mathematics

MDPI

Article

Magnetic Geodesic in (Almost) Cosymplectic Lie Groups of Dimension 3

Marian Ioan Munteanu

Faculty of Mathematics, Alexandru Ioan Cuza University of Iasi, Bd. Carol I, n. 11, 700506 Iasi, Romania; marian.ioan.munteanu@gmail.com or munteanu@uaic.ro

Abstract: In this paper, we study contact magnetic geodesics in a 3-dimensional Lie group G endowed with a left invariant almost cosymplectic structure. We distinguish the two cases: G is unimodular, and G is nonunimodular. We pay a careful attention to the special case where the structure is cosymplectic, and we write down explicit expressions of magnetic geodesics and corresponding magnetic Jacobi fields.

Keywords: magnetic Jacobi field; cosymplectic manifold; magnetic geodesic; 3-dimensional Lie group

MSC: 53B21; 53C15; 53C25; 53C80

check for updates

Citation: Munteanu, M.I. Magnetic Geodesic in (Almost) Cosymplectic Lie Groups of Dimension 3. *Mathematics* **2022**, *10*, 544. https://doi.org/10.3390/math10040544

Academic Editor: Vicente Muñoz

Received: 23 January 2022
Accepted: 2 February 2022
Published: 10 February 2022

Publisher's Note: MDPI stays neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Copyright: © 2022 by the authors. Licensee MDPI, Basel, Switzerland. This article is an open access article distributed under the terms and conditions of the Creative Commons Attribution (CC BY) license (https://creativecommons.org/licenses/by/4.0/).

1. Introduction

The study of magnetic trajectories has its origin in physics, where a magnetic field is given by a divergence-free vector field. Magnetic curves in physics represent the trajectories of charged particles moving on a Riemannian manifold under the influence of magnetic forces. The Landau–Hall problem refers to the study of the motion of charged particles on a Riemannian surface in the presence of a constant and static magnetic field. Later on, the problem was extended for higher dimensions. Since on a 3-dimensional Riemannian manifold vector fields and differential 2-forms may be identified, a magnetic field can be thought as a closed 2-form. Magnetic fields and their corresponding trajectories represent an important field of study, since this topic is situated at the interplay of Riemannian geometry and dynamical systems. Magnetic trajectories generalize geodesics (and for this reason they are also known as magnetic geodesics); therefore, they are used as a tool for better understanding of the geometry of ambient spaces. As a matter of fact, one of the most beautiful and difficult problems in dynamical systems on manifolds is finding closed trajectories. The existence of closed trajectories is closely related to topology and geometric structures of the manifolds.

The structure of this paper is the following. Sections 2 and 3 give some basic aspects on the geometry of (almost)cosymplectic manifolds, magnetic trajectories, and magnetic Jacobi fields. Theorem 1, due to Perrone, is the result that inspired us to write this paper. In Section 4, we study magnetic geodesics in a 3-dimensional Lie group G. For this purpose, we distinguish two cases: G is unimodular and G is nonunimodular. Particular attention is paid to the case when the almost contact metric structure is cosymplectic. In the unimodular case, we write the expression of contact magnetic geodesics in Euclidean motion group $\tilde{E}_2$. Moreover, we obtain magnetic Jacobi fields in Lie group $G(\lambda_0, \lambda_0)$. In the nonunimodular case, we study contact magnetic geodesics and the corresponding magnetic Jacobi fields in Lie group $G(\alpha, 0)$. We consider a multiplication low on semidirect product $\mathbb{R} \ltimes \mathbb{R}^2$, obtaining a nonunimodular Lie group, and we write the expression of contact magnetic geodesics corresponding to the cosymplectic structure naturally defined on this group.

2. Homogeneous Almost Cosymplectic 3-Manifolds

Let M be a smooth manifold. On M, consider the following geometric objects:

- a $(1,1)$ tensor field φ, such that $\varphi^2 = -I + \eta \otimes \xi$, where I is the identity,
- a vector field ξ,
- a 1-form η,

satisfying the conditions:

$$\varphi\xi = 0, \ \ \eta \circ \varphi = 0, \ \ \eta(\xi) = 1.$$

Consequently, the dimension of M is odd; let it be $2n+1$. Then, (M, φ, ξ, η) is an *almost contact manifold*. When a Riemannian metric g, compatible with the almost contact structure defined above, is considered, manifold M is called an *almost contact metric manifold*. The compatibility condition writes as

$$g(\varphi X, \varphi Y) = g(X,Y) - \eta(X)\eta(Y), \ \ \forall X, Y \in \mathfrak{X}(M).$$

An almost contact metric structure on an orientable $(2n+1)$-dimensional manifold M is a reduction in the structure group of M to $U(n) \times 1$. An almost contact metric manifold $(M, \varphi, \xi, \eta, g)$ is said to be *homogeneous* if there exists a connected Lie group G of isometries acting transitively on M and leaving η invariant.

The fundamental 2-form Φ on M is defined by:

$$\Phi(X,Y) = g(X, \varphi Y), \ \ \forall X, Y \in \mathfrak{X}(M).$$

An almost contact Riemannian manifold $(M, \varphi, \xi, \eta, g)$ with $d\eta = 0$ and $d\Phi = 0$ is an *almost cosymplectic* manifold. The almost contact structure is called *normal* if $N_\varphi + 2d\eta \otimes \xi = 0$, where N_φ is the Nijenhius tensor defined by:

$$N_\varphi(X,Y) = [\varphi X, \varphi Y] - \varphi[\varphi X, Y] - \varphi[X, \varphi Y] + \varphi^2[X,Y], \quad \forall X, Y \in \mathfrak{X}(M).$$

If an almost cosymplectic structure is normal, we obtain a *cosymplectic* manifold. The notion of cosymplectic manifold was independently introduced by Blair (in his Ph.D. thesis [1]) and by Ogiue (in [2]) under the name *cocomplex*. The authors in [3] showed that the cosymplectic structure is characterized among the almost contact metric structures, by the parallelism of φ, that is, $\nabla\varphi = 0$, where ∇ is the Levi-Civita connection on M. It follows that η and ξ are also parallel.

Recall the following important result:

Theorem 1 ([4])**.** *Let M be a simply connected homogeneous almost cosymplectic 3-manifold. Then M is*

(1) either one of the product Riemannian symmetric spaces

$$\mathbb{S}^2(\bar{c}) \times \mathbb{R} \qquad \text{and} \qquad \mathbb{H}^2(\bar{c}) \times \mathbb{R},$$

where $\mathbb{S}^2(\bar{c})$ and $\mathbb{H}^2(\bar{c})$ are the sphere of curvature $\bar{c} > 0$ and the hyperbolic plane of curvature $\bar{c} < 0$, respectively,

(2) or M is a Lie group equipped with a left invariant almost cosymplectic structure.

Magnetic curves in $\mathbb{S}^2 \times \mathbb{R}$ and $\mathbb{H}^2 \times \mathbb{R}$ are investigated in [5,6]. See also [7]. The study of magnetic Jacobi fields in 3-dimensional cosymplectic manifolds (in particular in $\mathbb{S}^2 \times \mathbb{R}$ and $\mathbb{H}^2 \times \mathbb{R}$) was developed in a very recent paper [8].

In this paper, we study magnetic curves and magnetic Jacobi fields in a Lie group of dimension 3 endowed with a left invariant almost cosymplectic structure. All the necessary tools to understand the ambient space (the case 2 of Theorem 1) are described in the same paper [4].

3. Magnetic Jacobi Fields

Magnetic curves represent an important topic in differential geometry not only because the problem originates from physics (in 3-dimensional Euclidean ambient), but also because a magnetic curve is a natural generalization of a geodesic. A magnetic field on M is defined by a closed 2-form F on M. A curve $\gamma : I \to M$ is called a *magnetic curve* or a *magnetic geodesic* if it is a solution of Lorentz equation

$$\frac{D}{ds}\gamma'(s) = \phi\gamma'(s),$$

where $\dfrac{D}{ds}$ stands for the covariant derivative along γ, and ϕ is the Lorentz force corresponding to the magnetic field F defined as $g(\phi\cdot, \cdot) = F$. Since any magnetic geodesic has constant speed, it is natural to work with arc-length parametrized magnetic curves; the case when γ is known as a *normal magnetic geodesic*.

Magnetic geodesics are solutions of a variational problem, i.e., they are the critical points of the Landau–Hall functional $\mathtt{LH}$ (on $C^\infty([a,b])$)

$$\mathtt{LH}(\gamma) = \mathtt{E}(\gamma) - q\int_a^b A(\gamma'(s))ds,$$

where $\mathtt{E}(\gamma) = \displaystyle\int_a^b \frac{1}{2}g(\gamma'(s), \gamma'(s))ds$ is the Dirichlet energy of γ and A is the potential 1-form generating the magnetic field F. A second variational formula for integral $\mathtt{LH}$ gives rise to the concept of a magnetic Jacobi field. W is a *magnetic Jacobi field* along the magnetic geodesic γ if it satisfies the following second-order differential equation:

$$\frac{D^2}{ds^2}W - R(\dot{\gamma}, W)\dot{\gamma} - \phi\left(\frac{D}{ds}W\right) - (\nabla_W\phi)\dot{\gamma} = 0. \tag{1}$$

Here, R denotes the Riemannian curvature tensor of M. See, e.g., [9,10].

To solve the magnetic Jacobi equation is a true challenge. The major difficulty is the presence of the last term in (1). When the covariant derivative of the Lorentz force has a particular concrete expression, we can think about solving Equation (1).

Let us indicate some particular situations.

- A first example of a magnetic Jacobi field is velocity vector field $\gamma'(s)$ of a magnetic geodesic $\gamma(s)$ (see, e.g., [11]).
- In a Kähler manifold (M, g, J), one considers (Kähler) magnetic fields that are uniform, meaning that Lorentz force $\phi = qJ$, $q \in \mathbb{R}$ is parallel; thus, the last term in (1) vanishes (see, e.g., [12,13]).
- In a Sasakian manifold $(M, \varphi, \xi, \eta, g)$, covariant derivative $\nabla\phi$ can be expressed by a concrete formula. More precisely, The Lorentz force is defined by $\phi = q\varphi$ and $(\nabla_X\varphi)Y = g(X,Y)\xi - \eta(Y)X$. The classification of nonuniform Jacobi magnetic fields is given in [14,15].
- In a cosymplectic manifold $(M, \varphi, \xi, \eta, g)$, the magnetic field is again uniform. The Lorentz force $\phi = q\varphi$ is parallel; hence, the last term of the Equation (1) vanishes. See [8].

In the case when the magnetic field is uniform, that is, the Lorentz force ϕ is parallel, i.e., $\nabla\varphi = 0$, we retrieve the equation of a magnetic Jacobi field given by Gouda in [10]

$$\frac{D^2}{ds^2}W - R(\dot{\gamma}, W)\dot{\gamma} - \phi\left(\frac{D}{ds}W\right) = 0. \tag{2}$$

In [8], the authors solved (2) in order to find all uniform magnetic fields in 3-dimensional cosymplectic space forms. In the same paper [8], a characterization of magnetic Jacobi fields in the product spaces $\mathbb{S}^2 \times \mathbb{R}$ and $\mathbb{H}^2 \times \mathbb{R}$ was also given. Equation (2) is still of interest to study the magnetic Jacobi fields in three-dimensional Lie groups endowed with a left invariant (almost) cosymplectic structure.

