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Analytic Resolving Families for Equations with Distributed Riemann–Liouville Derivatives

Vladimir E. Fedorov [1,2,*], Wei-Shih Du [3], Marko Kostić [4] and Aliya A. Abdrakhmanova [5]

[1] Department of Mathematical Analysis, Mathematics Faculty, Chelyabinsk State University, Kashirin Brothers Str. 129, 454001 Chelyabinsk, Russia
[2] Laboratory of Functional Materials, South Ural State University, Lenin Av. 76, 454080 Chelyabinsk, Russia
[3] Department of Mathematics, National Kaohsiung Normal University, Kaohsiung 82444, Taiwan; wsdu@mail.nknu.edu.tw
[4] Faculty of Technical Sciences, University of Novi Sad, Trg D. Obradovića 6, 21125 Novi Sad, Serbia; marco.s@verat.net
[5] Department of Mathematics, Ufa State Aviation Technical University, Karl Marks Str. 12, 450077 Ufa, Russia; abdrahmanova-a@mail.ru
* Correspondence: kar@csu.ru; Tel.: +7-952-514-1719

Abstract: Some new necessary and sufficient conditions for the existence of analytic resolving families of operators to the linear equation with a distributed Riemann–Liouville derivative in a Banach space are established. We study the unique solvability of a natural initial value problem with distributed fractional derivatives in the initial conditions to corresponding inhomogeneous equations. These abstract results are applied to a class of initial boundary value problems for equations with distributed derivatives in time and polynomials with respect to a self-adjoint elliptic differential operator in spatial variables.

Keywords: Riemann–Liouville derivative; distributed order equation; analytic resolving family of operators; generator of resolving family; perturbation theorem

MSC: 47D99; 34G10; 35R11; 47G20

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1. Introduction

The main goal of this work is the study of the unique solvability issues for a special initial value problem to a class of equations with a distributed Riemann–Liouville derivative. The concept of distributed derivative is firstly encountered, apparently, in the works of A.M. Nakhushev [1,2]. Equations with distributed fractional derivatives appear in various fields of investigations applied to the mathematical modelling of some real processes, when an order of a fractional derivative in a model continuously depends on the process parameters: in the kinetic theory [3], in the theory of viscoelasticity [4] and so on [5–7]. Numerical methods of solving such equations were developed in the last decades; see [8,9] and the references therein. The qualitative properties of equations with distributed fractional derivatives are investigated in the works of A.M. Nakhushev [1,2], A.V. Pskhu [10,11], S. Umarov and R. Gorenflo [12], T.M. Atanacković, Lj. Oparnica and S. Pilipović [13], A.N. Kochubei [14] and others.

Consider the distributed order equation

$$\int_b^c \omega(\alpha) D_t^\alpha z(t) d\alpha = Az(t) + g(t), \quad t \in (0, T], \tag{1}$$

with the Riemann–Liouville derivative D_t^α and with a closed linear operator A in a Banach space $\mathcal{Z}$, where $-\infty < b \le 0 \le m-1 < c \le m \in \mathbb{N}$, $\omega : (b,c) \to \mathbb{C}$, $\omega \not\equiv 0$, $T > 0$,

$g : [0, T] \to \mathcal{Z}$. The Cauchy problem for such an equation with the Gerasimov–Caputo distributed derivative was studied in the case of a bounded operator A in [15]. A special initial value problem

$$\int_{m-1-k}^{c} \omega(\alpha) D_t^{\alpha-m+k} z(0) d\alpha = z_k, \quad k = 0, 1, \dots, m-1, \tag{2}$$

for Equation (1) with the Riemann–Liouville distributed derivative was researched in [16] with a bounded operator A. Similar results for initial value problems to equations with a degenerate linear operator at the distributed derivative are also obtained in [15].

Necessary and sufficient conditions on a closed operator A for the existence of an analytic in a sector resolving operators family are obtained for homogeneous Equation (1) with the Gerasimov–Caputo distributed derivative in [17] with $c \in (0, 1]$ and in [18] with $c > 1$. In [19] analogous result was obtained for Equation (1) with a discretely distributed Gerasimov–Caputo derivative; Reference [20] is devoted to the existence issues for strongly continuous resolving operators family of the homogeneous Equation (1) with the Gerasimov–Caputo derivative. The obtained results on resolving operators families allowed, in [17–20], the research of the unique solvability of inhomogeneous Equation (1) and to investigate some properties of the equation, such as the continuity in the operator norm at zero of a resolving family, conditions for the boundedness of a generating operator A, a perturbation theorem for a class of generators A and others.

