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First-Order Approximate Mei Symmetries and Invariants of the Lagrangian

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Abstract: In this article, the formulation of first-order approximate Mei symmetries and Mei invariants of the corresponding Lagrangian is presented. Theorems and determining equations are given to evaluate approximate Mei symmetries, as well as approximate first integrals corresponding to each symmetry of the associated Lagrangian. The formulated procedure is explained with the help of the linear equation of motion of a damped harmonic oscillator (DHO). The Mei symmetries corresponding to the Lagrangian and Hamiltonian of DHO are compared.

Keywords: exact Mei symmetries; Lie symmetries; Lagrangian

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1. Introduction

The symmetry methods and conserved quantities are highly significant in the different fields of studies, such as, mathematics, social sciences, natural sciences, engineering, etc. Noether showed a connection between symmetries and the conservation laws in her famous Noether theory [1] in 1918. Furthermore, an action integral of a functional (Lagrangian) is invariant under the infinitesimal transformation of a group. This transformation, generated by a differential operator, referred as the Noether symmetry in this case. In 2000, Feng-Xiang [2] introduced an invariance of the equations of motion under infinitesimal transformation of a group, called the form invariance, also known as Mei symmetries. In Mei symmetries, the dynamical functions including Lagrangian, Hamiltonian, etc., are replaced by the transformed dynamical functions. Moreover, the equations of motion are satisfied, after doing some infinitesimal transformation of a group. More specifically, the form of *equations of motion* is preserved in Mei symmetries.

The form invariance of the Appell equations are discussed under infinitesimal transformation of a groups [3]. The Noether symmetries are calculated from the Lagrangian of Appell equations. After that, Noether symmetries are compared with the form invariance and different conserved quantities are obtained. Shu-Yong and Feng-Xiang [4] presented form invariance and the Lie symmetries of the non-holonomic system. In this paper, structure equations and form invariance are deduced, which have similarity with the Lie symmetries. Mei symmetries of the rotational relativistic mass variable system are discussed in [5], with a focus on the relationship between Lie and Mei symmetries. Jiang et al. [6] constructed the Mei symmetries for non-material volumes. A single-degree-of-freedom non-material volume is taken as an example to determine the conserved quantities. In [7,8], the Mei symmetries of the Lagrangian and Birkhoffian system on time scale are calculated. Its relation to the Noether symmetries is thoroughly discussed here. In the formulation of Mei symmetries of the Birkhoffian system, the Hamiltonian canonical equations are considered as a special case.

The perturbed part in differential equations arises in a variety of applications. The standard Lie and Noether theorems are not applied on these equations. Therefore, Baikov et al. in [9,10] introduced the approximate symmetry methods. It is the most efficient method for obtaining approximate solutions to the perturbed partial differential equations (PDEs). Feroze and Kara [11] made use of a group theoretic approach to construct the approximate Lie point symmetries and invariants of the ordinary differential equations (ODEs). In addition, the theory presented in [9,10] is used to calculate approximate Lie symmetries of PDEs. Ali and Feroze in [12] investigated the Noether symmetries of time conformal plane symmetric spacetimes corresponding to geodetic Lagrangian. In this paper, the time conformal factor is used to calculate approximate Noether symmetries. The general time conformal factor generates a perturbation of first order in the general plane symmetric static spacetimes.

Paliathanasis [13] developed an association between Noether symmetries related to the class of perturbed Lagrangian and elements of Homothetic algebra of the metric. Furthermore, a relation between approximate Noether symmetries and collineations of the underlying manifolds is also discussed here. The authors of [14] discussed the formulation of approximate Mei symmetries and invariants relative to a Hamiltonian. The determining equations, criterion and theorems to construct Mei symmetries of the Hamiltonian is presented here. The approximate Noether and Mei symmetries are compared. Through comparison, different approximate symmetries are obtained which lead to conserved quantities. Recently, Gorgane and Oliveri [15] presented an approximate Noether theorem that leads to the approximate conservation laws.

