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Some New Versions of Integral Inequalities for Left and Right Preinvex Functions in the Interval-Valued Settings

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Abstract: The principles of convexity and symmetry are inextricably linked. Because of the considerable association that has emerged between the two in recent years, we may apply what we learn from one to the other. In this paper, our aim is to establish the relation between integral inequalities and interval-valued functions (*IV-Fs*) based upon the pseudo-order relation. Firstly, we discuss the properties of left and right preinvex interval-valued functions (left and right preinvex *IV-Fs*). Then, we obtain Hermite–Hadamard ($\boldsymbol{\mathcal{H}}$-$\boldsymbol{\mathcal{H}}$) and Hermite–Hadamard–Fejér ($\boldsymbol{\mathcal{H}}$-$\boldsymbol{\mathcal{H}}$-Fejér) type inequality and some related integral inequalities with the support of left and right preinvex *IV-Fs* via pseudo-order relation and interval Riemann integral. Moreover, some exceptional special cases are also discussed. Some useful examples are also given to prove the validity of our main results.

Keywords: left and right preinvex interval-valued function; interval Riemann integral; Hermite–Hadamard type inequality; Hermite–Hadamard–Fejér type inequality

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1. Introduction

Hanson [1] defined the class of invex functions as one of the most significant extensions of convex functions. Weir and Mond [2], in 1988, used the notion of preinvex functions to demonstrate adequate optimality criteria and duality in nonlinear programming. For a differentiable mapping, the concept of fractional integral identities involving Riemann–Liouville fractional and Hadamard fractional integrals integrals was considered by Wang et al. [3], who identified some inequalities using standard convex, r-convex, m-convex, S-convex, (s, m)-convex, and $(\beta,\ m)$-convex. Moreover, Işcan [4] also used fractional integrals for preinvex functions to obtain various $\boldsymbol{\mathcal{H}}$-$\boldsymbol{\mathcal{H}}$ type inequalities. See [5–8] for other generalizations of the $\boldsymbol{\mathcal{H}}$-$\boldsymbol{\mathcal{H}}$ inequality.

For accurate solutions to various problems in practical mathematics, Moore [9] used interval arithmetic, *IV-Fs*, and integrals of *IV-Fs* to establish arbitrarily sharp upper and lower limits. Moore [9] showed that, if a real-valued mapping $Y(\varkappa)$ meets an ordinary Lipschitz condition in Y, $|Y(\varkappa) - Y(\omega)| \le L|\varkappa - \omega|$, for $\omega,\ \varkappa \in Y$, then, the united extension is a Lipschitz interval extension in Y. To combine the study of discrete and continuous dynamical systems, Hilger [10] introduced a time scales theory. The widespread use of dynamic equations and integral inequalities on time scales, in domains as diverse as electrical engineering, quantum physics, heat transfer, neural networks, combinatorics, and population dynamics [11], has highlighted the need for this theory. Young's inequality, Minkoswki's inequality, Jensen's inequality, Hölder's inequality, $\mathcal{H}$-$\mathcal{H}$ inequality, Steffensen's inequality, Opial type inequality and Čhebyšhev's inequality were all explored by Agarwal et al. [11]. Srivastava et al. [12] discovered some generic time scale weighted Opial type inequalities in 2010. Srivastava et al. [13] also proposed several time-based expansions and generalizations of Maroni's inequality. Under certain proper conditions, some new local fractional integral analogue of Anderson's inequality on fractal space was introduced by Wei et al. [14], demonstrating that for classical Anderson's inequality, it was a novel extension on fractal space. Tunç et al. [15] also constructed an identity for local fractional integrals and derived numerous modifications of the well-known Steffensen's inequality for fractional integrals. The papers [11,16] and the references therein might be consulted for further information. Bhurjee and Panda [17] identified the parametric form of an *IV-F* and devised a technique to investigate the existence of a generic interval optimization issue solution. Using the notion of the generalized Hukuhara difference, Lupulescu [18] developed differentiability and integrability for *IV-Fs* on time scales. Cano et al. [19] developed a novel form of the Ostrowski inequality for gH differentiable *IV-Fs* in 2015 and achieved an extension of the class of real functions that are not always differentiable. For gH-differentiable *IV-Fs*, Cano et al. [19] found error limitations to quadrature rules. In addition, Roy and Panda [20] developed the idea of the -monotonic property of *IV-Fs* in the higher dimension and used extended Hukuhara differentiability to obtain various conclusions. We refer to [21–25], and the references therein, for further information on *IV-Fs*. An et al. [26] and Zhao et al. [27] recently proposed an (h1, h2)-convex *IV-F* and harmonically h-convex *IV-F*, respectively. Moreover, they found certain interval $\mathcal{H}$-$\mathcal{H}$ type inequalities. Budak et al. [28] also created the $\mathcal{H}$-$\mathcal{H}$ inequality for a convex *IV-F* and its product. For more information related to generalized convex functions and fractional inequalities in interval-valued settings, see [29–53] and the references therein.

Inspired by the ongoing research, we introduce the concept of left and right preinvex *IV-F* and establish the $\mathcal{H}$-$\mathcal{H}$ and $\mathcal{H}$-$\mathcal{H}$-Fejér inequality for left and right preinvex *IV-Fs* and the product of two left and right preinvex *IV-Fs* using Riemann integrals in interval-valued settings, which are motivated by the above studies and ideas. We also provide some examples to support our ideas.

2. Preliminaries

First, we offer some background information on interval-valued functions, the theory of convexity, interval-valued integration, and interval-valued fractional integration, which will be utilized throughout the article.

We offer some fundamental arithmetic regarding interval analysis in this paragraph, which will be quite useful throughout the article.

$$\mathcal{Z} = [\mathcal{Z}_*,\ \mathcal{Z}^*],\ \mathcal{Q} = [\mathcal{Q}_*,\ \mathcal{Q}^*]\ (\mathcal{Z}_* \le \varkappa \le\ \mathcal{Z}^*\ \textit{and}\ \mathcal{Q}_* \le z \le\ \mathcal{Q}^*\varkappa\ ,\ z \in \mathbb{R})$$
$$\mathcal{Z} + \mathcal{Q} = [\mathcal{Z}_*,\ \mathcal{Z}^*] + [\mathcal{Q}_*,\ \mathcal{Q}^*] = [\mathcal{Z}_* + \mathcal{Q}_*,\ \mathcal{Z}^* + \mathcal{Q}^*],$$
$$\mathcal{Z} - \mathcal{Q} = [\mathcal{Z}_*,\ \mathcal{Z}^*] - [\mathcal{Q}_*,\ \mathcal{Q}^*] = [\mathcal{Z}_* - \mathcal{Q}_*,\ \mathcal{Z}^* - \mathcal{Q}^*],$$
$$\min\mathcal{X} = \min\{\mathcal{Z}_*\mathcal{Q}_*,\ \mathcal{Z}^*\mathcal{Q}_*,\ \mathcal{Z}_*\mathcal{Q}^*,\ \mathcal{Z}^*\mathcal{Q}^*\},\ \max\mathcal{X} = \max\{\mathcal{Z}_*\mathcal{Q}_*,\ \mathcal{Z}^*\mathcal{Q}_*,\ \mathcal{Z}_*\mathcal{Q}^*,\ \mathcal{Z}^*\mathcal{Q}^*\}$$
$$\nu.[\mathcal{Z}_*,\ \mathcal{Z}^*] = \begin{cases} [\nu\mathcal{Z}_*,\ \nu\mathcal{Z}^*] \text{ if } \nu > 0, \\ \{0\} \text{ if } \nu = 0, \\ [\nu\mathcal{Z}^*, \nu\mathcal{Z}_*] \text{ if } \nu < 0. \end{cases}$$

Let $\mathcal{K}_C$, $\mathcal{K}_C^+$, $\mathcal{K}_C^-$ be the set of all closed intervals of $\mathbb{R}$, the set of all closed positive intervals of $\mathbb{R}$ and the set of all closed negative intervals of $\mathbb{R}$. Then, $\mathcal{K}_C$, $\mathcal{K}_C^+$, and $\mathcal{K}_C^-$ are defined as

$$\mathcal{K}_C = \{[\mathcal{Z}_*,\ \mathcal{Z}^*] : \mathcal{Z}_*,\ \mathcal{Z}^* \in \mathbb{R} \text{ and } \mathcal{Z}_* \le\ \mathcal{Z}^*\}$$
$$\mathcal{K}_C^+ = \{[\mathcal{Z}_*,\ \mathcal{Z}^*] : \mathcal{Z}_*,\ \mathcal{Z}^* \in \mathcal{K}_C \text{ and } \mathcal{Z}_* > 0\}$$
$$\mathcal{K}_C^- = \{[\mathcal{Z}_*,\ \mathcal{Z}^*] : \mathcal{Z}_*,\ \mathcal{Z}^* \in \mathcal{K}_C \text{ and} \mathcal{Z}^* < 0\}$$

For $[\mathcal{Z}_*,\ \mathcal{Z}^*]$, $[\mathcal{Q}_*,\ \mathcal{Q}^*] \in \mathcal{K}_C$, the inclusion " $\subseteq$ " is defined by $[\mathcal{Z}_*,\ \mathcal{Z}^*] \subseteq [\mathcal{Q}_*,\ \mathcal{Q}^*]$, if and only if, $\mathcal{Q}_* \le \mathcal{Z}_*$, $\mathcal{Z}^* \le \mathcal{Q}^*$.