All the mentioned results were obtained for $b = 0$; here, we will consider case $b \le 0$, but this will not bring any significant changes to our reasoning.

In the second section, the statement of initial value problem (2) for Equation (1) with the Riemann–Liouville derivative is obtained and properties of functions, which arise when applying the Laplace transform to the distributed fractional derivative, are investigated. In the third section, the theorem on analytic in a sector inverse Laplace transforms is generalized to the case of functions with a power singularity at zero. A theorem on conditions for the operator A, which are necessary and sufficient for the existence of analytic in a sector resolving family of operators of homogeneous Equation (1) is proved in the fourth section. This result was applied to studying problem (1), (2) in the fifth section. The last section contains an application of obtained abstract results to a study of a class of initial boundary value problems for equations with a distributed fractional derivative in time and polynomials with respect to a self-adjoint elliptic differential operator in spatial variables.

2. Equation with Distributed Riemann–Liouville Derivative

Let $\mathcal{Z}$ be a Banach space. Denote at $\beta > 0$, $t > 0$, $h : \mathbb{R}_+ \to \mathcal{Z}$, for example, $h \in C(\mathbb{R}_+; \mathcal{Z})$, the fractional Riemann–Liouville integral is defined by

$$J_t^{\beta} h(t) := \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} h(s) ds,$$

where $\Gamma(\cdot)$ is the Euler gamma function. Let $m - 1 < \alpha \le m \in \mathbb{N}$, $D_t^m h(t)$ be the usual derivative of the m-th order of h, $D_t^{\alpha} h(t) := D_t^m J_t^{m-\alpha} h(t)$ be the Riemann–Liouville fractional derivative.

The Laplace transform of a function $h : \mathbb{R}_+ \to \mathcal{Z}$ will be denoted by $\widehat{h}$ or $\mathrm{Lap}[h]$, if an expression h is too long. By $\widehat{\mathcal{Z}}$ denote the set of functions $h : \mathbb{R}_+ \to \mathcal{Z}$, such that the Laplace transform $\widehat{h}$ is defined. The Laplace transform of the Riemann–Liouville fractional derivative of an order $\alpha > 0$ satisfies the equality (see [21]):

$$\widehat{D_t^{\alpha} h}(\lambda) = \lambda^{\alpha} \widehat{h}(\lambda) - \sum_{k=0}^{m-1} D_t^{\alpha-1-k} h(0) \lambda^k. \tag{3}$$

Here and further $D_t^\beta h(0) := \lim_{t\to 0+} D_t^\beta h(t)$.

Denote by $\mathcal{L}(\mathcal{Z})$ the Banach space of all linear continuous operators from $\mathcal{Z}$ to $\mathcal{Z}$; $\mathcal{C}l(\mathcal{Z})$ stands for the set of all linear closed operators, densely defined in $\mathcal{Z}$, acting to the space $\mathcal{Z}$. We supply the domain D_A of an operator $A \in \mathcal{C}l(\mathcal{Z})$ by the norm of its graph. Thus, we have the Banach space D_A.

Consider a distributed order equation:

$$\int_b^c \omega(\alpha) D_t^\alpha z(t) d\alpha = Az(t), \quad t > 0, \tag{4}$$

where $-\infty < b \leq 0 \leq m-1 < c \leq m \in \mathbb{N}$, $\omega : (b,c) \to \mathbb{C}$, $\omega \not\equiv 0$, $A \in Cl(\mathcal{Z})$. Note that, due to equality (3),

$$\int_b^c \omega(\alpha) \widehat{D_t^\alpha z}(\lambda) d\alpha = \int_b^c \omega(\alpha) \lambda^\alpha \widehat{z}(\lambda) d\alpha - \sum_{l=1}^{m-1} \int_{l-1}^{l} \omega(\alpha) \sum_{k=0}^{l-1} D_t^{\alpha-1-k} z(0) \lambda^k d\alpha -$$