In this paper, first-order approximate Mei symmetries and invariants of a Lagrangian are formulated. A comparison of approximate Mei symmetries corresponding to a Lagrangian and Hamiltonian is given. The paper is ordered as follows. Section 2 comprises the review of method for determining the exact Mei symmetries corresponding to the Lagrangian, definition and criterion of constructing these symmetries. Some new development, i.e., approximate Mei symmetries and invariants corresponding to a Lagrangian are given in Section 3. This section is based upon the theorems to obtain the determining equation of the approximate Mei symmetries and first integrals. An example to illustrate the method is given in Section 4. The comparison between Mei symmetries of a Lagrangian and Hamiltonian is presented in Section 4.4. The conclusion of paper is given in Section 5.

2. The Mei Symmetries

Zhai and Zhang [7] found a method of determining the Mei symmetries of the Euler Lagrange equation of the Lagrangian system. The method presented in this paper is used to describe the exact Mei symmetries. Consider the equation of motion corresponding to the Lagrangian $\mathcal{L}(t, x_k, \dot{x}_k)$

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}_k}\right) - \frac{\partial \mathcal{L}}{\partial x_k} = 0, \qquad (k = 1, 2, \ldots, n). \tag{1}$$

Writing

$$E_k = \frac{d}{dt}\left(\frac{\partial}{\partial \dot{x}_k}\right) - \frac{\partial}{\partial x_k}, \qquad (k = 1, 2, \ldots, n). \tag{2}$$

Equation (1) takes the form

$$E_k(\mathcal{L}) = 0, \qquad (k = 1, 2, \ldots, n). \tag{3}$$

Consider an infinitesimal group of transformations corresponding to one parameter

$$t^* = T(t, x_i(t), \varepsilon) = t + \varepsilon\xi(t, x_i(t)), \tag{4}$$

$$x_k^* = X_k(t, x_i(t), \varepsilon) = x_k(t) + \varepsilon\beta_k(t, x_i(t)),\ (k, i = 1, 2, \ldots, n), \tag{5}$$

the corresponding generator $\mathbf{X}$ is given as

$$\mathbf{X} = \xi\frac{\partial}{\partial t} + \beta_k\frac{\partial}{\partial x_k}. \tag{6}$$

Under the transformation, the Lagrangian $\mathcal{L}$ is replaced by a new Lagrangian $\mathcal{L}^*$

$$\mathcal{L}^* = \mathcal{L}^*(t^*, x_k^*, \frac{dx_k^*}{dt^*}) = \mathcal{L}^*\left(t + \varepsilon\xi, x_k + \varepsilon\beta_k, \frac{\dot{x}_k + \varepsilon\dot{\beta}_k}{1 + \varepsilon\dot{\xi}}\right). \tag{7}$$

The Taylor series expansion at $\varepsilon = 0$, yields

$$\mathcal{L}^* = \mathcal{L}(t, x_k, \dot{x}_k) + \varepsilon\mathbf{X}^{[1]}\mathcal{L} + O(\varepsilon^2), \tag{8}$$

where

$$\mathbf{X}^{[1]} = \xi\frac{\partial}{\partial t} + \beta_k\frac{\partial}{\partial x_k} + (\dot{\beta}_k - \dot{x}_k\dot{\xi})\frac{\partial}{\partial \dot{x}_k}, \tag{9}$$

is first prolongation of the infinitesimal generator.

Definition 1. *The Mei symmetries corresponding to a Lagrangian are defined mathematically as*

$$E_k(\mathcal{L}^*) = 0, \qquad (k = 1, 2, ..., n), \tag{10}$$

when the Lagrangian $\mathcal{L}$ is replaced with a new Lagrangian $\mathcal{L}^*$. If the form of Equation (3) remains unchanged, then it is called the Mei symmetries of the corresponding Lagrangian [7].

Criterion: If ξ and β_k satisfy the given condition

$$E_k[\mathbf{X}^{[1]}\mathcal{L}] = 0, \qquad (k = 1, 2, ..., n), \tag{11}$$

then the corresponding generator is called the Mei symmetry generator and the transformation is the Mei symmetry transformation.

After applying the Euler operator E_k on $\mathbf{X}^1\mathcal{L}$, an equation containing different powers of x' is obtained. Separating coefficients of different powers of x', yields a system of PDEs. The solution of this system gives ξ and β_k which satisfy the given criterion in Equation (11) of the Mei symmetries [7].

3. Approximate Mei Symmetries

The approximate Mei symmetries and the associated first integrals are formulated in Theorem 1 and Theorem 2, respectively.

