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Article

Upper Bounds for the Distance between Adjacent Zeros of First-Order Linear Differential Equations with Several Delays

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Abstract: The distance between successive zeros of all solutions of first-order differential equations with several delays is studied in this work. Many new estimations for the upper bound of the distance between zeros are obtained. Our results improve many-well known results in the literature. We also obtain some fundamental results for the lower bound of the distance between adjacent zeros. Some illustrative examples are introduced to show the accuracy and efficiency of the obtained results.

Keywords: distance between zeros; several delays; oscillation

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1. Introduction

In this paper, we study the distribution of zeros of the first-order differential equation with several delays:

$$x'(t) + \sum_{l=1}^{m} q_l(t)x(t - v_l) = 0, \qquad t \geq t_0, \tag{1}$$

where $q_l \in C([t_0, \infty), [0, \infty))$, $0 < v_1 \leq v_2 \leq \cdots \leq v_m$, $l = 1, 2, \ldots, m$. With Equation (1), we associate an initial function $\phi(t)$ where $\phi \in C([t_0 - v_m, t_0], \mathbb{R})$.

The qualitative properties of functional differential equations have attracted the attention of many researchers; see [1–31]. In particular, the oscillation theory of Equation (1) has received increasing interest in recent years; see for example [1,2,8–10,12,16–18]. However, only a few works have considered the distance between zeros of Equation (1) and its general forms. For more details about this topic, we refer to the works of El-Morshedy and Attia [14] and McCalla [20]. This encourages us to study this property for Equation (1) and clarify the influence of the several delays in the distribution of zeros of Equation (1).

The distance between zeros of all solutions of the equation

$$x'(t) + q(t)x(t - v) = 0, \qquad t \geq t_0, \tag{2}$$

where $v > 0$, $q \in C([t_0, \infty), [0, \infty))$, has attracted the interest of many mathematicians; for example, [4–7,11–14,20–22,24–30]. The purpose of most of these works was to obtain new estimations for the upper bound (UB) between successive zeros of all solutions. McCalla [20] proved that the upper and lower bounds for the distance between consecutive zeros of Equation (2) are determined by the first zero of the fundamental solution of Equation (2). Motivated by the ideas of [15,31], Zhou [30] obtained estimations of UB of all solutions of Equation (2) by using the upper and lower bounds of the ratio $\frac{x(t-v)}{x(t)}$, where $x(t)$ is a positive solution of Equation (2) on a bounded interval. Since then, many efforts have been made to obtain new results by improving the bounds of the ratio $\frac{x(t-v)}{x(t)}$; for example, see [13,14,22,25,27,28]. Furthermore, the authors [13,14,26] obtained new criteria of iterative types for UB of all solutions of Equation (2).

On the other hand, some studies have obtained some fundamental results for the lower bound (LB) between successive zeros of Equation (2); see [5–7,13,20,21]. Barr [5] and El-Morshedy [13] proved that the zeros of a solution of Equation (2) with an initial function that has a finite number of zeros do not accumulate. In addition, McCalla [20] and El-Morshedy [13] showed that any solution of Equation (2) with an initial function of a constant sign has at most one zero in any interval of length v. In the following example, we show that the LB of a solution of Equation (1) with an initial function of a constant sign cannot be greater than any one of the delays. Therefore, the latter result of McCalla [20] and El-Morshedy [13] cannot be extended to Equation (1).

Example 1. *Consider the differential equation*

$$x'(t) + q_1(t)x(t-1) + q_2(t)x(t-4) = 0, \quad t \geq 0, \tag{3}$$

with the initial function

$$\phi(t) = t, \qquad t \in [-4, 0],$$

where

$$q_1(t) = \begin{cases} \alpha_1, & \text{if } t \in [0,2], \\ 1000(\beta_1 - \alpha_1)(t-2) + \alpha_1, & \text{if } t \in [2, 2.001], \\ \beta_1, & \text{if } t \geq 2.001, \end{cases}$$

$$q_2(t) = \begin{cases} \alpha_2, & \text{if } t \in [0,2], \\ 1000(\beta_2 - \alpha_2)(t-2) + \alpha_2, & \text{if } t \in [2, 2.001], \\ \beta_2, & \text{if } t \geq 2.001. \end{cases}$$

For $t \in [0,1]$, we have

$$\begin{aligned} x(t) &= \phi(0) - \int_0^t q_1(\omega)x(\omega-1)d\omega - \int_0^t q_2(\omega)x(\omega-4)d\omega \\ &= -\alpha_1 \int_0^t \phi(\omega-1)d\omega - \alpha_2 \int_0^t \phi(\omega-4)d\omega = (\alpha_1 + 4\alpha_2)t - \frac{1}{2}(\alpha_1+\alpha_2)t^2. \end{aligned}$$

Let $x_1(t) = (\alpha_1 + 4\alpha_2)t - \frac{1}{2}(\alpha_1+\alpha_2)t^2$, $t \in [0,1]$. Furthermore, for $t \in [1,2]$, it follows that

$$\begin{aligned} x(t) &= x(1) - \int_1^t q_1(\omega)x(\omega-1)d\omega - \int_1^t q_2(\omega)x(\omega-4)d\omega \\ &= x_1(1) - \alpha_1 \int_1^t x_1(\omega-1)d\omega - \alpha_2 \int_1^t \phi(\omega-4)d\omega \\ &= \frac{1}{6}\left(\alpha_1^2 + \alpha_1\alpha_2\right)t^3 - \left(\alpha_1{}^2 + \frac{5\alpha_1\alpha_2}{2} + \frac{\alpha_2}{2}\right)t^2 + \left(\frac{3\alpha_1{}^2}{2} + \frac{9\alpha_1\alpha_2}{2} + 4\,\alpha_2\right)t \\ &\quad + \frac{\alpha_1}{2} - \frac{2\alpha_1^2}{3} - \frac{13\,\alpha_1\,\alpha_2}{6}. \end{aligned}$$

Let

$$\begin{aligned} x_2(t) &= \frac{1}{6}\left(\alpha_1^2 + \alpha_1\alpha_2\right)t^3 - \left(\alpha_1{}^2 + \frac{5\alpha_1\alpha_2}{2} + \frac{\alpha_2}{2}\right)t^2 + \left(\frac{3\alpha_1{}^2}{2} + \frac{9\alpha_1\alpha_2}{2} + 4\,\alpha_2\right)t \\ &\quad + \frac{\alpha_1}{2} - \frac{2\alpha_1^2}{3} - \frac{13\,\alpha_1\,\alpha_2}{6}, \quad t \in [1,2]. \end{aligned}$$