$$- \int_{m-1}^{c} \omega(\alpha) \sum_{k=0}^{m-1} D_t^{\alpha-1-k} z(0) \lambda^k d\alpha = \int_b^c \omega(\alpha) \lambda^\alpha \widehat{z}(\lambda) d\alpha - \sum_{k=0}^{m-1} \int_k^c \omega(\alpha) D_t^{\alpha-1-k} z(0) \lambda^k d\alpha =$$

$$= \int_b^c \omega(\alpha) \lambda^\alpha \widehat{z}(\lambda) d\alpha - \sum_{k=0}^{m-1} \int_{m-1-k}^{c} \omega(\alpha) D_t^{\alpha-m+k} z(0) \lambda^{m-1-k} d\alpha. \tag{5}$$

Therefore, the natural initial value conditions for Equation (4) are:

$$\int_{m-1-k}^{c} \omega(\alpha) D_t^{\alpha-m+k} z(0) d\alpha = z_k, \quad k = 0, 1, \ldots, m-1, \tag{6}$$

where

$$\int_{m-1-k}^{c} \omega(\alpha) D_t^{\alpha-m+k} z(0) d\alpha := \lim_{t\to 0+} \int_{m-1-k}^{c} \omega(\alpha) D_t^{\alpha-m+k} z(t) d\alpha, \quad k = 0, 1, \ldots, m-1.$$

By a solution of problem (4), (6) we mean a function $z \in C(\mathbb{R}_+; D_A)$, such that there exists $\int_b^c \omega(\alpha) D_t^\alpha z(t) d\alpha \in C(\mathbb{R}_+; \mathcal{Z})$, $\lim_{t\to 0+} \int_{m-1-k}^{c} \omega(\alpha) D_t^{\alpha-m+k} z(t) d\alpha$, $k = 0, 1, \ldots, m-1$, and equalities (4) and (6) are fulfilled.

Denote $S_{\theta,a} := \{\mu \in \mathbb{C} : |\arg(\mu - a)| < \theta, \mu \neq a\}$ at $\theta \in (\pi/2, \pi]$, $a \in \mathbb{R}$,

$$W(\lambda) := \int_b^c \omega(\alpha) \lambda^\alpha d\alpha, \quad W_k(\lambda) := \int_k^c \omega(\alpha) \lambda^\alpha d\alpha, \quad k = 0, 1, \ldots, m-1.$$

The following properties of these functions are available.

Lemma 1. ([17,20]). *Let $-\infty < b \leq 0 \leq m-1 < c \leq m \in \mathbb{N}$, $\omega \in L_1(b,c)$. Then W, W_k, $k = 0, 1, \ldots, m-1$, are analytic on the set $S_{\pi,0}$.*

Lemma 2. *Let $-\infty < b \leq 0 \leq m-1 < c \leq m \in \mathbb{N}$, $\omega \in L_1(b,c)$, ω be continuous from the left at the point c, $\lim_{\alpha\to c-} \omega(\alpha) \neq 0$. Then,*

$$\exists \varepsilon_0 > 0 \; \forall \varepsilon \in (0, \varepsilon_0) \; \exists C > 0 \; \exists \varrho > 0 \; \forall \lambda \in S_{\pi,0} \setminus \{\lambda \in \mathbb{C} : |\lambda| < \varrho\} \; |W(\lambda)| \geq C|\lambda|^{c-\varepsilon}. \tag{7}$$

Proof. For $c_1 \in (b, c)$, which is close enough to c, take arbitrary $\varepsilon_1 \in (0, c - c_1)$; then for large enough $|\lambda|$, the mean value theorem implies:

$$\left|\int_{c_1}^{c} \omega(\alpha)\lambda^\alpha d\alpha\right| = |\omega(\xi)| \int_{c_1}^{c} |\lambda|^\alpha d\alpha = |\omega(\xi)| \frac{|\lambda|^c - |\lambda|^{c_1}}{\ln|\lambda|} \geq C_1|\lambda|^{c-\varepsilon_1},$$

with some $\xi \in (c_1, c)$ and $C_1 = C_1(\varepsilon_1)$. Therefore, for every $\varepsilon \in (\varepsilon_1, c - c_1)$ there exists $C = C(\varepsilon) > 0$ such that for large enough $|\lambda|$

$$\left|\int_{b}^{c} \omega(\alpha)\lambda^\alpha d\alpha\right| \geq C_1|\lambda|^{c-\varepsilon_1} - |\lambda|^{c_1} \int_{b}^{c_1} |\omega(\alpha)| d\alpha \geq C|\lambda|^{c-\varepsilon}.$$