Remark 1. *[36] The relation "$\leq_p$" defined on $\mathcal{K}_C$ by*

$$[\mathcal{Q}_*,\ \mathcal{Q}^*] \leq_p [\mathcal{Z}_*,\ \mathcal{Z}^*] \text{ if and only if } \mathcal{Q}_* \leq \mathcal{Z}_*,\ \mathcal{Q}^* \leq \mathcal{Z}^*, \tag{1}$$

for all $[\mathcal{Q}_*,\ \mathcal{Q}^*],\ [\mathcal{Z}_*,\ \mathcal{Z}^*] \in \mathcal{K}_C$, is a pseudo-order relation.

Theorem 1. *[9] If $\Upsilon : [\mu,\ v] \subset \mathbb{R} \to \mathcal{K}_C$ is an IV-F, such that $\Upsilon(\omega) = [\Upsilon_*(\omega),\ \Upsilon^*(\omega)]$, then, Υ is Riemann integrable over $[\mu, v]$ if and only if, $\Upsilon_*(\omega)$ and $\Upsilon^*(\omega)$ are both Riemann integrable over $[\mu, v]$, such that*

$$(IR)\int_{\mu}^{v} \Upsilon(\omega)d\omega = \left[(R)\int_{\mu}^{v} \Upsilon_*(\omega)d\omega,\ (R)\int_{\mu}^{v} \Upsilon^*(\omega)d\omega\right] \tag{2}$$

where $\Upsilon_*, \Upsilon^* : [\mu,\ v] \to \mathbb{R}$.

The collection of all Riemann integrable real valued functions and Riemann integrable *IV-Fs* is denoted by $\mathcal{R}_{[\mu,v]}$ and $\mathcal{IR}_{[\mu,v]}$, respectively.

Definition 1. *A set $K \subset \mathbb{R}^n$ is said to be a convex set, if, for all $\omega, \varkappa \in K,\ \mathrm{t} \in [0,\ 1]$, we have*

$$\mathrm{t}\varkappa + (1-\mathrm{t})\omega \in K, \text{ or } \mathrm{t}\omega + (1-\mathrm{t})\varkappa \in K.$$

Definition 2. *[36] Let K be a convex set. Then, IV-F $\Upsilon : K \to \mathcal{K}_C^+$ is said to be left and right convex on K if*

$$\Upsilon(\mathrm{t}\omega + (1-\mathrm{t})\varkappa) \leq_p \mathrm{t}\Upsilon(\omega) + (1-\mathrm{t})\Upsilon(\varkappa), \tag{3}$$

for all $\omega, \varkappa \in K,\ \mathrm{t} \in [0,\ 1]$. Υ is called left and right concave on K if Equation (3) is reversed.

Definition 3. *[7] A set $A \subset \mathbb{R}^n$ is said to be an invex set, if, for all $\omega, \varkappa \in A,\ \mathrm{t} \in [0,\ 1]$, we have*

$$\omega + (1-\mathrm{t})\zeta(\varkappa, \omega) \in A \text{ or } \omega + \mathrm{t}\zeta(\varkappa, \omega) \in A,$$

where $\zeta : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$.

Definition 4. *[6] Let A be an invex set. Then, IV-F $\Upsilon : A \to \mathcal{K}_C^+$ is said to be left and right preinvex on A with respect to ζ if*

$$\Upsilon(\omega + (1-\mathrm{t})\zeta(\varkappa, \omega)) \leq_p \mathrm{t}\Upsilon(\omega) + (1-\mathrm{t})\Upsilon(\varkappa), \tag{4}$$

for all $\omega, \varkappa \in A,\ \mathrm{t} \in [0,\ 1]$, where $\zeta : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$. Υ is called left and right preincave on A with respect to ζ if inequality (4) is reversed. Υ is called affine if Υ is both convex and concave.

Remark 2. *The left and right preinvex IV-Fs have some very nice properties similar to left and right convex IV-F:*

- *if Υ is left and right preinvex IV-F, then, $\theta\Upsilon$ is also left and right preinvex for $\theta \geq 0$.*
- *if Υ and $\mathfrak{D}$ both are left and right preinvex IV-Fs, then, $max(\Upsilon(\omega), \mathfrak{D}(\omega))$ is also left and right preinvex IV-Fs.*

  In the case of $\zeta(\varkappa, \omega) = -\omega$, we obtain (4) from (3).

The following outcome is very important in the field of interval-valued calculus because, by using this result, we can easily handle *IV-Fs*. Basically, Theorem 2 establishes the relation between *IV-F* $\Upsilon(\omega)$ and lower function $\Upsilon_*(\omega)$ and upper function $\Upsilon^*(\omega)$.

The following assumption will be required to prove the next result regarding the bifunction $\zeta : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$, which is known as:

Condition C. *[7] Let A be an invex set with respect to ζ. For any $\varkappa, \omega \in A$ and $\mathrm{t} \in [0,\ 1]$,*

$$\zeta(\omega, \omega + \mathrm{t}\zeta(\varkappa, \omega)) = -\mathrm{t}\zeta(\varkappa, \omega),$$
$$\zeta(\varkappa, \omega + \mathrm{t}\zeta(\varkappa, \omega)) = (1-\mathrm{t})\zeta(\varkappa, \omega).$$

Clearly for $\mathrm{t} = 0$, we have $\zeta(\varkappa, \omega) = 0$ if and only if, $\varkappa = \omega$, for all $\varkappa, \omega \in A$. For the applications of Condition C, see [26,30,34,35].

Theorem 2. *[6] Let A be an invex set and $Y : A \to \mathcal{K}_C^+$ be a IV-F such that*

$$Y(\omega) = [Y_*(\omega),\ Y^*(\omega)],\ \forall\ \omega \in A, \tag{5}$$

for all $\omega \in A$. Then, Y is left and right preinvex IV-F on A, if and only if, $Y_*(\omega)$ and $Y^*(\omega)$ both are preinvex functions.

Remark 3. *If $Y_*(\omega) = Y^*(\omega)$, then, from (4), one can acquire the following inequality, see [2]:*

$$Y(\omega + (1-\mathrm{t})\zeta(\varkappa, \omega)) \le \mathrm{t}Y(\omega) + (1-\mathrm{t})Y(\varkappa), \tag{6}$$

for all $\omega,\ \in A,\ \mathrm{t} \in [0,\ 1]$, where $\zeta : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$.

If $Y_*(\omega) = Y^*(\omega)$ with $\zeta(\varkappa, \omega) = \varkappa - \omega$, then, from (4), one can acquire the following inequality:

$$Y(\mathrm{t}\omega + (1-\mathrm{t})\varkappa) \le \mathrm{t}Y(\omega) + (1-\mathrm{t})Y(\varkappa), \tag{7}$$

for all $\omega, \varkappa \in K,\ \mathrm{t} \in [0,\ 1]$.

Example 1. *We consider the IV-F $Y : [0,1] \to \mathcal{K}_C^+$ defined by $Y(\omega) = [2,\ 4\ ]\omega^2$. Since end point functions $Y_*(\omega), Y^*(\omega)$ are preinvex functions with respect to $\zeta(\varkappa, \omega) = \varkappa - \omega$. Hence, $Y(\omega)$ is left and right preinvex IV-F.*

3. Main Results

In this section, we derive interval $\boldsymbol{\mathcal{H}}$-$\boldsymbol{\mathcal{H}}$ type inequalities for left and right preinvex functions in interval-valued settings. Moreover, we provide some nontrivial examples to verify the validity of the theory developed in this study.

Theorem 3. *Let $Y : [v,\ v + \zeta(\mu,\ v)] \to \mathcal{K}_C^+$ be a left and right preinvex IV-F such that $Y(\omega) = [Y_*(\omega),\ Y^*(\omega)]$ for all $\omega \in [v,\ v + \zeta(\mu,\ v)]$. If $Y \in \mathfrak{IR}_{([v,\ v+\zeta(\mu,\ v)])}$, then*

$$Y\left(\frac{2v + \zeta(\mu,\ v)}{2}\right) \le_p \frac{1}{\zeta(\mu,\ v)}\ (IR)\int_v^{v+\zeta(\mu,\ v)} Y(\omega)d\omega \le_p \frac{Y(v) + Y(v + \zeta(\mu,\ v))}{2} \le_p \frac{Y(v) + Y(\mu)}{2} \tag{8}$$

If Y is left and right preincave, then, we achieve the following coming inequality:

$$Y\left(\frac{2v + \zeta(\mu,\ v)}{2}\right) \ge_p \frac{1}{\zeta(\mu,\ v)}\ (IR)\int_v^{v+\zeta(\mu,\ v)} Y(\omega)d\omega \ge_p \frac{Y(v) + Y(v + \zeta(\mu,\ v))}{2} \le_p \frac{Y(v) + Y(\mu)}{2} \tag{9}$$

Proof. Let $Y : [v,\ v + \zeta(\mu,\ v)] \to \mathcal{K}_C^+$ be a left and right preinvex *IV-F*. Then, by hypothesis, we have

$$2Y\left(\frac{2v + \zeta(\mu,\ v)}{2}\right) \le_p Y(v + (1-\mathrm{t})\zeta(\mu,\ v)) + Y(v + \mathrm{t}\zeta(\mu,\ v)).$$