Finally, assume for $t \in [2,3]$ that

$$\begin{aligned} x_3(t) &= x(2) - \int_2^t q_1(\omega)x(\omega-1)d\omega - \int_2^t q_2(\omega)x(\omega-4)d\omega \\ &= x_2(2) - \int_2^{2.001}(1000(\beta_1-\alpha_1)(\omega-2)+\alpha_1)x_2(\omega-1)d\omega - \beta_1\int_{2.001}^t x_2(\omega-1)d\omega \\ &\quad - \int_2^{2.001}(1000(\beta_2-\alpha_2)(\omega-2)+\alpha_2)\phi(\omega-4)d\omega - \beta_2\int_{2.001}^t \phi(\omega-4)d\omega. \end{aligned}$$

Assume that $\alpha_1 = \beta_2 = 3$ and $\alpha_2 = \beta_1 = 0.001$, using Maple software, we obtain

$$x_2(1.5) = 0.5659375, \qquad x_2(1.7) = -0.1879135000, \qquad x_3(2.3) = 0.159833338.$$

Consequently, $x(t)$ has at least two zeros in the interval $[1.5, 2.3]$, and hence the LB of Equation (3) with the initial function $\phi(t)$ cannot be greater than 1.

Motivated by the recent contributions of [3,13,14,28], in this work, we obtain new estimations for the UB of all solutions of Equation (1). Furthermore, our results improve upon many previous results for both Equations (1) and (2). In addition, we show that a fundamental result for the LU of some solutions of Equation (2) is valid for the case of several delays. Finally, two illustrative examples are given to demonstrate the effectiveness and improvement of our results.

2. Main Results

Let $D(x)$ be the UB of all solutions of Equation (1) on the interval $[t_0, \infty)$. The following result is an extension of ([5], Lemma 5) and ([13], Lemma 2.2) for Equation (1).

Theorem 1. *If ϕ and q_l have, respectively, a finite number of zeros in $[t_0 - v_m, t_0]$ and any bounded subinterval of $[t_0, \infty)$, for all $l = 1, 2, \ldots, m$, then the solution $x(t)$ of Equation (1) associated with ϕ has only a finite number of zeros in any bounded subinterval of $[t_0, \infty)$.*

Proof. Assume, for the sake of contradiction, that $x(t)$ has infinitely many zeros in $[T_0 - v_1, T_0]$, for some $T_0 \geq t_0$. Then, $x'(t)$ has also infinitely many zeros in $[T_0 - v_1, T_0]$. In view of Equation (1), $x(t)$ has infinitely many zeros in $[T_0 - v_1 - v_m, T_0 - v_1]$. Thus there exists $T_1 \leq T_0 - v_1$ such that $x(t)$ has infinitely many zeros in $[T_1 - v_1, T_1]$. Continuing this process k times such that $t_0 - v_m \leq T_k - v_1 \leq t_0$ and $x(t)$ has infinitely many zeros in $[T_k - v_1, T_k]$, we have the following two cases:

Case 1: $x(t)$ has infinitely many zeros in $[T_k - v_1, t_0]$. This contradicts our assumption that $\phi(t)$ has a finite number of zeros in $[t_0 - v_m, t_0]$.

Case 2: $x(t)$ has infinitely many zeros in $[t_0, T_k]$. Then $x'(t)$ has infinitely many zeros in $[t_0, T_k]$. Therefore $x(t)$ has infinitely many zeros in $[t_0 - v_m, T_k - v_1]$. In view of $T_k - v_1 \leq t_0$, so $[t_0 - v_m, T_k - v_1] \subseteq [t_0 - v_m, t_0]$. Therefore, $\phi(t)$ has infinitely many zeros in $[t_0 - v_m, t_0]$, which is a contradiction. The proof of the theorem is complete. □

Let $l, j \in \{1, 2, \ldots, m\}$, $t - v_j \leq \omega \leq t$, and

$$\begin{aligned} Q^1_{l,j}(\omega) &= q_l(\omega), && \text{for } t \geq t_0, \\ Q^n_{l,j}(\omega) &= q_l(\omega - nv_j)\int_{t-v_j}^{\omega} Q^{n-1}_{j,j}(\omega_1)d\omega_1, && \text{for } t \geq t_0 + nv_m, \quad n = 2, 3, \ldots. \end{aligned}$$

Furthermore, let $j \in \{1, 2, \ldots, m\}$, and $\{W_j^n\}_{n \geq 1}$ be a sequence of positive numbers such that

$$\frac{1 + \sum_{\substack{l=1 \\ l \neq j}}^{m} \int_{t-v_j}^{t} Q_{l,j}^{1}(\omega) d\omega}{1 - \int_{t-v_j}^{t} Q_{j,j}^{1}(\omega) d\omega} \geq W_j^1, \qquad \text{for } t \geq t_0 + v_m,$$

$$\frac{1 + \sum_{k=1}^{n} \sum_{\substack{l=1 \\ l \neq j}}^{m} \int_{t-v_j}^{t} Q_{l,j}^{k}(\omega) d\omega}{1 - \sum_{k=1}^{n} \prod_{l=2}^{k} \left(W_j^{n-(l-1)} \right) \int_{t-v_j}^{t} Q_{j,j}^{k}(\omega) d\omega} \geq W_j^n, \quad \text{for } t \geq t_0 + n v_m \quad n = 2, 3, \ldots.$$

Lemma 1. *Assume that $n \in \mathbb{N}$, $j \in \{1, 2, \ldots, m\}$ and $x(t)$ is a positive solution of Equation (1) on $[T_0, T_1]$, $T_0 \geq t_0$, $T_1 \geq T_0 + 2v_m + n v_j$. Then,*

$$\frac{x(t - v_j)}{x(t)} \geq W_j^n, \qquad \text{for } t \in [T_0 + 2v_m + n v_j, T_1], \tag{4}$$

and

$$\sum_{k=1}^{n} \prod_{l=2}^{k} \left(W_j^{n-(l-1)} \right) \int_{t-v_j}^{t} Q_{j,j}^{k}(\omega) d\omega < 1, \qquad \text{for } t \in [T_0 + 2v_m + n v_j, T_1].$$

Proof. Integrating Equation (1) from $t - v_j$ to t, we get

$$x(t) - x(t - v_j) + \int_{t-v_j}^{t} \sum_{l=1}^{m} q_l(\omega) x(\omega - v_l) d\omega = 0. \tag{5}$$