□

Lemma 3. *Let* $-\infty < b \leq 0 \leq m - 1 < c \leq m \in \mathbb{N}$, $\omega \in L_1(b, c)$. *Then, for all* $k = 0, 1, \dots, m - 1$

$$\exists C > 0 \quad \forall \lambda \in S_{\pi,0} \setminus \{\lambda \in \mathbb{C} : |\lambda| < 1\} \quad |W(\lambda) - W_k(\lambda)| \leq C|\lambda|^k;$$

$$\forall \lambda \in S_{\pi,0} \setminus \{\lambda \in \mathbb{C} : |\lambda| < 1\} \quad |W(\lambda)| \leq \|\omega\|_{L_1(b,c)}|\lambda|^c.$$

Proof. We have at $|\lambda| \geq 1$ the evident inequalities $|W(\lambda)| \leq \|\omega\|_{L_1(b,c)}|\lambda|^c$ and

$$|W(\lambda) - W_k(\lambda)| \leq \left|\int_{b}^{k} \omega(\alpha)\lambda^\alpha d\alpha\right| \leq \int_{b}^{k} |\omega(\alpha)| d\alpha |\lambda|^k = C|\lambda|^k$$

for all $k = 0, 1, \dots, m - 1$. □

3. Analytic in a Sector Function with a Power Singularity at Zero

Let us introduce the notation $\Sigma_\psi := \{t \in \mathbb{C} : |\arg t| < \psi,\ t \neq 0\}$ for $\psi \in (0, \pi/2]$ and prove an important for further considerations assertion.

Theorem 1. *Let* $\theta_0 \in (\pi/2, \pi]$, $a \in \mathbb{R}$, $\beta \in [0, 1)$, $\mathcal{X}$ *be a Banach space, a map* $H : (a, \infty) \to \mathcal{X}$ *be set. The next assertions are equivalent.*

(i) *There exists an analytic function* $F : \Sigma_{\theta_0 - \pi/2} \to \mathcal{X}$, *for every* $\theta \in (\pi/2, \theta_0)$ *there exists such* $C(\theta) > 0$, *that for all* $t \in \Sigma_{\theta - \pi/2}$ *the inequality* $\|F(t)\|_\mathcal{X} \leq C(\theta)|t|^{-\beta}e^{a\mathrm{Re}\, t}$ *is satisfied;* $\widehat{F}(\lambda) = H(\lambda)$ *at* $\lambda > a$.

(ii) *The map* H *is analytically continued on* $S_{\theta_0,a}$; *for every* $\theta \in (\pi/2, \theta_0)$ *there exists* $K(\theta) > 0$, *such that for all* $\lambda \in S_{\theta,a}$

$$\|H(\lambda)\|_\mathcal{X} \leq \frac{K(\theta)}{|\lambda - a|^{1-\beta}}.$$

Proof. For $\beta = 0$ this statement was proved in ([22], Theorem 0.1, p. 5), ([23], Theorem 2.6.1, p. 84) directly, using properties of analytic functions and estimates for the Laplace transform and contour integrals. We will carry out arguments similar to the proof of Theorem 2.6.1 in [23], but in the case $\beta \in (0, 1)$.

