mathematics

Article

New Sharp Double Inequality of Becker–Stark Type

Ling Zhu

Department of Mathematics, Zhejiang Gongshang University, Hangzhou 310018, China; zhuling@zjgsu.edu.cn

Abstract: In this paper, we establish new sharp double inequality of Becker–Stark type by using a role of the monotonicity criterion for the quotient of power series and the estimation of the ratio of two adjacent even-indexed Bernoulli numbers. The inequality results are better than those in the existing literature.

Keywords: Becker–Stark type inequality; circular functions; the ratio of two adjacent even-indexed Bernoulli numbers

Citation: Zhu, L. New Sharp Double Inequality of Becker–Stark Type. *Mathematics* **2022**, *10*, 558. https://doi.org/10.3390/math10040558

Academic Editor: Juan Benigno Seoane-Sepúlveda

Received: 26 January 2022
Accepted: 10 February 2022
Published: 11 February 2022

1. Introduction

Becker and Stark [1] (or see Kuang [2] (5.1.102, p. 398)) obtained the following two-sided rational approximation for $(\tan x)/x$:

Proposition 1. *Let $0 < x < \pi/2$, then*

$$\frac{8}{\pi^2 - 4x^2} < \frac{\tan x}{x} < \frac{\pi^2}{\pi^2 - 4x^2}. \tag{1}$$

Furthermore, 8 and π^2 are the best constants in (1).

In [3], Zhu and Hua obtained the following further result.

Proposition 2. *Let $0 < x < \pi/2$, then*

$$\frac{\pi^2 + 4(8-\pi^2)x^2/\pi^2}{\pi^2 - 4x^2} < \frac{\tan x}{x} < \frac{\pi^2 + (\pi^2 - 12)x^2/3}{\pi^2 - 4x^2}. \tag{2}$$

Furthermore, $\alpha = 4(8-\pi^2)/\pi^2$ and $\beta = (\pi^2 - 12)/3$ are the best constants in (2).

Moreover, ref. [3] established a general refinement of the Becker–Stark inequalities as follows.

Proposition 3. *Let $0 < x < \pi/2$, $n, N \geq 0$ be natural numbers, B_{2n} be the even-indexed Bernoulli numbers,*

$$p_n = \frac{2^{2n+2}(2^{2n+2}-1)\pi^2}{(2n+2)!}|B_{2n+2}| - \frac{2^{2n+2}(2^{2n}-1)}{(2n)!}|B_{2n}|,$$

and

$$\begin{aligned} \alpha &= \frac{8 - p_0 - p_1(\pi/2)^2 - \cdots - p_N(\pi/2)^{2N}}{(\pi/2)^{2N+2}}, \\ \beta &= p_{N+1}. \end{aligned} \tag{3}$$

Then

$$\frac{P_{2N}(x) + \alpha x^{2N+2}}{\pi^2 - 4x^2} < \frac{\tan x}{x} < \frac{P_{2N}(x) + \beta x^{2N+2}}{\pi^2 - 4x^2} \tag{4}$$

holds, where $P_{2N}(x) = a_0 + a_1x^2 + \cdots + a_N x^{2N}$. *Furthermore,* β *and* α *are the best constants in (4).*

Zhu [4] obtained a refinement of the Becker–Stark inequalities (1) in another way as follows.

Proposition 4. *Let* $0 < x < \pi/2$, *then*

$$\frac{8}{\pi^2-4x^2}+\frac{2}{\pi^2}-\frac{\pi^2-9}{6\pi^4}\left(\pi^2-4x^2\right) \quad < \quad \frac{\tan x}{x} < \frac{8}{\pi^2-4x^2}+\frac{2}{\pi^2}-\frac{10-\pi^2}{\pi^4}\left(\pi^2-4x^2\right). \tag{5}$$

Furthermore, $-\left(\pi^2-9\right)/\left(6\pi^4\right)$ *and* $-\left(10-\pi^2\right)/\pi^4$ *are the best constants in (5).*

Obviously, letting $N = 0$ in (4) gives (2). The double inequalities (2) and (5) can deduce the inequalities (1). Moreover, the left inequalities of (2) and (5) are not included each other while the upper estimate in (5) is less than the one in (2).