4. Magnetic Geodesics in a 3-Dimensional Lie Group

Let $M = G$ be a Lie group of dimension 3. In fact, Lie group G is diffeomorphic to M via map $\pi : G \to M, a \mapsto ax_0$, where x_0 is a certain fixed point of M. All geometric structures can be moved from M to G via π_* (and π^*); hence, we consider M as the Lie group G endowed with a left invariant almost cosymplectic structure (φ, ξ, η, g). We have $h = \varphi\nabla\xi$, where ∇ is the Levi-Civita connection of g and $h := \frac{1}{2}\mathcal{L}_\xi\varphi$. We know from [4] that h commutes with left invariant translations, and the eigenvectors of h are left invariant. Indeed, if $\mathfrak{g} \equiv T_{x_0}G$ (the Lie algebra of G) and $e_{x_0} \in \mathfrak{g}$ such that $h_{x_0}e_{x_0} = \lambda e_{x_0}$, it follows that vector field $e_x = (L_a)_{*,x_0}e_{x_0}$, where $x = ax_0$, is left invariant and satisfies $h_x e_x = \lambda e_x$. As a consequence, eigenvalues λ and $-\lambda$ of h are constant.

Due to Milnor [16], Lie group G can be either unimodular or nonunimodular. A Lie group G is called unimodular if its left invariant Haar measure is also right invariant. In terms of Lie algebra $\mathfrak{g}$, Lie group G is unimodular if and only if linear transformation ad_X is traceless for any $X \in \mathfrak{g}$.

4.1. Unimodular Case

We can consider an orthonormal basis $\{e_1, e_2 = \varphi e_1, e_3 = \xi\}$ of $\mathfrak{g}$ given by a φ basis, such that $he_1 = \lambda e_1$, $he_2 = -\lambda e_2$, which satisfies

$$[e_1, e_2] = \lambda_3 e_3, \quad [e_2, e_3] = \lambda_1 e_1, \quad [e_3, e_1] = \lambda_2 e_2,$$

where λ_1, λ_2 are constants. Since the structure is almost cosymplectic, we must have $\lambda_3 = 0$. Therefore, from now on, we denote G by $G(\lambda_1, \lambda_2)$. The Lie brackets are now given by

$$[e_1, e_2] = 0, \quad [e_2, e_3] = \lambda_1 e_1, \quad [e_3, e_1] = \lambda_2 e_2. \tag{3}$$

The Levi-Civita connection on $G(\lambda_1, \lambda_2)$ is expressed by

$$\begin{cases} \nabla_{e_1}e_1 = 0, & \nabla_{e_1}e_2 = -\frac{\lambda_1-\lambda_2}{2}e_3, & \nabla_{e_1}e_3 = \frac{\lambda_1-\lambda_2}{2}e_2, \\ \nabla_{e_2}e_1 = -\frac{\lambda_1-\lambda_2}{2}e_3, & \nabla_{e_2}e_2 = 0, & \nabla_{e_2}e_3 = \frac{\lambda_1-\lambda_2}{2}e_1, \\ \nabla_{e_3}e_1 = \frac{\lambda_1+\lambda_2}{2}e_2, & \nabla_{e_3}e_2 = -\frac{\lambda_1+\lambda_2}{2}e_1, & \nabla_{e_3}e_3 = 0. \end{cases} \tag{4}$$

Remark 1. *ξ is parallel if and only if $\lambda_1 = \lambda_2$.*

We can compute the eigenvalue λ of h:

$$he_1 = \varphi\nabla_{e_1}\xi = \varphi(\frac{\lambda_1 - \lambda_2}{2}e_2) = \frac{\lambda_2 - \lambda_1}{2}e_1.$$

It follows $\lambda = \frac{\lambda_2-\lambda_1}{2}$ and hence $||h||^2 = \frac{(\lambda_1-\lambda_2)^2}{2}$.

For later use, we give the expressions of Riemannian curvature R, Ricci tensor Ric, and Ricci operator Q:

$$\begin{cases} R(e_1,e_2)e_1 = -\frac{(\lambda_1-\lambda_2)^2}{4}e_2, & R(e_1,e_2)e_2 = \frac{(\lambda_1-\lambda_2)^2}{4}e_1, \\ R(e_1,e_3)e_1 = -\frac{(\lambda_1-\lambda_2)(\lambda_1+3\lambda_2)}{4}e_3, & R(e_1,e_3)e_3 = \frac{(\lambda_1-\lambda_2)(\lambda_1+3\lambda_2)}{4}e_1, \\ R(e_2,e_3)e_2 = \frac{(\lambda_1-\lambda_2)(3\lambda_1+\lambda_2)}{4}e_3, & R(e_2,e_3)e_3 = -\frac{(\lambda_1-\lambda_2)(3\lambda_1+\lambda_2)}{4}e_2, \\ R(e_1,e_2)e_3 = 0, \quad R(e_1,e_3)e_2 = 0, & R(e_2,e_3)e_1 = 0, \end{cases} \tag{5}$$

$$\begin{cases} Ric(e_1e_1) = \frac{\lambda_1^2-\lambda_2^2}{2}, & Ric(e_1,e_2) = 0, \quad Ric(e_1,e_3) = 0 \\ Ric(e_2,e_2) = -\frac{\lambda_1^2-\lambda_2^2}{2}, & Ric(e_2,e_3) = 0, \quad Ric(e_3,e_3) = -\frac{(\lambda_1-\lambda_2)^2}{2}, \end{cases} \tag{6}$$

$$Qe_1 = \frac{\lambda_1^2-\lambda_2^2}{2}e_1, \quad Qe_2 = -\frac{\lambda_1^2-\lambda_2^2}{2}e_2, \quad Qe_3 = -\frac{(\lambda_1-\lambda_2)^2}{2}e_3. \tag{7}$$

Remark 2. *The metric g is flat if and only if $\lambda_1 = \lambda_2$.*

Remark 3. *The basis $\{e_1, e_2, e_3\}$ diagonalizes the Ricci operator Q.*

The covariant derivative of φ is given by

$$\begin{cases} (\nabla_{e_1}\varphi)e_1 = \frac{\lambda_2-\lambda_1}{2}e_3, & (\nabla_{e_1}\varphi)e_2 = 0, & (\nabla_{e_1}\varphi)e_3 = \frac{\lambda_1-\lambda_2}{2}e_1, \\ (\nabla_{e_2}\varphi)e_1 = 0, & (\nabla_{e_2}\varphi)e_2 = \frac{\lambda_1-\lambda_2}{2}e_3, & (\nabla_{e_2}\varphi)e_3 = -\frac{\lambda_1-\lambda_2}{2}e_2, \\ (\nabla_{e_3}\varphi)e_1 = 0, & (\nabla_{e_3}\varphi)e_2 = 0, & (\nabla_{e_3}\varphi)e_3 = 0. \end{cases} \tag{8}$$

Proposition 1. *$G(\lambda_1, \lambda_2)$ is cosymplectic if and only if $\lambda_1 = \lambda_2$.*

The previous Formula (8) can be summarized as

$$(\nabla_X\varphi)Y = g(hX,Y)\xi - \eta(Y)hX, \tag{9}$$

for all X, Y tangent to M. See also [17].

Recall also the following fact due to Olszak [17].

Proposition 2. *An almost contact Riemannian 3-manifold is cosymplectic if and only if ξ is parallel.*

As a matter of fact, recall an interesting result obtained in [18].

Proposition 3. *The almost cosymplectic unimodular Lie group $G(\lambda_1, \lambda_2)$ has vanishing characteristic Jacobi operator $\ell = R(\bullet, \xi)\xi$ if and only if $\lambda_1 = \lambda_2$.*

4.1.1. Contact Magnetic Geodesics in $G(\lambda_1, \lambda_2)$

We study arc-length parametrized curves $\gamma : I \to G(\lambda_1, \lambda_2)$ in the almost cosymplectic Lie group $G(\lambda_1, \lambda_2)$, which satisfy the Lorentz equation

$$\nabla_{\dot\gamma}\dot\gamma = q\varphi\dot\gamma, \quad q \in \mathbb{R} \setminus \{0\}.$$

First, we decompose $\dot\gamma$ in basis $\{e_1, e_2, e_3\}$, that is,

$$\dot\gamma = T_1e_1 + T_2e_2 + T_3e_3,$$

where T_1, T_2 and T_3 are smooth functions on I. Second, we compute the acceleration of γ using (4)

$$\nabla_{\dot\gamma}\dot\gamma = (T_1' - \lambda_2 T_2 T_3)e_1 + (T_2' + \lambda_1 T_1 T_3)e_2 + (T_3' + (\lambda_2 - \lambda_1)T_1 T_2)e_3.$$

Thus, the Lorentz equation can be written as the following ordinary differential equation (ODE) system:

$$\begin{cases} T_1' = \lambda_2 T_2 T_3 - qT_2, \\ T_2' = -\lambda_1 T_1 T_3 + qT_1, \\ T_3' = (\lambda_1 - \lambda_2)T_1 T_2. \end{cases} \tag{10}$$

Remark 4. *The contact angle θ, which is the angle between the velocity vector $\dot\gamma(s)$ and the characteristic vector $\xi(\gamma(s))$, is not automatically constant. However, if $\lambda_1 = \lambda_2$, then contact angle θ is constant (see also [7]).*

Remark 5. *The first curvature of a normal contact magnetic curve in an almost cosymplectic manifold is $\kappa_1 = |q|\sin\theta$. Hence, it is not always constant. A normal contact magnetic curve in a cosymplectic manifold is a helix (see also [7]).*

In the sequel, we study this system by distinguishing two cases:

Case U1 $\lambda_1 = \lambda_2 := \lambda_0$. The unimodular Lie group $G(\lambda_0, \lambda_0)$ is locally isometric either to $\tilde{\mathbb{E}}_2$ (when $\lambda_0 \neq 0$), or to the Euclidean space $\mathbb{E}_3$ (when $\lambda_0 = 0$).

One obtains that $T_3 = \cos\theta$; hence, $T_1^2(s) + T_2^2(s) = \sin^2\theta$. Then, the first two equations in (10) become

$$\begin{cases} T_1'(s) = -(q - \lambda_0\cos\theta)T_2(s), \\ T_2'(s) = (q - \lambda_0\cos\theta)T_1(s). \end{cases}$$

Denote $\omega := q - \lambda_0\cos\theta$. It follows that the solution of the system above is

$$\begin{cases} T_1(s) = c_1\cos(\omega s) + c_2\sin(\omega s), \\ T_2(s) = c_1\sin(\omega s) - c_2\cos(\omega s), \end{cases} \tag{11}$$

where $c_1, c_2 \in \mathbb{R}$ such that $c_1^2 + c_2^2 = \sin^2\theta$.

Case U2 $\lambda_1 \neq \lambda_2$ As we saw, the Lie group $G(\lambda_1, \lambda_2)$ is not a cosymplectic manifold. The possible situations are

- the Heisenberg group H^3 when the Perrone invariant p vanishes;
- the universal covering $\tilde{E}(2)$ of the group motions of Euclidean 2-space, when $p > 0$;
- the group $E(1,1)$ of rigid motions of Minkowski 2-space when $p < 0$.