Theorem 1. *Let $\mathbf{X} = \mathbf{X}_0 + \epsilon\mathbf{X}_1$ be an approximate symmetry generator and $\mathcal{L} = \mathcal{L}_0 + \epsilon\mathcal{L}_1$ be the first-order approximate Lagrangian, where $\mathbf{X}_0 = \xi_0\frac{\partial}{\partial t} + \beta_0^k\frac{\partial}{\partial x^k}$ and $\mathbf{X}_1 = \xi_1\frac{\partial}{\partial t} + \beta_1^k\frac{\partial}{\partial x^k}$. Then,*

$$E_k(\mathbf{X}_0^{[1]}\mathcal{L}_0) = 0, \qquad k = 1, 2, \ldots, n. \tag{12}$$

$$E_k(\mathbf{X}_0^{[1]}\mathcal{L}_1 + \mathbf{X}_1^{[1]}\mathcal{L}_0) = 0, \qquad k = 1, 2, \ldots, n. \tag{13}$$

Proof of Theorem 1. With $\mathbf{X} = \mathbf{X}_0 + \epsilon\mathbf{X}_1$ and $\mathcal{L} = \mathcal{L}_0 + \epsilon\mathcal{L}_1$, we have

$$\mathbf{X}^{[1]}L = \mathbf{X}_0^{[1]}\mathcal{L}_0 + \epsilon(\mathbf{X}_0^{[1]}\mathcal{L}_1 + \mathbf{X}_1^{[1]}\mathcal{L}_0) + O(\epsilon^2), \tag{14}$$

neglecting higher powers of ϵ, we get

$$\mathbf{X}^{[1]}L = \mathbf{X}_0^{[1]}\mathcal{L}_0 + \epsilon(\mathbf{X}_0^{[1]}\mathcal{L}_1 + \mathbf{X}_1^{[1]}\mathcal{L}_0). \tag{15}$$

Comparing powers of ϵ, and applying the operator given in Equation (2), we get Equations (12) and (13). This completes the proof. □

Theorem 2. *Let the symmetry generator* $\mathbf{X} = \xi\frac{\partial}{\partial t} + \beta^k\frac{\partial}{\partial x^k}$ *satisfying Theorem 1 and* $\mathcal{L} = \mathcal{L}_0 + \epsilon\mathcal{L}_1$ *be the first-order perturbed Lagrangian. Then, the first integrals take the following form*

$$I_0 = \xi_0(\mathbf{X}_0^{[1]}\mathcal{L}_0) + (\beta_0^k - \dot{x}^k\xi_0)\frac{\partial(\mathbf{X}_0^{[1]}\mathcal{L}_0)}{\partial\dot{x}^k}, \tag{16}$$

$$I_1 = \xi_0(\mathbf{X}_0^{[1]}\mathcal{L}_1 + \mathbf{X}_1^{[1]}\mathcal{L}_0) + \xi_1(\mathbf{X}_0^{[1]}\mathcal{L}_0) + (\beta_1^k - \dot{x}^k\xi_1)\frac{\partial(\mathbf{X}_0^{[1]}\mathcal{L}_0)}{\partial\dot{x}^k}. \tag{17}$$
$$+ (\beta_0^k - \dot{x}^k\xi_0)\frac{\partial(\mathbf{X}_0^{[1]}\mathcal{L}_1 + \mathbf{X}_1^{[1]}\mathcal{L}_0)}{\partial\dot{x}^k}$$

Proof of Theorem 2. To prove the above expression, consider the invariant [7]

$$I = \xi(\mathbf{X}^{[1]}\mathcal{L}) + (\beta^k - \dot{x}^k\xi)\frac{\partial(\mathbf{X}^{[1]}\mathcal{L})}{\partial\dot{x}^k}. \tag{18}$$

Now, introducing the first-order perturbed invariant by taking $I = I_0 + \epsilon I_1$, $\mathbf{X}^{[1]} = \mathbf{X}_0^{[1]} + \epsilon\mathbf{X}_1^{[1]}$ and $\mathcal{L} = \mathcal{L}_0 + \epsilon\mathcal{L}_1$ in Equation (18), we obtain

$$I_0 + \epsilon I_1 = (\xi_0 + \epsilon\xi_1)[(\mathbf{X}_0^{[1]} + \epsilon\mathbf{X}_1^{[1]})(\mathcal{L}_0 + \epsilon\mathcal{L}_1)] + [(\beta_0^k + \epsilon\beta_1^k) - \dot{x}^k(\xi_0 + \epsilon\xi_1)]$$
$$\frac{\partial(\mathbf{X}_0^{[1]} + \epsilon\mathbf{X}_1^{[1]})}{\partial\dot{x}^k}(\mathcal{L}_0 + \epsilon\mathcal{L}_1). \tag{19}$$