Numerous discussions on Becker–Stark inequality can be found in [5–12], as well as references therein. In 2015, Banjac, Markragić and Malešević [11] obtained the following results about the function $(\tan x)/x$.

Proposition 5. *Let* $0 < x < \pi/2$, *then*

$$\frac{\pi^2+\left(\frac{\pi^2}{3}-4\right)x^2+\left(\frac{\pi^2}{18}-\frac{2}{3}\right)x^4}{\pi^2-4x^2} < \frac{\tan x}{x} < \frac{\pi^2-\frac{\pi^2}{16}x^2+\frac{1}{2}x^4-\frac{1}{\pi^2}x^6}{\pi^2-4x^2} \tag{6}$$

and

$$\frac{\pi^2+\frac{\pi^2-12}{3}x^2+\frac{384-4\pi^4}{3\pi^4}x^4}{\pi^2-4x^2} < \frac{\tan x}{x} < \frac{\pi^2+\frac{72-8\pi^2}{\pi^2}x^2+\frac{16\pi^2-160}{\pi^4}x^4}{\pi^2-4x^2}. \tag{7}$$

Bagul and Chesneau looked closely at the Becker–Stark inequality from another perspective, and in [12] they got the following result.

Proposition 6. *For any* $x \in (0, \pi/2)$, *we have*

$$\sqrt{1+\frac{128}{\pi^4}\frac{x^2\left(5\pi^2-12x^2\right)}{\left(\pi^2-4x^2\right)^2}} < \frac{\tan x}{x} < \sqrt{1+\frac{2\pi^2}{5}\frac{x^2\left(5\pi^2-12x^2\right)}{\left(\pi^2-4x^2\right)^2}} \tag{8}$$

holds with the best constants $128/\pi^4$ *and* $2\pi^2/5$.

Inspired by the above ideas, this paper considers the power series expansion of the following function

$$\begin{aligned}\frac{\left[\left(\frac{\tan x}{x}\right)^2-1\right]\left(\pi^2-4x^2\right)^2}{x^2\left(\lambda-x^2\right)} \quad = \quad & \frac{2}{3}\frac{\pi^4}{\lambda}+\frac{\pi^2}{45}\frac{30\pi^2-\lambda\left(240-17\pi^2\right)}{\lambda^2}x^2 \\ & +\frac{1}{315}\frac{\left[\begin{array}{c}210\pi^4+\lambda\left(119\pi^4-1680\pi^2\right)\\+\lambda^2\left(62\pi^4-952\pi^2+3360\right)\end{array}\right]}{\lambda^3}x^4 \\ & +O\left(x^6\right).\end{aligned}$$

Letting $\lambda = 30\pi^2/\left(240-17\pi^2\right)$ we can obtain the expression of the above function without x^2, and draw the following inequality conclusion by using the property for the ratio of two adjacent even-indexed Bernoulli numbers and a role of the monotonicity criterion for the quotient of power series.

Theorem 1. *Let* $0 < x < \pi/2$. *Then the double inequality*

$$\sqrt{1+\frac{(240-17\pi^2)\pi^2}{45}\frac{x^2\left(\frac{30\pi^2}{240-17\pi^2}-x^2\right)}{(\pi^2-4x^2)^2}}<\frac{\tan x}{x}<\sqrt{1+\frac{(240-17\pi^2)1024}{\pi^4(17\pi^2-120)}\frac{x^2\left(\frac{30\pi^2}{240-17\pi^2}-x^2\right)}{(\pi^2-4x^2)^2}} \tag{9}$$

holds with the best constants $(240-17\pi^2)\pi^2/45$ *and* $(240-17\pi^2)1024/\left[\pi^4(17\pi^2-120)\right]$.

2. Lemmas

In order to prove the main conclusion of this paper, the following three lemmas are needed.