The Perrone invariant is defined by

$$p := ||\mathcal{L}_\xi h|| - 2||h||^2 = |\lambda_1 - \lambda_2|\left(\sqrt{\lambda_1^2 + \lambda_2^2} - |\lambda_1 - \lambda_2|\right).$$

In ODE System (10), multiply the first equation by $\lambda_1 T_1$, and the second by $\lambda_2 T_2$; then, add the two obtained equations to obtain

$$\lambda_1 T_1 T_1' + \lambda_2 T_2 T_2' = -(\lambda_1 - \lambda_2)qT_1 T_2.$$

Using the third equation in (10), we get

$$\frac{1}{2}(\lambda_1 T_1^2 + \lambda_2 T_2^2)' = -qT_3'.$$

Hence, the expression $\frac{1}{2}\left(\lambda_1 T_1^2 + \lambda_2 T_2^2\right) + qT_3$ is constant. Denote it by $\frac{1}{2}\left(\frac{\lambda_1+\lambda_2}{2} + c_0\right)$. Taking into account the arc-length condition, we immediately obtain

$$\begin{cases} T_1^2 = \dfrac{1}{\lambda_1 - \lambda_2}\left[\lambda_2 T_3^2 - 2qT_3 + \dfrac{\lambda_1 - \lambda_2}{2} + c_0\right], \\ T_2^2 = \dfrac{1}{\lambda_2 - \lambda_1}\left[\lambda_1 T_3^2 - 2qT_3 + \dfrac{\lambda_2 - \lambda_1}{2} + c_0\right]. \end{cases} \tag{12}$$

$T_3 = \cos\theta$ and can be obtained by integration from the third equation of System (10), which can be written in the following form:

$$T_3'(s) = \sqrt{P(T_3(s))}, \tag{13}$$

where P is a polynomial of degree four if $\lambda_1\lambda_2 \neq 0$, respectively of degree three if $\lambda_1\lambda_2 = 0$.

The integration in (13) involves elliptic functions that are not easy to work with.

Remark 6. *Milnor's table shows that $G(\lambda_1, \lambda_2)$ is represented by the Heisenberg group when $\lambda_1 = 0$ or $\lambda_2 = 0$ (but not both). The Heisenberg group can carry one almost contact structure that leads to a Sasakian space form, and another almost contact structure that leads to an almost cosymplectic manifold (see, e.g., [18] Example 5.3). Analogously, the space Sol_3 admits both a left invariant contact metric structure and a left invariant almost cosymplectic structure.*

We already indicated that a magnetic curve in a cosymplectic 3-manifold is slant, that is, its contact angle θ is constant. Moreover, it is a helix, meaning that curvatures κ_1 and κ_2 are also constant. Since Case **U1** leads to a cosymplectic structure on $G(\lambda_0, \lambda_0)$, we are interested to obtain the situations when a magnetic curve in $G(\lambda_1, \lambda_2)$ (with $\lambda_1 \neq \lambda_2$) is slant. If this is the case, we must locally have $T_1 T_2 = 0$. This leads to the following possibilities:

(i) $\gamma'(s) = \pm e_3$;
(ii) $\gamma'(s) = \pm\sin\theta e_1 + \cos\theta e_3$ and $q = \lambda_1\cos\theta$, $\theta \neq 0, \pi$;
(iii) $\gamma'(s) = \pm\sin\theta e_2 + \cos\theta e_3$ and $q = \lambda_2\cos\theta$, $\theta \neq 0, \pi$.

As a consequence, we can easily prove the following proposition.

Proposition 4. *Let γ be a slant contact normal magnetic curve in the almost cosymplectic unimodular Lie group $G(\lambda_1, \lambda_2)$ (with $\lambda_1 \neq \lambda_2$). Then*

(i) either γ is an integral curve of ξ case when it is a geodesic,
(ii) or γ is a helix with curvatures $\kappa_1 = |q|\sin\theta$ and $\kappa_2 = \frac{1}{2}|\lambda_2 + \lambda_1\cos 2\theta|$,
(iii) or γ is a helix with curvatures $\kappa_1 = |q|\sin\theta$ and $\kappa_2 = \frac{1}{2}|\lambda_1 + \lambda_2\cos 2\theta|$.

Remark 7. *In the case (ii) of the previous proposition, curvature κ_2 may be rewritten as $\kappa_2 = \left|q\cos\theta - \frac{\lambda_1-\lambda_2}{2}\right|$. In Case (iii), we rewrite $\kappa_2 = \left|q\cos\theta + \frac{\lambda_1-\lambda_2}{2}\right|$. See the analogy with the expression of the curvature κ_2 obtained in [7].*

Remark 8. *The study in [19] is an interesting survey on slant curves in 3-dimensional almost contact metric geometry where the slant curves in 3-dimensional solvable Lie groups equipped with natural left invariant almost contact metric structure are studied as well. Slant magnetic curves in almost cosymplectic space Sol_3 are studied in [20]. Lie group Sol_3 may be realized as the unimodular Lie group with $\lambda_2 = -\lambda_1 \neq 0$ (see, for example, §3 in [21]).*

We continue our investigation in the Lie group $G(\lambda_0, \lambda_0)$ endowed with its cosymplectic structure. In [7], we focused on cosymplectic manifolds $M^2 \times \mathbb{R}$, where M^2 is a complex space form of complex dimension 1. In particular, we obtained important geometric properties for normal contact magnetic curves in $\mathbb{S}^2 \times \mathbb{R}$ and $\mathbb{H}^2 \times \mathbb{R}$ (see also [5,6]).

Example 1 (Contact magnetic geodesics in the Euclidean space $\mathbb{E}^3$)**.** *For $\lambda_0 = 0$, the Lie group $G(0,0)$ is nothing but the Euclidean 3-space $\mathbb{E}^3$. If we denote the global coordinates by x, y and z, the multiplication law is the standard one:*

$$(x_1, y_1, z_1) * (x_2, y_2, z_2) = (x_1 + x_2, y_1 + y_2, z_1 + z_2)$$

and the left invariant vector fields (that appear, previously, in the general description) are

$$e_1 = \frac{\partial}{\partial x}\,,\; e_2 = \frac{\partial}{\partial y} \text{ and } e_3 = \frac{\partial}{\partial z}.$$

Thus, we find $\omega = q$ and

$$\dot{\gamma}(s) = (c_1 \cos(qs) + c_2 \sin(qs), c_1 \sin(qs) - c_2 \cos(qs), \cos\theta),$$

with $c_1 + c_2 = \sin^2\theta$.

Setting $\gamma(0) = (0,0,0)$ and $\dot{\gamma}(0) = (u_0, v_0, \cos\theta)$, such that $u_0^2 + v_0^2 = \sin^2\theta$, we immediately find

$$\gamma(s) = \left(\frac{u_0}{q}\sin(qs) - \frac{v_0}{q}(1 - \cos(qs)), \frac{u_0}{q}(1 - \cos(qs)) + \frac{v_0}{q}\sin(qs), s\cos\theta\right).$$

See also [22].

4.1.2. Contact Magnetic Geodesics in the Euclidean Motion Group $\tilde{E}_2$

This space is realized as $\mathbb{R}^3$, on which we denote the global coordinates by x, y, z and set the multiplication

$$(x_1, y_1, z_1) * (x_2, y_2, z_2) = (x_1 + x_2 \cos z_1 - y_2 \sin z_1, y_1 + x_2 \sin z_1 + y_2 \cos z_1, z_1 + z_2).$$

Let $a, b, c > 0$ and define three left invariant vector fields

$$e_1 = \frac{1}{a}\left(\cos z\frac{\partial}{\partial x} + \sin z\frac{\partial}{\partial y}\right), e_2 = \frac{1}{b}\left(-\sin z\frac{\partial}{\partial x} + \cos z\frac{\partial}{\partial y}\right), e_3 = \frac{1}{c}\frac{\partial}{\partial z},$$

whose Lie brackets are given by

$$[e_1, e_2] = 0,\; [e_2, e_3] = \frac{a}{bc}e_1,\; [e_3, e_1] = \frac{b}{ca}e_2.$$

Hence, the Lie group $\tilde{E}_2$ is the unimodular Lie group $G(\lambda_1, \lambda_2)$ with $\lambda_1 = \frac{a}{bc}$ and $\lambda_2 = \frac{b}{ca}$. The dual basis is given by

$$\eta_1 = a(\cos z dx + \sin z dy),\; \eta_2 = b(-\sin z dx + \cos z dy),\; \eta = c dz.$$

The left invariant metric g is determined by the condition that $\{e_1, e_2, e_3\}$ is an orthonormal basis with respect to it, and it can be expressed as

$$g = \eta_1 \otimes \eta_1 + \eta_2 \otimes \eta_2 + \eta \otimes \eta.$$

Defining $\xi = e_3$ and φ as before, i.e., $\varphi e_1 = e_2$, $\varphi e_2 = -e_1$, $\varphi e_3 = 0$, we obtain a homogeneous almost contact structure which is almost cosymplectic. It can be easily check that $(\tilde{E}_2, \varphi, \xi, \eta, g)$ is cosymplectic if and only if $b = a$. In this case the metric g is flat and takes an easier form

$$g = a^2(dx^2 + dy^2) + c^2 dz^2.$$

In order to obtain an explicit expression for contact normal magnetic geodesics $\gamma : I \to \tilde{E}_2$, $\gamma(s) = (x(s), y(s), z(s))$ we need to express velocity vector $\dot{\gamma}$ in terms of the basis $\{e_1, e_2, e_3\}$

and then to use the obtained results in Case **U1**. For this end, we consider $b = a$; hence, $\tilde{E}_2$ is obtained as $G(\lambda_0, \lambda_0)$ with $\lambda_0 = \frac{1}{c} \neq 0$. Then we obtain

$$\dot{\gamma} = a(\dot{x}\cos z + \dot{y}\sin z)e_1 + a(-\dot{x}\sin z + \dot{y}\cos z)e_2 + c\dot{z}e_3,$$

where we denoted $\dot{x} = \frac{dx}{ds}$, $\dot{y} = \frac{dy}{ds}$ and so on. Subsequently, component functions x, y and z satisfy the following ordinary differential equation system:

$$\begin{cases} \dot{x}\cos z + \dot{y}\sin z &= \frac{1}{a}\big(c_1\cos(\omega s) + c_2\sin(\omega s)\big), \\ -\dot{x}\sin z + \dot{y}\cos z &= \frac{1}{a}\big(c_1\sin(\omega s) - c_2\cos(\omega s)\big), \\ \dot{z} &= \frac{\cos\theta}{c}, \end{cases} \tag{14}$$

where $c_1^2 + c_2^2 = \sin^2\theta$ and $\omega = q - \lambda_0\cos\theta$.

ODE System (14) can be rewritten as

$$\begin{cases} \dot{x} &= \frac{1}{a}\big[c_1\cos\big(\omega s + z(s)\big) + c_2\sin\big(\omega s + z(s)\big)\big], \\ \dot{y} &= \frac{1}{a}\big[c_1\sin\big(\omega s + z(s)\big) - c_2\cos\big(\omega s + z(s)\big)\big], \\ z &= \frac{\cos\theta}{c}s + z_0, \quad z_0 := z(0). \end{cases}$$

Taking into account that $\omega + \frac{\cos\theta}{c} = q$, we obtain

$$\begin{cases} \dot{x} &= \frac{1}{a}[c_1\cos(qs + z_0) + c_2\sin(qs + z_0)], \\ \dot{y} &= \frac{1}{a}[c_1\sin(qs + z_0) - c_2\cos(qs + z_0)], \\ z &= \frac{\cos\theta}{c}s + z_0. \end{cases} \tag{15}$$

Setting initial conditions $\gamma(0) = (0, 0, 0)$, $\dot{\gamma}(0) = (u_0, v_0, \frac{1}{c}\cos\theta)$, with $u_0^2 + v_0^2 = \frac{1}{a^2}\sin^2\theta$, we conclude with the explicit expression for the magnetic curve

$$\gamma(s) = \left(\frac{u_0}{q}\sin(qs) - \frac{v_0}{q}(1 - \cos(qs)), \frac{u_0}{q}(1 - \cos(qs)) + \frac{v_0}{q}\sin(qs), \lambda_0 s\cos\theta\right).$$

See the analogy with the expression of magnetic geodesics in Euclidean space $\mathbb{E}^3$.