Therefore, we have

$$2Y_*\left(\tfrac{2v+\zeta(\mu,\ v)}{2}\right) \le Y_*(v + (1-\mathrm{t})\zeta(\mu,\ v)) + Y_*(v + \mathrm{t}\zeta(\mu,\ v)),$$
$$2Y^*\left(\tfrac{2v+\zeta(\mu,\ v)}{2}\right) \le Y^*(v + (1-\mathrm{t})\zeta(\mu,\ v)) + Y^*(v + \mathrm{t}\zeta(\mu,\ v)).$$

Then

$$2\int_0^1 Y_*\left(\frac{2v+\zeta(\mu,\ v)}{2}\right)dt \le \int_0^1 Y_*(v+(1-t)\zeta(\mu,\ v))dt + \int_0^1 Y_*(v+t\zeta(\mu,\ v))dt,$$
$$2\int_0^1 Y^*\left(\frac{2v+\zeta(\mu,\ v)}{2}\right)dt \le \int_0^1 Y^*(v+(1-t)\zeta(\mu,\ v))dt + \int_0^1 Y^*(v+t\zeta(\mu,\ v))dt.$$

It follows that

$$Y_*\left(\frac{2v+\zeta(\mu,\ v)}{2}\right) \le \frac{1}{\zeta(\mu,\ v)} \int_v^{v+\zeta(\mu,\ v)} Y_*(\omega)d\omega,$$
$$Y^*\left(\frac{2v+\zeta(\mu,\ v)}{2}\right) \le \frac{2}{\zeta(\mu,\ v)} \int_v^{v+\zeta(\mu,\ v)} Y^*(\omega)d\omega.$$

That is

$$\left[Y_*\left(\frac{2v+\zeta(\mu,\ v)}{2}\right),\ Y^*\left(\frac{2v+\zeta(\mu,\ v)}{2}\right)\right] \quad \le_p \frac{1}{\zeta(\mu,\ v)}\left[\int_v^{v+\zeta(\mu,\ v)} Y_*(\omega)d\omega,\ \int_v^{v+\zeta(\mu,\ v)} Y^*(\omega)d\omega\right].$$

Thus,

$$Y\left(\frac{2v+\zeta(\mu,\ v)}{2}\right) \le_p \frac{1}{\zeta(\mu,\ v)}\ (IR)\int_v^{v+\zeta(\mu,\ v)} Y(\omega)d\omega. \tag{10}$$

In a similar way to the above, we have

$$\frac{1}{\zeta(\mu,\ v)}\ (IR)\int_v^{v+\zeta(\mu,\ v)} Y(\omega)d\omega \le_p \frac{Y(v)+Y(\mu)}{2}. \tag{11}$$

Combining (10) and (11), we have

$$Y\left(\frac{2v+\zeta(\mu,\ v)}{2}\right) \le_p \frac{1}{\zeta(\mu,\ v)}\ (IR)\int_v^{v+\zeta(\mu,\ v)} Y(\omega)d\omega \le_p \frac{Y(v)+Y(\mu)}{2}.$$

This completes the proof. □

Remark 4. *If $\zeta(\mu,v) = \mu - v$, then Theorem 3 reduces to the result for left and right convex IV-F, see [29]:*

$$Y\left(\frac{v+\mu}{2}\right) \le_p \frac{1}{\mu-v}\ (IR)\int_v^{\mu} Y(\omega)d\omega \le_p \frac{Y(v)+Y(\mu)}{2}. \tag{12}$$

If $Y_*(\omega) = Y^*(\omega)$, then Theorem 3 reduces to the result for the preinvex function, see [30]:

$$Y\left(\frac{2v+\zeta(\mu,\ v)}{2}\right) \le \frac{1}{\zeta(\mu,\ v)}\ (R)\int_v^{v+\zeta(\mu,\ v)} Y(\omega)d\omega \le [Y(v)+Y(\mu)]\int_0^1 t dt. \tag{13}$$

If $Y_*(\omega) = Y^*(\omega)$ with $\zeta(\mu,v) = \mu - v$, then Theorem 3 reduces to the result for the convex function, see [31,32]:

$$Y\left(\frac{v+\mu}{2}\right) \le \frac{1}{\mu-v}\ (R)\int_v^{\mu} Y(\omega)d\omega \le \frac{Y(v)+Y(\mu)}{2}. \tag{14}$$

Example 2. *We consider the IV-F $Y : [v,\ v+\zeta(\mu,\ v)] = [0,\ \zeta(2,\ 0)] \to \mathcal{K}_C^+$ defined by $Y(\omega) = \left[2\omega^2, 4\omega^2\right]$. Since end point functions $Y_*(\omega) = 2\omega^2$, $Y^*(\omega) = 4\omega^2$ are preinvex functions with respect to $\zeta(\mu,\ v) = \mu - v$. Hence, $Y(\omega)$ is left and right preinvex IV-F with respect to $\zeta(\mu,\ v) = \mu - v$. We now compute the following*

$$Y\left(\frac{2v+\zeta(\mu,\ v)}{2}\right) \le_p \frac{1}{\zeta(\mu,\ v)}\ (IR)\int_v^{v+\zeta(\mu,\ v)} Y(\omega)d\omega \le_p \frac{Y(v)+Y(\mu)}{2}.$$
$$Y_*\left(\frac{2v+\zeta(\mu,\ v)}{2}\right) = Y_*(1) = 2,$$
$$\frac{1}{\zeta(\mu,\ v)} \int_v^{v+\zeta(\mu,\ v)} Y_*(\omega)d\omega = \frac{1}{2}\int_0^2 2\omega^2 d\omega = \frac{8}{3},$$
$$\frac{Y_*(v)+Y_*(\mu)}{2} = 4,$$

that means

$$2 \leq \frac{8}{3} \leq 4.$$

Similarly, it can be easily shown that

$$Y^*\left(\frac{2v+\zeta(\mu,\ v)}{2}\right) \leq \frac{1}{\zeta(\mu,\ v)} \int_v^{v+\zeta(\mu,\ v)} Y^*(\omega)d\omega \leq \frac{Y^*(v)+Y^*(\mu)}{2}$$

such that

$$Y^*\left(\tfrac{2v+\zeta(\mu,\ v)}{2}\right) = Y_*(1) = 4,$$
$$\tfrac{1}{\zeta(\mu,\ v)} \int_v^{v+\zeta(\mu,\ v)} Y^*(\omega)d\omega = \tfrac{1}{2} \int_0^2 4\omega^2 d\omega = \tfrac{16}{3},$$
$$\tfrac{Y^*(v)+Y^*(\mu)}{2} = 8.$$

From which, it follows that

$$4 \leq \frac{16}{3} \leq 8,$$

that is

$$[2,\ 4] \quad \leq_p \left[\frac{8}{3},\ \frac{16}{3}\right] \quad \leq_p [4,\ 8]$$

hence,

$$Y\left(\frac{2v+\zeta(\mu,\ v)}{2}\right) \leq_p \frac{1}{\zeta(\mu,\ v)}\ (IR)\int_v^{v+\zeta(\mu,\ v)} Y(\omega)d\omega \leq_p \frac{Y(v)+Y(\mu)}{2}.$$

Theorem 4. *Let* $Y, \mathfrak{D} : [v,\ v+\zeta(\mu,\ v)] \to \mathcal{K}_C^+$ *be two left and right preinvex IV-F such that* $Y(\omega) = [Y_*(\omega),\ Y^*(\omega)]$ *and* $\mathfrak{D}(\omega) = [\mathfrak{D}_*(\omega),\ \mathfrak{D}^*(\omega)]$ *for all* $\omega \in [v,\ v+\zeta(\mu,\ v)]$. *If* Y, $\mathfrak{D}$ *and* $Y \times \mathfrak{D} \in \mathfrak{TR}_{([v,\ v+\zeta(\mu,\ v)])}$, *then*

$$\frac{1}{\zeta(\mu,\ v)}\ (IR)\int_v^{v+\zeta(\mu,\ v)} Y(\omega) \times \mathfrak{D}(\omega)d\omega \leq_p \frac{\mathcal{A}(v,\mu)}{3} + \frac{\mathcal{C}(v,\mu)}{6}, \tag{15}$$

where $\mathcal{A}(v,\mu) = Y(v) \times \mathfrak{D}(v) + Y(\mu) \times \mathfrak{D}(\mu)$, $\mathcal{C}(v,\mu) = Y(v) \times \mathfrak{D}(\mu) + Y(\mu) \times \mathfrak{D}(v)$, *and* $\mathcal{A}(v,\mu) = [\mathcal{A}_*((v,\mu)),\ \mathcal{A}^*((v,\mu))]$ *and* $\mathcal{C}(v,\mu) = [\mathcal{C}_*((v,\mu)),\ \mathcal{C}^*((v,\mu))]$.