Lemma 1 ([13]). *Let* B_{2n} *be the even-indexed Bernoulli numbers, we have the following power series expansion*

$$\tan x=\sum_{n=1}^{\infty}\frac{2^{2n}-1}{(2n)!}2^{2n}|B_{2n}|x^{2n-1},$$

holds for all $x\in(-\pi/2,\pi/2)$.

Lemma 2 ([14–17]). *Let* B_{2n} *be the even-indexed Bernoulli numbers, we have*

$$\frac{2^{2n-1}-1}{2^{2n+1}-1}\frac{(2n+2)(2n+1)}{\pi^2}<\frac{|B_{2n+2}|}{|B_{2n}|}<\frac{2^{2n}-1}{2^{2n+2}-1}\frac{(2n+2)(2n+1)}{\pi^2}. \tag{10}$$

Lemma 3 ([18]). *Let* a_n *and* b_n $(n=0,1,2,\cdots)$ *be real numbers, and let the power series* $A(x)=\sum_{n=0}^{\infty}a_nx^n$ *and* $B(x)=\sum_{n=0}^{\infty}b_nx^n$ *be convergent for* $|x|<R$ $(R\le+\infty)$. *If* $b_n>0$ *for* $n=0,1,2,\cdots$, *and if* $\varepsilon_n=a_n/b_n$ *is strictly increasing (or decreasing) for* $n=0,1,2,\cdots$, *then the function* $A(x)/B(x)$ *is strictly increasing (or decreasing) on* $(0,R)$ $(R\le+\infty)$.

3. Proof of Theorem 1

Proof. Let

$$F(x)=\frac{\left[\left(\frac{\tan x}{x}\right)^2-1\right](\pi^2-4x^2)^2}{x^2\left(\frac{30\pi^2}{240-17\pi^2}-x^2\right)},\ 0<x<\frac{\pi}{2}.$$

Then we can rewrite $F(x)$ as

$$F(x)=\frac{\left(\frac{\tan x}{x}\right)^2-1}{\frac{x^2\left(\frac{30\pi^2}{240-17\pi^2}-x^2\right)}{(\pi^2-4x^2)^2}}:=\frac{A(x)}{B(x)},$$

where

$$\begin{aligned}A(x)&=\left(\frac{\tan x}{x}\right)^2-1,\\ B(x)&=\frac{x^2\left(\frac{30\pi^2}{240-17\pi^2}-x^2\right)}{(\pi^2-4x^2)^2}.\end{aligned}$$

By Lemma 1 we have

$$\begin{aligned}\tan^2 x&=\sec^2 x-1=(\tan x)'-1=\sum_{n=1}^{\infty}\frac{2^{2n}(2^{2n}-1)(2n-1)}{(2n)!}|B_{2n}|x^{2n-2}-1\\ &=\sum_{n=2}^{\infty}\frac{2^{2n}(2^{2n}-1)}{(2n)!}|B_{2n}|x^{2n-2},\end{aligned}$$

and

$$\begin{aligned} A(x) &= \sum_{n=2}^{\infty} \frac{2^{2n}(2^{2n}-1)(2n-1)}{(2n)!}|B_{2n}|x^{2n-4} - 1 \\ &= \sum_{n=3}^{\infty} \frac{2^{2n}(2^{2n}-1)(2n-1)}{(2n)!}|B_{2n}|x^{2n-4} \\ &= : \sum_{n=3}^{\infty} a_n x^{2n-4}, \end{aligned}$$

where

$$a_n = \frac{2^{2n}(2^{2n}-1)(2n-1)}{(2n)!}|B_{2n}|,\ n \geq 3.$$

At the same time, since

$$\frac{1}{(\pi^2 - 4x^2)^2} = \sum_{n=3}^{\infty} \frac{(n-2)4^{n-3}}{\pi^{2n-2}} x^{2n-6},$$