4.1.3. Magnetic Jacobi Fields on $G(\lambda_0, \lambda_0)$

We have at least two advantages in considering Lie group $G(\lambda_0, \lambda_0)$ (endowed with its homogeneous almost cosymplectic structure defined before) as the background for this study. The first is the parallelism of Lorentz force $\phi = q\varphi$, since the space is cosymplectic. The second is the flatness of the metric. Therefore, the Jacobi magnetic equation reduces a lot. More precisely, if $W(s)$ is a magnetic Jacobi field along the normal contact magnetic geodesic $\gamma(s)$, then it satisfies the following ODE:

$$\frac{D^2W}{ds^2} - q\varphi\frac{DW}{ds} = 0. \tag{16}$$

To find explicit solutions of this equation, we use basis $\{e_1, e_2, e_3\}$.

Let $\frac{DW}{ds} = A_1e_1 + A_2e_2 + A_3e_3$, where A_1, A_2, A_3 are smooth functions on I. Even if velocity vector $\dot{\gamma}(s)$ does not appear explicitly in the Equation (16), it is obtained from (11) togeher with the condition for γ of being slant. Equation (16) becomes

$$\begin{cases} A_1'(s) = -\omega A_2(s), \\ A_2'(s) = \ \omega A_1(s), \\ A_3'(s) = 0, \end{cases}$$

where $\omega = q - \lambda_0 \cos\theta$, as before. The general solution of this system is

$$A_1(s) = a_1 \cos(\omega s) + a_2 \sin(\omega s),\ A_2(s) = a_1 \sin(\omega s) - a_2 \cos(\omega s),\ A_3(s) = a_3,$$

where a_1, a_2, a_3 are real constants.

To find magnetic Jacobi field $W(s) = W_1(s)e_1 + W_2(s)e_2 + W_3(s)e_3$, where W_1, W_2, W_3 are smooth function on I, we integrate equation $\frac{DW}{ds} = A_1e_1 + A_2e_2 + A_3e_3$. We have

$$\begin{cases} W_1'(s) - \lambda_0 \cos\theta\, W_2(s) = A_1(s), \\ W_2'(s) + \lambda_0 \cos\theta\, W_1(s) = A_2(s), \\ W_3'(s) = A_3(s). \end{cases}$$

It is straightforward to show that the solution of this system is given by

$$\begin{cases} W_1(s) = \frac{1}{q}\big(a_1 \sin(\omega s) - a_2 \cos(\omega s)\big) + \zeta_1 \cos(\lambda_0 s \cos\theta) + \zeta_2 \sin(\lambda_0 s \cos\theta), \\ W_2(s) = -\frac{1}{q}\big(a_1 \cos(\omega s) + a_2 \sin(\omega s)\big) - \zeta_1 \sin(\lambda_0 s \cos\theta) + \zeta_2 \cos(\lambda_0 s \cos\theta), \\ W_3(s) = a_3 s + \zeta_3, \end{cases}$$

where ζ_1, ζ_2 and ζ_3 are real constants.

Remark 9. *In the special case when $\omega = \frac{q}{2}$, equivalently to $\omega = \lambda_0 \cos\theta$, the above expressions simplify. Moreover, functions* $\sin(\omega s)$, $\cos(\omega s)$, $\sin(\lambda_0 s \cos\theta)$ *and* $\cos(\lambda_0 s \cos\theta)$ *are no longer independent.*

Example 2. *In the Euclidean case, we have $\lambda_0 = 0$; hence, $\omega = q$. Consequently, magnetic Jacobi field $W(s)$ can be expressed as*

$$\begin{cases} W_1(s) = \frac{1}{q}(a_1 \sin(qs) - a_2 \cos(qs)) + \zeta_1, \\ W_2(s) = -\frac{1}{q}(a_1 \cos(qs) + a_2 \sin(qs)) + \zeta_2, \\ W_3(s) = a_3 s + \zeta_3, \end{cases}$$

where ζ_1, ζ_2 and ζ_3 are real constants. See also [8] (Theorem 3.1).

Remark 10. *When magnetic geodesic γ in unimodular Lie group $G(\lambda_0, \lambda_0)$ is not an integral curve of the characteristic vector field, that is, if $\dot\gamma$ is not collinear to ξ, magnetic Jacobi field $W(s)$ may be expressed in the basis $\{\dot\gamma, \varphi\dot\gamma, \xi\}$. It is straightforward to compute coefficients of decomposition $W(s) = A(s)\dot\gamma + B(s)\varphi\dot\gamma + C(s)\xi(\gamma(s))$. We find*

$$\begin{aligned} \sin^2\theta\, A(s) &= (c_1\zeta_1 - c_2\zeta_2)\cos(qs) + (c_1\zeta_2 + c_2\zeta_1)\sin(qs) + \tfrac{c_2a_1 - c_1a_2}{q} \\ \sin^2\theta\, B(s) &= (c_1\zeta_2 + c_2\zeta_1)\cos(qs) - (c_1\zeta_1 - c_2\zeta_2)\sin(qs) - \tfrac{a_1c_1 + a_2c_2}{q} \\ C(s) &= a_3 s + \zeta_3 - \cos\theta A(s), \end{aligned}$$

where constants c_1, c_2, c_3, a_1, a_2, a_3, ζ_1, ζ_2 and ζ_3 are those obtained before. See also [8].

4.2. Nonunimodular Case

Let us now consider a (simply connected) three dimensional nonunimodular Lie group G equipped with a left invariant almost cosymplectic structure. We briefly describe a φ basis and emphasize the commutation relations in the corresponding Lie algebra $\mathfrak{g}$ following the construction given by Perrone [4].

A first remark is that ξ belongs to kernel $\mathfrak{u} = \{X \in \mathfrak{g} \ :\ \mathrm{tr}\ ad_X = 0\}$, which is two-dimensional and unimodular. Then, we consider an orthonormal basis $\{e_2, e_3 = \xi\}$ of $\mathfrak{u}$.

Hence, $e_1 := -\varphi e_2$ belongs to orthogonal complement $\mathfrak{u}^\perp$ of $\mathfrak{u}$. Since ad_{e_1} preserves $\mathfrak{u}$, and taking into account that η is closed and $\nabla_\xi \xi = 0$, we obtain that

$$[e_1, e_2] = \alpha e_2, \quad [e_1, e_3] = \beta e_2, \quad [e_2, e_3] = 0, \tag{17}$$

where α and β are constants with $\alpha \neq 0$. We denote this Lie algebra by $\mathfrak{g}(\alpha, \beta)$. Explicit descriptions of $\mathfrak{g}(\alpha, \beta)$ and of corresponding simply connected Lie group $G(\alpha, \beta) = \exp\mathfrak{g}(\alpha, \beta)$ are presented in [18].

The Levi-Civita connection on $G(\alpha, \beta)$ is given by

$$\begin{cases} \nabla_{e_1} e_1 = 0, & \nabla_{e_1} e_2 = -\frac{\beta}{2} e_3, \quad \nabla_{e_1} e_3 = \frac{\beta}{2} e_2, \\ \nabla_{e_2} e_1 = -\alpha e_2 - \frac{\beta}{2} e_3, & \nabla_{e_2} e_2 = \alpha e_1, \quad \nabla_{e_2} e_3 = \frac{\beta}{2} e_1, \\ \nabla_{e_3} e_1 = -\frac{\beta}{2} e_2, & \nabla_{e_3} e_2 = \frac{\beta}{2} e_1, \quad \nabla_{e_3} e_3 = 0. \end{cases} \tag{18}$$

Remark 11. *Characteristic vector field ξ is parallel if and only if $\beta = 0$.*

We give the expressions of operator h, Riemannian curvature R, Ricci tensor Ric and Ricci operator Q:

$$he_1 = -\frac{\beta}{2} e_1, \quad he_2 = \frac{\beta}{2} e_2, \quad he_3 = 0, \tag{19}$$

$$\begin{cases} R(e_1, e_2)e_1 = \left(\alpha^2 - \frac{\beta^2}{4}\right) e_2 + \alpha\beta e_3, & R(e_1, e_2)e_2 = -\left(\alpha^2 - \frac{\beta^2}{4}\right) e_1, \\ R(e_1, e_2)e_3 = -\alpha\beta e_1, & R(e_1, e_3)e_1 = \alpha\beta e_2 + \frac{3\beta^2}{4} e_3, \\ R(e_1, e_3)e_2 = -\alpha\beta e_1, & R(e_1, e_3)e_3 = -\frac{3\beta^2}{4} e_1, \\ R(e_2, e_3)e_1 = 0, \quad R(e_2, e_3)e_2 = -\frac{\beta^2}{4} e_3, & R(e_2, e_3)e_3 = \frac{\beta^2}{4} e_2, \end{cases} \tag{20}$$

$$\begin{cases} Ric(e_1, e_1) = -(\alpha^2 + \frac{\beta^2}{2}), & Ric(e_1, e_2) = 0, \quad Ric(e_1, e_3) = 0, \\ Ric(e_2, e_2) = \frac{\beta^2}{2} - \alpha^2, & Ric(e_2, e_3) = -\alpha\beta, \quad Ric(e_3, e_3) = -\frac{\beta^2}{2}, \end{cases} \tag{21}$$

$$Qe_1 = -\left(\alpha^2 + \frac{\beta^2}{2}\right) e_1, \; Qe_2 = \left(\frac{\beta^2}{2} - \alpha^2\right) e_2 - \alpha\beta e_3, \; Qe_3 = -\alpha\beta e_2 - \frac{\beta^2}{2} e_3. \tag{22}$$

Remark 12. *Since $\alpha \neq 0$, the metric g cannot be flat.*

Remark 13. *The basis $\{e_1, e_2, e_3\}$ diagonalizes Ricci tensor Q if and only if $\beta = 0$.*

Lastly, the covariant derivatives of φ are computed as

$$\begin{cases} (\nabla_{e_1} \varphi) e_1 = -\frac{\beta}{2} e_3, & (\nabla_{e_1} \varphi) e_2 = 0, \quad (\nabla_{e_1} \varphi) e_3 = \frac{\beta}{2} e_1, \\ (\nabla_{e_2} \varphi) e_1 = 0, & (\nabla_{e_2} \varphi) e_2 = \frac{\beta}{2} e_3, \quad (\nabla_{e_2} \varphi) e_3 = -\frac{\beta}{2} e_2, \quad \nabla_{e_3} \varphi = 0. \end{cases} \tag{23}$$

As a consequence, φ is parallel if and only if $\beta = 0$. Hence, we state:

Proposition 5. *Almost cosymplectic nonunimodular Lie group $G(\alpha, \beta)$ is cosymplectic if and only if $\beta = 0$.*

In analogy with Proposition 3 we recall the following result obtained in [18].