After simplifying Equation (19), we get

$$I_0 + \epsilon I_1 = \xi_0(\mathbf{X}_0^{[1]}\mathcal{L}_0) + \epsilon[\xi_0(\mathbf{X}_0^{[1]}\mathcal{L}_1 + \mathbf{X}_1^{[1]}\mathcal{L}_0) + \xi_1(\mathbf{X}_0^{[1]}\mathcal{L}_0)] + (\beta_0^k - \dot{x}^k\xi_0)\frac{\partial(\mathbf{X}_0^{[1]}\mathcal{L}_0)}{\partial\dot{x}^k}$$
$$+ \epsilon\left[(\beta_1^k - \dot{x}^k\xi_1)\frac{\partial(\mathbf{X}_0^{[1]}\mathcal{L}_0)}{\partial\dot{x}^k} + (\beta_0^k - \dot{x}^k\xi_0)\frac{\partial(\mathbf{X}_0^{[1]}\mathcal{L}_1 + \mathbf{X}_1^{[1]}\mathcal{L}_0)}{\partial\dot{x}^k}\right]. \tag{20}$$

Comparing powers of ϵ up to first order and neglecting higher powers, gives Equations (16) and (17). This completes the proof. □

4. Example

Consider the DHO equation, which is linear in this case.

$$x'' + 2\epsilon x' + x = 0. \tag{21}$$

The Lagrangian of the DHO is $\mathcal{L} = \frac{1}{2}e^{2\epsilon t}(x'^2 - x^2)$ [16–18], with $\mathcal{L} = \mathcal{L}_0 + \epsilon\mathcal{L}_1$, then $\mathcal{L}$ takes the following form

$$\mathcal{L} = \frac{1}{2}(x'^2 - x^2) + \epsilon t(x'^2 - x^2). \tag{22}$$

Now, writing $\mathcal{L}$ by separating the powers of ϵ, we obtain

$$\mathcal{L}_0 = \frac{1}{2}(x'^2 - x^2), \tag{23}$$
$$\mathcal{L}_1 = t(x'^2 - x^2). \tag{24}$$

4.1. The Mei Symmetries of DHO

First of all, the first-order prolonged infinitesimal generator $\mathbf{X}^{[1]}$ defined in Equation (9) is applied to the first-order Lagrangian given in Equation (23). This yields

$$\mathbf{X}_0^{[1]}\mathcal{L} = -x\beta_0 + x'\beta_{0,t} + x'^2\beta_{0,x} - x'^2\xi_{0,t} - x'^3\xi_{0,x}. \tag{25}$$

Then, applying the Euler operator defined in Equation (2) for $k = 1$, Equation (25) gives

$$E_1(\mathbf{X}_0^{[1]}\mathcal{L}) = 0. \tag{26}$$

Alternatively,

$$\frac{d}{dt}\left(\frac{\partial \mathbf{X}_0^{[1]}\mathcal{L}_0}{\partial x'}\right) - \left(\frac{\partial \mathbf{X}_0^{[1]}\mathcal{L}_0}{\partial x}\right) = 0. \tag{27}$$

Equation (27) gives the following expression

$$\begin{aligned} &\beta_{0,tt} + 2x'\beta_{0,tx} + 2x''\beta_{0,x} - 2x''\xi_{0,t} + x'^2\beta_{0,xx} - 2x'\xi_{0,tt} - 4x'^2\xi_{0,tx} - 6x''x'\xi_{0,x} \\ &- 2x'^3\xi_{0,xx} + \beta_0 + x\beta_{0,x} = 0. \end{aligned} \tag{28}$$

Substituting $x'' + x = 0$ in Equation (28), we obtain

$$\begin{aligned} &\beta_{0,tt} + 2x'\beta_{0,tx} - x\beta_{0,x} + 2x\xi_{0,t} + x'^2\beta_{0,xx} - 2x'\xi_{0,tt} - 4x'^2\xi_{0,tx} + 6xx'\xi_{0,x} \\ &- 2x'^3\xi_{0,xx} + \beta_0 = 0. \end{aligned} \tag{29}$$