we have

$$\begin{aligned} B(x) &= x^2 \left(\frac{30\pi^2}{240 - 17\pi^2} - x^2 \right) \sum_{n=3}^{\infty} \frac{(n-2)4^{n-3}}{\pi^{2n-2}} x^{2n-6} \\ &= \frac{30\pi^2}{240 - 17\pi^2} \sum_{n=3}^{\infty} \frac{(n-2)4^{n-3}}{\pi^{2n-2}} x^{2n-4} - \sum_{n=3}^{\infty} \frac{(n-2)4^{n-3}}{\pi^{2n-2}} x^{2n-2} \\ &= \frac{30}{\pi^2(240 - 17\pi^2)} x^2 + \frac{30\pi^2}{240 - 17\pi^2} \sum_{n=4}^{\infty} \frac{(n-2)4^{n-3}}{\pi^{2n-2}} x^{2n-4} \\ &\quad - \sum_{n=3}^{\infty} \frac{(n-2)4^{n-3}}{\pi^{2n-2}} x^{2n-2} \\ &= \frac{30}{\pi^2(240 - 17\pi^2)} x^2 + \frac{30\pi^2}{240 - 17\pi^2} \sum_{n=4}^{\infty} \frac{(n-2)4^{n-3}}{\pi^{2n-2}} x^{2n-4} \\ &\quad - \sum_{n=4}^{\infty} \frac{(n-3)4^{n-4}}{\pi^{2n-4}} x^{2n-4} \\ &= \frac{30}{\pi^2(240 - 17\pi^2)} x^2 + \sum_{n=4}^{\infty} \left[\frac{30\pi^2}{240 - 17\pi^2} \frac{(n-2)4^{n-3}}{\pi^{2n-2}} - \frac{(n-3)4^{n-4}}{\pi^{2n-4}} \right] x^{2n-4} \\ &= : \sum_{n=3}^{\infty} b_n x^{2n-4}, \end{aligned}$$

where

$$\begin{aligned} b_3 &= \frac{30}{\pi^2(240 - 17\pi^2)}, \\ b_n &= \frac{30\pi^2}{240 - 17\pi^2} \frac{(n-2)4^{n-3}}{\pi^{2n-2}} - \frac{(n-3)4^{n-4}}{\pi^{2n-4}},\ n \geq 4. \end{aligned}$$

We find that

$$\begin{aligned} \frac{a_3}{b_3} &= \frac{a_4}{b_4} = \frac{\pi^2(240 - 17\pi^2)}{45} \approx 15.839, \\ \frac{a_5}{b_5} &= \frac{31\pi^6}{1260} \frac{240 - 17\pi^2}{17\pi^2 - 60} \approx 15.848, \\ \frac{a_6}{b_6} &= \frac{691\pi^8}{340,200} \frac{240 - 17\pi^2}{17\pi^2 - 80} \approx 15.855, \end{aligned}$$

and can prove the fact that $\{a_n/b_n\}_{n\geq 6}$ is increasing, which is equivalent to: for all $n \geq 6$,

$$\begin{aligned} \frac{a_n}{b_n} &< \frac{a_{n+1}}{b_{n+1}} \\ &\Longleftrightarrow \\ \frac{\frac{2^{2n}(2^{2n}-1)(2n-1)}{(2n)!}|B_{2n}|}{\frac{30\pi^2}{240-17\pi^2}\frac{(n-2)4^{n-3}}{\pi^{2n-2}}-\frac{(n-3)4^{n-4}}{\pi^{2n-4}}} &< \frac{\frac{2^{2n+2}(2^{2n+2}-1)(2n+1)}{(2n+2)!}|B_{2n+2}|}{\frac{30\pi^2}{240-17\pi^2}\frac{(n-1)4^{n-2}}{\pi^{2n}}-\frac{(n-2)4^{n-3}}{\pi^{2n-2}}} \\ &\Longleftrightarrow \\ \frac{|B_{2n+2}|}{|B_{2n}|} &> \frac{\frac{2^{2n}(2^{2n}-1)(2n-1)}{(2n)!}}{\frac{2^{2n+2}(2^{2n+2}-1)(2n+1)}{(2n+2)!}} \frac{\left[\frac{30\pi^2}{240-17\pi^2}\frac{(n-1)4^{n-2}}{\pi^{2n}} - \frac{(n-2)4^{n-3}}{\pi^{2n-2}}\right]}{\left[\frac{30\pi^2}{240-17\pi^2}\frac{(n-2)4^{n-3}}{\pi^{2n-2}} - \frac{(n-3)4^{n-4}}{\pi^{2n-4}}\right]}. \end{aligned}$$