It is clear that

$$\int_{t-v_j}^{t} \sum_{l=1}^{m} q_l(\omega) x(\omega - v_l) d\omega = \int_{t-v_j}^{t} Q_{j,j}^{1}(\omega) x(\omega - v_j) d\omega + \sum_{\substack{l=1 \\ l \neq j}}^{m} \int_{t-v_j}^{t} Q_{l,j}^{1}(\omega) x(\omega - v_l) d\omega. \tag{6}$$

On the other hand, using the integration by parts, it follows that

$$\begin{aligned} \int_{t-v_j}^{t} Q_{j,j}^{1}(\omega) x(\omega - v_j) d\omega &= \int_{t-v_j}^{t} d\left(\int_{t-v_j}^{\omega} Q_{j,j}^{1}(\omega_1) d\omega_1 \right) x(\omega - v_j) d\omega \\ &= x(t - v_j) \int_{t-v_j}^{t} Q_{j,j}^{1}(\omega) d\omega \\ &\quad - \int_{t-v_j}^{t} x'(\omega - v_j) \int_{t-v_j}^{\omega} Q_{j,j}^{1}(\omega_1) d\omega_1 d\omega. \end{aligned}$$

This together with Equation (1) leads to

$$\begin{aligned} \int_{t-v_j}^{t} Q_{j,j}^{1}(\omega) x(\omega - v_j) d\omega &= x(t - v_j) \int_{t-v_j}^{t} Q_{j,j}^{1}(\omega) d\omega \\ &\quad + \int_{t-v_j}^{t} \sum_{l=1}^{m} x(\omega - v_j - v_l) q_l(\omega - v_j) \int_{t-v_j}^{\omega} Q_{j,j}^{1}(\omega_1) d\omega_1 d\omega, \end{aligned}$$

that is

$$\begin{aligned} \int_{t-v_j}^{t} Q_{j,j}^{1}(\omega) x(\omega - v_j) d\omega &= x(t - v_j) \int_{t-v_j}^{t} Q_{j,j}^{1}(\omega) d\omega + \int_{t-v_j}^{t} x(\omega - 2v_j)\, Q_{j,j}^{2}(\omega) d\omega \\ &\quad + \int_{t-v_j}^{t} \sum_{\substack{l=1 \\ l \neq j}}^{m} x(\omega - v_j - v_l)\, Q_{l,j}^{2}(\omega) d\omega. \end{aligned}$$

Substituting into (6), we get

$$\begin{aligned}\int_{t-v_j}^{t}\sum_{l=1}^{m} q_l(\omega)x(\omega-v_l)d\omega &= x(t-v_j)\int_{t-v_j}^{t} Q_{j,j}^{1}(\omega)d\omega+\sum_{\substack{l=1\\ l\neq j}}^{m}\int_{t-v_j}^{t} Q_{l,j}^{1}(\omega)x(\omega-v_l)d\omega \\ &+\int_{t-v_j}^{t} x(\omega-2v_j)\, Q_{j,j}^{2}(\omega)d\omega \\ &+\int_{t-v_j}^{t}\sum_{\substack{l=1\\ l\neq j}}^{m} x(\omega-v_j-v_l)\, Q_{l,j}^{2}(\omega)d\omega.\end{aligned} \tag{7}$$

Since $x(t)>0$ for $t\in[T_0,T_1]$, we have $x'(t)\leq 0$, for $t\in[T_0+v_m,T_1]$. Then, for $t\in[T_0+2v_m+v_j,\ T_1]$, it follows that

$$\int_{t-v_j}^{t} x(\omega-2v_j)\, Q_{j,j}^{2}(\omega)d\omega+\int_{t-v_j}^{t}\sum_{\substack{l=1\\ l\neq j}}^{m} x(\omega-v_j-v_l)\, Q_{l,j}^{2}(\omega)d\omega\geq 0.$$

In view of this and the nonincreasing nature of $x(t)$, (7) gives

$$\int_{t-v_j}^{t}\sum_{l=1}^{m} q_l(\omega)x(\omega-v_l)d\omega\geq x(t-v_j)\int_{t-v_j}^{t} Q_{j,j}^{1}(\omega)d\omega+x(t)\sum_{\substack{l=1\\ l\neq j}}^{m}\int_{t-v_j}^{t} Q_{l,j}^{1}(\omega)d\omega.$$

This together with (5) leads to

$$x(t)-x(t-v_j)+x(t-v_j)\int_{t-v_j}^{t} Q_{j,j}^{1}(\omega)d\omega+x(t)\sum_{\substack{l=1\\ l\neq j}}^{m}\int_{t-v_j}^{t} Q_{l,j}^{1}(\omega)d\omega\leq 0,$$

for $t\in[T_0+2v_m+v_j,\ T_1]$. That is

$$\left(1-\int_{t-v_j}^{t} Q_{j,j}^{1}(\omega)d\omega\right)x(t-v_j)\geq x(t)\left(1+\sum_{\substack{l=1\\ l\neq j}}^{m}\int_{t-v_j}^{t} Q_{l,j}^{1}(\omega)\right)>0,$$

for $t\in[T_0+2v_m+v_j,\ T_1]$.
Consequently

$$\frac{x(t-v_j)}{x(t)}\geq\frac{1+\sum_{\substack{l=1\\ l\neq j}}^{m}\int_{t-v_j}^{t} Q_{l,j}^{1}(\omega)d\omega}{1-\int_{t-v_j}^{t} Q_{j,j}^{1}(\omega)d\omega}\geq W_j^1,\quad \text{for } t\in[T_0+2v_m+v_j,\ T_1], \tag{8}$$

and

$$\int_{t-v_j}^{t} Q_{j,j}^{1}(\omega)d\omega<1,\quad \text{for } t\in[T_0+2v_m+v_j,\ T_1].$$

