t_i)\|$$

If $|z|_{1-var:[s,t]} < +\infty$, z is bounded variation or finite 1-variation on $[s,t]$. $C^{1-var}([0,T],E)$ is the space of continuous maps of bounded variation on $[0,T]$.

Theorem 10. *Let $I = [0,1]$ and $E = \mathbf{R}^e$. Let $t \mapsto A(t) : D(A(t)) \to ccl(E)$ be a time-dependent m-accretive operator satisfying*
$(\mathcal{H}^A_1)$ *there exists a nonnegative real number c such that*

$$\|A^0(t,x)\| \le c(1+\|x\|) \ \text{ for } \ t \in I, \ x \in D(A(t)),$$

$(\mathcal{H}^A_2)$ *$t \mapsto D(A(t))$ has closed graph, $Gr(D(A))$,*
$(\mathcal{H}^A_3)$ *$(t,x) \mapsto A(t,x) : Gr(D(A)) \to ccl(E)$ is scalar upper semicontinuous: for $t_n \to t$, for $x_n \to x$ with $x_n \in D(A(t_n))$ and $x \in D(A(t))$,*

$$\forall x^* \in E^*, \limsup_n \delta^*(x^*, A(t_n,x_n)) \le \delta^*(x^*, A(t,x)),$$

$(\mathcal{H}^A_4)'$ *There exists a nondecreasing and absolutely continuous function $\beta : I \to [0,\infty[$ with $\dot{\beta} \in L^2$, such that for $t < \tau \subset I$, for $\lambda > 0$ and $x \in D(A(t))$*

$$\|x - J^{A(\tau)}_\lambda(x)\| \le (\beta(\tau) - \beta(t))(1 + ||A^0(t,x)||).$$

Let $z \in C^{1-var}(I, \mathbf{R}^d)$ the space of continuous functions of bounded variation defined on I with values in $\mathbf{R}^d$. Let $\mathcal{L}(\mathbf{R}^d, \mathbf{R}^e)$ the space of linear mappings f from $\mathbf{R}^d$ to $\mathbf{R}^e$ endowed with the operator norm

$$|f| := \sup_{x \in \mathbf{R}^d, ||x||_{\mathbf{R}^d}=1} |f(x)|_{\mathbf{R}^e}.$$

Let us consider a class of continuous integrand operator $b : I \times \mathbf{R}^e \to \mathcal{L}(\mathbf{R}^d, \mathbf{R}^e)$ satisfying

(a) $|b(t,x)| \leq M, \quad \forall (t,x) \in I \times \mathbf{R}^e,$

(b) $|b(t,x) - b(t,y)| \leq M||x-y||_{\mathbf{R}^e}, \quad \forall (t,x,y) \in I \times \mathbf{R}^e \times \mathbf{R}^e,$

where M is a positive constant and $\int_0^t b(\tau, x(\tau)) dz_\tau$ is the Riemann-Stieltjes integral defined on $x \in C(I, \mathbf{R}^e)$.

Let $g : I \times \mathbf{R}^e \to \mathbf{R}^e$ be a continuous mapping satisfying:

(i) $||g(t,x)|| \leq M$ *for all* $(t,x) \in I \times \mathbf{R}^e$,

(ii) $\|g(t,x) - g(t,y)\| \leq M(||x-y||)$ *for all* $(t,x,y) \in I \times \mathbf{R}^e \times \mathbf{R}^e$ *for some constant* $M > 0$.

Let $a \in D(A(0))$. Then there exist a BVC function $x : I \to \mathbf{R}^e$ and BVC function $h : I \to \mathbf{R}^e$ and AC functions $k : I \to \mathbf{R}^e$, $u : I \to \mathbf{R}^e$ satisfying

$$\begin{cases} x(0) = u(0) = a \\ x(t) = h(t) + k(t) + u(t), \ \forall t \in I \\ h(t) = \int_0^t b(\tau, x(\tau)) dz_\tau, \ \forall t \in I \\ k(t) = \int_0^t g(s, x(s)) ds, \ \forall t \in I \\ u(t) \in D(A(t)), \ \forall t \in I \\ -\frac{du}{dt}(t) \in A(t, u(t)) + k(t), \ \ a.e., \ t \in I. \end{cases}$$