From Lemma 2, the last inequality above holds as long as the following inequality is proved

$$\begin{aligned} \frac{2^{2n-1}-1}{2^{2n+1}-1}\frac{(2n+2)(2n+1)}{\pi^2} &> \frac{\frac{2^{2n}(2^{2n}-1)(2n-1)}{(2n)!}}{\frac{2^{2n+2}(2^{2n+2}-1)(2n+1)}{(2n+2)!}} \frac{\frac{30\pi^2}{240-17\pi^2}\frac{(n-1)4^{n-2}}{\pi^{2n}} - \frac{(n-2)4^{n-3}}{\pi^{2n-2}}}{\frac{30\pi^2}{240-17\pi^2}\frac{(n-2)4^{n-3}}{\pi^{2n-2}} - \frac{(n-3)4^{n-4}}{\pi^{2n-4}}} \\ &\Longleftrightarrow \\ \Delta \quad : &= \frac{2^{2n-1}-1}{(2^{2n+1}-1)\pi^2} \\ &> \frac{(2^{2n}-1)(2n-1)}{4(2^{2n+2}-1)(2n+1)} \frac{\frac{30\pi^2}{240-17\pi^2}\frac{(n-1)4^{n-2}}{\pi^{2n}} - \frac{(n-2)4^{n-3}}{\pi^{2n-2}}}{\frac{30\pi^2}{240-17\pi^2}\frac{(n-2)4^{n-3}}{\pi^{2n-2}} - \frac{(n-3)4^{n-4}}{\pi^{2n-4}}} \\ &= : \nabla. \end{aligned}$$

We compute to obtain

$$\Delta - \nabla = \frac{1}{2}\frac{h(n)}{\pi^2(4\times 2^{2n}-1)(2\times 2^{2n}-1)(2n+1)(17\pi^2 n - 120n - 51\pi^2 + 480)},$$

where

$$\begin{aligned} h(n) &= 20\left(168-17\pi^2\right)\left[2^{2n}-u(n)\right]\times 2^{2n} + \left(1680-170\pi^2\right). \\ u(n) &= \frac{3\left[2n^2(17\pi^2-120)+5n(168-17\pi^2)-221\pi^2+2160\right]}{20(168-17\pi^2)}, \end{aligned}$$

Since $1680 - 170\pi^2 > 0$, the fact $h(n) > 0$ holds for all $n \geq 6$ when proving

$$2^{2n} > u(n),\ n \geq 6. \tag{11}$$

It is not difficult to prove (11) by mathematical induction. First, the above formula (11) is true for $n = 6$ due to

$$2^{12} - u(6) = \frac{1}{20}\frac{1{,}394{,}119\pi^2 - 13{,}766{,}880}{17\pi^2 - 168} \approx 1725 > 0.$$

Second, suppose (11) holds for m, that is

$$2^{2m} > u(m),\ m \geq 6. \tag{12}$$

Next, by (12) we have

$$2^{2m+2} = 4\cdot 2^{2m} > 4\cdot u(m).$$

The inequality (11) is proved when proving

$$4 \cdot u(m) > u(m+1),$$

that is

$$4u(m) - u(m+1) > 0.$$

In fact,

$$\begin{aligned} & 4u(m) - u(m+1) \\ = & \frac{3}{20} \frac{\left(102\pi^2 - 720\right)(m-6)^2 + \left(901\pi^2 - 5640\right)(m-6) + \left(1122\pi^2 - 2040\right)}{168 - 17\pi^2} \\ > & 0 \end{aligned}$$

holds for all $m \geq 6$.