Proposition 6. *Almost cosymplectic nonunimodular Lie group $G(\alpha, \beta)$ has a vanishing characteristic Jacobi operator if and only if $\beta = 0$.*

4.2.1. Contact Magnetic Geodesics in $G(\alpha, \beta)$

We study arc-length parametrized curves $\gamma : I \to G(\alpha, \beta)$ in almost cosymplectic Lie group $G(\alpha, \beta)$ that satisfies Lorentz equation

$$\nabla_{\dot{\gamma}}\dot{\gamma} = q\varphi\dot{\gamma}, \qquad q \in \mathbb{R} \setminus \{0\}.$$

In order to compute the acceleration of γ, we need to decompose its speed $\dot{\gamma}$ in basis $\{e_1, e_2, e_3\}$ as

$$\dot{\gamma} = T_1 e_1 + T_2 e_2 + T_3 e_3, \quad T_1, T_2, T_3 \in C^\infty(I).$$

Thus, we write

$$\nabla_{\dot{\gamma}}\dot{\gamma} = (T_1' + \alpha T_2^2 + \beta T_2 T_3)e_1 + (T_2' - \alpha T_1 T_2)e_2 + (T_3' - \beta T_1 T_2)e_3.$$

Hence, the Lorentz equation leads to the following ODE system:

$$\begin{cases} T_1' + (\alpha T_2 + q)T_2 + \beta T_2 T_3 = 0, \\ T_2' - (\alpha T_2 + q)T_1 = 0, \\ T_3' - \beta T_1 T_2 = 0. \end{cases} \tag{24}$$

Remark 14. *The contact angle θ is, as expected, not automatically constant. Nevertheless, when $\beta = 0$, the third equation in (24) implies that $T_3 = \cos\theta$ is constant. This reminds us that a contact magnetic curve in a cosymplectic manifold is slant.*

Let us study Case **NU1**: $\beta = 0$.

A special case is furnished by the integral curves of ξ, the case in which γ is a geodesic. Suppose now that $\sin\theta \neq 0$. Since $T_1(s)^2 + T_2(s)^2 = \sin^2\theta$ it follows that there exists (locally) a function u, such that

$$T_1(s) = \sin\theta\cos u(s) \quad \text{and} \quad T_2(s) = \sin\theta\sin u(s).$$

Plugging these expressions in the first two equations of (24), we obtain

$$\begin{cases} \big(u' - \alpha\sin\theta\sin u - q\big)\sin u = 0, \\ \big(u' - \alpha\sin\theta\sin u - q\big)\cos u = 0. \end{cases}$$

A particular solution is $u = \pm\frac{\pi}{2}$ which can be obtained when and only when $q = \mp\alpha\sin\theta$.

In this situation $\dot{\gamma}(s) = \pm\sin\theta e_2 + \cos\theta e_3$.

Other constant solution (for u) may be obtained as $u = u_0 := -\arcsin\frac{q}{\alpha\sin\theta}$; in this case, we need to have $\left|\frac{q}{\alpha\sin\theta}\right| \leq 1$.

In this situation $\dot{\gamma}(s) = \sin\theta\cos u_0 e_1 + \sin\theta\sin u_0 e_2 + \cos\theta e_3$.

Suppose that u is not a constant function. The general solution of the ODE

$$u' = \alpha\sin\theta\sin u + q$$

is $u(s) = 2\arctan t(s)$, where

(i) if $\alpha^2\sin^2\theta < q^2$, then $t(s) = \zeta\tan(\frac{q\zeta s}{2} + c_0) - \frac{\alpha\sin\theta}{q}$, where $\zeta = \sqrt{1 - \frac{\alpha^2\sin^2\theta}{q^2}}$;

(ii) if $q = \varepsilon\alpha\sin\theta$, then $t(s) = -\frac{2}{qs+c_0} - \varepsilon$, where $\varepsilon = \pm 1$;

(iii) if $\alpha^2\sin^2\theta > q^2$, then $t(s) = -\frac{\alpha\sin\theta}{q} + \begin{cases} \text{either} & -\zeta\coth(\frac{q\zeta s}{2} + c_0), \\ \text{or} & \zeta, \\ \text{or} & -\zeta\tanh(\frac{q\zeta s}{2} + c_0), \end{cases}$

where $\zeta = \sqrt{\frac{\alpha^2\sin^2\theta}{q^2} - 1}$.

Here $c_0 \in \mathbb{R}$. Hence, $T_1(s) = \sin\theta \,\dfrac{1-t(s)^2}{1+t(s)^2}$ and $T_2(s) = \sin\theta \,\dfrac{2t(s)}{1+t(s)^2}$.

Almost cosymplectic nonunimodular Lie group $G(\alpha,\beta)$ is cosymplectic if and only if $\beta = 0$. Since every contact magnetic curve in a cosymplectic manifold is slant, we are interested to find when a magnetic curve in $G(\alpha,\beta)$ (with $\beta \neq 0$) is slant. The third equation in (24) yields $T_1 T_2 = 0$ (locally). Thus, we obtain the following possibilities:

(i) $\dot\gamma(s) = \pm e_3$;

(ii) $\dot\gamma'(s) = \pm\sin\theta e_2 + \cos\theta e_3$ and $q = \mp\alpha\sin\theta - \beta\cos\theta$, $\theta \neq 0, \pi$.

Compare with the unimodular case.

As a consequence, we can easily prove the following proposition.

Proposition 7. *Let γ be a slant contact normal magnetic curve in the almost cosymplectic nonunimodular Lie group $G(\alpha,\beta)$ with $\beta \neq 0$. Then,*

(i) either γ is an integral curve of the ξ case when it is a geodesic;

(ii) or γ is a helix with curvatures $\kappa_1 = |q|\sin\theta$ and $\kappa_2 = |q\cos\theta + \frac{\beta}{2}|$.

Again, it is interesting to compare κ_2 from (ii) with the expression of curvature κ_2, obtained in [7].

Example 3. Let $G = \mathbb{R}^3(x,y,z)$, on which we consider the following multiplication law:

$$(x,y,z) * (x',y',z') = (x + x', y + e^{\alpha x} y', z + z'), \quad \alpha \in \mathbb{R} \setminus \{0\}.$$

In fact, G is semidirect product $\mathbb{R} \ltimes \mathbb{R}^2$. The set of left invariant vector fields on G is generated by

$$\frac{\partial}{\partial x}, \quad e^{\alpha x}\frac{\partial}{\partial y}, \quad \frac{\partial}{\partial z}.$$

We can also identify G with the set of all matrices of the following form:

$$X_\alpha(x,y,z) := \begin{pmatrix} 1 & 0 & 0 & x \\ 0 & e^{\alpha x} & 0 & y \\ 0 & 0 & 1 & z \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad x,y,z \in \mathbb{R},\ \alpha > 0.$$

In this interpretation, the set of left invariant vector fields is generated by the three following matrices:

$$E_1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & \alpha e^{\alpha x} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \; E_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & e^{\alpha x} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \; E_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Under identification $\mathbb{R}^3 \ni (x,y,z) \leftrightarrow X_\alpha(x,y,z)$, we obtain the correspondence for the left invariant vector fields and

$$\frac{\partial}{\partial x} \equiv E_1, \quad e^{\alpha x}\frac{\partial}{\partial y} \equiv E_2, \quad \frac{\partial}{\partial z} \equiv E_3.$$

With no danger of confusion, we denote by

$$E_1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & \alpha & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \; E_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \; E_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

the generators for Lie algebra $\mathfrak{g}$.

It is easy to check that the commuting relations are

$$[E_1, E_2] = \alpha E_2, \quad [E_2, E_3] = 0, \quad [E_3, E_1] = 0;$$

hence, G is a nonunimodular Lie group on that we can define, as usual, the cosymplectic structure. Let us indicate that the left invariant metric is

$$g = dx^2 + e^{2\alpha x} dy^2 + dz^2.$$

Consider now a normal contact magnetic geodesic $\gamma : I \to G$, $\gamma(s) = (x(s), y(s), z(s))$. Since velocity vector field $\dot{\gamma}$ was computed, we must solve the following ODE system in order to obtain explicit parametrization. for γ

$$\begin{cases} x'(s) = T_1(s), \\ y'(s) = e^{\alpha x(s)} T_2(s), \\ z'(s) = T_3 = \cos\theta. \end{cases} \tag{25}$$

The third coordinate z can be easily obtained as $z(s) = z(0) + s\cos\theta$.

Let us now obtain $x(s)$ and $y(s)$.

Then, we draw, in a series of figures, magnetic curve γ (in $\mathbb{R}^3$) and its projection on the hyperbolic plane. We need to indicate that this model is nothing but the product space $\mathbb{H}^2(-\alpha^2) \times \mathbb{R}$. The hyperbolic metric on $\mathbb{R}^2(x, y)$ is $dx^2 + e^{2\alpha x} dy^2$. We also consider the upper-half plane model for the hyperbolic plane via transformation $u = \alpha y$ and $v = e^{\alpha x}$. Thus, for each projection curve, we draw two pictures, one (central) lying in $\mathbb{R}^2(x, y)$ and one (on the right) lying in the upper-half plane model of $\mathbb{H}^2(-\alpha^2)$ with coordinates u and $v > 0$.

We have $T_1(s) = \sin\theta \, \dfrac{1 - t(s)^2}{1 + t(s)^2}$ and $T_2(s) = \sin\theta \, \dfrac{2t(s)}{1 + t(s)^2}$.

Case (i) $t(s) = \zeta \tan(\frac{q\zeta s}{2} + c_0) - \frac{\alpha \sin\theta}{q}$, where $\zeta = \sqrt{1 - \frac{\alpha^2 \sin^2\theta}{q^2}}$

Set $\psi \in (0, \pi)$ such that $\cos\psi = \frac{\alpha \sin\theta}{q}$; hence $\zeta = \sin\psi$. Then,

$$t(s) = -\frac{\cos(\psi + \frac{q\zeta s}{2} + c_0)}{\cos(\frac{q\zeta s}{2} + c_0)}$$

and

$$T_1(s) = \frac{\zeta \sin\theta \sin(\psi + q\zeta s + 2c_0)}{1 + \cos\psi \cos(\psi + q\zeta s + 2c_0)}.$$

Integrating, we obtain

$$x(s) = -\frac{1}{\alpha} \log\left(1 + \cos\psi \cos(\psi + q\zeta s + 2c_0)\right) + c_1, \quad c_1 \in \mathbb{R}. \tag{26}$$

Since

$$T_2(s) = -\sin\theta \, \frac{\cos\psi + \cos(\psi + q\zeta s + 2c_0)}{1 + \cos\psi \cos(\psi + q\zeta s + 2c_0)},$$

we find

$$y(s) = -\frac{e^{\alpha c_1} \cot\psi}{\alpha} \, \frac{\sin(\psi + q\zeta s + 2c_0)}{1 + \cos\psi \cos(\psi + q\zeta s + 2c_0)} + c_2, \quad c_2 \in \mathbb{R}. \tag{27}$$

In Figure 1 we draw the magnetic curve in $\mathbb{R}^3$ (left), its corresponding projection in $\mathbb{R}^2$ (center) and the projection on the hyperbolic plane $\mathbb{H}^2$ (right).