Again, the integration by parts leads to

$$\begin{aligned}\int_{t-v_j}^{t} x(\omega-2v_j)\, Q_{j,j}^{2}(\omega)d\omega &= x(t-2v_j)\int_{t-v_j}^{t} Q_{j,j}^{2}(\omega)d\omega \\ &+\int_{t-v_j}^{t}\sum_{\substack{l=1\\ l\neq j}}^{m} x(\omega-2v_j-v_l)q_j(\omega-2v_j)\int_{t-v_j}^{\omega} Q_{j,j}^{2}(\omega_1)d\omega_1 d\omega \\ &+\int_{t-v_j}^{t} x(\omega-3v_j)q_l(\omega-2v_j)\int_{t-v_j}^{\omega} Q_{j,j}^{2}(\omega_1)d\omega_1 d\omega,\end{aligned}$$

that is,

$$\begin{aligned}\int_{t-v_j}^{t} x(\omega-2v_j)\, Q_{j,j}^{2}(\omega)d\omega &= x(t-2v_j)\int_{t-v_j}^{t} Q_{j,j}^{2}(\omega)d\omega \\ &+\int_{t-v_j}^{t}\sum_{\substack{l=1\\ l\neq j}}^{m} x(\omega-2v_j-v_l)\, Q_{l,j}^{3}(\omega)d\omega \\ &+\int_{t-v_j}^{t} x(\omega-3v_j)\, Q_{j,j}^{3}(\omega)d\omega.\end{aligned}$$

Substituting into (7), we obtain

$$\begin{aligned}\int_{t-v_j}^{t}\sum_{l=1}^{m} q_l(\omega)x(\omega-v_l)d\omega &= \sum_{k=1}^{2} x(t-kv_j)\int_{t-v_j}^{t} Q_{j,j}^{k}(\omega)d\omega \\ &+\sum_{k=1}^{2}\sum_{\substack{l=1\\ l\neq j}}^{m}\int_{t-v_j}^{t} x(\omega-(k-1)v_j-v_l)\, Q_{l,j}^{k}(\omega)d\omega \\ &+\int_{t-v_j}^{t} x(\omega-3v_j)\, Q_{j,j}^{3}(\omega)d\omega \\ &+\sum_{\substack{l=1\\ l\neq j}}^{m}\int_{t-v_j}^{t} x(\omega-2v_j-v_l)\, Q_{l,j}^{3}(\omega)d\omega.\end{aligned}$$

As before, using the positivity of $x(t)$ on $[T_0,\, T_1]$, we obtain

$$\begin{aligned}\int_{t-v_j}^{t}\sum_{l=1}^{m} q_l(\omega)x(\omega-v_l)d\omega &\geq \sum_{k=1}^{2} x(t-kv_j)\int_{t-v_j}^{t} Q_{j,j}^{k}(\omega)d\omega \\ &+x(t)\sum_{k=1}^{2}\sum_{\substack{l=1\\ l\neq j}}^{m}\int_{t-v_j}^{t} Q_{l,j}^{k}(\omega)d\omega,\end{aligned} \tag{9}$$

for $t\in[T_0+2v_m+2v_j,\, T_1]$. In view of (8), we have

$$x(t-2v_j)\geq x(t-v_j)\, W_j^{1},\quad \text{for } t\in[T_0+2v_m+2v_j,\, T_1].$$

From this and (9), we get

$$\begin{aligned}\int_{t-v_j}^{t}\sum_{l=1}^{m} q_l(\omega)x(\omega-v_l)d\omega &\geq x(t-v_j)\left(\int_{t-v_j}^{t} Q_{j,j}^{1}(\omega)d\omega+W_j^{1}\int_{t-v_j}^{t} Q_{j,j}^{2}(\omega)d\omega\right) \\ &+x(t)\sum_{k=1}^{2}\sum_{\substack{l=1\\ l\neq j}}^{m}\int_{t-v_j}^{t} Q_{l,j}^{k}(\omega)d\omega.\end{aligned}$$

Substituting into (5), it follows that

$$\frac{x(t-v_j)}{x(t)}\left(1-\int_{t-v_j}^{t} Q_{j,j}^{1}(\omega)d\omega-W_j^{1}\int_{t-v_j}^{t} Q_{j,j}^{2}(\omega)d\omega\right)\geq\left(1+\sum_{k=1}^{2}\sum_{\substack{l=1\\ l\neq j}}^{m}\int_{t-v_j}^{t} Q_{l,j}^{k}(\omega)d\omega\right)>0,$$

for $t\in[T_0+2v_m+2v_j,\, T_1]$. Therefore,

$$\frac{x(t-v_j)}{x(t)}\geq\frac{1+\sum_{k=1}^{2}\sum_{\substack{l=1\\ l\neq j}}^{m}\int_{t-v_j}^{t} Q_{l,j}^{k}(\omega)d\omega}{1-\int_{t-v_j}^{t} Q_{j,j}^{1}(\omega)d\omega-W_j^{1}\int_{t-v_j}^{t} Q_{j,j}^{2}(\omega)d\omega}\geq W_j^{2},\quad \text{for } t\in[T_0+2v_m+2v_j,\, T_1],$$

and

$$\int_{t-v_j}^{t} Q^1_{j,j}(\omega)d\omega + W^1_j \int_{t-v_j}^{t} Q^2_{j,j}(\omega)d\omega < 1, \quad \text{for } t \in [T_0 + 2v_m + 2v_j,\, T_1].$$

Continuing in this process n times, we obtain

$$\begin{aligned}\int_{t-v_j}^{t} \sum_{l=1}^{m} q_l(\omega)x(\omega - v_l)d\omega &= \sum_{k=1}^{n} x(t-kv_j) \int_{t-v_j}^{t} Q^k_{j,j}(\omega)d\omega \\ &+ \sum_{k=1}^{n} \sum_{\substack{l=1\\ l\neq j}}^{m} \int_{t-v_j}^{t} x(\omega-(k-1)v_j - v_l)\, Q^k_{l,j}(\omega)d\omega \\ &+ \int_{t-v_j}^{t} x(\omega-(n+1)v_j)\, Q^{n+1}_{j,j}(\omega)d\omega \\ &+ \int_{t-v_j}^{t} \sum_{\substack{l=1\\ l\neq j}}^{m} x(\omega - nv_j - v_l)\, Q^{n+1}_{l,j}(\omega)d\omega,\end{aligned}$$

and

$$\frac{x(t-v_j)}{x(t)} \geq W^{n-1}_j, \quad \text{for } t \in [T_0 + 2v_m + (n-1)v_j,\, T_1]. \tag{10}$$

Using the positivity of $x(t)$ on $[T_0,\, T_1]$, we obtain

$$\int_{t-v_j}^{t} \sum_{l=1}^{m} q_l(\omega)x(\omega - v_l)d\omega \geq \sum_{k=1}^{n} x(t-kv_j) \int_{t-v_j}^{t} Q^k_{j,j}(\omega)d\omega + x(t) \sum_{k=1}^{n} \sum_{\substack{l=1\\ l\neq j}}^{m} \int_{t-v_j}^{t} Q^k_{l,j}(\omega)d\omega, \tag{11}$$

for $t \in [T_0 + 2v_m + nv_j,\, T_1]$.
Clearly

$$x(t-kv_j) = x(t-v_j) \prod_{l=2}^{k} \frac{x(t-lv_j)}{x(t-(l-1)v_j)}, \qquad k = 1, 2, \ldots. \tag{12}$$

