*mathematics*

*Article*

# Soliton-Type Equations on a Riemannian Manifold

**Nasser Bin Turki [1,†], Adara M. Blaga [2,*,†] and Sharief Deshmukh [1,†]**

1 Department of Mathematics, College of Science, King Saud University, P.O. Box 2455, Riyadh 11451, Saudi Arabia; nassert@ksu.edu.sa (N.B.T.); shariefd@ksu.edu.sa (S.D.)
2 Department of Mathematics, Faculty of Mathematics and Computer Science, West University of Timisoara, 300223 Timisoara, Romania
* Correspondence: adara.blaga@e-uvt.ro
† These authors contributed equally to this work.

**Abstract:** We study some particular cases of soliton-type equations on a Riemannian manifold. We give an estimation of the first nonzero eigenvalue of the Laplace operator and provide necessary and sufficient conditions for the manifold to be isometric to a sphere. Finally, we characterize trivial generalized gradient Ricci solitons.

**Keywords:** generalized gradient soliton; unit geodesic vector field

**MSC:** 35Q51; 53B25; 53B50

check for updates

**Citation:** Bin Turki, N.; Blaga, A.M.; Deshmukh, S. Soliton-Type Equations on a Riemannian Manifold. *Mathematics* **2022**, *10*, 633. https://doi.org/10.3390/math10040633

Academic Editor: Juan Benigno Seoane-Sepúlveda

Received: 21 January 2022
Accepted: 17 February 2022
Published: 18 February 2022

**Publisher's Note:** MDPI stays neutral with regard to jurisdictional claims in published maps and institutional affiliations.



## 1. Introduction

Regarded as stationary solutions to the Ricci flow [1], *Ricci solitons* have been intensively studied in various frameworks and from different points of view. Properties of Ricci solitons deal with aspects concerning the curvature of the manifold on which they are defined as well as provide information on the behavior of the flow. Recently, generalizations of this notion have been used and soliton-type equations have been studied. Such a generalization was considered in [2]. Indeed, a generalized gradient Ricci soliton on a smooth manifold $M$ is given by the data $(g, \nabla f, \alpha, \beta)$ fulfilling

$$\mathrm{Hess}(f) + \alpha\,\mathrm{Ric} = \beta g, \tag{1}$$

where $g$ and Ric are the Riemannian metric and the Ricci curvature with respect to $g$ and $f$, $\alpha$ and $\beta$ are smooth functions on $M$. If $(\alpha, \beta) = \left(-f, -\frac{1}{n-1}(1 + rf)\right)$, then the metric is said to satisfy the *Miao–Tam equation* [3], and if $(\alpha, \beta) = (-f, \Delta(f))$, then $g$ is said to satisfy the *Fischer–Marsden equation* [4], where $r$ denotes the scalar curvature of $(M, g)$.

It is interesting to note that an $n$-sphere $S^n(c)$ is a generalized gradient Ricci soliton $(g, \nabla f, \alpha, \beta)$, where $g$ is the canonical metric on $S^n(c)$, $f$ is an eigenfunction of the Laplace operator corresponding to the first nonzero eigenvalue and $\alpha = -f$ and $\beta = -nfc$. This example initiates the question of finding conditions under which a generalized gradient Ricci soliton $(g, \nabla f, \alpha, \beta)$ on an $n$-dimensional compact smooth manifold $M$ is isometric to $S^n(c)$.

In the present paper, we treat this kind of *soliton*, finding necessary and sufficient conditions for the manifold to be isometric to a sphere and also characterizing the so-called trivial solitons, i.e., solitons with Killing potential vector fields.

## 2. Generalized Gradient Ricci Solitons

Let $(g, \nabla f, \alpha, \beta)$ be a generalized gradient Ricci soliton on an $n$-dimensional smooth manifold $M$. From the soliton in Equation (1), we have

$$H_f + \alpha Q = \beta I, \tag{2}$$

where $H_f$ is the Hessian operator defined by $g\left(H_f X, Y\right) := \mathrm{Hess}(f)(X, Y)$ and $Q$ is the Ricci operator defined by $g(QX, Y) := \mathrm{Ric}(X, Y)$. Taking the trace in Equation (2), we have