So $\{a_n/b_n\}_{n\geq 3}$ is increasing. From Lemma 3 we have that the function $F(x)$ is increasing on $(0, \pi/2)$. Considering the reasons

$$\begin{aligned} \lim_{x\to 0^+} F(x) &= \frac{\pi^2\left(240 - 17\pi^2\right)}{45} \approx 15.839, \\ \lim_{x\to \left(\frac{\pi}{2}\right)^-} F(x) &= \frac{1024\left(240 - 17\pi^2\right)}{\pi^4\left(17\pi^2 - 120\right)} \approx 15.888, \end{aligned}$$

the proof of Theorem 1 is completed. □

4. Remarks

Through the following analysis, we conclude that the left-hand side inequality of (9) does not match the one of (8) while the right-hand side inequality of (9) is better than the one of (8).

Remark 1. *The left-hand side inequality of* (9) *does not match the one of* (8) *due to*

$$\begin{aligned} \sqrt{1 + \frac{\left(240 - 17\pi^2\right)\pi^2}{45} \frac{x^2\left(\frac{30\pi^2}{240-17\pi^2} - x^2\right)}{\left(\pi^2 - 4x^2\right)^2}} &> \sqrt{1 + \frac{128}{\pi^4} \frac{x^2\left(5\pi^2 - 12x^2\right)}{\left(\pi^2 - 4x^2\right)^2}} \\ &\iff \\ \frac{\left(240 - 17\pi^2\right)\pi^2}{45} \frac{x^2\left(\frac{30\pi^2}{240-17\pi^2} - x^2\right)}{\left(\pi^2 - 4x^2\right)^2} &> \frac{128}{\pi^4} \frac{x^2\left(5\pi^2 - 12x^2\right)}{\left(\pi^2 - 4x^2\right)^2} \\ &\iff \\ \frac{\left(240 - 17\pi^2\right)\pi^2}{45}\left(\frac{30\pi^2}{240 - 17\pi^2} - x^2\right) &> \frac{128}{\pi^4}\left(5\pi^2 - 12x^2\right) \\ &\iff \\ \left(240\pi^6 - 17\pi^8 - 69,120\right)x^2 &< 30\pi^8 - 28,800\pi^2 \\ &\iff \end{aligned}$$

$$x < \sqrt{\frac{30\pi^8 - 28,800\pi^2}{240\pi^6 - 17\pi^8 - 69,120}} \approx 1.1549.$$

Remark 2. *Since*

$$\sqrt{1+\frac{(240-17\pi^2)1024}{\pi^4(17\pi^2-120)}\frac{x^2\left(\frac{30\pi^2}{240-17\pi^2}-x^2\right)}{(\pi^2-4x^2)^2}} < \sqrt{1+\frac{2\pi^2}{5}\frac{x^2(5\pi^2-12x^2)}{(\pi^2-4x^2)^2}}$$
$$\Longleftrightarrow$$
$$\frac{(240-17\pi^2)1024}{\pi^4(17\pi^2-120)}\left(\frac{30\pi^2}{240-17\pi^2}-x^2\right) < \frac{2\pi^2(5\pi^2-12x^2)}{5}$$
$$\Longleftrightarrow$$
$$4x^2\left(10,880\pi^2-360\pi^6+51\pi^8-153,600\right) < 85\pi^{10}-600\pi^8-76,800\pi^2$$
$$\Longleftrightarrow$$
$$x<\frac{1}{2}\sqrt{\frac{-76,800\pi^2-600\pi^8+85\pi^{10}}{(10,880\pi^2-360\pi^6+51\pi^8-153,600)}}\approx 2.0294,$$

we can obtain that the right-hand side inequality of (9) *is better than the one of* (8) *on* $(0,\pi/2)$.