In view of $t-(l-1)v_j \in [T_0 + 2v_m + (n-(l-1))v_j,\, T_1 - (l-1)v_j]$, for $t \in [T_0 + 2v_m + nv_j,\, T_1]$ and $l = 2, 3, \ldots, n$, it follows from (10) that

$$\frac{x(t-lv_j)}{x(t-(l-1)v_j)} \geq W^{n-(l-1)}_j, \quad \text{for } t \in [T_0 + 2v_m + nv_j,\, T_1].$$

From this and (12), we get

$$x(t-kv_j) \geq x(t-v_j) \prod_{l=2}^{k} W^{n-(l-1)}_j, \quad \text{for } t \in [T_0 + 2v_m + nv_j,\, T_1], \qquad k = 1, 2, \ldots.$$

Substituting into (11), we have

$$\begin{aligned}\int_{t-v_j}^{t} \sum_{l=1}^{m} q_l(\omega)x(\omega - v_l)d\omega &\geq x(t-v_j) \sum_{k=1}^{n} \prod_{l=2}^{k} \left(W^{n-(l-1)}_j\right) \int_{t-v_j}^{t} Q^k_{j,j}(\omega)d\omega \\ &+ x(t) \sum_{k=1}^{n} \sum_{\substack{l=1\\ l\neq j}}^{m} \int_{t-v_j}^{t} Q^k_{l,j}(\omega)d\omega, \quad \text{for } t \in [T_0 + 2v_m + nv_j,\, T_1].\end{aligned}$$

Substituting into (5), it follows that

$$x(t-v_j)\left(1 - \sum_{k=1}^{n} \prod_{l=2}^{k} \left(W^{n-(l-1)}_j\right) \int_{t-v_j}^{t} Q^k_{j,j}(\omega)d\omega\right) \geq x(t)\left(1 + \sum_{k=1}^{n} \sum_{\substack{l=1\\ l\neq j}}^{m} \int_{t-v_j}^{t} Q^k_{l,j}(\omega)d\omega\right) > 0,$$

for $t \in [T_0 + 2v_m + nv_j,\ T_1]$. Therefore

$$\frac{x(t-v_j)}{x(t)} \geq \frac{1 + \sum_{k=1}^{n} \sum_{\substack{l=1 \\ l\neq j}}^{m} \int_{t-v_j}^{t} Q_{l,j}^{k}(\omega)d\omega}{1 - \sum_{k=1}^{n} \prod_{l=2}^{k} \left(W_j^{n-(l-1)}\right) \int_{t-v_j}^{t} Q_{j,j}^{k}(\omega)d\omega}, \qquad \text{for } t \in [T_0 + 2v_m + nv_j,\ T_1],$$

and

$$\sum_{k=1}^{n} \prod_{l=2}^{k} \left(W_j^{n-(l-1)}\right) \int_{t-v_j}^{t} Q_{j,j}^{k}(\omega)d\omega < 1, \qquad \text{for } t \in [T_0 + 2v_m + nv_j,\ T_1].$$

The proof of the lemma is complete. □

Theorem 2. *Assume that $n \in \mathbb{N}$ and $j \in \{1, 2, \ldots, m\}$. If*

$$\sum_{k=1}^{n} \prod_{l=2}^{k} \left(W_j^{n-(l-1)}\right) \int_{t-v_j}^{t} Q_{j,j}^{k}(\omega)d\omega \geq 1, \qquad \text{for all } t \geq t_0 + nv_m, \tag{13}$$

then every solution of Equation (1) is oscillatory and $D(x) \leq 2v_m + nv_j$.

Proof. Assume, for the sake of contradiction, that there exists a solution $x(t)$ of Equation (1) such that $x(t) > 0$ on $[T_0,\ T_1]$, for some $T_0 \geq t_0$, $T_1 \geq T_0 + 2v_m + nv_j$. In view of Lemma 1, we have

$$\sum_{k=1}^{n} \prod_{l=2}^{k} \left(W_j^{n-(l-1)}\right) \int_{t-v_j}^{t} Q_{j,j}^{k}(\omega)d\omega < 1, \qquad \text{for } t \in [T_0 + 2v_m + nv_j,\ T_1],$$

which contradicts (13). The proof of the theorem is complete. □

Theorem 3. *Assume that $n \in \mathbb{N}$ and $j \in \{1, 2, \ldots, m\}$. If*

$$\int_{t-v_1}^{t} \sum_{l=1}^{m} q_l(\omega) e^{\int_{\omega-v_l}^{t-v_1} \left(\sum_{l_1=1}^{j-1} q_{l_1}(\omega_1) W_{l_1}^{n} + W_j^{n} \sum_{l_1=j}^{m} q_{l_1}(\omega_1)\right) d\omega_1} d\omega \geq 1, \qquad \text{for all } t \geq t_0 + v_1 + v_m,$$

then every solution of Equation (1) is oscillatory and $D(x) \leq 3v_m + v_1 + nv_j$.

Proof. As before, let $x(t)$ be a positive solution of Equation (1) on $[T_0,\ T_1]$, for some $T_0 \geq t_0$, $T_1 \geq T_0 + 3v_m + v_1 + nv_j$. In view of Equation (1), we get

$$x(t) - x(t-v_1) + \int_{t-v_1}^{t} \sum_{l=1}^{m} q_l(\omega) x(\omega - v_l) d\omega = 0. \tag{14}$$

Dividing Equation (1) by $x(t)$, and integrating the resulting equation from $\omega - v_l$ to $t - v_1$, we obtain

$$x(\omega - v_l) = x(t - v_1) e^{\int_{\omega-v_l}^{t-v_1} \sum_{l_1=1}^{m} q_{l_1}(\omega_1) \frac{x(\omega_1 - v_l)}{x(\omega_1)} d\omega_1 = 0}.$$

From this and (14), it follows that

$$x(t) - x(t-v_1) + x(t-v_1) \int_{t-v_1}^{t} \sum_{l=1}^{m} q_l(\omega) e^{\int_{\omega-v_l}^{t-v_1} \sum_{l_1=1}^{m} q_{l_1}(\omega_1) \frac{x(\omega_1 - v_{l_1})}{x(\omega_1)} d\omega_1} d\omega = 0. \tag{15}$$