$$\Delta(f) = n\beta - r\alpha. \tag{3}$$

On taking the inner product with $H_f$ in (2) and using (3), one obtains

$$\|H_f\|^2 + \alpha\langle H_f, Q\rangle = \beta\Delta(f) = \beta(n\beta - r\alpha)$$

and taking the inner product with $Q$, we have

$$\langle H_f, Q\rangle + \alpha\|Q\|^2 = r\beta.$$

Comparing the above relations, we obtain

$$\|H_f\|^2 = \alpha^2\|Q\|^2 + \beta(n\beta - 2r\alpha) = \alpha^2\left(\|Q\|^2 - \frac{r^2}{n}\right) + \frac{1}{n}(n\beta - r\alpha)^2 \tag{4}$$

and we can state:

**Proposition 1.** *If $(g, \nabla f, \alpha, \beta)$ is a generalized gradient Ricci soliton on an $n$-dimensional smooth manifold $M$ and $\|H_f\|^2 \leq \alpha^2\left(\|Q\|^2 - \frac{r^2}{n}\right)$, then $f$ is a harmonic function; hence, $\beta = \frac{r\alpha}{n}$.*

**Proof.** The hypothesis implies $(n\beta - r\alpha)^2 \leq 0$; therefore, $\beta = \frac{r\alpha}{n}$. □

As a consequence, we obtain:

**Corollary 1.** *Let $f$ be a smooth non-constant function on an $n$-dimensional Riemannian manifold $(M, g)$ and assume that $\|H_f\|^2 \leq f^2\left(\|Q\|^2 - \frac{r^2}{n}\right)$. If the Riemannian metric $g$ satisfies the Fischer–Marsden equation, then the scalar curvature is zero and $H_f = 0$; hence, $M$ is a Ricci flat manifold.*

Next, we prove the following lemmas, which will be useful for our main results.

**Lemma 1.** *Let $(g, \nabla f, \alpha, \beta)$ be a generalized gradient Ricci soliton on an $n$-dimensional smooth manifold $M$. Then*

$$Q(\nabla f + \nabla\alpha) = -(n-1)\nabla\beta + r\nabla\alpha + \frac{1}{2}\alpha\nabla r.$$

**Proof.** We have

$$H_f X = \beta X - \alpha Q X,$$

and differentiating the above equation, we obtain

$$\left(\nabla_X H_f\right)Y = X(\beta)Y - X(\alpha)QY - \alpha(\nabla_X Q)Y.$$

Now, using the above equation in

$$R(X, Y)\nabla f = \left(\nabla_X H_f\right)Y - \left(\nabla_Y H_f\right)X,$$

we conclude

$$R(X, Y)\nabla f = X(\beta)Y - Y(\beta)X - X(\alpha)QY + Y(\alpha)QX - \alpha((\nabla_X Q)Y - (\nabla_Y Q)X).$$

Taking the trace in the above equation, while using the symmetry of the Ricci operator $Q$ and the formula

$$\frac{1}{2}\nabla r=\sum_{i=1}^{n}(\nabla_{e_i}Q)e_i,$$

we obtain

$$\mathrm{Ric}(Y,\nabla f)=-(n-1)Y(\beta)-\mathrm{Ric}(\nabla\alpha,Y)+rY(\alpha)-\frac{1}{2}\alpha Y(r)+\alpha Y(r),$$

that is,

$$\mathrm{Ric}(\nabla f+\nabla\alpha,Y)=-(n-1)Y(\beta)+rY(\alpha)+\frac{1}{2}\alpha Y(r),$$

which implies the conclusion. □

For $\alpha=-f$, we obtain

$$(n-1)\nabla\beta=-r\nabla f-\frac{1}{2}f\nabla r,$$

from Lemma 1. Now, for the Miao–Tam equation, we obtain

$$f\nabla r=0,$$

by means of (3). Thus, we have the following:

**Corollary 2.** *If $f$ is a non-trivial solution of the Miao–Tam equation on a complete Riemannian manifold $(M,g)$, then the scalar curvature $r$ is a constant.*

In order to give an estimation for the first nonzero eigenvalue of the Laplace operator, we prove the following result.