Proof. Let $a \in D(A(0))$. Let us set for all $t \in I$

$$x^0(t) = a, \ h^1(t) = \int_0^t b(\tau, a) dz_\tau.$$

Then by Proposition 2.2 in Friz-Victoir [37], we have

$$|\int_0^t b(\tau, a) dz_\tau| \leq |b(., a)|_{\infty:I} |z|_{1-var:[0,t]}. \tag{22}$$

Moreover

$$\int_0^t b(\tau, a) dz_\tau - \int_0^s b(\tau, a) dz_\tau = \int_s^t b(\tau, a) dz_\tau$$

so that by condition (a)

$$||h^1(t) - h^1(s)|| \leq M|z|_{1-var:[s,t]}, \tag{23}$$

for all $0 \leq s \leq t \leq 1$ and in particular

$$||h^1(t)|| \leq M|z|_{1-var:[0,t]} \leq M|z|_{1-var:I}$$

for all $t \in I$. Let us set for all $t \in I = [0,1]$,

$$x^0(t) = a, \ k^1(t) = \int_0^t g(s, x^0(s)) ds,$$

then k^1 is continuous with $\|k^1(t)\| \leq M$ for all $t \in I$. By an easy computation, using conditions (i) and (ii) we have the estimate $||k^1(t) - k^1(\tau)|| \leq M|t - \tau|$, for all $\tau, t \in I$. By Theorem 3 there is a unique AC solution $u^1 : I \to E$ to the problem

$$\begin{cases} u^1(0) = a, \ u^1(t) \in D(A(t)), \ \forall t \in I; \\ -\dfrac{du^1}{dt}(t) \in A(t, u^1(t)) + k^1(t), \quad a.e. \end{cases}$$

with

$$u^1(t) = a + \int_{]0,t]} \frac{du^1}{ds}(s)ds, \ \forall t \in I$$

and $\|\frac{du^1}{dt}\|_{L^2_E} \leq L$, where L is a positive constant depending on the data. Set

$$x^1(t) = h^1(t) + k^1(t) + u^1(t) = \int_0^t b(\tau, x^0(\tau)dz_\tau + \int_0^t g(s, x^0(s))ds + u^1(t).$$

Then x^1 is BVC with $x^1(0) = a$. Now we construct x^n by induction as follows. Let for all $t \in I$,

$$h^n(t) = \int_0^t b(\tau, x^{n-1}(\tau))dz_\tau,$$

$$k^n(t) = \int_0^t g\big(s, x^{n-1}(s)\big)ds.$$

Then k^n is equi-Lipschitz: $||k^n(t) - k^n(\tau)|| \leq M|t - \tau|$, for all $\tau, t \in I$ with $\|k^n(t)\| \leq M$ for all $t \in I$. By Proposition 2.2 in Friz-Victoir [37] we have the estimate

$$||h^n(t) - h^n(s)|| \leq M|z|_{1-var:[s,t]}$$

for all $0 \leq s \leq t \leq 1$ and in particular,

$$||h^n(t)|| \leq M|z|_{1-var:[0,t]} \leq M|z|_{1-var:I},$$

for all $0 \leq t \leq 1$. By Theorem 3 there is a unique AC solution $u^n : I \to E$ to the problem

$$\begin{cases} u^n(0) = a, \ u^n(t) \in D(A(t)), \ \forall t \in I, \\ -\dfrac{du^n}{dt}(t) \in A(t, u^n(t)) + k^n(t), \quad a.e. \end{cases}$$

with

$$u^n(t) = a + \int_{]0,t]} \frac{du^n}{ds}(s)\, ds, \ \forall t \in I$$

and $\|\frac{du^n}{dt}\|_{L^2_E} \leq L$ where L is a positive constant depending on the data. Set for all $t \in I$,

$$x^n(t) = h^n(t) + k^n(t) + u^n(t) = \int_0^t b(\tau, x^{n-1}(\tau))dz_\tau + \int_0^t g\big(s, x^{n-1}(s)\big)ds + u^n(t),$$