Comparing the coefficients of different powers of x', we obtain a system of PDEs

$$\beta_{0,tt} + \beta_0 - x\beta_{0,x} + 2x\xi_{0,t} = 0, \tag{30}$$

$$\beta_{0,tx} - \xi_{0,tt} + 3x\xi_{0,x} = 0, \tag{31}$$

$$\beta_{0,xx} - 4\xi_{0,tx} = 0, \tag{32}$$

$$\xi_{0,xx} = 0. \tag{33}$$

Solving Equations (30)–(33), we obtain

$$\begin{aligned} &\xi_0 = C_1 + \sin\sqrt{2}tC_2 + \cos\sqrt{2}tC_3, \\ &\beta_0 = -\sqrt{2}x\sin\sqrt{2}tC_3 + \sqrt{2}x\cos\sqrt{2}tC_2 + \cos tC_4 + \sin tC_5 + C_6x, \end{aligned} \tag{34}$$

and the corresponding Mei symmetries are listed below

$$\mathbf{X}_0^1 = \frac{\partial}{\partial t}, \tag{35}$$

$$\mathbf{X}_0^2 = \sin\sqrt{2}t\frac{\partial}{\partial t} + \sqrt{2}x\cos\sqrt{2}t\frac{\partial}{\partial x}, \tag{36}$$

$$\mathbf{X}_0^3 = \cos\sqrt{2}t\frac{\partial}{\partial t} - \sqrt{2}x\sin\sqrt{2}t\frac{\partial}{\partial x}, \tag{37}$$

$$\mathbf{X}_0^4 = \cos t\frac{\partial}{\partial x}, \tag{38}$$

$$\mathbf{X}_0^5 = \sin t\frac{\partial}{\partial x}, \tag{39}$$

$$\mathbf{X}_0^6 = x\frac{\partial}{\partial x}. \tag{40}$$

4.2. Approximate Mei Symmetries of DHO

Now, we calculate the approximate Mei symmetries up to first order of ϵ by using the exact symmetries given in Equations (35)–(40). First of all, we use $\mathbf{X}_0^2 = sin\sqrt{2}t\frac{\partial}{\partial t} + \sqrt{2}xcos\sqrt{2}t\frac{\partial}{\partial x}$, where $\mathbf{X}_0^{[1]}\mathcal{L}_1 + \mathbf{X}_1^{[1]}\mathcal{L}_0$ is expressed as

$$\begin{aligned} \mathbf{X}_0^{[1]}\mathcal{L}_1 + \mathbf{X}_1^{[1]}\mathcal{L}_0 = &-x\beta_1 + x'\beta_{1,t} + x'^2\beta_{1,x} - x'^2\xi_{1,t} - x'^3\xi_{1,x} + x'^2\sin\sqrt{2}t - x^2\sin\sqrt{2}t \\ &- 2\sqrt{2}x^2t\cos\sqrt{2}t - 4txx'\sin\sqrt{2}t. \end{aligned} \tag{41}$$

Now, using Equation (41) in Equation (13), we have

$$\frac{d}{dt}\left(\frac{\partial(\mathbf{X}_0^{[1]}\mathcal{L}_1+\mathbf{X}_1^{[1]}\mathcal{L}_0)}{\partial x'}\right)-\left(\frac{\partial(\mathbf{X}_0^{[1]}\mathcal{L}_1+\mathbf{X}_1^{[1]}\mathcal{L}_0)}{\partial x}\right)=0. \tag{42}$$

Putting Equations (23) and (24) into Equation (13), we obtain

$$\begin{aligned}&\beta_{1,tt}+2x'\beta_{1,tx}+2x''\beta_{1,x}-2x''\xi_{1,t}+x'^2\beta_{1,xx}-2x'\xi_{1,tt}-4x'^2\xi_{1,tx}-6x''x'\xi_{1,x}\\&-2x'^3\xi_{1,xx}+\beta_1+x\beta_{1,x}+2x''\sin\sqrt{2}t+2\sqrt{2}\cos\sqrt{2}tx'-2x\sin\sqrt{2}t=0.\end{aligned} \tag{43}$$