**Lemma 2.** *Let $(g,\nabla f,\alpha,\beta)$ be a generalized gradient Ricci soliton on an $n$-dimensional smooth manifold $M$. If $\nabla f+\nabla\alpha$ is an eigenvector of the Ricci operator $Q$ corresponding to the eigenvalue $\frac{r}{n}$, then*

$$\nabla\left(\Delta(f)+\frac{r}{n-1}f\right)=\frac{1}{n-1}\left(f-\left(\frac{n-2}{2}\right)\alpha\right)\nabla r.$$

**Proof.** Assume $Q(\nabla f+\nabla\alpha)=\frac{r}{n}(\nabla f+\nabla\alpha)$. Using Lemma 1, we have

$$\frac{r}{n}(\nabla f+\nabla\alpha)=-(n-1)\nabla\beta+r\nabla\alpha+\frac{1}{2}\alpha\nabla r,$$

that is,

$$\begin{aligned}\frac{r}{n}\nabla f &= -(n-1)\nabla\beta+\frac{n-1}{n}r\nabla\alpha+\frac{1}{2}\alpha\nabla r\\ &= -(n-1)\nabla\beta+\frac{n-1}{n}\nabla(r\alpha)-\frac{n-1}{n}\alpha\nabla r+\frac{1}{2}\alpha\nabla r\\ &= -\frac{n-1}{n}\nabla(n\beta-r\alpha)-\frac{n-2}{2n}\alpha\nabla r.\end{aligned}$$

Now, using Equation (3), we obtain

$$\frac{n-1}{n}\nabla(\Delta(f))+\frac{1}{n}\nabla(rf)-\frac{1}{n}f\nabla r=-\frac{n-2}{2n}\alpha\nabla r,$$

which gives

$$\frac{n-1}{n}\nabla\left(\Delta(f)+\frac{r}{n-1}f\right)=\frac{1}{n}\left(f-\frac{n-2}{2}\alpha\right)\nabla r,$$

and implies the conclusion. □

As a consequence, we have the following result, which gives an estimate of the first nonzero eigenvalue of the Laplace operator.

**Proposition 2.** *Let $(g, \nabla f, \alpha, \beta)$ be a generalized gradient Ricci soliton on an $n$-dimensional smooth compact and connected manifold $M$ of constant scalar curvature $r$. If $\nabla f + \nabla\alpha$ is an eigenvector of the Ricci operator $Q$ corresponding to the eigenvalue $\frac{r}{n}$, then the first nonzero eigenvalue $\lambda_1$ of the Laplace operator satisfies $\lambda_1 \leq \frac{r}{n-1}$.*

**Proof.** Since the scalar curvature is constant, Lemma 2 implies

$$\nabla\left(\Delta(f) + \frac{r}{n-1}f\right) = 0,$$

that is, $\Delta(f) + \frac{r}{n-1}f$ is a constant and we have

$$\Delta(f) = -\frac{r}{n-1}(f - \bar{c})$$

for a constant $\bar{c}$. Denoting $\overline{f} := f - \bar{c}$, the above equation becomes

$$\Delta(\overline{f}) = -\frac{r}{n-1}\overline{f}. \tag{5}$$

Note that as $f$ is non-constant, the function $\overline{f}$ is also non-constant and it is an eigenfunction of the Laplace operator corresponding to the eigenvalue $\frac{r}{n-1}$. Since $M$ is compact, we conclude

$$\lambda_1 \leq \frac{r}{n-1}. \quad \square$$

## 3. Characterization of Spheres

We prove the following result for further use.

**Lemma 3.** *Let $(g, \nabla f, \alpha, \beta)$ be a generalized gradient Ricci soliton on an $n$-dimensional smooth compact manifold $M$. Then*

$$\int_M \alpha^2\left(\|Q\|^2 - \frac{r^2}{n}\right) = \int_M \left(\left\|H_{\overline{f}}\right\|^2 - \frac{1}{n}\left(\Delta(\overline{f})\right)^2\right),$$

*for $\overline{f} = f - \bar{c}$, with $\bar{c}$ a constant.*

**Proof.** Using Equation (2) and the fact that $H_f = H_{\overline{f}}$, we have

$$\left\|H_{\overline{f}}\right\|^2 = n\beta^2 + \alpha^2\|Q\|^2 - 2\alpha\beta r \tag{6}$$

which gives

$$\alpha^2\left(\|Q\|^2 - \frac{r^2}{n}\right) = \left\|H_{\overline{f}}\right\|^2 - \frac{1}{n}\left(n^2\beta^2 - 2n\alpha\beta r + \alpha^2 r^2\right) = \left\|H_{\overline{f}}\right\|^2 - \frac{1}{n}(n\beta - r\alpha)^2.$$