Proof. Since $Y,\ \mathfrak{D} \in \mathcal{IR}_{([v,\ v+\zeta(\mu,\ v)])}$, then we have

$$Y_*(v+(1-t)\zeta(\mu,\ v)) \leq tY_*(v) + (1-t)Y_*(\mu),$$
$$Y^*(v+(1-t)\zeta(\mu,\ v)) \leq tY^*(v) + (1-t)Y^*(\mu).$$

And

$$\mathfrak{D}_*(v+(1-t)\zeta(\mu,\ v)) \leq t\mathfrak{D}_*(v) + (1-t)\mathfrak{D}_*(\mu),$$
$$\mathfrak{D}^*(v+(1-t)\zeta(\mu,\ v)) \leq t\mathfrak{D}^*(v) + (1-t)\mathfrak{D}^*(\mu).$$

From the definition of left and right preinvex *IV-F*, it follows that $0 \leq_p Y(\omega)$ and $0 \leq_p \mathfrak{D}(\omega)$, so

$$\begin{gathered}
Y_*(v+(1-t)\zeta(\mu,\ v)) \times \mathfrak{D}_*(v+(1-t)\zeta(\mu,\ v)) \\
\leq \left(\ tY_*(v) + (1-t)Y_*(\mu)\ \right)\left(\ t\mathfrak{D}_*(v) + (1-t)\mathfrak{D}_*(\mu)\ \right) \\
= Y_*(v) \times \mathfrak{D}_*(v)t^2 + Y_*(\mu) \times \mathfrak{D}_*(\mu)t^2 + Y_*(v) \times \mathfrak{D}_*(\mu)t(1-t) \\
+Y_*(\mu) \times \mathfrak{D}_*(v)t(1-t), \\
Y^*(v+(1-t)\zeta(\mu,\ v)) \times \mathfrak{D}^*(v+(1-t)\zeta(\mu,\ v)) \\
\leq \left(\ tY^*(v) + (1-t)Y^*(\mu)\ \right)\left(\ t\mathfrak{D}^*(v) + (1-t)\mathfrak{D}^*(\mu)\ \right) \\
= Y^*(v) \times \mathfrak{D}^*(v)t^2 + Y^*(\mu) \times \mathfrak{D}^*(\mu)t^2 + Y^*(v) \times \mathfrak{D}^*(\mu)t(1-t) \\
+Y^*(\mu) \times \mathfrak{D}^*(v)t(1-t),
\end{gathered}$$

Integrating both sides of the above inequality over [0,1], we obtain

$$\begin{gathered}
\int_0^1 Y_*(v+(1-\mathrm{t})\zeta(\mu,\ v))\mathfrak{D}_*(v+(1-\mathrm{t})\zeta(\mu,\ v)) \\
=\frac{1}{\zeta(\mu,\ v)}\int_v^{v+\zeta(\mu,\ v)} Y_*(\omega)\mathfrak{D}_*(\omega)d\omega \\
\le (Y_*(v)\mathfrak{D}_*(v)+Y_*(\mu)\mathfrak{D}_*(\mu))\int_0^1 \mathrm{t}^2 d\mathrm{t} \\
+(Y_*(v)\mathfrak{D}_*(\mu)+Y_*(\mu)\mathfrak{D}_*(v))\int_0^1 \mathrm{t}(1-\mathrm{t})d\mathrm{t}, \\
\int_0^1 Y^*(v+(1-\mathrm{t})\zeta(\mu,\ v))\mathfrak{D}^*(v+(1-\mathrm{t})\zeta(\mu,\ v)) \\
=\frac{1}{\zeta(\mu,\ v)}\int_v^{v+\zeta(\mu,\ v)} Y^*(\omega)\mathfrak{D}^*(\omega)d\omega \\
\le (Y^*(v)\mathfrak{D}^*(v)+Y^*(\mu)\mathfrak{D}^*(\mu))\int_0^1 \mathrm{t}^2 d\mathrm{t} \\
+(Y^*(v)\mathfrak{D}^*(\mu)+Y^*(\mu)\mathfrak{D}^*(v))\int_0^1 \mathrm{t}(1-\mathrm{t})d\mathrm{t}.
\end{gathered}$$

It follows that,

$$\begin{gathered}
\frac{1}{\zeta(\mu,\ v)}\int_v^{v+\zeta(\mu,\ v)} Y_*(\omega)\mathfrak{D}_*(\omega)d\omega \le \mathcal{A}_*((v,\mu))\int_0^1 \mathrm{t}^2 d\mathrm{t}\ +\mathcal{C}_*((v,\mu))\int_0^1 \mathrm{t}(1-\mathrm{t})d\mathrm{t}, \\
\frac{1}{\zeta(\mu,\ v)}\int_v^{v+\zeta(\mu,\ v)} Y^*(\omega)\mathfrak{D}^*(\omega)d\omega \le\ \mathcal{A}^*((v,\mu))\int_0^1 \mathrm{t}^2 d\mathrm{t}\ +\ \mathcal{C}^*((v,\mu))\int_0^1 \mathrm{t}(1-\mathrm{t})d\mathrm{t},
\end{gathered}$$

that is

$$\begin{gathered}
\frac{1}{\zeta(\mu,\ v)}\left[\int_v^{v+\zeta(\mu,\ v)} Y_*(\omega)\mathfrak{D}_*(\omega)d\omega,\ \int_v^{v+\zeta(\mu,\ v)} Y^*(\omega)\mathfrak{D}^*(\omega)d\omega\right] \\
\le_p \left[\frac{\mathcal{A}_*((v,\mu))}{3},\ \frac{\mathcal{A}^*((v,\mu))}{3}\right]+\left[\frac{\mathcal{C}_*((v,\mu))}{6},\ \frac{\mathcal{C}^*((v,\mu))}{6}\right].
\end{gathered}$$

Thus,

$$\frac{1}{\zeta(\mu,\ v)}\ (IR)\int_v^{v+\zeta(\mu,\ v)} Y(\omega)\mathfrak{D}(\omega)d\omega \le_p \frac{\mathcal{A}(v,\mu)}{3}+\frac{\mathcal{C}(v,\mu)}{6},$$

and the theorem has been established. □

Example 3. *We consider the IV-Fs* $Y,\ \mathfrak{D}:[v,\ v+\zeta(\mu,\ v)]=[0,\ \zeta(1,\ 0)]\to\mathcal{K}_C^+$ *defined by* $Y(\omega)=[2\omega^2,4\omega^2]$ *and* $\mathfrak{D}(\omega)=[\omega,2\omega]$*. Since end point functions* $Y_*(\omega)=2\omega^2$, $Y^*(\omega)=4\omega^2$ *and* $\mathfrak{D}_*(\omega)=\omega$, $\mathfrak{D}^*(\omega)=2\omega$ *are preinvex functions with respect to* $\zeta(\mu,\ v)=\mu-v$*. Hence* Y, $\mathfrak{D}$ *both are left and right preinvex IV-Fs. We now compute the following*

$$\begin{gathered}
\frac{1}{\zeta(\mu,\ v)}\int_v^{v+\zeta(\mu,\ v)} Y_*(\omega)\times\mathfrak{D}_*(\omega)d\omega=\frac{1}{2}, \\
\frac{1}{\zeta(\mu,\ v)}\int_v^{v+\zeta(\mu,\ v)} Y^*(\omega)\times\mathfrak{D}^*(\omega)d\omega=2, \\
\frac{\mathcal{A}_*((v,\mu))}{3}=\frac{1}{3}, \\
\frac{\mathcal{A}^*((v,\mu))}{3}=\frac{8}{3}, \\
\frac{\mathcal{C}_*((v,\mu))}{6}=0, \\
\frac{\mathcal{C}^*((v,\mu))}{6}=0,
\end{gathered}$$

that means

$$\frac{1}{2}\le\frac{2}{3},\ 2\le\frac{8}{3}.$$

Hence, Theorem 4 is verified.

Theorem 5. *Let* $Y,\mathfrak{D}:[v,\ v+\zeta(\mu,\ v)]\to\mathcal{K}_C^+$ *be two left and right preinvex IV-Fs, such that* $Y(\omega)=[Y_*(\omega),\ Y^*(\omega)]$ *and* $\mathfrak{D}(\omega)=[\mathfrak{D}_*(\omega),\ \mathfrak{D}^*(\omega)]$ *for all* $\omega\in[v,\ v+\zeta(\mu,\ v)]$*. If* Y, $\mathfrak{D}$ *and* $Y\times\mathfrak{D}\in\mathfrak{T}\mathcal{R}_{([v,\ v+\zeta(\mu,\ v)])}$ *and condition C hold for* ζ*, then*

$$2\,Y\left(\frac{2v+\zeta(\mu,\ v)}{2}\right)\times\mathfrak{D}\left(\frac{2v+\zeta(\mu,\ v)}{2}\right)\le_p\frac{1}{\zeta(\mu,\ v)}\ (IR)\int_v^{v+\zeta(\mu,\ v)} Y(\omega)\times\mathfrak{D}(\omega)d\omega+\frac{\mathcal{A}(v,\mu)}{6}+\frac{\mathcal{C}(v,\mu)}{3}, \tag{16}$$

where $\mathcal{A}(v,\mu)=Y(v)\times\mathfrak{D}(v)+Y(\mu)\times\mathfrak{D}(\mu)$, $\mathcal{C}(v,\mu)=Y(v)\times\mathfrak{D}(\mu)+Y(\mu)\times\mathfrak{D}(v)$, and $\mathcal{A}(v,\mu)=[\mathcal{A}_*((v,\mu)),\ \mathcal{A}^*((v,\mu))]$ and $\mathcal{C}(v,\mu)=[\mathcal{C}_*((v,\mu)),\ \mathcal{C}^*((v,\mu))]$.