5. Conclusions

This paper established a new sharp double inequality of Becker–Stark type

$$\sqrt{1+\frac{(240-17\pi^2)\pi^2}{45}\frac{x^2\left(\frac{30\pi^2}{240-17\pi^2}-x^2\right)}{(\pi^2-4x^2)^2}} < \frac{\tan x}{x} < \sqrt{1+\frac{(240-17\pi^2)1024}{\pi^4(17\pi^2-120)}\frac{x^2\left(\frac{30\pi^2}{240-17\pi^2}-x^2\right)}{(\pi^2-4x^2)^2}}, \tag{13}$$

which holds for $x \in (0,\pi/2)$, where $(240-17\pi^2)\pi^2/45$ and $(240-17\pi^2)1024/\left[\pi^4(17\pi^2-120)\right]$ are the best possible.

Funding: This research received no external funding.

Acknowledgments: The author is thankful to reviewers for reviewers' careful corrections to and valuable comments on the original version of this paper.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Becker, M.; Stark, E.L. On a hierarchy of quolynomial inequalities for tan *x*. *Publ. Elektrotehničkog Fak. Ser. Mat. i Fiz.* **1978**, *602–633*, 133–138.
2. Kuang, J.-C. *Applied Inequalities*, 5th ed.; Shangdong Science and Technology Press: Jinan, China, 2021.
3. Zhu, L.; Hua, J.-K. Sharpening the Becker-Stark inequalities. *J. Inequal. Appl.* **2010**, *2010*, 931275. [CrossRef]
4. Zhu, L. A refinement of the Becker-Stark inequalities. *Math. Notes* **2013**, *93*, 421–425. [CrossRef]
5. Chen, C.-P.; Cheung, W.-S. Sharp Cusa and Becker-Stark inequalities. *J. Inequal. Appl.* **2011**, *2011*, 136. [CrossRef]
6. Wu, Y.T.; Bercu, G. New refinements of Becker-Stark and Cusa-Huygens inequalities via trigonometric polynomials method. *Rev. Real Acad. Cienc. Exactas Físicas Nat. Ser. A Mat.* **2021**, *115*, 87. [CrossRef]
7. Nenezić, M.; Zhu, L. Some improvements of Jordan-Stečkin and Becker-Stark Inequalities. *Appl. Anal. Discrete Math.* **2018**, *12*, 244–256. [CrossRef]
8. Zhu, L. Sharp Becker-Stark-Type inequalities for Bessel functions. *J. Inequal. Appl.* **2010**, *2010*, 838740. [CrossRef]
9. Nishizawa, Y. Sharp Becker-Stark-type inequalities with power exponential functions. *J. Inequal. Appl.* **2015**, *2015*, 402. [CrossRef]
10. Chouikha, A.R. New sharp inequalities related to classical trigonometric inequalities. *J. Inequal. Spec. Func.* **2020**, *11*, 27–35.
11. Banjac, B.; Makragić, M.; Maleševixcx, B. Some notes on a method for proving inequalities by computer. *Results Math.* **2016**, *69*, 161–176. [CrossRef]
12. Bagul, Y.J.; Chesneau, C. New sharp bounds for tangent function. *Bull. Alla. Math. Soc.* **2020**, *34*, 277–282.
13. Jeffrey, A. *Handbook of Mathematical Formulas and Integrals*, 3rd ed.; Elsevier: San Diego, CA, USA, 2004.
14. Qi, F. A double inequality for the ratio of two non-zero neighbouring Bernoulli numbers. *J. Comput. Appl. Math.* **2019**, *351*, 1–5. [CrossRef]
15. Zhu, L. New bounds for the ratio of two adjacent even-indexed Bernoulli numbers. *Rev. Real Acad. Cienc. Exactas Físicas y Nat. Ser. A Mat.* **2020**, *114*, 83. [CrossRef]
16. Zhu, L. Monotonicities of some functions involving multiple logarithm function and their applications. *Rev. Real Acad. Cienc. Exactas FíSicas Nat. Ser. A Mat.* **2020**, *114*, 139. [CrossRef]

17. Yang, Z.H.; Tian, J.F. Sharp bounds for the ratio of two zeta functions. *J. Comput. Appl. Math.* **2020**, 364, 112359. [CrossRef]
18. Biernacki, M.; Krzyz, J. On the monotonicity of certain functionals in the theory of analytic functions. *Can. Math. Bull.* **1955**, *2*, 134–145.