Integrating the above equation and using Equation (3), we obtain

$$\int_M \alpha^2\left(\|Q\|^2 - \frac{r^2}{n}\right) = \int_M \left(\left\|H_{\overline{f}}\right\|^2 - \frac{1}{n}\left(\Delta(\overline{f})\right)^2\right). \quad \square$$

Next, we see that the tools developed above give us the following characterization of a sphere $\mathbf{S}^n(c)$.

**Theorem 1.** *Let $(g, \nabla f, \alpha, \beta)$ be a generalized gradient Ricci soliton on an $n$-dimensional smooth compact and connected manifold $M$ of constant scalar curvature $r$. If $\nabla f + \nabla \alpha$ is an eigenvector of the Ricci operator $Q$ corresponding to the eigenvalue $\frac{r}{n}$, then*

$$\mathrm{Ric}(\nabla f, \nabla f) \geq \frac{r}{n}\|\nabla f\|^2$$

*if and only if $r > 0$ and $M$ is isometric to the sphere $\mathbf{S}^n(c)$ with $r = n(n-1)c$.*

**Proof.** Assume that the conditions in the statement hold. Then using Equation (5), we have

$$\overline{f}\Delta(\overline{f}) = -\frac{r}{n-1}\overline{f}^2,$$

where $\overline{f} = f - \overline{c}$. Integrating the above equation yields

$$\int_M \left\|\nabla\overline{f}\right\|^2 = \frac{r}{n-1}\int_M \overline{f}^2,$$

that is,

$$\int_M \|\nabla f\|^2 = \frac{r}{n-1}\int_M \overline{f}^2, \tag{7}$$

and as $\overline{f}$ is non-constant, it implies that $r > 0$. Note that $\nabla f = \nabla\overline{f}$, $H_f = H_{\overline{f}}$ and $\Delta(f) = \Delta(\overline{f})$. Thus, from Bochner's formula [5]

$$\int_M \left(\mathrm{Ric}\left(\nabla\overline{f}, \nabla\overline{f}\right) + \left\|H_{\overline{f}}\right\|^2 - \left(\Delta(\overline{f})\right)^2\right) = 0$$

we have

$$\int_M \left(\mathrm{Ric}(\nabla f, \nabla f) + n\beta^2 + \alpha^2\|Q\|^2 - 2\alpha\beta r - \frac{r^2}{(n-1)^2}\overline{f}^2\right) = 0,$$

by means of (6). Using $n\beta^2 - 2\alpha\beta r = \frac{1}{n}\left[(n\beta - r\alpha)^2 - \alpha^2 r^2\right]$ and Equation (3), from the above relation, we obtain

$$\int_M \left(\mathrm{Ric}(\nabla f, \nabla f) + \frac{1}{n}(\Delta(f))^2 + \alpha^2\left(\|Q\|^2 - \frac{r^2}{n}\right) - \frac{r^2}{(n-1)^2}\overline{f}^2\right) = 0.$$

By Equation (5), we have

$$\int_M \left(\mathrm{Ric}(\nabla f, \nabla f) + \frac{r^2}{n(n-1)^2}\overline{f}^2 + \alpha^2\left(\|Q\|^2 - \frac{r^2}{n}\right) - \frac{r^2}{(n-1)^2}\overline{f}^2\right) = 0,$$

that is,

$$\int_M \left(\mathrm{Ric}(\nabla f, \nabla f) - \frac{r^2}{n(n-1)}\overline{f}^2 + \alpha^2\left(\|Q\|^2 - \frac{r^2}{n}\right)\right) = 0.$$

Now, using Equation (7), we have

$$\int_M \left(\left(\mathrm{Ric}(\nabla f, \nabla f) - \frac{r}{n}\|\nabla f\|^2\right) + \alpha^2\left(\|Q\|^2 - \frac{r^2}{n}\right)\right) = 0.$$