Proof. Using condition C, we can write

$$v+\frac{1}{2}\zeta(\mu,\ v)=v+\mathrm{t}\zeta(\mu,\ v)+\frac{1}{2}\zeta(v+(1-\mathrm{t})\zeta(\mu,v),\ v+\mathrm{t}\zeta(\mu,v)).$$

By hypothesis, we have

$$\begin{aligned}
&\Upsilon_*\left(\tfrac{2v+\zeta(\mu,\ v)}{2}\right)\times\mathfrak{D}_*\left(\tfrac{2v+\zeta(\mu,\ v)}{2}\right)\\
&\Upsilon^*\left(\tfrac{2v+\zeta(\mu,\ v)}{2}\right)\times\mathfrak{D}^*\left(\tfrac{2v+\zeta(\mu,\ v)}{2}\right)\\
&=\Upsilon_*\left(v+\mathrm{t}\zeta(\mu,\ v)+\tfrac{1}{2}\zeta(v+(1-\mathrm{t})\zeta(\mu,v),\ v+\mathrm{t}\zeta(\mu,v))\right)\\
&\times\mathfrak{D}_*\left(v+\mathrm{t}\zeta(\mu,\ v)+\tfrac{1}{2}\zeta(v+(1-\mathrm{t})\zeta(\mu,v),\ v+\mathrm{t}\zeta(\mu,v))\right)\\
&=\Upsilon^*\left(v+\mathrm{t}\zeta(\mu,\ v)+\tfrac{1}{2}\zeta(v+(1-\mathrm{t})\zeta(\mu,v),\ v+\mathrm{t}\zeta(\mu,v))\right)\\
&\times\mathfrak{D}^*\left(v+\mathrm{t}\zeta(\mu,\ v)+\tfrac{1}{2}\zeta(v+(1-\mathrm{t})\zeta(\mu,v),\ v+\mathrm{t}\zeta(\mu,v))\right)\\
&\leq\tfrac{1}{4}\begin{bmatrix}\Upsilon_*(v+(1-\mathrm{t})\zeta(\mu,\ v))\times\mathfrak{D}_*(v+(1-\mathrm{t})\zeta(\mu,\ v))\\+\Upsilon_*(v+(1-\mathrm{t})\zeta(\mu,\ v))\times\mathfrak{D}_*(v+\mathrm{t}\zeta(\mu,\ v))\end{bmatrix}\\
&\quad+\tfrac{1}{4}\begin{bmatrix}\Upsilon_*(v+\mathrm{t}\zeta(\mu,\ v))\times\mathfrak{D}_*(v+(1-\mathrm{t})\zeta(\mu,\ v))\\+\Upsilon_*(v+\mathrm{t}\zeta(\mu,\ v))\times\mathfrak{D}_*(v+\mathrm{t}\zeta(\mu,\ v))\end{bmatrix},\\
&\leq\tfrac{1}{4}\begin{bmatrix}\Upsilon^*(v+(1-\mathrm{t})\zeta(\mu,\ v))\times\mathfrak{D}^*(v+(1-\mathrm{t})\zeta(\mu,\ v))\\+\Upsilon^*(v+(1-\mathrm{t})\zeta(\mu,\ v))\times\mathfrak{D}^*(v+\mathrm{t}\zeta(\mu,\ v))\end{bmatrix}\\
&\quad+\tfrac{1}{4}\begin{bmatrix}\Upsilon^*(v+\mathrm{t}\zeta(\mu,\ v))\times\mathfrak{D}^*(v+(1-\mathrm{t})\zeta(\mu,\ v))\\+\Upsilon^*(v+\mathrm{t}\zeta(\mu,\ v))\times\mathfrak{D}^*(v+\mathrm{t}\zeta(\mu,\ v))\end{bmatrix},\\
&\leq\tfrac{1}{4}\begin{bmatrix}\Upsilon_*(v+(1-\mathrm{t})\zeta(\mu,\ v))\times\mathfrak{D}_*(v+(1-\mathrm{t})\zeta(\mu,\ v))\\+\Upsilon_*(v+\mathrm{t}\zeta(\mu,\ v))\times\mathfrak{D}_*(v+\mathrm{t}\zeta(\mu,\ v))\end{bmatrix}\\
&+\tfrac{1}{4}\begin{bmatrix}(\mathrm{t}\Upsilon_*(v)+(1-\mathrm{t})\Upsilon_*(\mu))\times((1-\mathrm{t})\mathfrak{D}_*(v)+\mathrm{t}\mathfrak{D}_*(\mu))\\+((1-\mathrm{t})\Upsilon_*(v)+\mathrm{t}\Upsilon_*(\mu))\times(\mathrm{t}\mathfrak{D}_*(v)+(1-\mathrm{t})\mathfrak{D}_*(\mu))\end{bmatrix},\\
&\leq\tfrac{1}{4}\begin{bmatrix}\Upsilon^*(v+(1-\mathrm{t})\zeta(\mu,\ v))\times\mathfrak{D}^*(v+(1-\mathrm{t})\zeta(\mu,\ v))\\+\Upsilon^*(v+\mathrm{t}\zeta(\mu,\ v))\times\mathfrak{D}^*(v+\mathrm{t}\zeta(\mu,\ v))\end{bmatrix}\\
&+\tfrac{1}{4}\begin{bmatrix}(\mathrm{t}\Upsilon^*(v)+(1-\mathrm{t})\Upsilon^*(\mu))\times((1-\mathrm{t})\mathfrak{D}^*(v)+\mathrm{t}\mathfrak{D}^*(\mu))\\+((1-\mathrm{t})\Upsilon^*(v)+\mathrm{t}\Upsilon^*(\mu))\times(\mathrm{t}\mathfrak{D}^*(v)+(1-\mathrm{t})\mathfrak{D}^*(\mu))\end{bmatrix},\\
&=\tfrac{1}{4}\begin{bmatrix}\Upsilon_*(v+(1-\mathrm{t})\zeta(\mu,\ v))\times\mathfrak{D}_*(v+(1-\mathrm{t})\zeta(\mu,\ v))\\+\Upsilon_*(v+\mathrm{t}\zeta(\mu,\ v))\times\mathfrak{D}_*(v+\mathrm{t}\zeta(\mu,\ v))\end{bmatrix}\\
&\quad+\tfrac{1}{2}\begin{bmatrix}\left\{\mathrm{t}^2+(1-\mathrm{t})^2\right\}\mathcal{C}_*((v,\mu))\\+\{\mathrm{t}(1-\mathrm{t})+(1-\mathrm{t})\mathrm{t}\}\mathcal{A}_*((v,\mu))\end{bmatrix},\\
&=\tfrac{1}{4}\begin{bmatrix}\Upsilon^*(v+(1-\mathrm{t})\zeta(\mu,\ v))\times\mathfrak{D}^*(v+(1-\mathrm{t})\zeta(\mu,\ v))\\+\Upsilon^*(v+\mathrm{t}\zeta(\mu,\ v))\times\mathfrak{D}^*(v+\mathrm{t}\zeta(\mu,\ v))\end{bmatrix}\\
&\quad+\tfrac{1}{2}\begin{bmatrix}\left\{\mathrm{t}^2+(1-\mathrm{t})^2\right\}\mathcal{C}^*((v,\mu))\\+\{\mathrm{t}(1-\mathrm{t})+(1-\mathrm{t})\mathrm{t}\}\mathcal{A}^*((v,\mu))\end{bmatrix}.
\end{aligned}$$

Integrating over $[0,\ 1]$, we have

$$\begin{aligned}
2\,\Upsilon_*\left(\tfrac{2v+\zeta(\mu,\ v)}{2}\right)\times\mathfrak{D}_*\left(\tfrac{2v+\zeta(\mu,\ v)}{2}\right)&\leq\tfrac{1}{\zeta(\mu,\ v)}\int_v^{v+\zeta(\mu,\ v)}\Upsilon_*(\omega)\times\mathfrak{D}_*(\omega)d\omega+\tfrac{\mathcal{A}_*((v,\mu))}{6}+\tfrac{\mathcal{C}_*((v,\mu))}{3},\\
2\,\Upsilon^*\left(\tfrac{2v+\zeta(\mu,\ v)}{2}\right)\times\mathfrak{D}^*\left(\tfrac{2v+\zeta(\mu,\ v)}{2}\right)&\leq\tfrac{1}{\zeta(\mu,\ v)}\int_v^{v+\zeta(\mu,\ v)}\Upsilon^*(\omega)\times\mathfrak{D}^*(\omega)d\omega+\tfrac{\mathcal{A}^*((v,\mu))}{6}+\tfrac{\mathcal{C}^*((v,\mu))}{3},
\end{aligned}$$

from which, we have

$$\begin{aligned}
&2\left[\Upsilon_*\left(\tfrac{2v+\zeta(\mu,\ v)}{2}\right)\times\mathfrak{D}_*\left(\tfrac{2v+\zeta(\mu,\ v)}{2}\right),\ \Upsilon^*\left(\tfrac{2v+\zeta(\mu,\ v)}{2}\right)\times\mathfrak{D}^*\left(\tfrac{2v+\zeta(\mu,\ v)}{2}\right)\right]\\
&\leq\ {}_p\tfrac{1}{\zeta(\mu,\ v)}\left[\int_v^{v+\zeta(\mu,\ v)}\Upsilon_*(\omega)\times\mathfrak{D}_*(\omega)d\omega\ ,\ \int_v^{v+\zeta(\mu,\ v)}\Upsilon^*(\omega)\times\mathfrak{D}^*(\omega)d\omega\right]\\
&\quad+\left[\tfrac{\mathcal{A}_*((v,\mu))}{6},\ \tfrac{\mathcal{A}^*((v,\mu))}{6}\right]+\left[\tfrac{\mathcal{C}_*((v,\mu))}{3},\ \tfrac{\mathcal{C}^*((v,\mu))}{3}\right],
\end{aligned}$$

that is

$$2\,Y\left(\frac{2v+\zeta(\mu,\,v)}{2}\right)\times\mathfrak{D}\left(\frac{2v+\zeta(\mu,\,v)}{2}\right)\leq_p\frac{1}{\zeta(\mu,\,v)}\,(IR)\int_v^{v+\zeta(\mu,\,v)}Y(\omega)\times\mathfrak{D}(\omega)d\omega+\frac{\mathcal{A}(v,\mu)}{6}+\frac{\mathcal{C}(v,\mu)}{3}.$$