so that x^n is BVC, and

$$-\frac{du^n}{dt}(t) \in A(t, u^n(t)) + k^n(t), \quad a.e. \tag{24}$$

As (u^n) is equi-absolutely continuous and for all $t \in I$ $u^n(t) \in D(A(t))$, we may assume that (u^n) converges uniformly to an AC mapping $u : I \to H$ with $u(t) \in D(A(t)), \forall t \in I$, using the estimate $\|\frac{du^n}{dt}(t)\|_{L^2_E} \leq L$, we may also assume that $(\frac{du^n}{dt})$ weakly converges in $L^2_H(I, dt)$ to $\frac{du}{dt}$, and by Ascoli theorem we may assume that k^n converges uniformly to a continuous mapping $k : I \to H$. Now, recall that

$$||h^n(t) - h^n(s)|| \leq M|z|_{1-var:[s,t]}$$

for all $0 \leq s \leq t \leq T$ using Proposition 2.2 in Friz-Victoir [37], and our assumption (a) on the mapping b. So h^n is bounded and equicontinuous. By Ascoli theorem, we may assume that h^n converge uniformly to a continuous mapping h. Similarly, k^n is bounded and equi-Lipschitz. By Ascoli theorem, we may assume that k^n converge uniformly to a continuous mapping k. Hence $x^n(t) = h^n(t) + k^n(t) + u^n(t)$ converge uniformly to $x(t) := h(t) +$

$k(t)+u(t)$, and $b(.,x^{n-1}(.))$ converges uniformly to $b(.,x(.))$ using the Lipschitz condition (b). Then by Friz-Victoir [37] (Proposition 2.7) $\int_0^t b(\tau,x^{n-1}(\tau))dz_\tau$ converges uniformly to $\int_0^t b(\tau,x(\tau))dz_\tau$. By hypothesis (i), $g(s,x^{n-1}(s))$ pointwise converge to $g(s,x(s))$. Hence $\int_0^t g(s,x^{n-1}(s))ds \to \int_0^t g(s,x(s))ds$ for each $t \in I$ by Lebesgue theorem. So, by identifying the limit

$$\lim_{n\to\infty} x^n(t) = \lim_{n\to\infty}\int_0^t b(\tau,x^{n-1}(\tau))dz_\tau + \lim_{n\to\infty}\int_0^t g(s,x^{n-1}(s))ds + \lim_{n\to\infty} u^n(t)$$
$$= \int_0^t b(\tau,x(\tau))dz_\tau + \int_0^t g(s,x(s))ds + u(t) = x(t).$$

Now, by $(\mathcal{H}_3^A)$ it is easily seen that J_λ is continuous on $I\times E$. From equation (24), applying the m-accretive (equivalent maximal monotone) extension (cf Lemma 4) we obtain

$$-\frac{du}{dt}(t) \in A(t,u(t)) + k(t), \quad a.e.\ t \in I.$$

The proof is therefore complete. □

In Theorems 5-9 we have provided existence results of solution for a class of integral equation of Volterra type coupled with a m-accretive operator. Our tools allow the statement of several variants of Theorem 10 according to the nature of the control z, the perturbation and the operator, e.g., A is an m-accretive operator satisfying $(\mathcal{H}_1)$ and $(\mathcal{H}_2)$ with Volterra integral perturbation (Cf Theorem 2). It is a challenge to obtain the uniqueness. In this setting, our result is quite new by comparison with the sole classical integral equation $x(t) = \int_0^t g(t,s,x(s))ds + \int_0^t b(\tau,x(\tau))dz_\tau$.

In this vein we provide below some application in the problem of Optimal Control. In the following $E = \mathbf{R}^e$ and $dx_t = V(x_t)dz_t,\ t \in [0,T], x_0 = \psi \in \mathbf{R}^e$ denotes rough differential equation ([37], Theorem 3.4) with $V : \mathbf{R}^e \to \mathcal{L}(\mathbf{R}^d,\mathbf{R}^e)$ bounded continuous.