After plugging $x''+x=0$, we get

$$\begin{aligned}&\beta_{1,tt}+2x'\beta_{1,tx}-x\beta_{1,x}+2x\xi_{1,t}+x'^2\beta_{1,xx}-2x'\xi_{1,tt}-4x'^2\xi_{1,tx}+6xx'\xi_{1,x}\\&-2x'^3\xi_{1,xx}+\beta_1+2\sqrt{2}x'\cos\sqrt{2}t-2x\sin\sqrt{2}t=0.\end{aligned} \tag{44}$$

Again, using the standard procedure of comparing coefficients of different powers of x', the obtained system of PDEs is

$$\beta_{1,tt}+\beta_1-x\beta_{1,x}+2x\xi_{1,t}-4x\sin\sqrt{2}t=0, \tag{45}$$

$$\beta_{1,tx}-\xi_{1,tt}+3x\xi_{0,x}+\sqrt{2}\cos\sqrt{2}t=0, \tag{46}$$

$$\beta_{1,xx}-4\xi_{1,tx}=0, \tag{47}$$

$$\xi_{1,xx}=0. \tag{48}$$

Solving the above system yields

$$\xi_1=C_1+\sin\sqrt{2}tC_2+\cos\sqrt{2}tC_3-\frac{1}{2\sqrt{2}}\cos\sqrt{2}t-\frac{1}{2}t\sin\sqrt{2}t, \tag{49}$$

$$\begin{aligned}\beta_1=&-\sqrt{2}x\sin\sqrt{2}tC_3+\sqrt{2}x\cos\sqrt{2}tC_2+\cos tC_4+\sin tC_5+xC_6\\&-\frac{1}{\sqrt{2}}tx\cos\sqrt{2t}-x\sin\sqrt{2}t.\end{aligned} \tag{50}$$

Substituting any constant equal to one, say, $C_2=1$, and all the remaining constants are equal to zero, gives $\mathbf{X}_0^3$, and $\mathbf{X}_1^3$ is given below

$$\mathbf{X}_1^3=\left(-\frac{1}{2\sqrt{2}}\cos\sqrt{2}t-\frac{1}{2}t\sin\sqrt{2}t\right)\frac{\partial}{\partial t}+\epsilon\left(-\frac{1}{\sqrt{2}}tx\cos\sqrt{2t}-x\sin\sqrt{2}t\right)\frac{\partial}{\partial x}. \tag{51}$$

The nontrivial approximate Mei symmetry of Equation (13) has the form

$$\begin{aligned}\mathbf{X}^3=\mathbf{X}_0^3+\epsilon\mathbf{X}_1^3=&\left(\sin\sqrt{2}t\frac{\partial}{\partial t}+\sqrt{2}x\cos\sqrt{2}t\frac{\partial}{\partial x}\right)+\epsilon\left[\left(-\frac{1}{2\sqrt{2}}\cos\sqrt{2}t-\frac{1}{2}t\sin\sqrt{2}t\right)\frac{\partial}{\partial t}\right.\\&\left.+\left(-\frac{1}{\sqrt{2}}tx\cos\sqrt{2t}-x\sin\sqrt{2}t\right)\frac{\partial}{\partial x}\right].\end{aligned} \tag{52}$$

The remaining approximate Mei symmetries are obtained in a similar way as described above. The list of symmetries is given as

$$\mathbf{X}^1 = \mathbf{X}_0^1 + \epsilon \mathbf{X}_1^1 = \frac{\partial}{\partial t} - \epsilon x \frac{\partial}{\partial t}, \tag{53}$$

$$\mathbf{X}^2 = \mathbf{X}_0^2 + \epsilon \mathbf{X}_1^2 = \left(\cos \sqrt{2}t \frac{\partial}{\partial t} - \sqrt{2} x \sin \sqrt{2}t \frac{\partial}{\partial x} \right) + \epsilon \left[\left(-\frac{1}{2} t \cos \sqrt{2}t + \frac{1}{\sqrt{2}} \sin \sqrt{2}t \right) \frac{\partial}{\partial t} \right. \tag{54}$$

$$\left. + \left(\frac{1}{\sqrt{2}} t x \sin \sqrt{2t} - \frac{1}{2} x \cos \sqrt{2}t \right) \frac{\partial}{\partial x} \right],$$

$$\mathbf{X}^4 = \mathbf{X}_0^4 + \epsilon \mathbf{X}_1^4 = \sin t \frac{\partial}{\partial x} - \epsilon t \sin t \frac{\partial}{\partial x}, \tag{55}$$

$$\mathbf{X}^5 = \mathbf{X}_0^5 + \epsilon \mathbf{X}_1^5 = \cos t \frac{\partial}{\partial x} - \epsilon t \cos t \frac{\partial}{\partial x}, \tag{56}$$

$$\mathbf{X}^6 = \mathbf{X}_0^6 + \epsilon \mathbf{X}_1^6 = x \frac{\partial}{\partial x} - \epsilon 2 t x \frac{\partial}{\partial x}. \tag{57}$$