From the hypothesis $\mathrm{Ric}(\nabla f, \nabla f) \geq \frac{r}{n}\|\nabla f\|^2$ and Schwartz's inequality $\|Q\|^2 \geq \frac{r^2}{n}$, the above equation implies

$$\alpha^2\left(\|Q\|^2 - \frac{r^2}{n}\right) = 0.$$

Inserting the above equation into Lemma 3, we have

$$\int_M \left( \left\| H_{\overline{f}} \right\|^2 - \frac{1}{n} \left( \Delta(\overline{f}) \right)^2 \right) = 0.$$

Using Schwartz's inequality, we obtain that the equality $\left\| H_{\overline{f}} \right\|^2 = \frac{1}{n} \left( \Delta(\overline{f}) \right)^2$ holds if and only if $H_{\overline{f}} = \frac{\Delta(\overline{f})}{n} I$. Now, from Equation (5), we arrive at

$$H_{\overline{f}} = -\frac{r}{n(n-1)} \overline{f} I = -c\overline{f} I,$$

where $c$ is a positive constant. Hence, by the Theorem of Obata, $M$ is isometric to $\mathbf{S}^n(c)$.

Conversely, for the sphere $\mathbf{S}^n(c)$, its Ricci tensor and scalar curvature are given by

$$\mathrm{Ric} = (n-1)cg, \quad r = n(n-1)c.$$

Moreover, there exists a smooth function $f$ (the eigenfunction corresponding to the first nonzero eigenvalue $\lambda_1 = nc$) on $\mathbf{S}^n(c)$ that satisfies

$$H_f = -cfI, \quad \Delta(f) = -ncf.$$

Thus, we see that

$$\mathrm{Hess}(f) + (-f)\,\mathrm{Ric} = (-ncf)g,$$

that is, $(g, \nabla f, \alpha, \beta)$ is a generalized gradient Ricci soliton on $\mathbf{S}^n(c)$, with $\alpha = -f$ and $\beta = -ncf$. We see that all the conditions in the hypothesis are satisfied by this generalized gradient Ricci soliton on the sphere $\mathbf{S}^n(c)$. □

Finally, we prove the following characterization of the sphere $\mathbf{S}^n(c)$.

**Theorem 2.** *A generalized gradient Ricci soliton $(g, \nabla f, \alpha, \beta)$ on an $n$-dimensional smooth compact and connected manifold $M$ is isometric to the sphere $\mathbf{S}^n(c)$ if and only if the positive constant $c$ satisfies*

$$\mathrm{Ric}(\nabla f, \nabla f) \geq (n-1)c\|\nabla f\|^2$$

*and*

$$\int_M (n\beta - r\alpha + cf)(n\beta - r\alpha + ncf) \leq 0.$$

**Proof.** Using Equation (3), we have

$$\begin{aligned}
\int_M (n\beta - r\alpha + cf)(n\beta - r\alpha + ncf) &= \int_M \left( (\Delta(f))^2 + (n+1)cf\Delta(f) + nc^2 f^2 \right) \\
&= \int_M \left( \mathrm{Ric}(\nabla f, \nabla f) + \left\| H_f \right\|^2 + (n+1)cf\Delta(f) + nc^2 f^2 \right).
\end{aligned}$$

Now, using $\mathrm{Ric}(\nabla f, \nabla f) \geq (n-1)c\|\nabla f\|^2$ and $\int_M (n\beta - r\alpha + cf)(n\beta - r\alpha + ncf) \leq 0$ in the above equation, we conclude

$$\int_M \left( (n-1)c\|\nabla f\|^2 + \left\| H_f \right\|^2 + (n+1)cf\Delta(f) + nc^2 f^2 \right) \leq 0.$$

Inserting $\int_M f\Delta(f) = -\int_M \|\nabla f\|^2$ into the above inequality, we have

$$\int_M \left( -(n-1)cf\Delta(f) + \left\| H_f \right\|^2 + (n+1)cf\Delta(f) + nc^2 f^2 \right) \leq 0,$$

that is,

$$\int_M \left( \left\| H_f \right\|^2 + 2cf\Delta(f) + nc^2 f^2 \right) \leq 0$$

and we conclude

$$\int_M \left\| H_f + cfI \right\|^2 \leq 0.$$

This proves that $H_f = -cfI$, that is, $M$ is isometric to the sphere $\mathbf{S}^n(c)$.