Let assertion (i) hold, $\pi/2 < \theta < \theta_0 \leq \pi$, $\gamma_\pm^\delta = (0, \delta] \cup \{\delta + re^{\pm i(\theta - \pi/2)} : r \in (0, \infty)\}$. By the Cauchy theorem for all $\lambda > a$

$$\widehat{F}(\lambda) = \int_{0}^{\infty} F(t)e^{-\lambda t} dt = \int_{\gamma_\pm^\delta} F(\tau)e^{-\lambda\tau} d\tau =$$

$$= \int_0^{\delta} F(t)e^{-\lambda t}dt + e^{\pm i(\theta-\pi/2)} \int_0^{\infty} F(\delta + re^{\pm i(\theta-\pi/2)})e^{-\lambda(\delta+re^{\pm i(\theta-\pi/2)})}dr.$$

If $\delta \to 0+$, then

$$\widehat{F}(\lambda) = e^{\pm i(\theta-\pi/2)} \int_0^{\infty} F(re^{\pm i(\theta-\pi/2)})e^{-\lambda re^{\pm i(\theta-\pi/2)}}dr := H_{\pm}(\lambda),$$

since $\|F(\delta + re^{\pm i(\theta-\pi/2)})e^{-\lambda(\delta+re^{\pm i(\theta-\pi/2)})}\|_{\mathcal{X}} \leq C(\theta)C_1 r^{-\beta}e^{(a-\lambda)r\cos(\theta-\pi/2)}$ for $\delta \in [0,1]$,

$$\left\| \int_0^{\delta} F(t)e^{-\lambda t}dt \right\|_{\mathcal{X}} \leq C_1\delta^{1-\beta} \to 0 \quad \text{as} \quad \delta \to 0+.$$

Take $\varepsilon \in (0, \theta_0 - \pi/2)$ and $\lambda \in \mathbb{C}$ such that $\arg(\lambda - a) \in (-\theta + \varepsilon, \pi - \theta - \varepsilon)$; then $\arg((\lambda - a)e^{i(\theta-\pi/2)}) \in (-\pi/2 + \varepsilon, \pi/2 - \varepsilon)$, hence, $\mathrm{Re}((\lambda - a)e^{i(\theta-\pi/2)}) \geq |\lambda - a|\sin\varepsilon$, $\|F(re^{i(\theta-\pi/2)})e^{-\lambda re^{i(\theta-\pi/2)}}\|_{\mathcal{X}} \leq C(\theta)r^{-\beta}e^{-r|\lambda-a|\sin\varepsilon}$. So, the integral $H_+(\lambda)$ converges absolutely and defines an analytic function in the sector $\{\lambda \in \mathbb{C} : \arg(\lambda - a) \in (-\theta + \varepsilon, \pi - \theta - \varepsilon),\ \lambda \neq a\}$, where

$$\|H_+(\lambda)\|_{\mathcal{X}} \leq C(\theta) \int_0^{\infty} r^{-\beta}e^{-r|\lambda-a|\sin\varepsilon}dr = \frac{C(\theta)\sin^{\beta-1}\varepsilon\,\Gamma(1-\beta)}{|\lambda - a|^{1-\beta}} := \frac{K(\theta)}{|\lambda - a|^{1-\beta}}.$$

Analogously it can be shown that $H_-(\lambda)$ defines an analytic function in $\{\lambda \in \mathbb{C} : \arg(\lambda - a) \in (-\pi + \theta + \varepsilon, \theta - \varepsilon),\ \lambda \neq a\}$ with the estimate $\|H_-(\lambda)\|_{\mathcal{X}} \leq K(\theta)|\lambda - a|^{\beta-1}$. Since H_+ and H_- are extensions of $\widehat{F}$, which is defined on $(a, +\infty)$, due to the analytic continuation theorem they define an analytic function H on $S_{\theta-\varepsilon,a}$, satisfying the inequality $\|H(\lambda)\|_{\mathcal{X}} \leq K(\theta)|\lambda - a|^{\beta-1}$. Since $\theta \in (\pi/2, \theta_0)$ and $\varepsilon \in (0, \theta_0 - \pi/2)$ are arbitrary, the assertion (ii) is valid.