Proposition 1. *Let A be an m-accretive operator $A : D(A) \subset \mathbf{R}^e \rightrightarrows \mathbf{R}^e$ satisfying $(\mathcal{H}_1)$ $||A^0x|| \le c(1+||x||)$ for all $x \in D(A)$ where c is a positive constant. Let $f : [0,T]\times[0,T]\times \mathbf{R}^e\times\mathbf{R}^e \to \mathbf{R}^e$ satisfying to the conditions*
$(\mathcal{H}_3)$ $(t,s) \to f(t,s,x,y)$ is Lebesgue measurable on $[0,T]\times[0,T]$, $\forall (x,y)\in \mathbf{R}^e\times\mathbf{R}^e$,
$(\mathcal{H}_4)$ $(x,y) \to f(t,s,x,y)$ is continuous, $\forall (t,s) \in [0,T]\times[0,T]$,
$(\mathcal{H}_5)$ $\|f(t,s,z,x) - f(t,s,z,y)\| \le M||x-y||,\ \forall t,s \in [0,T], \forall x,y,z, \in \mathbf{R}^e$,
$(\mathcal{H}_6)||f(t,s,x,y)|| \le M(1+||x||), \forall t,s,x,y \in [0,T]\times[0,T]\times\mathbf{R}^e\times\mathbf{R}^e$,
where M is positive constant.
Let $L : [0,T]\times\mathbf{R}^e\times\mathbf{R}^e\times\mathbf{R}^e \to [0,\infty[$ be a lower semicontinuous integrand such that $L(t,x,y,.)$ is convex on $\mathbf{R}^e$ for every $(t,x,y) \in [0,T]\times\mathbf{R}^e\times\mathbf{R}^e$. Then the problem of minimizing the cost function $\int_0^T L(t,x(t),y(t),\dot{y}(t))dt$ subject to

$$\begin{cases} (4.1.1) : x_0 = \psi \in \mathbf{R}^e, dx_t = V(x_t)dz_t,\ t \in [0,T] \\ (4.1.2) : y(0) = y_0 \in D(A), -\dot{y}(t) \in Ay(t) + \int_0^t f(t,s,x(s),y(s))ds, \quad a.e.\, t \in [0,T]. \end{cases}$$

has an optimal solution.

Proof. Let us consider a minimizing sequence (x_n,y_n) that is

$$\lim_{n\to\infty}\int_0^T L(t,x_n(t),y_n(t),\dot{y}_n(t))dt = \inf_{(u,v)\in\mathcal{X}}\int_0^T L(t,u(t),v(t),\dot{v}(t))dt$$

where $\mathcal{X}$ is the solutions set (u,v) to the above dynamical system. First by ([37], Theorem 3.4) we assert that the $C^{1-var}([0,T],\mathbf{R}^e)$-solution set to $(4.1.1)$ is compact in $C([0,T],\mathbf{R}^e)$ and so is the $W^{1,\infty}([0,T], \mathbf{R}^e)$-solution set to (4.1.2) (cf Theorem 2) or ([20], Theorem 4.6). We may ensure that

(i) $x_n \to x \in C^{1-var}([0,T], \mathbf{R}^e)$ with $x_t = \psi + \int_0^t V(x_s)dz_s$.

(ii) $y_n \to y \in W^{1,\infty}([0,T], \mathbf{R}^e)$ and $\dot{y}_n \to \dot{y}$ weakly in $L^1_{\mathbf{R}^e}([0,T])$.

Applying the lower semicontinuity of the integral functional ([38] , Theorem 8.1.6) gives

$$\liminf_n \int_0^T L(t, x_n(t), y_n(t), \dot{y}_n(t))dt \geq \int_0^T L(t, x(t), y(t), \dot{y}(t))dt.$$

From the inclusion

$$-\dot{y}_n(t) \in Ay_n(t) + \int_0^t f(t, s, x_n(s), y_n(s))ds$$

and (i) and (ii), we conclude by repeating the limit argument via the m-accretive extension $\mathcal{A}$ of A that

$$-\dot{y}(t) \in Ay(t) + \int_0^t f(t, s, x(s), y(s))ds\ a.e.$$

□

5. Conclusions

In this paper, we present several existences of BV and AC (absolutely continuous) solutions to some class of evolution inclusion with various perturbations governed by m-accretive operators. Several applications such as the Skorohod problem, fractional differential equation, optimal control problem, and relaxation are provided. Our results contain novelties. However, there remain several issues that need development.

C.1 Our techniques have some importance in further applications such as the periodic solution (in the line of Theorem 3).

C.2 We can deal with optimal control theory involving the dynamics under consideration in the same vein as [38]. In such a new setting, we will study evolution inclusions with time- and state-dependent maximal monotone/m-accretive operators and Young measure control

$$-\dot{u}(t) \in A(t, x(t))u(t) + \int_Z f(t, x(t), u(t), z)\lambda_t(dz)$$

C.3 In this spirit, the asymptotic behavior of solutions in these dynamics is an open problem.

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