This completes the proof. □

Example 4. *We consider the IV-Fs* $Y,\,\mathfrak{D}:[v,\,v+\zeta(\mu,\,v)]=[0,\,\zeta(1,\,0)]\rightarrow\mathcal{K}_C^+$ *defined by,* $Y(\omega)=\left[2\omega^2,4\omega^2\right]$ *and* $\mathfrak{D}(\omega)=[1,2]\omega$, *and these functions fulfill all the assumptions of Theorem 5. Since* $Y(\omega)$, $\mathfrak{D}(\omega)$ *both are left and right preinvex IV-Fs with respect to* $\zeta(\mu,\,v)=\mu-v$, *we have* $Y_*(\omega)=2\omega^2$, $Y^*(\omega)=4\omega^2$ *and* $\mathfrak{D}_*(\omega)=\omega$, $\mathfrak{D}^*(\omega)=2\omega$. *We now compute the following*

$$\begin{gathered}2\,Y_*\left(\tfrac{2v+\zeta(\mu,\,v)}{2}\right)\times\mathfrak{D}_*\left(\tfrac{2v+\zeta(\mu,\,v)}{2}\right)=\tfrac{1}{2},\\ 2\,Y^*\left(\tfrac{2v+\zeta(\mu,\,v)}{2}\right)\times\mathfrak{D}^*\left(\tfrac{2v+\zeta(\mu,\,v)}{2}\right)=2,\\ \tfrac{1}{\zeta(\mu,\,v)}\int_v^{v+\zeta(\mu,\,v)}Y_*(\omega)\times\mathfrak{D}_*(\omega)d\omega=\tfrac{1}{2},\\ \tfrac{1}{\zeta(\mu,\,v)}\int_v^{v+\zeta(\mu,\,v)}Y^*(\omega)\times\mathfrak{D}^*(\omega)d\omega=2,\\ \tfrac{\mathcal{A}_*((v,\mu))}{6}=\tfrac{1}{3},\\ \tfrac{\mathcal{A}^*((v,\mu))}{6}=\tfrac{4}{3},\\ \tfrac{\mathcal{C}_*((v,\mu))}{3}=0,\\ \tfrac{\mathcal{C}^*((v,\mu))}{3}=0,\end{gathered}$$

that means

$$\begin{gathered}\tfrac{1}{2}\leq\tfrac{1}{2}+0+\tfrac{1}{3}=\tfrac{5}{6},\\ 2\leq 2+0+\tfrac{4}{3}=\tfrac{10}{3}.\end{gathered}$$

Hence, Theorem 5 is verified.

It is well known that classical $\boldsymbol{\mathcal{H}}$-$\boldsymbol{\mathcal{H}}$-Fejér inequality is a generalization of classical $\boldsymbol{\mathcal{H}}$-$\boldsymbol{\mathcal{H}}$ inequality. Now we derive $\boldsymbol{\mathcal{H}}$-$\boldsymbol{\mathcal{H}}$-Fejér inequality for left and right preinvex *IV-Fs* and then we will obtain the validity of this inequality with the help of a non-trivial example. Firstly, we obtain the second $\boldsymbol{\mathcal{H}}$-$\boldsymbol{\mathcal{H}}$-Fejér inequality for left and right preinvex *IV-F*.

Theorem 6. *Let* $Y:[v,\,v+\zeta(\mu,\,v)]\rightarrow\mathcal{K}_C^+$ *be a left and right preinvex IV-F with* $v<v+\zeta(\mu,\,v)$ *such that* $Y(\omega)=[Y_*(\omega),\,Y^*(\omega)]$ *for all* $\omega\in[v,\,v+\zeta(\mu,\,v)]$. *If* $Y\in\mathfrak{TR}_{([v,\,v+\zeta(\mu,\,v)])}$ *and* $\mathcal{S}:[v,\,v+\zeta(\mu,\,v)]\rightarrow\mathbb{R}$, $\mathcal{S}(\omega)\geq 0$, *symmetric with respect to* $v+\frac{1}{2}\zeta(\mu,\,v)$, *then*

$$\frac{1}{\zeta(\mu,\,v)}\,(IR)\int_v^{v+\zeta(\mu,\,v)}Y(\omega)\mathcal{S}(\omega)d\omega\leq_p[Y(v)+Y(\mu)]\int_0^1\mathrm{t}\mathcal{S}(v+\mathrm{t}\zeta(\mu,\,v))d\mathrm{t}. \quad (17)$$

Proof. Let Y be a left and right preinvex *IV-F*. Then, we have

$$\begin{gathered}Y_*(v+(1-\mathrm{t})\zeta(\mu,v))\mathcal{S}(v+(1-\mathrm{t})\zeta(\mu,v))\\ \leq(\mathrm{t}Y_*(v)+(1-\mathrm{t})Y_*(\mu))\mathcal{S}(v+(1-\mathrm{t})\zeta(\mu,v)),\\ Y^*(v+(1-\mathrm{t})\zeta(\mu,v))\mathcal{S}(v+(1-\mathrm{t})\zeta(\mu,v))\\ \leq(\mathrm{t}Y^*(v)+(1-\mathrm{t})Y^*(\mu))\mathcal{S}(v+(1-\mathrm{t})\zeta(\mu,v)).\end{gathered} \quad (18)$$

And

$$\begin{gathered}Y_*(v+\mathrm{t}\zeta(\mu,\,v))\mathcal{S}(v+\mathrm{t}\zeta(\mu,\,v))\leq((1-\mathrm{t})Y_*(v)+\mathrm{t}Y_*(\mu))\mathcal{S}(v+\mathrm{t}\zeta(\mu,\,v)),\\ Y^*(v+\mathrm{t}\zeta(\mu,\,v))\mathcal{S}(v+\mathrm{t}\zeta(\mu,\,v))\leq((1-\mathrm{t})Y^*(v)+\mathrm{t}Y^*(\mu))\mathcal{S}(v+\mathrm{t}\zeta(\mu,\,v)).\end{gathered} \quad (19)$$

After adding (18) and (19), and integrating over $[0,\ 1]$, we get

$$\begin{gathered}\int_0^1 Y_*(v+(1-\mathrm{t})\zeta(\mu,v))\mathcal{S}(v+(1-\mathrm{t})\zeta(\mu,v))d\mathrm{t}\\ +\int_0^1 Y_*(v+\mathrm{t}\zeta(\mu,\ v))\mathcal{S}(v+\mathrm{t}\zeta(\mu,\ v))d\mathrm{t}\\ \leq \int_0^1\left[\begin{array}{c} Y_*(v)\{\mathrm{t}\mathcal{S}(v+(1-\mathrm{t})\zeta(\mu,v))+(1-\mathrm{t})\mathcal{S}(v+\mathrm{t}\zeta(\mu,\ v))\}\\ +Y_*(\mu)\{(1-\mathrm{t})\mathcal{S}(v+(1-\mathrm{t})\zeta(\mu,v))+\mathrm{t}\mathcal{S}(v+\mathrm{t}\zeta(\mu,\ v))\}\end{array}\right]d\mathrm{t},\\ \int_0^1 Y^*(v+\mathrm{t}\zeta(\mu,\ v))\mathcal{S}(v+\mathrm{t}\zeta(\mu,\ v))d\mathrm{t}\\ +\int_0^1 Y^*(v+(1-\mathrm{t})\zeta(\mu,v))\mathcal{S}(v+(1-\mathrm{t})\zeta(\mu,v))d\mathrm{t}\\ \leq \int_0^1\left[\begin{array}{c} Y^*(v)\{\mathrm{t}\mathcal{S}(v+(1-\mathrm{t})\zeta(\mu,v))+(1-\mathrm{t})\mathcal{S}(v+\mathrm{t}\zeta(\mu,\ v))\}\\ +Y^*(\mu)\{(1-\mathrm{t})\mathcal{S}(v+(1-\mathrm{t})\zeta(\mu,v))+\mathrm{t}\mathcal{S}(v+\mathrm{t}\zeta(\mu,\ v))\}\end{array}\right]d\mathrm{t}.\\ =2Y_*(v)\int_0^1\ \mathrm{t}\mathcal{S}(v+(1-\mathrm{t})\zeta(\mu,v))\ d\mathrm{t}+2Y_*(\mu)\int_0^1\ \mathrm{t}\mathcal{S}(v+\mathrm{t}\zeta(\mu,\ v))\ d\mathrm{t},\\ =2Y^*(v)\int_0^1\ \mathrm{t}\mathcal{S}(v+(1-\mathrm{t})\zeta(\mu,v))\ d\mathrm{t}+2Y^*(\mu)\int_0^1\ \mathrm{t}\mathcal{S}(v+\mathrm{t}\zeta(\mu,\ v))\ d\mathrm{t}.\end{gathered}$$