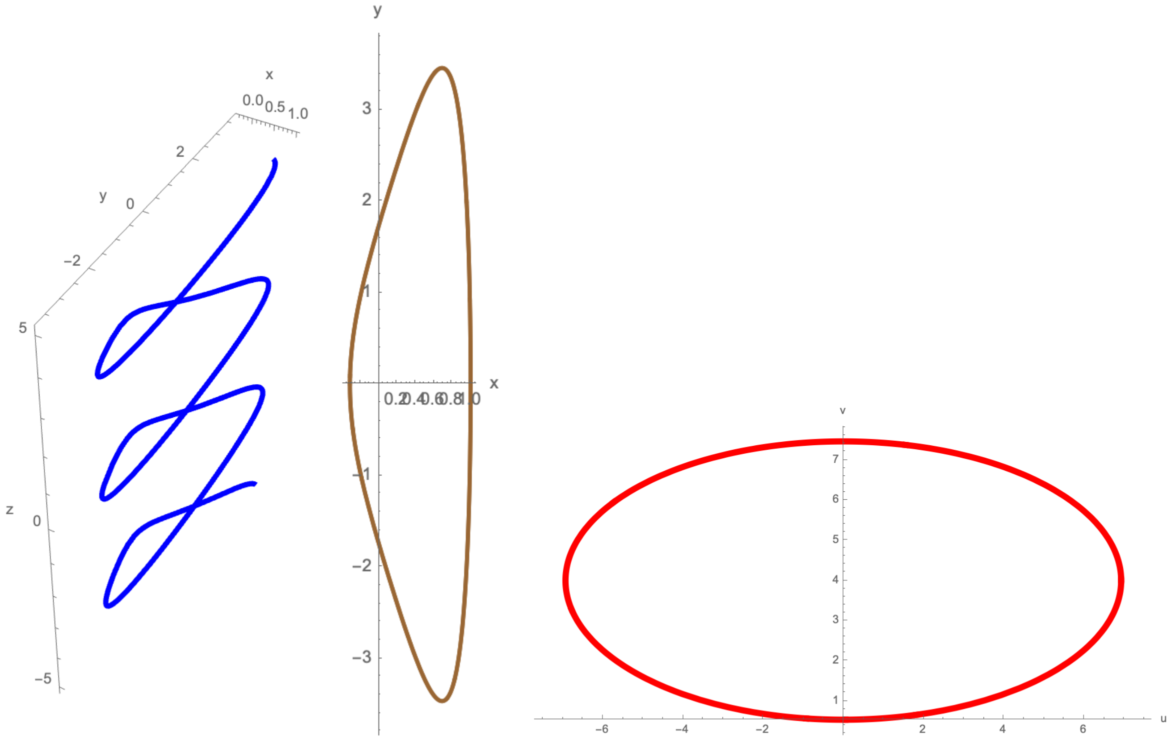

Figure 1. $\alpha = 2, q = 2, \theta = \frac{\pi}{3}, c_0 = 0, c_1 = 0, c_2 = 0.$

Case (ii) $t(s) = -\frac{2}{qs+c_0} - \varepsilon$
Integrating, we obtain

$$x(s) = -\frac{1}{\alpha} \log\left(2(qs+c_0)^2 + 4\varepsilon(qs+c_0) + 4\right) + c_1, \quad c_1 \in \mathbb{R},$$
$$y(s) = \frac{e^{\alpha c_1}(qs+c_0+\varepsilon)}{\alpha\left[2(qs+c_0)^2 + 4\varepsilon(qs+c_0) + 4\right]} + c_2, \quad c_2 \in \mathbb{R}.$$

In Figure 2 we draw the magnetic curve in $\mathbb{R}^3$ (left), its corresponding projection in $\mathbb{R}^2$ (center) and the projection on the hyperbolic plane $\mathbb{H}^2$ (right).

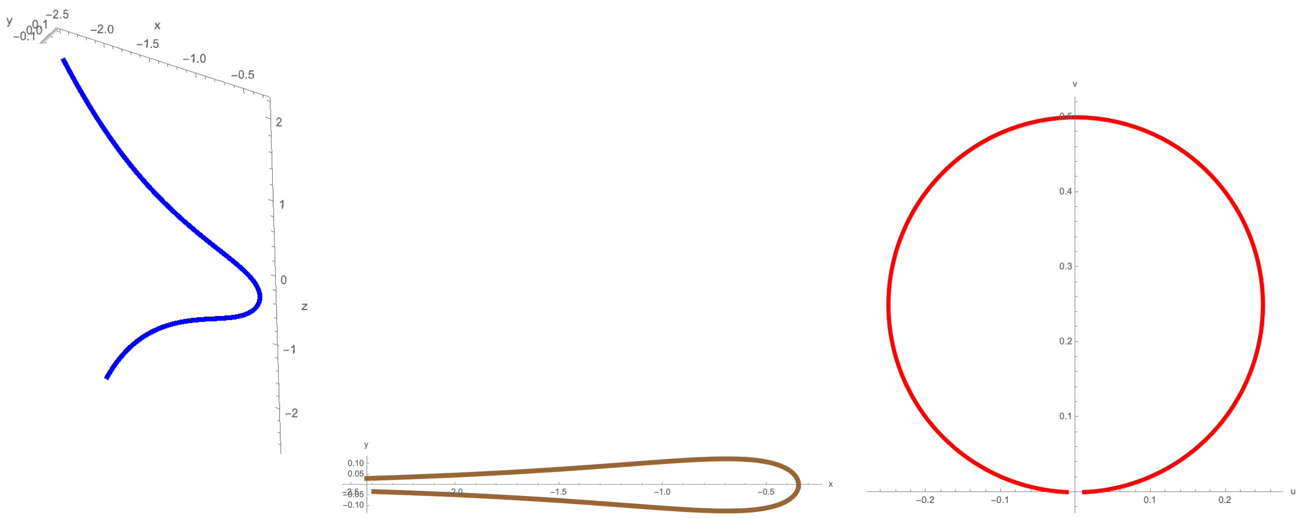

Figure 2. $\alpha = 2, \varepsilon = 1, q = \sqrt{3}, \theta = \frac{\pi}{3}, c_0 = 0, c_1 = 0, c_2 = 0.$

Case (iii) We need to distinguish three subcases. Before this, let us introduce $\psi > 0$, such that $\frac{\alpha \sin\theta}{q} = \varepsilon \cosh\psi$, $\varepsilon = \pm 1$ and $\zeta = \sinh\psi$.

Subcase (iii)-1 $t(s) = -\zeta \coth(\frac{q\zeta s}{2} + c_0) - \frac{\alpha \sin\theta}{q} = -\frac{\varepsilon \sinh(\frac{q\zeta s}{2} + c_0 + \varepsilon\psi)}{\sinh(\frac{q\zeta s}{2} + c_0)}$

Integrating, we obtain

$$x(s) = -\frac{1}{\alpha} \log\left(\cosh\psi \cosh(\varepsilon\psi + q\zeta s + 2c_0) - 1\right) + c_1, \quad c_1 \in \mathbb{R}$$
$$y(s) = \frac{e^{\alpha c_1} \coth\psi}{\alpha} \frac{\sinh(\varepsilon\psi + q\zeta s + 2c_0)}{\cosh\psi \cosh(\varepsilon\psi + q\zeta s + 2c_0) - 1} + c_2, \quad c_2 \in \mathbb{R}.$$

As before, in Figure 3 we draw the magnetic curve in $\mathbb{R}^3$ (left), its corresponding projection in $\mathbb{R}^2$ (center) and the projection on the hyperbolic plane $\mathbb{H}^2$ (right).

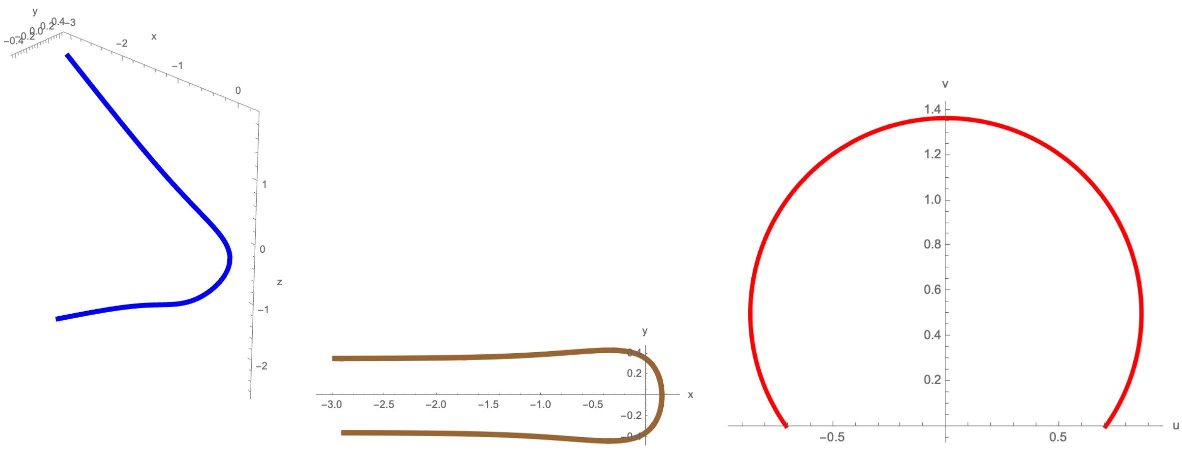

Figure 3. $\alpha = 2$, $\varepsilon = 1$, $q = 1$, $\theta = \frac{\pi}{3}$, $c_0 = 0$, $c_1 = 0$, $c_2 = 0$.

Subcase (iii)-2 $t(s) = -\frac{\alpha \sin\theta}{q} + \zeta$ (this leads to a constant solution for u)

It follows that $T_1 \neq 0$ and T_2 are constant; hence,

$$x(s) = T_1 s + c_1, \quad y(s) = \frac{T_2}{\alpha T_1} e^{\alpha(T_1 s + c_1)} + c_2, \quad c_1, c_2 \in \mathbb{R}.$$

Figure 4 represents the magnetic curve in $\mathbb{R}^3$ (left) and its corresponding projection in $\mathbb{R}^2$ (center) and on the hyperbolic plane $\mathbb{H}^2$ (right), respectively.

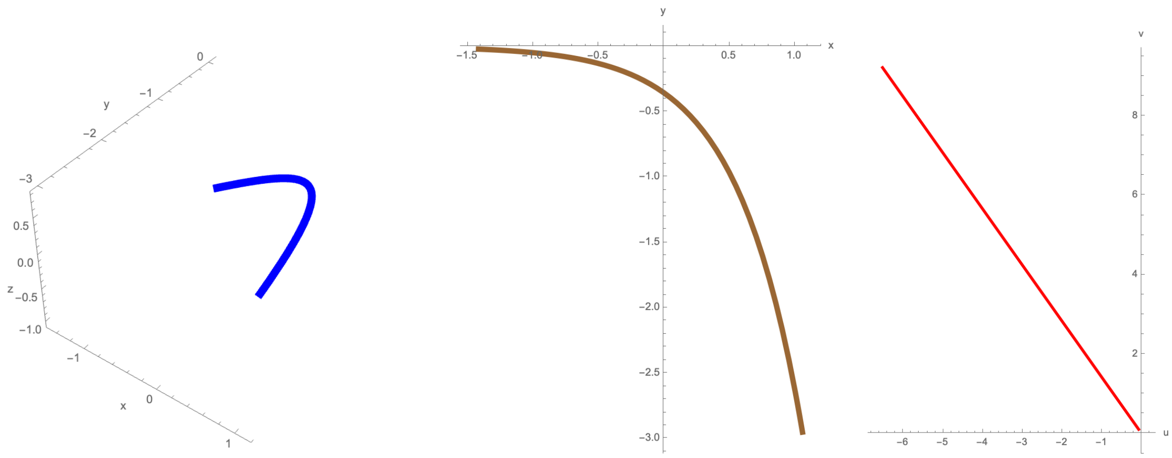

Figure 4. $\alpha = 2$, $\varepsilon = 1$, $q = 1$, $\theta = \frac{\pi}{3}$, $c_0 = 0$, $c_1 = 0$, $c_2 = 0$.