Conversely, as seen in the proof of Theorem 1, we know that $(g, \nabla f, \alpha, \beta)$ is a generalized gradient Ricci soliton on the sphere $\mathbf{S}^n(c)$, where $f$ is the eigenfunction of the Laplace operator $\Delta$ corresponding to the first nonzero eigenvalue $nc$ and $\alpha = -f$ and $\beta = -ncf$. In addition, we have

$$\int_M (n\beta - r\alpha + cf)(n\beta - r\alpha + ncf) = \int_M (-ncf + cf)(-ncf + ncf) = 0.$$

Hence, all the conditions in the hypothesis are satisfied. $\square$

## 4. Trivial Solitons

Following the ideas from [2,6,7], we shall further provide some characterizations for trivial generalized gradient Ricci solitons $(g, \nabla f, \alpha, \beta)$ with unit geodesic potential vector fields, i.e., $\nabla_{\nabla f} \nabla f = 0$. Note that it is not a unit vector field, but to distinguish between a geodesic vector field (whose integral curves are conformal geodesics) and those whose integral curves are geodesics, we use the term unit geodesic vector field.

**Theorem 3.** *Let $(g, \nabla f, \alpha, \beta)$ be a generalized gradient Ricci soliton on an $n$-dimensional compact and connected smooth manifold $M$ ($n > 2$) with a unit geodesic potential vector field and nonzero scalar curvature. Assume that $\alpha$ and $\beta$ are constant, $\alpha \neq 0$. Then $\nabla f$ is an eigenvector of the Ricci operator with constant eigenvalue $\frac{\beta}{\alpha}$ satisfying $(n\beta - r\alpha)r\alpha \geq 0$ if and only if the soliton is trivial.*

**Proof.** The proof follows the same steps as [6,7]. The converse implication is trivial. For the direct implication, if we assume that $Q(\nabla f) = \sigma \nabla f$, $\sigma \in \mathbb{R}^*$, then taking the inner product with $\nabla f$ implies $\sigma = \frac{\beta}{\alpha}$, so $\left(g, \nabla(\frac{f}{\alpha}), \frac{\beta}{\alpha}\right)$ is a gradient Ricci soliton. Then from (1), (3) and Lemma 1, we obtain

$$\nabla r = \frac{2\beta}{\alpha^2} \nabla f,$$

$$\mathrm{Hess}(r) = \frac{2\beta}{\alpha^2} \mathrm{Hess}(f) = \frac{2\beta}{\alpha^2}(\beta g - \alpha \,\mathrm{Ric}),$$

$$\Delta(r) = \frac{2\beta}{\alpha^2} \Delta(f) = \frac{2\beta}{\alpha^2}(n\beta - r\alpha),$$

$$\mathrm{Ric}(\nabla r, \nabla r) = \frac{\beta}{\alpha} \|\nabla r\|^2.$$

In this case, Bochner's formula

$$\int_M \left( \mathrm{Ric}(\nabla r, \nabla r) + \| \mathrm{Hess}(r) \|^2 - (\Delta(r))^2 \right) = 0$$

becomes

$$\int_M \alpha^2 \left( \|Q\|^2 - \frac{r^2}{n} \right) = \int_M \left( \frac{n-1}{n} (n\beta - r\alpha)^2 - \frac{\alpha^3}{4\beta} \|\nabla r\|^2 \right).$$

However, $\Delta(r) = \frac{2\beta}{\alpha^2}(n\beta - r\alpha)$ and $\mathrm{div}(r\nabla r) = r\Delta(r) + \|\nabla r\|^2$ imply

$$\int_M (n\beta - r\alpha) = 0, \quad \int_M \|\nabla r\|^2 = -\frac{2\beta}{\alpha^2} \int_M (n\beta - r\alpha)r,$$

which, replaced in the previous relation, gives

$$\int_M \left( \|Q\|^2 - \frac{r^2}{n} \right) = -\frac{n-2}{2n\alpha} \int_M (n\beta - r\alpha) r.$$

Using Schwartz's inequality, we deduce that $\|Q\|^2 = \frac{r^2}{n}$; hence, $Q = \frac{r}{n} I$. Moreover, since $r$ is nonzero, we obtain $n\beta = r\alpha$; therefore, $\mathrm{Hess}(f) = 0$ by (4), i.e., the soliton is trivial. □