Since $\mathcal{S}$ is symmetric, then

$$\begin{aligned}&=2[Y_*(v)+Y_*(\mu)]\int_0^1\ \mathrm{t}\mathcal{S}(v+\mathrm{t}\zeta(\mu,\ v))\ d\mathrm{t},\\ &=2[Y^*(v)+Y^*(\mu)]\int_0^1\ \mathrm{t}\mathcal{S}(v+\mathrm{t}\zeta(\mu,\ v))\ d\mathrm{t}.\end{aligned}\tag{20}$$

Since

$$\begin{gathered}\int_0^1 Y_*(v+(1-\mathrm{t})\zeta(u,v))\mathcal{S}(v+(1-\mathrm{t})\zeta(\mu,v))d\mathrm{t}\\ =\int_0^1 Y_*(v+\mathrm{t}\zeta(u,v))\mathcal{S}(v+\mathrm{t}\zeta(\mu,v))d\mathrm{t}\\ =\frac{1}{\zeta(\mu,\ v)}\int_v^{v+\zeta(\mu,\ v)} Y_*(\omega)\mathcal{S}(\omega)d\omega,\\ \int_0^1 Y^*(v+\mathrm{t}\zeta(u,v))\mathcal{S}(v+\mathrm{t}\zeta(\mu,v))d\mathrm{t}\\ =\int_0^1 Y^*(v+(1-\mathrm{t})\zeta(u,v))\mathcal{S}(v+(1-\mathrm{t})\zeta(\mu,v))d\mathrm{t}\\ =\frac{1}{\zeta(\mu,\ v)}\int_v^{v+\zeta(\mu,\ v)} Y^*(\omega)\mathcal{S}(\omega)d\omega.\end{gathered}\tag{21}$$

From (21), we have

$$\begin{aligned}\frac{1}{\zeta(\mu,\ v)}\ \int_v^{v+\zeta(\mu,\ v)} Y_*(\omega)\mathcal{S}(\omega)d\omega &\leq [Y_*(v)+Y_*(\mu)]\int_0^1\ \mathrm{t}\mathcal{S}(v+\mathrm{t}\zeta(\mu,\ v))\ d\mathrm{t},\\ \frac{1}{\zeta(\mu,\ v)}\ \int_v^{v+\zeta(\mu,\ v)} Y^*(\omega)\mathcal{S}(\omega)d\omega &\leq [Y^*(v)+Y^*(\mu)]\int_0^1\ \mathrm{t}\mathcal{S}(v+\mathrm{t}\zeta(\mu,\ v))\ d\mathrm{t},\end{aligned}$$

that is

$$\begin{gathered}\left[\frac{1}{\zeta(\mu,\ v)}\ \int_v^{v+\zeta(\mu,\ v)} Y_*(\omega)\mathcal{S}(\omega)d\omega,\ \frac{1}{\zeta(\mu,\ v)}\ \int_v^{v+\zeta(\mu,\ v)} Y^*(\omega)\mathcal{S}(\omega)d\omega\right]\\ \leq_p [Y_*(v)+Y_*(\mu),\ Y^*(v)+Y^*(\mu)]\int_0^1\ \mathrm{t}\mathcal{S}(v+\mathrm{t}\zeta(\mu,\ v))\ d\mathrm{t}\end{gathered}$$

hence

$$\frac{1}{\zeta(\mu,\ v)}\ (IR)\int_v^{v+\zeta(\mu,\ v)} Y(\omega)\mathcal{S}(\omega)d\omega \leq_p [Y(v)+Y(\mu)]\int_0^1 \mathrm{t}\mathcal{S}(v+\mathrm{t}\zeta(\mu,\ v))d\mathrm{t}.$$

□

Now, we present the succeeding reformative version of the generalized version of first $\boldsymbol{\mathcal{H}}$-$\boldsymbol{\mathcal{H}}$-Fejér inequalities for left and right preinvex *IV-Fs*.

Theorem 7. *Let* $Y:[v,\ v+\zeta(\mu,\ v)]\to\mathcal{K}_C^+$ *be a left and right preinvex IV-F with* $v<v+\zeta(\mu,\ v)$ *such that* $Y(\omega)=[Y_*(\omega),\ Y^*(\omega)]$ *for all* $\omega\in[v,\ v+\zeta(\mu,\ v)]$*. If* $Y\in\mathfrak{IR}_{([v,\ v+\zeta(\mu,\ v)])}$ *and* $\mathcal{S}:[v,\ v+\zeta(\mu,\ v)]\to\mathbb{R}$, $\mathcal{S}(\omega)\geq 0$, *symmetric with respect to* $v+\frac{1}{2}\zeta(\mu,\ v)$, *and* $\int_v^{v+\zeta(\mu,\ v)}\mathcal{S}(\omega)d\omega>0$, *and Condition C for* ζ, *then*

$$Y\left(v+\frac{1}{2}\zeta(\mu,\ v)\right)\leq_p \frac{1}{\int_v^{v+\zeta(\mu,\ v)}\mathcal{S}(\omega)d\omega}\ (IR)\int_v^{v+\zeta(\mu,\ v)} Y(\omega)\mathcal{S}(\omega)d\omega.\tag{22}$$

Proof. Using condition C, we can write

$$v + \frac{1}{2}\zeta(\mu,\ v) = v + \mathrm{t}\zeta(\mu,\ v) + \frac{1}{2}\zeta(v + (1-\mathrm{t})\zeta(\mu, v),\ v + \mathrm{t}\zeta(\mu, v)).$$

Since Y is a left and right preinvex, we have

$$\begin{aligned} Y_*\left(v + \tfrac{1}{2}\zeta(\mu,\ v)\right) &= Y_*\left(v + \mathrm{t}\zeta(\mu,\ v) + \tfrac{1}{2}\zeta(v + (1-\mathrm{t})\zeta(\mu, v),\ v + \mathrm{t}\zeta(\mu, v))\right) \\ &\leq \tfrac{1}{2}(Y_*(v + (1-\mathrm{t})\zeta(\mu, v)) + Y_*(v + \mathrm{t}\zeta(\mu, v))), \\ Y^*\left(v + \tfrac{1}{2}\zeta(\mu,\ v)\right) &= Y^*\left(v + \mathrm{t}\zeta(\mu,\ v) + \tfrac{1}{2}\zeta(v + (1-\mathrm{t})\zeta(\mu, v),\ v + \mathrm{t}\zeta(\mu, v))\right) \\ &\leq (Y^*(v + (1-\mathrm{t})\zeta(\mu, v)) + Y^*(v + \mathrm{t}\zeta(\mu, v))). \end{aligned} \tag{23}$$

By multiplying (23) by $\mathcal{S}(v + (1-\mathrm{t})\zeta(\mu, v)) = \mathcal{S}(v + \mathrm{t}\zeta(\mu, v))$ and integrating it by t over $[0,\ 1]$, we obtain

$$\begin{aligned} & Y_*\left(v + \tfrac{1}{2}\zeta(\mu,\ v)\right)\int_0^1 \mathcal{S}(v + \mathrm{t}\zeta(\mu,\ v))d\mathrm{t} \\ \leq \tfrac{1}{2} & \left(\begin{array}{c} \int_0^1 Y_*(v + (1-\mathrm{t})\zeta(\mu, v))\mathcal{S}(v + (1-\mathrm{t})\zeta(\mu,\ v))d\mathrm{t} \\ + \int_0^1 Y_*(v + \mathrm{t}\zeta(\mu, v))d\mathrm{t}\mathcal{S}(v + \mathrm{t}\zeta(\mu,\ v)) \end{array}\right) \\ & Y^*\left(v + \tfrac{1}{2}\zeta(\mu,\ v)\right)\int_0^1 \mathcal{S}(v + \mathrm{t}\zeta(\mu,\ v))d\mathrm{t} \\ \leq \tfrac{1}{2} & \left(\begin{array}{c} \int_0^1 Y^*(v + (1-\mathrm{t})\zeta(\mu, v))\mathcal{S}(v + (1-\mathrm{t})\zeta(\mu,\ v))d\mathrm{t} \\ + \int_0^1 Y^*(v + \mathrm{t}\zeta(\mu, v))\mathcal{S}(v + \mathrm{t}\zeta(\mu,\ v))d\mathrm{t} \end{array}\right). \end{aligned} \tag{24}$$