Since $x'(t) \leq 0$, for $t \in [T_0 + v_m,\ T_1]$, then for $j \leq l_1 \leq m$, we have

$$x(\omega_1 - v_{l_1}) \geq x(\omega_1 - v_j), \qquad \omega - v_l \leq \omega_1 \leq t - v_1, \quad t - v_1 \leq \omega \leq t, \quad l = 1, 2, \ldots, m,$$

for $t \in [T_0 + 3v_m + v_1,\ T_1]$. From this and (15), we obtain

$$x(t) - x(t - v_1) + x(t - v_1)$$
$$\int_{t-v_1}^{t} \sum_{l=1}^{m} q_l(\omega) \mathrm{e}^{\int_{\omega-v_l}^{t-v_1} \left(\sum_{l_1=1}^{j-1} q_{l_1}(\omega_1) \frac{x(\omega_1 - v_{l_1})}{x(\omega_1)} + \frac{x(\omega_1 - v_j)}{x(\omega_1)} \sum_{l_1=j}^{m} q_{l_1}(\omega_1) \right) d\omega_1} d\omega \le 0, \quad (16)$$

$t \in [T_0 + 3v_m + v_1,\ T_1]$. For $l = 1, 2, \ldots, m$, it follows from (4) that

$$\frac{x(\omega_1 - v_r)}{x(\omega_1)} \ge W_r^n, \quad \omega - v_l \le \omega_1 \le t - v_1, \quad t - v_1 \le \omega \le t, \quad \text{for } r = 1, 2, \ldots, j,$$

for $t \in [T_0 + 3v_m + v_1 + nv_r,\ T_1]$. This together with (16) implies that

$$x(t) - x(t - v_1) + x(t - v_l) \int_{t-v_1}^{t} \sum_{l=1}^{m} q_l(\omega) \mathrm{e}^{\int_{\omega-v_l}^{t-v_1} \left(\sum_{l_1=1}^{j-1} q_{l_1}(\omega_1) W_{l_1}^n + W_j^n \sum_{l_1=j}^{m} q_{l_1}(\omega_1) \right) d\omega_1} d\omega \le 0,$$

for $t \in [T_0 + 3v_m + v_1 + nv_j,\ T_1]$.
Therefore

$$x(t) + x(t - v_1) \left(\int_{t-v_1}^{t} \sum_{l=1}^{m} q_l(\omega) \mathrm{e}^{\int_{\omega-v_l}^{t-v_1} \left(\sum_{l_1=1}^{j-1} q_{l_1}(\omega_1) W_{l_1}^n + W_j^n \sum_{l_1=j}^{m} q_{l_1}(\omega_1) \right) d\omega_1} d\omega - 1 \right) \le 0,$$

for $t \in [T_0 + 3v_m + v_1 + nv_j,\ T_1]$. This contradiction completes the proof. □

Let $j \in \{1, 2, \ldots, m\}$, and

$$\Omega_j^1(t) = \sum_{l=1}^{m} q_l(t) \int_{t-v_l}^{t} q_j(\omega) \mathrm{e}^{\int_{\omega-v_j}^{t} \sum_{l_1=1}^{m} q_{l_1}(\omega_1) d\omega_1} d\omega, \qquad t \ge t_0 + v_m + v_j,$$
$$\Omega_j^n(t) = \sum_{l=1}^{m} \Omega_l^{n-1}(t) \int_{t-v_l}^{t} \Omega_j^{n-1}(\omega) \mathrm{e}^{\int_{\omega-v_j}^{t} \sum_{l_1=1}^{m} \Omega_{l_1}^{n-1}(\omega_1) d\omega_1} d\omega, \quad n = 2, 3, \ldots,$$

for $t \ge t_0 + n(v_m + v_j)$.

Theorem 4. *Assume that $n \in \mathbb{N}$. If*

$$\prod_{l=1}^{m} \left(\prod_{j=1}^{m} \int_{t-v_l}^{t} \Omega_j^n(\omega) d\omega \right)^{\frac{1}{m}} \ge m^m, \qquad \text{for all } t \ge t_0 + (2n+1)v_m, \quad (17)$$

then, every solution of Equation (1) is oscillatory and $D(x) \le (2n+3)v_m$.

Proof. Let $x(t)$ be a positive solution of Equation (1) on $[T_0, T_1]$, for some $T_0 \ge t_0$, $T_1 \ge T_0 + (2n+3)v_m$. It follows from Equation (1) that

$$x(t) - x(t - v_l) + \int_{t-v_l}^{t} \sum_{j=1}^{m} q_j(\omega) x(\omega - v_j) d\omega = 0, \qquad l = 1, 2, \ldots, m.$$

Multiplying both sides by $q_l(t)$ and summing up from 1 to m, we get

$$x(t) \sum_{l=1}^{m} q_l(t) - \sum_{l=1}^{m} q_l(t) x(t - v_l) + \sum_{l=1}^{m} q_l(t) \int_{t-v_l}^{t} \sum_{j=1}^{m} q_j(\omega) x(\omega - v_j) d\omega = 0.$$

From Equation (1), we obtain

$$x'(t)+x(t)\sum_{l=1}^{m}q_l(t)+\sum_{l=1}^{m}q_l(t)\int_{t-v_l}^{t}\sum_{j=1}^{m}q_j(\omega)x(\omega-v_j)d\omega=0.$$

Let $V_1(t)=x(t)\mathrm{e}^{\int_{t_0}^{t}\sum_{l_1=1}^{m}q_{l_1}(\omega)d\omega}$. Then $V_1(t)>0$ on $[T_0,T_1]$, and

$$V_1'(t)+\sum_{l=1}^{m}q_l(t)\int_{t-v_l}^{t}\sum_{j=1}^{m}q_j(\omega)V_1(\omega-v_j)\mathrm{e}^{\int_{\omega-v_j}^{t}\sum_{l_1=1}^{m}q_{l_1}(\omega_1)d\omega_1}d\omega=0. \tag{18}$$

In view of $x'(t)\le 0$ on $[T_0+v_m,T_1]$, we have

$$x'(t)+x(t)\sum_{l=1}^{m}q_l(t)\le 0, \qquad \text{for } t\in[T_0+2v_m,T_1].$$

Then

$$V_1'(t)=\left(x'(t)+x(t)\sum_{l_1=1}^{m}q_{l_1}(t)\right)\mathrm{e}^{\int_{t_0}^{t}\sum_{l_1=1}^{m}q_{l_1}(\omega)d\omega}\le 0, \qquad \text{for } t\in[T_0+2v_m,T_1]. \tag{19}$$