**Theorem 4.** *Let $(g, \nabla f, \alpha, \beta)$ be a generalized gradient Ricci soliton on an n-dimensional compact and connected smooth manifold M ($n > 2$) with a unit geodesic potential vector field. Then*

$$\mathrm{Ric}(\nabla f, \nabla f) \geq \frac{n-1}{n} (n\beta - r\alpha)^2$$

*if and only if the soliton is trivial.*

**Proof.** The converse implication is trivial. For the direct implication, from (2), we obtain

$$\| \mathrm{Hess}(f) \|^2 = \alpha^2 \left( \|Q\|^2 - \frac{r^2}{n} \right) + \frac{(n\beta - r\alpha)^2}{n}.$$

Using (3) and Bochner's formula

$$\int_M \left( \mathrm{Ric}(\nabla f, \nabla f) + \| \mathrm{Hess}(f) \|^2 - (\Delta(f))^2 \right) = 0$$

we obtain

$$\int_M \alpha^2 \left( \|Q\|^2 - \frac{r^2}{n} \right) = \int_M \left( \frac{n-1}{n} (n\beta - r\alpha)^2 - \mathrm{Ric}(\nabla f, \nabla f) \right).$$

By using Schwartz's inequality, we deduce $\|Q\|^2 = \frac{r^2}{n}$; hence, $Q = \frac{r}{n} I$. Therefore,

$$\frac{r}{n} \nabla f = Q(\nabla f) = \frac{\beta}{\alpha} \nabla f - \frac{1}{\alpha} \nabla_{\nabla f} \nabla f = \frac{\beta}{\alpha} \nabla f$$

which implies $n\beta = r\alpha$, and we deduce that $\mathrm{Hess}(f) = 0$, i.e., the soliton is trivial. □

For particular cases, we can state:

**Corollary 3.** *Let $(M, g)$ be an n-dimensional compact and connected Riemannian manifold M ($n > 2$) and $\nabla f$ a unit geodesic vector field.*

*(i) If g satisfies the Miao–Tam equation and* $\mathrm{Ric}(\nabla f, \nabla f) \geq \frac{1}{n(n-1)} (n + rf)^2$, *then M is an Einstein manifold.*

*(ii) If g satisfies the Fischer–Marsden equation and* $\mathrm{Ric}(\nabla f, \nabla f) \geq \frac{1}{n(n-1)} (rf)^2$, *then M is a Ricci flat manifold.*

If $\alpha$ is nowhere zero and $\nabla f$ is a conformal vector field with $\pounds_{\nabla f} g = 2\beta g$, then $\mathrm{Hess}(f) = \beta g$ and $M$ is an Einstein manifold, provided $n \geq 3$. If $\alpha = 0$, Equation (1) becomes

$$\mathrm{Hess}(f) = \beta g, \tag{8}$$

hence, $\| \mathrm{Hess}(f) \|^2 = n\beta^2$ and $\Delta(f) = n\beta$, which implies the equality case in Schwartz's inequality. Note that in [8–12], the authors proved that a non-constant function $f$ on a complete $n$-dimensional Riemannian manifold $(M, g)$ satisfies Equation (8) for $\beta$ a negative constant if and only if $M$ is isometric to the $n$-dimensional Euclidean space. In [13], the authors proved that if Equation (8) holds with $\beta$ a function, then $(M, g)$ is locally a warped product $(a, b) \times_h N^{n-1}$. If $\beta$ is a non-constant function on $M$, we prove the following result.

**Proposition 3.** *Let $(M, g)$ be an $n$-dimensional compact Riemannian manifold and let $f$ be a smooth function on $M$ satisfying Equation (8). If $\mathrm{Ric}(\nabla f, \nabla f) \leq 0$, then $\nabla f \in \ker Q$ and $\mathrm{Hess}(f) = 0$.*