Subcase (iii)-3 $t(s) = -\zeta \tanh(\frac{q\zeta s}{2} + c_0) - \frac{\alpha \sin\theta}{q} = -\frac{\varepsilon \cosh(\frac{q\zeta s}{2} + c_0 + \varepsilon\psi)}{\cosh(\frac{q\zeta s}{2} + c_0)}$

Integrating, we obtain

$$x(s) = -\frac{1}{\alpha} \log\big(\cosh\psi \cosh(\varepsilon\psi + q\zeta s + 2c_0) + 1\big) + c_1, \quad c_1 \in \mathbb{R}$$
$$y(s) = -\frac{e^{\alpha c_1} \coth\psi}{\alpha} \frac{\sinh(\varepsilon\psi + q\zeta s + 2c_0)}{\cosh\psi \cosh(\varepsilon\psi + q\zeta s + 2c_0) + 1} + c_2, \quad c_2 \in \mathbb{R}.$$

Figure 5 represents the magnetic curve in $\mathbb{R}^3$ (left) and its corresponding projection in $\mathbb{R}^2$ (center) and on the hyperbolic plane $\mathbb{H}^2$ (right), respectively.

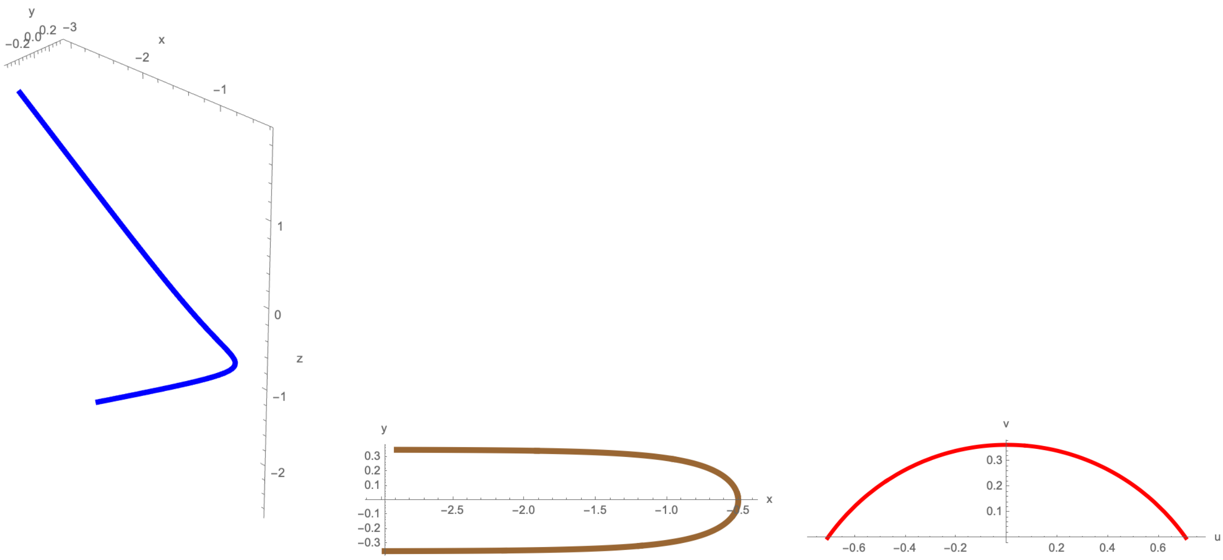

Figure 5. $\alpha = 2, \varepsilon = 1, q = 1, \theta = \frac{\pi}{3}, c_0 = 0, c_1 = 0, c_2 = 0$.

Remark 15. *The nonunimodular Lie group* $G(\alpha, 0) \overset{not.}{=} G(\alpha)$ *can be also considered as the set of all matrices of the following form:*

$$\begin{pmatrix} e^{(1+\alpha)x} & 0 & ye^x f(x) \\ 0 & e^x & ze^x \\ 0 & 0 & e^x \end{pmatrix}, \qquad x, y, z \in \mathbb{R},\ \alpha > 0,$$

where $f : \mathbb{R} \to \mathbb{R}$, $f(x) = \begin{cases} \frac{e^{\alpha x}-1}{\alpha x} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0. \end{cases}$

One can obtain a simply connected Lie group $G(\alpha)$ *whose Lie algebra* $\mathfrak{g}(\alpha)$ *is*

$$\mathfrak{g}(\alpha) = \left\{ \begin{pmatrix} (1+\alpha)u & 0 & v \\ 0 & u & w \\ 0 & 0 & u \end{pmatrix} \;\middle|\; u, v, w \in \mathbb{R} \right\}.$$

A basis in $\mathfrak{g}(\alpha)$ *is defined by*

$$e_1 = \begin{pmatrix} 1+\alpha & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, e_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, e_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix},$$

with the property

$$[e_1, e_2] = \alpha e_2\,, \quad [e_2, e_3] = 0 \quad \text{and} \quad [e_3, e_1] = 0.$$

In fact, Lie group $G(\alpha)$ *is simply connected Lie group* $\exp \mathfrak{g}(\alpha)$*.*

Identify $G(\alpha)$ with $\mathbb{R}^3$ with global coordinates (x, y, z) and set the following geometric objects:

- *the metric: $dx^2 + \big(\alpha y \varrho(x)dx + f(x)e^{-\alpha x}dy\big)^2 + dz^2$, where*

$$\varrho(x) = \begin{cases} \frac{e^{-\alpha x}+\alpha x-1}{\alpha^2 x^2}, & \text{if } x \neq 0 \\ \frac{1}{2} & \text{if } x = 0; \end{cases}$$

- *1-form $\eta = dz$;*
- *vector field $\xi = \frac{\partial}{\partial z}$.*

Left invariant vector fields obtained from e_1, e_2 and e_3 are

$$e_1 = \frac{\partial}{\partial x} - \alpha y e^{\alpha x}\frac{\varrho(x)}{f(x)}\frac{\partial}{\partial y}\,,\ e_2 = \frac{e^{\alpha x}}{f(x)}\frac{\partial}{\partial y}\,,\ e_3 = \frac{\partial}{\partial z}\,.$$

The construction above is from [18] (see also [23,24]).

4.2.2. Magnetic Jacobi Fields on $G(\alpha, 0)$

Contrary to the unimodular case, when the metric on $G(\lambda_0, \lambda_0)$ is flat, for nonunimodular group $G(\alpha, 0)$, curvature R plays its role in the magnetic Jacobi equation. Therefore, we must pay attention to term $R(\dot\gamma, W)\dot\gamma$, where γ is a contact normal magnetic geodesic on $G(\alpha, 0)$, and W is the magnetic Jacobi field along γ. We use Equation (20) with $\beta = 0$.

So, if we take $W = W_1e_1 + W_2e_2 + W_3e_3$ and $\dot\gamma = T_1e_1 + T_2e_2 + \cos\theta e_3$, we immediately find

$$R(\dot\gamma, W)\dot\gamma = \alpha^2(T_1W_2 - T_2W_1)(T_1e_2 - T_2e_1).$$

This can be rewritten as

$$R(\dot\gamma, W)\dot\gamma = \alpha^2 \det(\dot\gamma, W, \xi)\varphi\dot\gamma,$$

where the "det" product of the three vectors is considered with respect to the orthonormal basis $\{e_1, e_2, e_3\}$. Thus, the magnetic Jacobi equation becomes

$$\frac{D^2W}{ds^2} - \alpha^2 \det(\dot\gamma, W, \xi)\varphi\dot\gamma - q\varphi\frac{DW}{ds} = 0. \tag{28}$$

We compute $\frac{DW}{ds} = A_1e_1 + A_2e_2 + A_3e_3$, where

$$A_1 = W_1' + \alpha T_2W_2, \quad A_2 = W_2' - \alpha T_2W_1, \quad A_3 = W_3' \tag{29}$$

and $\frac{D^2W}{ds^2} = (A_1' + \alpha T_2A_2)e_1 + (A_2' - \alpha T_2A_1)e_2 + A_3'e_3$.

Equation (28) can be written as

$$\begin{cases} A_1' + \alpha T_2A_2 + \alpha^2T_2(T_1W_2 - T_2W_1) + qA_2 = 0, \\ A_2' - \alpha T_2A_1 - \alpha^2T_1(T_1W_2 - T_2W_1) - qA_1 = 0, \\ A_3' = 0. \end{cases} \tag{30}$$

In the following, we use (29) and (24) to obtain

$$\begin{cases} W_1'' + (2\alpha T_2 + q)W_2' + \alpha(2\alpha T_2 + q)(T_1W_2 - T_2W_1) = 0, \\ W_2'' - (2\alpha T_2 + q)W_1' - \alpha^2 \sin^2\theta W_2 - \alpha q(T_1W_1 + T_2W_2) = 0, \\ W_3'' = 0. \end{cases} \tag{31}$$

Particular solution $W_1 = T_1$, $W_2 = T_2$ and $W_3 = \cos\theta$ corresponds to $W = \dot\gamma$.

Remark 16. *Expression $A_1T_1 + A_2T_2$ is constant; this can be interpreted as a conservation law.*

ODE System (31) seems to be very complicated to be solved. Let us make some notations:

$$a = W_1 T_1 + W_2 T_2 \quad \text{and} \quad b = W_2 T_1 - W_1 T_2.$$

Compute

$$\begin{aligned} a' &= (A_1 + qW_2)T_1 + (A_2 - qW_1)T_2 = (A_1 T_1 + A_2 T_2) + qb, \\ b' &= (A_2 - qW_1)T_1 - (A_1 + qW_2)T_2 = (A_2 T_1 - A_1 T_2) - qa, \end{aligned}$$

to obtain

$$\begin{cases} a'' - qb' = 0, \\ b'' + qa' - \alpha^2 \sin^2\theta b = 0. \end{cases} \tag{32}$$

This ODE system is a simplified version of System (31).

The first equation yields $a' = qb + \mathfrak{c}_0$, where $\mathfrak{c}_0$ is the constant involved in Remark 16. Plugging a' in the second equation of System (32), we obtain

$$b'' + (q^2 - \alpha^2 \sin^2\theta)b + \mathfrak{c}_0 q = 0.$$

The solution of this ODE depends on the sign of expression $q^2 - \alpha^2 \sin^2\theta$. This sign was also used when we classified the normal contact magnetic geodesics in $G(\alpha)$.

Case (i) If $\alpha^2 \sin^2\theta < q^2$, let $\zeta = \sqrt{1 - \frac{\alpha^2 \sin^2\theta}{q^2}}$.

The equation becomes $b'' + q^2\zeta^2 b + \mathfrak{c}_0 q = 0$ with the general solution

$$b(s) = \mathfrak{c}_1 \cos(q\zeta s) + \mathfrak{c}_2 \sin(q\zeta s) - \frac{\mathfrak{c}_0}{q\zeta^2}, \quad \mathfrak{c}_1, \mathfrak{c}_2 \in \mathbb{R}.$$

We obtain

$$a(s) = \frac{\mathfrak{c}_1}{\zeta} \sin(q\zeta s) - \frac{\mathfrak{c}_2}{\zeta} \cos(q\zeta s) - \frac{\alpha^2 \mathfrak{c}_0 s \sin^2\theta}{q^2\zeta^2} + \mathfrak{c}_3, \quad \mathfrak{c}_3 \in \mathbb{R}.$$

Case (ii) If $\alpha^2 \sin^2\theta = q^2$, we obtain

$$\begin{aligned} b(s) &= -\frac{\mathfrak{c}_0 q s^2}{2} + \mathfrak{c}_1 s + \mathfrak{c}_2, \\ a(s) &= -\frac{\mathfrak{c}_0 q^2 s^3}{6} + \frac{\mathfrak{c}_1 q s^2}{2} + (\mathfrak{c}_0 + q\mathfrak{c}_2)s + \mathfrak{c}_3, \end{aligned}$$

where $\mathfrak{c}_1, \mathfrak{c}_2, \mathfrak{c}_3 \in \mathbb{R}$.