From this and (18), we get

$$V_1'(t)+\sum_{j=1}^{m}V_1(t-v_j)\sum_{l=1}^{m}q_l(t)\int_{t-v_l}^{t}q_j(\omega)\mathrm{e}^{\int_{\omega-v_j}^{t}\sum_{l_1=1}^{m}q_{l_1}(\omega_1)d\omega_1}d\omega\le 0,$$

for $t\in[T_0+4v_m,T_1]$; that is,

$$V_1'(t)+\sum_{j=1}^{m}\Omega_j^1(t)\,V_1(t-v_j)\le 0, \qquad \text{for } t\in[T_0+4v_m,T_1]. \tag{20}$$

Integrating from $t-v_l$ to t, $l=1,2,\dots,m$, we obtain

$$V_1(t)-V_1(t-v_l)+\int_{t-v_l}^{t}\sum_{j=1}^{m}\Omega_j^1(\omega)\,V_1(\omega-v_j)d\omega\le 0, \qquad \text{for } t\in[T_0+5v_m,T_1].$$

Multiplying by $\Omega_l^1(t)$ and summing up from 1 to m, we get

$$V_1'(t)+V_1(t)\sum_{l=1}^{m}\Omega_l^1(t)+\sum_{l=1}^{m}\Omega_l^1(t)\int_{t-v_l}^{t}\sum_{j=1}^{m}\Omega_j^1(\omega)\,V_1(\omega-v_j)d\omega\le 0,$$

for $t\in[T_0+5v_m,T_1]$.
Let $V_2(t)=V_1(t)\mathrm{e}^{\int_{t_0}^{t}\sum_{l_1=1}^{m}\Omega_{l_1}^1(\omega)d\omega}$. Clearly $V_2(t)>0$ on $[T_0,T_1]$, and

$$V_2'(t)+\sum_{l=1}^{m}\Omega_l^1(t)\int_{t-v_l}^{t}\sum_{j=1}^{m}\Omega_j^1(\omega)\,V_2(\omega-v_j)\mathrm{e}^{\int_{t-v_j}^{t}\sum_{l_1=1}^{m}\Omega_{l_1}^1(\omega_1)d\omega_1}d\omega\le 0,$$

for $t\in[T_0+5v_m,T_1]$. It follows from (19) and (20) that

$$V_2'(t)=\left(V_1'(t)+V_1(t)\sum_{l_1=1}^{m}\Omega_{l_1}^1(t)\right)\mathrm{e}^{\int_{t_0}^{t}\sum_{l_1=1}^{m}\Omega_{l_1}^1(\omega)d\omega}\le 0 \quad \text{for } t\in[T_0+4v_m,T_1].$$

Therefore

$$V_2'(t)+\sum_{j=1}^{m}\Omega_j^2(t)\,V_2(t-v_j)\le 0, \quad \text{for } t\in[T_0+6v_m,T_1].$$

Repeating the above procedure n times, we obtain

$$V_n'(t) + \sum_{j=1}^{m} \Omega_j^n(t)\, V_n(t - v_j) \le 0, \quad \text{for } t \in [T_0 + (2n+2)v_m,\, T_1],$$

where $V_n(t) = V_{n-1}(t)e^{\int_{t_0}^{t} \sum_{l_1=1}^{m} \Omega_{l_1}^{n-1}(\omega)d\omega}$ and $V_n'(t) \le 0$ for $t \in [T_0 + 2nv_m,\, T_1]$. Integrating the above inequality from $t - v_l$ to t, we get

$$V_n(t) - V_n(t - v_l) + \sum_{j=1}^{m} \int_{t-v_l}^{t} \Omega_j^n(\omega)\, V_n(\omega - v_j)d\omega \le 0, \quad \text{for } t \in [T_0 + (2n+3)v_m,\, T_1].$$

Using the positivity and nonincreasing nature of $V_n(t)$ on $[T_0 + 2nv_m,\, T_1]$, we have

$$V_n(t - v_l) > \sum_{j=1}^{m} V_n(t - v_j) \int_{t-v_l}^{t} \Omega_j^n(\omega)d\omega, \quad \text{for } t \in [T_0 + (2n+3)v_m,\, T_1].$$

By the arithmetic–geometric mean, we get

$$V_n(t - v_l) > m\left(\prod_{j=1}^{m} V_n(t - v_j)\right)^{\frac{1}{m}} \left(\prod_{j=1}^{m} \int_{t-v_l}^{t} \Omega_j^n(\omega)d\omega\right)^{\frac{1}{m}}, \quad \text{for } t \in [T_0 + (2n+3)v_m,\, T_1].$$

Taking the product of both sides, we have

$$\prod_{j=1}^{m} V_n(t - v_j) > m^m \prod_{j=1}^{m} V_n(t - v_j) \prod_{l=1}^{m} \left(\prod_{j=1}^{m} \int_{t-v_l}^{t} \Omega_j^n(\omega)d\omega\right)^{\frac{1}{m}},$$

for $t \in [T_0 + (2n+3)v_m,\, T_1]$, which in turn implies

$$\prod_{l=1}^{m} \left(\prod_{j=1}^{m} \int_{t-v_l}^{t} \Omega_j^n(\omega)d\omega\right)^{\frac{1}{m}} < m^m, \quad \text{for } t \in [T_0 + (2n+3)v_m,\, T_1].$$

This contradicts (17), and hence the proof of the theorem is complete. □

Remark 1.
(1) *Theorem 2 with $m = 1$ improves ([13], Theorem 2.6) and ([14], Corollary 2.24);*
(2) *Theorem 4 with $m = 1$ improves ([13], Theorem 2.3);*
(3) *The techniques used in this work can be extended to study the oscillation of first-order differential equations with several nonmonotonous delays.*

In the following, we introduce two illustrative examples to show the strength and accuracy of our results.