4.3. Approximate Mei Invariants of DHO

The approximate invariants $I = I_0 + \epsilon I_1$ are calculated by using Equations (16) and (17).

$$I^1 = I_0^1 + \epsilon I_1^1 = 0, \tag{58}$$

$$I^2 = I_0^2 + \epsilon I_1^2 = -3\sqrt{2} x^2 \sin \sqrt{2}t \cos \sqrt{2}t + \epsilon \tag{59}$$

$$\left(-3\sqrt{2} t x^2 \sin \sqrt{2}t \cos \sqrt{2}t - \frac{5}{2} x^2 \cos^2 \sqrt{2}t + 2x^2 \sin^2 \sqrt{2}t \right),$$

$$I^3 = I_0^3 + \epsilon I_1^3 = 3\sqrt{2} x^2 \sin \sqrt{2}t \cos \sqrt{2}t \tag{60}$$

$$+ \epsilon \left(3\sqrt{2} t x^2 \sin \sqrt{2}t \cos \sqrt{2}t - \frac{1}{2} x^2 \cos^2 \sqrt{2}t - x^2 \sin^2 \sqrt{2}t \right),$$

$$I^4 = I_0^4 + \epsilon I_1^4 = \sin t \cos t - \epsilon \sin^2 t, \tag{61}$$

$$I^5 = I_0^5 + \epsilon I_1^5 = -\sin t \cos t - \epsilon \cos^2 t, \tag{62}$$

$$I^6 = I_0^6 + \epsilon I_1^6 = 2 x x' - \epsilon(-4 t x x' - 2x^2). \tag{63}$$

4.4. Comparison between Mei Symmetries of the Hamiltonian and the Lagrangian

The Mei symmetries corresponding to the Hamiltonian and the Lagrangian of DHO are compared in Table 1. Both sets of symmetries give rise to different conserved quantities. The number of Mei symmetries corresponding to both Hamiltonian and the Lagrangian is the same. In the two sets of symmetries, $\mathbf{X}^1$ is common, approximate part of $\mathbf{X}^2$ is slightly different, whereas other Mei symmetries, i.e., $\mathbf{X}^3$, $\mathbf{X}^4$, $\mathbf{X}^5$ and $\mathbf{X}^6$ are completely different.

Table 1. Comparison between Mei Symmetries corresponding to the Hamiltonian and the Lagrangian.