Since

$$\begin{aligned} & \int_0^1 Y_*(v + (1-\mathrm{t})\zeta(\mu, v))\mathcal{S}(v + (1-\mathrm{t})\zeta(\mu, v))d\mathrm{t} \\ & = \int_0^1 Y_*(v + \mathrm{t}\zeta(\mu,\ v))\mathcal{S}(v + \mathrm{t}\zeta(\mu,\ v))d\mathrm{t} \\ & = \tfrac{1}{\zeta(\mu,\ v)}\int_v^{v+\zeta(\mu,\ v)} Y_*(\omega)\mathcal{S}(\omega)d\omega \\ & \int_0^1 Y^*(v + \mathrm{t}\zeta(\mu,\ v))\mathcal{S}(v + \mathrm{t}\zeta(\mu,\ v))d\mathrm{t} \\ & = \int_0^1 Y^*(v + (1-\mathrm{t})\zeta(\mu, v))\mathcal{S}(v + (1-\mathrm{t})\zeta(\mu, v))d\mathrm{t} \\ & = \tfrac{1}{\zeta(\mu,\ v)}\int_v^{v+\zeta(\mu,\ v)} Y^*(\omega)\mathcal{S}(\omega)d\omega. \end{aligned} \tag{25}$$

From (25), we have

$$\begin{aligned} Y_*\left(v + \tfrac{1}{2}\zeta(\mu,\ v)\right) &\leq \tfrac{1}{\int_v^{v+\zeta(\mu,\ v)}\mathcal{S}(\omega)d\omega}\int_v^{v+\zeta(\mu,\ v)} Y_*(\omega)\mathcal{S}(\omega)d\omega, \\ Y^*\left(v + \tfrac{1}{2}\zeta(\mu,\ v)\right) &\leq \tfrac{1}{\int_v^{v+\zeta(\mu,\ v)}\mathcal{S}(\omega)d\omega}\int_v^{v+\zeta(\mu,\ v)} Y^*(\omega)\mathcal{S}(\omega)d\omega. \end{aligned}$$

From which, we have

$$\begin{aligned} & \left[Y_*\left(v + \tfrac{1}{2}\zeta(\mu,\ v)\right),\ Y^*\left(v + \tfrac{1}{2}\zeta(\mu,\ v)\right)\right] \\ \leq \quad & {}_p\tfrac{1}{\int_v^{v+\zeta(\mu,\ v)}\mathcal{S}(\omega)d\omega}\left[\int_v^{v+\zeta(\mu,\ v)} Y_*(\omega)\mathcal{S}(\omega)d\omega,\ \int_v^{v+\zeta(\mu,\ v)} Y^*(\omega)\mathcal{S}(\omega)d\omega\right], \end{aligned}$$

that is

$$Y\left(v + \frac{1}{2}\zeta(\mu,\ v)\right) \leq_p \frac{1}{\int_v^{v+\zeta(\mu,\ v)}\mathcal{S}(\omega)d\omega}\,(IR)\int_v^{v+\zeta(\mu,\ v)} Y(\omega)\mathcal{S}(\omega)d\omega.$$

This completes the proof. □

Remark 5. *If one considers taking $\zeta(\mu,\ v) = \mu - v$, then, by combining inequalities (17) and (22), we achieve the expected inequality.*

If one considers taking $Y_*(\omega) = Y^*(\omega)$, then, by combining inequalities (17) and (22), we achieve the classical $\boldsymbol{\mathcal{H}}$-$\boldsymbol{\mathcal{H}}$-Fejér inequality, see [30].

If one considers taking $Y_*(\omega) = Y^*\omega$ and $\zeta(\mu,\ v) = \mu - v$, then, by combining inequalities (17) and (22), we acquire the classical $\mathcal{H}$-$\mathcal{H}$-Fejér inequality, see [33].

Example 5. *We consider the IV-F $Y : [1,\ 1+\zeta(4,\ 1)] \to \mathcal{K}_C^+$ defined by $Y(\omega) = [2,4]e^{\omega}$. Since end point functions $Y_*(\omega)$, $Y^*(\omega)$ are preinvex functions $\zeta(\varkappa,\omega) = \varkappa - \omega$, then, $Y(\omega)$ is left and right preinvex IV-F. If*

$$\mathcal{S}(\omega) = \begin{cases} \omega - 1,\ \sigma \in \left[1, \frac{5}{2}\right], \\ 4 - \omega,\ \sigma \in \left(\frac{5}{2},\ 4\right]. \end{cases}$$

Then, we have

$$\begin{aligned}
\frac{1}{\zeta(4,\ 1)} \int_1^{1+\ \zeta(4,\ 1)} Y_*(\omega)\mathcal{S}(\omega)d\omega &= \frac{1}{3}\int_1^4 Y_*(\omega)\mathcal{S}(\omega)d\omega \\
&= \frac{1}{3}\int_1^{\frac{5}{2}} Y_*(\omega)\mathcal{S}(\omega)d\omega + \frac{1}{3}\int_{\frac{5}{2}}^4 Y_*(\omega)\mathcal{S}(\omega)d\omega, \\
\frac{1}{\zeta(4,\ 1)} \int_1^{1+\ \zeta(4,\ 1)} Y^*(\omega)\mathcal{S}(\omega)d\omega &= \frac{1}{3}\int_1^4 Y^*(\omega)\mathcal{S}(\omega)d\omega \\
&= \frac{1}{3}\int_1^{\frac{5}{2}} Y^*(\omega)\mathcal{S}(\omega)d\omega + \frac{1}{3}\int_{\frac{5}{2}}^4 Y^*(\omega)\mathcal{S}(\omega)d\omega, \\
&= \frac{2}{3}\int_1^{\frac{5}{2}} e^{\omega}(\omega - 1)d\omega + \frac{2}{3}\int_{\frac{5}{2}}^4 e^{\omega}(4-\omega)d\omega \approx 22, \\
&= \frac{4}{3}\int_1^{\frac{5}{2}} e^{\omega}(\omega - 1)d\omega + \frac{4}{3}\int_{\frac{5}{2}}^4 e^{\omega}(4-\omega)d\omega \approx 44,
\end{aligned} \tag{26}$$

and

$$\begin{aligned}
&[Y_*(v) + Y_*(\mu)]\int_0^1\ \mathrm{t}\mathcal{S}(v + \mathrm{t}\zeta(\mu,\ v))\ d\mathrm{t} \\
&[Y^*(v) + Y^*(\mu)]\int_0^1\ \mathrm{t}\mathcal{S}(v + \mathrm{t}\zeta(\mu,\ v))\ d\mathrm{t} \\
&= 2\left[e + e^4\right]\left[\int_0^{\frac{1}{2}} 3\mathrm{t}^2 d\omega + \int_{\frac{1}{2}}^1 \mathrm{t}(3 - 3\mathrm{t})d\mathrm{t}\right] \approx 43. \\
&= 4\left[e + e^4\right]\left[\int_0^{\frac{1}{2}} 3\mathrm{t}^2 d\omega + \int_{\frac{1}{2}}^1 \mathrm{t}(3 - 3\mathrm{t})d\mathrm{t}\right] \approx 86.
\end{aligned} \tag{27}$$

From (26) and (27), we have

$$[22,\ 44] \quad \leq_p \quad [43,\ 86]$$

Hence, Theorem 6 is verified. For Theorem 7, we have

$$\begin{aligned}
Y_*\left(v + \tfrac{1}{2}\zeta(\mu,\ v)\right) &\approx \tfrac{122}{5}, \\
Y^*\left(v + \tfrac{1}{2}\zeta(\mu,\ v)\right) &\approx \tfrac{244}{5},
\end{aligned} \tag{28}$$

$$\int_v^{v+\zeta(\mu,\ v)} \mathcal{S}(\omega)d\omega = \int_1^{\frac{5}{2}} (\omega - 1)d\omega + \int_{\frac{5}{2}}^4 (4-\omega)d\omega = \frac{9}{4},$$

$$\begin{aligned}
\frac{1}{\int_v^{v+\zeta(\mu,\ v)} \mathcal{S}(\omega)d\omega}\ \int_1^4 Y_*(\omega)\mathcal{S}(\omega)d\omega &\approx \tfrac{146}{5} \\
\frac{1}{\int_v^{v+\zeta(\mu,\ v)} \mathcal{S}(\omega)d\omega}\ \int_1^4 Y^*(\omega)\mathcal{S}(\omega)d\omega &\approx \tfrac{293}{5}
\end{aligned} \tag{29}$$

From (28) and (29), we have

$$\left[\frac{122}{5},\ 49\right] \quad \leq_p \left[\frac{146}{5},\ \frac{293}{5}\right].$$

Hence, Theorem 7 is verified.

4. Conclusions and Prospective Results

In this study, the notion of left and right preinvex functions in interval-valued settings was presented. For left and right preinvex interval-valued functions, we constructed Hermite–Hadamard type inequalities, as well as for the product of two left and right preinvex interval-valued functions. We also established Hemite–Hadamard–Fejér type inequality. We also discussed some special cases and provided some examples to prove the validity of our main results. In future, we will seek to explore this concept by using different fractional integral operators, such as Riemann–Liouville fractional operators, Katugampola fractional operators and generalized K-fractional operators.

Finally, we think that our results may be relevant to other fractional calculus models having Mittag–Liffler functions in their kernels, such as Atangana–Baleanu and Prabhakar fractional operators. This consideration has been presented as an open problem for academics interested in this topic. Researchers who are interested might follow the steps outlined in the references [54,55].

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