Example 2. *Consider the differential equation*

$$x'(t) + 0.001\, x(t-1) + 0.667\, x(t-1.5) = 0, \qquad t \ge 1.5.$$

This equation has the form of Equation (1) with

$$q_1(t) = 0.001, \quad q_2(t) = 0.667, \quad v_1 = 1, \quad v_2 = 1.5.$$

Since

$$\sum_{k=1}^{1}\prod_{l=2}^{k}W_2^{1-(l-1)}\int_{t-v_2}^{t}Q_{2,2}^{k}(\omega)d\omega=\int_{t-v_2}^{t}Q_{2,2}^{1}(\omega)d\omega=\int_{t-v_2}^{t}q_2(\omega)d\omega=1.005,$$

for all t. *Theorem 2 with* $j=2$ *and* $n=1$ *implies that* $D(x)\le 3v_2=4.5$. *However, all the results of [14] cannot give this estimation, as we will show. Let* $P=0.001+0.667$, $h(t)=t-v_2$ *and* $g_r(t)=t-\delta_r v_1$, $r=1,2,3$. *Since*

$$\int_{g_1(t)}^{t}\mathrm{P}e^{\int_{g_1(\omega)}^{g_1(t)}P_1d\omega_1}\int_{g_1(\omega)}^{\omega}\mathrm{P}e^{\int_{g_1(\omega_1)}^{\omega}Pd\omega_2}d\omega_1d\omega>1.0006874,\quad \text{for } \delta_1=0.843,$$

where

$$P_1=\mathrm{P}e^{\int_{g_1(t)}^{t}Pd\omega}\int_{g_1(t)}^{t}Pd\omega.$$

Then, ([14], Theorem 2.17) with $n=1$ *implies that*

$$D(x)\le\sup\left\{\left(g_1^{-3}(h^{-2}(t))-t\right):\ t\ge 1.5\right\}=3\delta_1v_1+2v_2=5.529.$$

Also, it is clear for $\delta_2=0.88$ *that*

$$\int_{g_2(t)}^{t}Pd\omega=P\delta_2v_1.$$

Therefore

$$\int_{g_2(t)}^{t}Pd\omega+\frac{1}{1-P\delta_2v_1}\int_{g_2(t)}^{t}P\int_{g_2(t)}^{\omega}Pd\omega_1d\omega>1.007.$$

A direct application of ([14], Theorem 2.23) with $n=2$ *leads to*

$$D(x)\le\sup\left\{\left(g_2^{-2}(h^{-2}(t))-t\right):\ t\ge 1.5\right\}=2\delta_2v_1+2v_2=4.76.$$

Finally, since

$$D_1=Pe^{\int_{g_3(t)}^{t}Pd\omega_1}\int_{g_3(t)}^{t}Pd\omega_1>0.76313,$$

and

$$\int_{g_3(t)}^{t}D_1e^{\int_{g_3(\omega)}^{\omega}D_1d\omega_1}\int_{g_3(\omega)}^{\omega}D_1d\omega_1d\omega>1.003,$$

for $\delta_3=0.9231$. *Then, according to ([14], Corollary 2.15) with* $n=2$, *we obtain*

$$D(x)\le\sup\left\{\left(g_3^{-2}(h^{-2}(t))-t\right):\ t\ge 1.5\right\}=2\delta_3v_1+2v_2=4.8462.$$

Example 3. *Consider the differential equation*

$$x'(t)+\alpha\,x(t-0.3)+\beta\,x(t-1)=0,\qquad t\ge 1,\tag{21}$$

where $\alpha,\beta>0$. *It follows from Equation (1) that*

$$q_1(t)=\alpha,\quad q_2(t)=\beta,\quad v_1=0.3,\quad v_2=1.$$

Therefore

$$\begin{aligned}\prod_{l=1}^{2}\left(\prod_{j=1}^{2}\int_{t-v_l}^{t}\Omega_j^1(\omega)d\omega\right)^{\frac{1}{2}} &= \prod_{l=1}^{2}\Big(\sum_{l=1}^{2}q_l(t)\int_{t-v_l}^{t}\alpha e^{\int_{\omega-0.3}^{t}B\omega_1}d\omega \\ &\quad\times\sum_{l=1}^{2}q_l(t)\int_{t-v_l}^{t}\beta e^{\int_{\omega-1}^{t}B\omega_1}d\omega\Big) \\ &= \frac{3\alpha\beta}{10B^2}\Big(Be^{B}-\beta\, e^{2B}-\alpha\, e^{\frac{13}{10}B}\Big)\Big(Be^{\frac{3}{10}B}-\alpha\, e^{\frac{3}{5}B}-\beta\, e^{\frac{13}{10}B}\Big),\end{aligned}$$

where $B=\alpha+\beta$. *Consequently,*

$$\prod_{l=1}^{2}\left(\prod_{j=1}^{2}\int_{t-v_l}^{t}\Omega_j^1(\omega)d\omega\right)^{\frac{1}{2}}>0.25,$$

for all $\alpha\geq 0.845$ *and* $\beta\geq 0.3$. *According to Theorem 4 with* $n=1$, $D(x)\leq 5$. *We remark here that all corresponding results of [14] cannot give this estimation for* $\alpha=0.845$ *and* $\beta=0.3$. *For example, using Maple software, we have*

$$\sum_{k=1}^{11}\prod_{l=2}^{k}R_{11-(l-1)}(\zeta)\int_{g(t)}^{t}Z_k(\omega)d\omega\approx 0.48985718,$$

where $g(t)=t-0.3$, $\zeta=0.3(\alpha+\beta)$,

$$\begin{aligned}Z_1(\omega) &= \alpha+\beta, \\ Z_n(\omega) &= Z_1(\omega-0.3(n-1))\int_{g(t)}^{\omega}Z_{n-1}(\omega_1)d\omega_1, \qquad \omega\in(g(t),\,t),\quad n=2,3,\ldots,\end{aligned}$$

and

$$R_0(\zeta)=1,\quad R_1(\zeta)=\frac{1}{1-\zeta},\quad R_n(\zeta)=\frac{R_{n-2}(\zeta)}{R_{n-2}(\zeta)+1-e^{\zeta R_{n-2}(\zeta)}},\ n=2,3,\ldots.$$

Therefore, ([14], Theorem 2.23) with $n=11$ *fails to apply to (21) when* $\alpha=0.845$ *and* $\beta=0.3$.

3. Conclusions

In this work, the distance between consecutive zeros of all solutions of Equation (1) was studied. We have developed and generalized some methods used by [13,14,27] for Equation (2) and obtained many new approximations for the UB of all solutions of Equation (1). We also proved that some solutions of Equation (1) have separate zeros. The differences between the distribution of zeros for Equations (1) and (2) have been discussed. The techniques introduced in this work can be used to study the distribution of zeros for some other equations, such as differential equations with several variable delays and neutral differential equations.

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Abbreviations

The following abbreviations are used in this paper:

UB	The upper bound between successive zeros of a solution of a differential equation
LB	The lower bound between successive zeros of a solution of a differential equation

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