Mei Symmetries of the Hamiltonian	**Mei Symmetries of the Lagrangian**
$\mathbf{X}^1 = \frac{\partial}{\partial t} - \epsilon x \frac{\partial}{\partial x}$	$\mathbf{X}^1 = \frac{\partial}{\partial t} - \epsilon x \frac{\partial}{\partial x}$
$\mathbf{X}^2 = x \frac{\partial}{\partial x} - \epsilon 4 t x \frac{\partial}{\partial x}.$	$\mathbf{X}^2 = x \frac{\partial}{\partial x} - \epsilon 2 t x \frac{\partial}{\partial x}.$
$\mathbf{X}^3 = \left(e^{\sqrt{2}t} \frac{\partial}{\partial t} + x\sqrt{2} e^{\sqrt{2}t} \frac{\partial}{\partial x} \right) + \epsilon \left(-\frac{1}{2} t e^{\sqrt{2}t} \frac{\partial}{\partial t} \right) + \left(-\frac{3}{2} x e^{\sqrt{2}t} - \frac{1}{\sqrt{2}} x t e^{\sqrt{2}t} \right) \frac{\partial}{\partial x}$	$\mathbf{X}^3 = \left(\cos \sqrt{2}t \frac{\partial}{\partial t} - \sqrt{2} x \sin \sqrt{2}t \frac{\partial}{\partial x} \right) + \epsilon \left[\left(-\frac{1}{2} t \cos \sqrt{2}t + \frac{1}{\sqrt{2}} \sin \sqrt{2}t \right) \frac{\partial}{\partial t} + \left(\frac{1}{\sqrt{2}} t x \sin \sqrt{2t} - \frac{1}{2} x \cos \sqrt{2}t \right) \frac{\partial}{\partial x} \right]$
$\mathbf{X}^4 = \left(e^{-\sqrt{2}t} \frac{\partial}{\partial t} + x\sqrt{2} e^{-\sqrt{2}t} \frac{\partial}{\partial x} \right) + \epsilon \left(-\frac{1}{2} t e^{-\sqrt{2}t} \frac{\partial}{\partial t} \right) + \left(-\frac{3}{2} x e^{\sqrt{2}t} + \frac{1}{\sqrt{2}} x t e^{\sqrt{2}t} \right) \frac{\partial}{\partial x}$	$\mathbf{X}^4 = \left(\sin \sqrt{2}t \frac{\partial}{\partial t} + \sqrt{2} x \cos \sqrt{2}t \frac{\partial}{\partial x} \right) + \epsilon \left[\left(-\frac{1}{2\sqrt{2}} \cos \sqrt{2}t - \frac{1}{2} t \sin \sqrt{2}t \right) \frac{\partial}{\partial t} + \left(-\frac{1}{\sqrt{2}} t x \cos \sqrt{2t} - x \sin \sqrt{2}t \right) \frac{\partial}{\partial x} \right]$

Table 1. *Cont.*

Mei Symmetries of the Hamiltonian	Mei Symmetries of the Lagrangian
$\mathbf{X}^5 = e^{-t}\frac{\partial}{\partial x} - \epsilon t e^{-t}\frac{\partial}{\partial x}$ $\mathbf{X}^6 = e^{t}\frac{\partial}{\partial x} - \epsilon t e^{t}\frac{\partial}{\partial x}$	$\mathbf{X}^5 = \sin t\frac{\partial}{\partial x} - \epsilon t\sin t\frac{\partial}{\partial x}$ $\mathbf{X}^6 = \cos t\frac{\partial}{\partial x} - \epsilon t\cos t\frac{\partial}{\partial x}$

5. Conclusions

In this paper, approximate Mei symmetries and invariants corresponding to the Lagrangian are formulated. First of all, the definition and criterion to develop the Mei symmetries are explained [2,7]. Then, these exact Mei symmetries are used to construct approximate Mei symmetries and invariants, which are discussed in Theorems 1 and 2. At the end, approximate Mei symmetries and invariants of Lagrangian of DHO are obtained as an example. The exact and approximate Mei symmetries and invariants of DHO corresponding to the Hamiltonian are already calculated in [14], which are different from the approximate Mei symmetries obtained by using the Lagrangian. A comparison of approximate Mei symmetries corresponding to the Lagrangian and Hamiltonian are given in Table 1. From this comparison, it is noticed that

- $\mathbf{X}^1$ is common in both, i.e., related to the Hamiltonian and the Lagrangian;
- A minor difference in approximate part of $\mathbf{X}^2$ is noted;
- Mei symmetries $\mathbf{X}^3$, $\mathbf{X}^4$, $\mathbf{X}^5$, and $\mathbf{X}^6$ of both sets are completely different from each other. These new Mei symmetries related to the Lagrangian lead to new Mei invariants of DHO;
- Approximate Mei symmetries and invariants in both formalisms (Lagrangian and Hamiltonian) are related as

$$(\mathbf{X}_0 + \epsilon\mathbf{X}_1)[(H_0 + \epsilon H_1) + (\mathcal{L}_0 + \epsilon\mathcal{L}_1)] = (\mathbf{X}_0 + \epsilon\mathbf{X}_1)p_k\dot{q}^k, \tag{64}$$

$$I = -I_0 + [\xi_0\mathbf{X}_0 + (\beta_0^k - \dot{x}^k\xi_0)\frac{\partial}{\partial\dot{x}_k}\mathbf{X}_0]p_k\dot{q}^k. \tag{65}$$

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Abbreviations

The following abbreviations are used in this manuscript:

ODEs	Ordinary differential equations
PDEs	Partial differential equations
DHO	Damped harmonic oscillator

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