**Proof.** From (8), we obtain $\Delta(f) = n\beta$. Hence,

$$d\beta = \mathrm{div}(\mathrm{Hess}(f)) = d(\Delta(f)) + i_{Q(\nabla f)}g = nd\beta + i_{Q(\nabla f)}g$$

which implies

$$\nabla\beta = -\frac{1}{n-1}Q(\nabla f).$$

However, $\nabla\beta = \frac{1}{n}\nabla(\Delta(f))$. Therefore,

$$\nabla(\Delta(f)) = -\frac{n}{n-1}Q(\nabla f).$$

Replacing these relations in Bochner's formula

$$\frac{1}{2}\Delta(\|\nabla f\|^2) = \mathrm{Ric}(\nabla f, \nabla f) + \|\,\mathrm{Hess}(f)\|^2 + g(\nabla(\Delta(f)), \nabla f)$$

we obtain

$$\frac{1}{2}\Delta(\|\nabla f\|^2) = -\frac{1}{n-1}\mathrm{Ric}(\nabla f, \nabla f) + n\beta^2 = -\frac{1}{n-1}\mathrm{Ric}(\nabla f, \nabla f) + \frac{1}{n}(\Delta(f))^2,$$

which, by integration, in the compact case, gives

$$\int_M \mathrm{Ric}(\nabla f, \nabla f) = \frac{n-1}{n}\int_M (\Delta(f))^2$$

and using the hypothesis, we deduce that $\mathrm{Ric}(\nabla f, \nabla f) = 0$, $\Delta(f) = 0$, $\beta = 0$, $Q(\nabla f) = 0$ and $\mathrm{Hess}(f) = 0$. □

**Author Contributions:** All authors have contributed to the conceptualization and investigation. All authors have read and agreed to the published version of the manuscript.

**Funding:** This work is supported by the Researchers Supporting Project number (RSP2022R413), King Saud University, Riyadh, Saudi Arabia.

**Institutional Review Board Statement:** Not applicable.

**Informed Consent Statement:** Not applicable.

**Data Availability Statement:** Not applicable.

**Conflicts of Interest:** The authors declare no conflict of interest. The funder had no role in the design of the study; in the collection, analyses or interpretation of data; in the writing of the manuscript or in the decision to publish the results.

## References

1. Hamilton, R.S. Three-manifolds with positive Ricci curvature. *J. Differ. Geom.* **1982**, *17*, 255–306. [CrossRef]
2. Blaga, A.M.; Ishan, A.; Deshmukh, S. A note on solitons with generalized geodesic vector field. *Symmetry* **2021**, *13*, 1104. [CrossRef]
3. Miao, P.; Tam, L.-F. On the volume functional of compact manifolds with boundary with constant scalar curvature. *Calc. Var. Partial Differ. Equ.* **2009**, *36*, 141–171. [CrossRef]
4. Fischer, A.E.; Marsden, J.E. Manifolds of Riemannian metrics with prescribed scalar curvature. *Bull. Am. Math. Soc.* **1974**, *80*, 479–484. [CrossRef]

5. Yano, K. *Integral Formulas in Riemannian Geometry*; Marcel Dekker Inc.: New York, NY, USA, 1970.
6. Blaga, A.M.; Deshmukh, S. Einstein solitons with unit geodesic potential vector field. *AIMS Math.* **2021**, *6*, 7961–7970. [CrossRef]
7. Deshmukh, S.; Alsodais, H.; Bin Turki, N. Some Results on Ricci Almost Solitons. *Symmetry* **2021**, *13*, 430. [CrossRef]
8. Erkekoglu, F.; García-Río, E.; Kupeli, D.N.; Ünal, B. Characterizing specific Riemannian manifolds by differential equations. *Acta Appl. Math.* **2003**, *76*, 195–219. [CrossRef]
9. García-Río, E.; Kupeli, D.N.; Ünal, B. On differential equations characterizing Euclidean sphere. *J. Differ. Equ.* **2003**, *194*, 287–299. [CrossRef]
10. Pigola, S.; Rimoldi, M.; Setti, A.G. Remarks on non-compact gradient Ricci solitons. *Math. Z.* **2011**, *268*, 777–790. [CrossRef]
11. Tanno, S. Some differential equations on Riemannian manifolds. *J. Math. Soc. Jpn.* **1978**, *30*, 509–531. [CrossRef]
12. Tashiro, Y. Complete Riemannian manifolds and some vector fields. *Trans. Am. Math. Soc.* **1965**, *117*, 251–275. [CrossRef]
13. Cheeger, J.; Colding, T.H. Lower bounds on Ricci curvature and almost rigidity of warped products. *Ann. Math.* **1996**, *144*, 189–237. [CrossRef]