Case (iii) If $\alpha^2 \sin^2\theta > q^2$, let $\zeta = \sqrt{\frac{\alpha^2 \sin^2\theta}{q^2} - 1}$.

We find

$$\begin{aligned} b(s) &= \mathfrak{c}_1 \cosh(q\zeta s) + \mathfrak{c}_2 \sinh(q\zeta s) + \frac{\mathfrak{c}_0}{q\zeta^2}, \\ a(s) &= \frac{\mathfrak{c}_1}{\zeta} \sinh(q\zeta s) + \frac{\mathfrak{c}_2}{\zeta} \cosh(q\zeta s) + \frac{\alpha^2 \mathfrak{c}_0 s \sin^2\theta}{q^2\zeta^2} + \mathfrak{c}_3, \end{aligned}$$

where $\mathfrak{c}_1, \mathfrak{c}_2, \mathfrak{c}_3 \in \mathbb{R}$.

We obtained a and b in all the three situations. Returning to coefficients W_1 and W_2 is only one more step to do, that is, we need to take into account that

$$W_1 \sin^2\theta = aT_1 - bT_2 \quad \text{and} \quad W_2 \sin^2\theta = aT_2 + bT_1,$$

where $T_1(s)$ and $T_2(s)$ should be taken keeping in mind the sign of expression $q^2 - \alpha^2 \sin^2\theta$.

To finalize this section, we indicate that the third equation in (31) yielding W_3 is an affine function on s.

Remark 17. *Since $G(\alpha, 0)$ endowed with the cosymplectic structure is a cosymplectic space form with constant φ-sectional curvature $c = -\alpha^2$ we invite the reader to read our paper in [8].*

Remark 18. *In [25], the author was interested in magnetic trajectories in Lie groups equipped with bi-invariant Riemannian metric. We add some comments on this problem. Recall that (e.g., [26]) 3-dimensional Lie groups with bi-invariant metrics are: $SU(2) \equiv \mathbb{S}^3$, $SO(3) \equiv \mathrm{U}\mathbb{S}^2$ (both non-abelian) and commutative groups $\mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{S}^1$, $\mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{R}$, $\mathbb{S}^1 \times \mathbb{R}^2$ and $\mathbb{R}^3$. In this regard, three recent works studied magnetic trajectories: ref. [27] (3-dimensional Berger spheres, especially $SU(2) \equiv \mathbb{S}^3$), ref. [28] (unit tangent sphere bundles, in particular $\mathrm{U}\mathbb{S}^2$) and ref. [29] (3-dimensional torus $\mathbb{T}^3$).*

5. Conclusions

The study of magnetic fields and their corresponding trajectories on different Riemannian manifolds plays an important role in differential geometry, physics, and dynamical systems. With the origin in the classical problem (in a Euclidean 3-dimensional space), when magnetic geodesics describe trajectories of charged particles moving under the influence of a magnetic field, the problem was extended to several ambient spaces, the magnetic fields being defined by other geometrical structure compatible with the Riemannian metric on the manifold. The easiest example is to consider a constant vector (which is Killing) in a 3-dimensional Euclidean space that canonically defines a magnetic field. The corresponding magnetic geodesics are helices with that vector field as axis. Because helices are important tools in the study of the geometry of 3-dimensional spaces, many geometers have studied the geometric properties of magnetic geodesics in different ambient spaces. The study of magnetic geodesics in arbitrary Riemannian manifolds has developed since the 1990s, and some early works may be found in the literature. Several classifications of magnetic geodesics in Riemannian manifolds were obtained. The aim is a better understanding of the geometry of the underlying space. To conclude, we choose a few works from the huge list of papers studying this topic: [27,30–36] (see also references therein). In this paper, we investigated magnetic geodesics in a 3-dimensional Lie group endowed with a left invariant almost cosymplectic structure that defined the magnetic field. Explicit expressions of magnetic geodesics and the corresponding magnetic Jacobi fields were obtained.

Funding: The author was supported by a grant of the Romanian Ministry of Research and Innovation, CNCS–UEFISCDI, project number PN-III-P1-1.1-PD-2019-0253, within PNCDI III.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Acknowledgments: The author wishes to express his deepest thanks to Jun-ichi Inoguchi for several discussions that they have had on Lie groups over the course of the last five years (at least). The author gratefully thanks to all the referees for the constructive comments and recommendations that definitely helped in improving the readability and quality of the paper.

Conflicts of Interest: The author declares no conflict of interest.

References

1. Blair, D.E. The Theory of Quasi-Sasakian Structures. Ph.D. Thesis, University of Illinois, Champaign, IL, USA, 1966.
2. Ogiue, K. *G*-structures defined by tensor fields. *Kodai Math. Sem. Rep.* **1968**, *20*, 54–75. [CrossRef]
3. Blair, D.E. The theory of quasi-Sasakian structure. *J. Differ. Geom.* **1967**, *1*, 331–345. [CrossRef]
4. Perrone, D. Classification of homogeneous almost cosymplectic three-maifolds. *Diff. Geom. Appl.* **2012**, *30*, 49–58. [CrossRef]
5. Munteanu, M.I.; Nistor, A.I. The classification of Killing magnetic curves in $\mathbb{S}^2 \times \mathbb{R}$. *J. Geom. Phys.* **2012**, *62*, 170–182. [CrossRef]
6. Nistor, A.I. Motion of charged particles in a Killing magnetic field in $\mathbb{H}^2 \times \mathbb{R}$. *Rend. Sem. Mat. Univ. Politec. Torino* **2015**, *73*, 161–170.

7. Druţă-Romaniuc, S.L.; Inoguchi, J.; Munteanu, M.I.; Nistor, A.I. Magnetic curves in cosymplectic manifolds. *Rep. Math. Phys.* **2016**, *78*, 33–48. [CrossRef]
8. Munteanu, M.I.; Nistor, A.I. Magnetic Jacobi fields in 3-dimensional cosymplectic manifolds. *Mathematics* **2021**, *9*, 3220. [CrossRef]
9. Adachi, T. A comparison theorem on magnetic Jacobi fields. *Proc. Edinb. Math. Soc.* **1997**, *40*, 293–308. [CrossRef]
10. Gouda, N. The theorem of E. Hopf under uniform magnetic fields. *J. Math. Soc. Jpn.* **1998**, *50*, 767–779. [CrossRef]
11. Gouda, N. Magnetic flows of Anosov type. *Tohoku Math. J.* **1997**, *49*, 165–183. [CrossRef]
12. Adachi, T. Magnetic Jacobi fields for Kähler magnetic fields. In *Recent Progress in Differential Geometry and Its Related Fields*; Adachi, T., Hashimoto, H., Hristov, M., Eds.; World Scientific: Singapore, 2011; pp. 41–53.
13. Shi, O. Magnetic Jacobi fields for surface magnetic fields. In *Current Developments in Differential Geometry and Its Related Fields*; Adachi, T., Hashimoto, H., Hristov, M., Eds.; World Scientific: Singapore, 2016; pp. 215–224.
14. Inoguchi, J.; Munteanu, M.I. Magnetic Jacobi fields in 3-dimnesional Sasakian space forms. *J. Geom. Anal.* **2022**, *32*, 1–26. [CrossRef]
15. Inoguchi, J.; Munteanu, M.I. Magnetic Jacobi fields in Sasakian space forms. *submitted*.
16. Milnor, J. Curvature of left invariant metrics on Lie groups. *Adv. Math.* **1976**, *21*, 293–329. [CrossRef]
17. Olszak, Z. On almost cosymplectic manifold. *Kodai Math. J.* **1981**, *4*, 239–250. [CrossRef]
18. Inoguchi, J. Characteristic Jacobi operator an almost cosymplectic 3-manifolds. *Internat. Electron. J. Geom.* **2019**, *12*, 276–299. [CrossRef]
19. Inoguchi, J.; Lee, J.E. Slant curves in 3-dimensional almost contact metric geometry. *Internat. Electron. J. Geom.* **2015**, *8*, 106–146. [CrossRef]
20. Erjavec, Z.; Inoguchi, J. On Magnetic Curves in Almost Cosymplectic Sol Space. *Results Math.* **2020**, *75*, 1–16. [CrossRef]
21. Inoguchi, J.; Van der Veken, J. Parallel surfaces in the motion groups $E(1,1)$ and $E(2)$. *Bull. Belg. Math. Soc. Simon Stevin* **2007**, *14*, 321–332. [CrossRef]
22. Druţă-Romaniuc, S.L.; Munteanu, M.I. Magnetic curves corresponding to Killing magnetic fields in $\mathbb{E}^3$. *J. Math. Phys.* **2011**, *52*, 113506. [CrossRef]
23. Meeks, W.H., III; Mira, P.; Pérez, J. The geometry of stable minimal surfaces in metric Lie groups. *Trans. Amer. Math. Soc.* **2019**, *372*, 1023–1056. [CrossRef]
24. Inoguchi, J.; Naitoh, H. Grassmann geometry on the 3-dimensional non-unimodular Lie groups. *Hokkaido Math. J.* **2019**, *48*, 385–406. [CrossRef]
25. Turhan, T. Magnetic trajectories in three-dimensional Lie groups. *Math. Meth. Appl. Sci.* **2020**, *43*, 2747–2758. [CrossRef]
26. do Espírito-Santo, N.; Fornari, S.; Frensel, K.; Ripoll, J. Constant mean curvature hypersurfaces in a Lie group with a bi-invariant metric. *Manuscripta Math.* **2003**, *111*, 459–470. [CrossRef]
27. Inoguchi, J.; Munteanu, M.I. Periodic magnetic curves in Berger spheres. *Tohoku Math. J.* **2017**, *69*, 113–128. [CrossRef]
28. Inoguchi, J.; Munteanu, M.I. Magnetic curves on tangent sphere bundles. *RACSAM* **2019**, *113*, 2087–2112. [CrossRef]
29. Munteanu, M.I.; Nistor, A.I. On some closed magnetic curves on a 3-torus. *Math. Phys. Anal. Geom.* **2017**, *20*, 1–13. [CrossRef]
30. Barros, M.; Cabrerizo, J.L.; Fernández, M.; Romero, A. Magnetic vortex filament flows. *J. Math. Phys.* **2007**, *48*, 082904. [CrossRef]
31. Barros, M.; Romero, A.; Cabrerizo, J.L.; Fernández, M. The Gauss-Landau-Hall problem on Riemannian surfaces. *J. Math. Phys.* **2005**, *46*, 112905. [CrossRef]
32. Barros, M.; Romero, A. Magnetic vortices. *EPL* **2007**, *77*, 34002. [CrossRef]
33. Erjavec, Z. On Killing magnetic curves in $\mathrm{SL}(2,R)$ geometry. *Rep. Math. Phys.* **2019**, *84*, 333–350. [CrossRef]
34. Ginzburg, V.L. A charge in a magnetic field: Arnold's problems 1981–9, 1982–24, 1984–4, 1994–14, 1994–35, 1996–17, 1996–18. In *Arnold's Problems*; Arnold, V.I., Ed.; Springer: Berlin/Heidelberg, Germany, 2004; pp. 395–401.
35. Munteanu, M.I. Magnetic curves in a Euclidean space: One example, several approaches. *Publ. l'Institut Math. (Beograd)* **2013**, *94*, 141–150. [CrossRef]
36. Sunada, T. Magnetic flows on a Riemann surface. In Proceedings of the KAIST Mathematics Workshop: Analysis and Geometry, KAIST, Taejeon, Korea, 3–6 August 1993.