Assume that assertion (ii) holds. Take $\theta \in (\pi/2, \theta_0)$, $\delta > 0$ and an oriented contour $\Gamma = \Gamma_- \cup \Gamma_0 \cup \Gamma_+$, where $\Gamma_{\pm} := \{a + re^{\pm i\theta} : r \in [\delta, \infty)\}$, $\Gamma_0 := \{a + \delta e^{i\varphi} : \varphi \in (-\theta, \theta)\}$. At $\varepsilon \in (0, \theta - \pi/2)$, $t \in \Sigma_{\theta-\pi/2-\varepsilon}$, $\lambda \in \Gamma_{\pm}$

$$\mathrm{Re}(\lambda t) = a\mathrm{Re}\, t + r|t|\cos(\arg t \pm \theta) \leq a\mathrm{Re}\, t - r|t|\sin\varepsilon.$$

Therefore, $\|H(\lambda)e^{\lambda t}\|_{\mathcal{X}} \leq K(\theta)r^{\beta-1}e^{a\mathrm{Re}\, t}e^{-r|t|\sin\varepsilon}$ and the integral

$$F(t) := \frac{1}{2\pi i}\int_{\Gamma} H(\lambda)e^{\lambda t}d\lambda$$

is absolutely convergent, uniformly over compact subsets of $\Sigma_{\theta-\pi/2}$ and, consequently, defines an analytic function in the sector $\Sigma_{\theta_0-\pi/2}$.

Take $\theta \in (\pi/2, \theta_0)$, $t \in \Sigma_{\theta-\pi/2}$, $\varepsilon \in (0, \theta - \pi/2 - |\arg t|)$, $\delta = |t|^{-1}$, then

$$\left\| \frac{1}{2\pi i}\int_{\Gamma_0} H(\lambda)e^{\lambda t}d\lambda \right\|_{\mathcal{X}} \leq \frac{K(\theta)|t|^{-\beta}}{2\pi}\int_{-\theta}^{\theta} e^{a\mathrm{Re}t}e^{\cos(\arg t+\varphi)}d\varphi \leq K(\theta)|t|^{-\beta}e^{1+a\mathrm{Re}t},$$

$$\left\| \frac{1}{2\pi i}\int_{\Gamma_{\pm}} H(\lambda)e^{\lambda t}d\lambda \right\|_{\mathcal{X}} \leq \frac{K(\theta)}{2\pi}\int_{1/|t|}^{\infty} r^{\beta-1}e^{a\mathrm{Re}t}e^{-r|t|\sin\varepsilon}dr \leq \frac{K(\theta)\Gamma(\beta)e^{a\mathrm{Re}t}}{2\pi\sin^{\beta}\varepsilon|t|^{\beta}},$$

so, $\|F(t)\|_{\mathcal{X}} \leq C(\theta - \varepsilon)|t|^{-\beta}e^{a\mathrm{Re}t}$ for all $t \in \Sigma_{\theta-\varepsilon-\pi/2}$.

By the Fubini theorem and the Cauchy residue theorem we have for $\lambda > a$

$$\widehat{F}(\lambda) = \frac{1}{2\pi i}\int\limits_{\Gamma}\frac{H(\mu)d\mu}{\lambda-\mu} = H(\lambda) - \lim_{R\to\infty}\left(\int\limits_{\Gamma_R^0}\frac{H(\mu)d\mu}{\lambda-\mu} - \int\limits_{\Gamma_R^+}\frac{H(\mu)d\mu}{\lambda-\mu} - \int\limits_{\Gamma_R^-}\frac{H(\mu)d\mu}{\lambda-\mu}\right),$$

where $\Gamma_R^0 := \{a + Re^{i\varphi} : \varphi \in (-\theta,\theta)\}$, $\Gamma_R^{\pm} := \{a + re^{\pm i\theta} : r \in [R,\infty)\}$. Then,

$$\left\|\int\limits_{\Gamma_R^0}\frac{H(\mu)d\mu}{\lambda-\mu}\right\|_{\mathcal{X}} \le \int\limits_{-\theta}^{\theta}\frac{R^{\beta}K(\theta)d\varphi}{|a+Re^{i\varphi}-\lambda|} \to 0,$$

$$\left\|\int\limits_{\Gamma_R^{\pm}}\frac{H(\mu)d\mu}{\lambda-\mu}\right\|_{\mathcal{X}} \le \int\limits_{R}^{\infty}\frac{r^{\beta-1}K(\theta)dr}{|a+re^{\pm i\theta}-\lambda|} \le C_1\int\limits_{R}^{\infty} r^{\beta-2}dr \to 0,$$

as $R \to \infty$, since $\beta < 1$. Thus, $\widehat{F} \equiv H$. □

4. *k*-Resolving Families of Operators

A family of operators $\{S_l(t) \in \mathcal{L}(\mathcal{Z}) : t > 0\}$ is called *l-resolving*, $l \in \{0,1,\dots,m-1\}$, for Equation (4), if the next conditions are satisfied:

(i) $S_l(t)$ is strongly continuous at $t > 0$;
(ii) $S_l(t)[D_A] \subset D_A$, $S_l(t)Ax = AS_l(t)x$ for all $x \in D_A$, $t > 0$;
(iii) for every $z_l \in D_A$ $S_l(t)z_l$ is a solution of problem (4), (6) with $z_k = 0$, $k \in \{0,\dots,m-1\}\setminus\{l\}$

A k-resolving family of operators at $k \in \{0,1,\dots,m-1\}$ is called *analytic*, if it has the analytic continuation to a sector Σ_{ψ_0} at some $\psi_0 \in (0,\pi/2]$. An analytic k-resolving family of operators $\{S_k(t) \in \mathcal{L}(\mathcal{Z}) : t > 0\}$ has a type (ψ_0, a_0, β) at some $\psi_0 \in (0,\pi/2]$, $a_0 \in \mathbb{R}$, $\beta > 0$, if for all $\psi \in (0,\psi_0)$, $a > a_0$ there exists such $C(\psi,a)$, that for all $t \in \Sigma_{\psi}$ the inequality $\|S_k(t)\|_{\mathcal{L}(\mathcal{Z})} \le C(\psi,a)|t|^{-\beta}e^{a\mathrm{Re}\,t}$ is satisfied.

Proposition 1. *Let $\theta_0 \in (\pi/2,\pi]$, $a_0 \in \mathbb{R}$, $\beta \in [0,1)$, there exist an analytic 0-resolving family $\{S_0(t) \in \mathcal{L}(\mathcal{Z}) : t > 0\}$ of the type (θ_0, a_0, β) for Equation (4) and k-resolving families $\{S_k(t) \in \mathcal{L}(\mathcal{Z}) : t > 0\}$ for Equation (4), $k = 1,2,\dots,m-1$. Then for $k = 1,2,\dots,m-1$ the k-resolving families $\{S_k(t) \in \mathcal{L}(\mathcal{Z}) : t > 0\}$ are analytic with the type $(\theta_0, a_1, 0)$ at some $a_1 > 0$, $a_1 \ge a_0$. Moreover, for every $k \in \{0,1,\dots,m-1\}$ a k-resolving family is unique and $S_k(t) \equiv J_t^k S_0(t)$, $t > 0$.*

Proof. Take for $k \in \{1,2,\dots,m-1\}$ the family $\{S_k(t) := J_t^k S_0(t) \in \mathcal{L}(\mathcal{Z}) : t > 0\}$. Then condition (i) in the definition of a k-resolving family is satisfied. For $x \in D_A$, $t > 0$

$$J_t^k S_0(t)Ax = \int\limits_0^t \frac{(t-s)^{k-1}}{k!}S_0(s)Axds = \int\limits_0^t \frac{(t-s)^{k-1}}{k!}AS_0(s)xds = AJ_t^k S_0(t)x$$

due to the closedness of the operator A. So, condition (ii) holds.

Take $x_k \in D_A$, $S_0(t)x_k$ is a solution of problem (4), (6) with the initial values $z_0 = x_k$, $z_1 = z_2 = \dots = z_{m-1} = 0$, consequently, due to (5) $W(\lambda)\widehat{S}_0 x_k - \lambda^{m-1}x_k = A\widehat{S}_0 x_k$. Hence,

$$W(\lambda)\widehat{J_t^k S_0}x_k - \lambda^{m-1-k}x_k = W(\lambda)\lambda^{-k}\widehat{S}_0 x_k - \lambda^{m-1-k}x_k = A\lambda^{-k}\widehat{S}_0 x_k = A\widehat{J_t^k S_0}x_k.$$

Therefore, if there exists a k-resolving family for $k \in \{1,2,\dots,m-1\}$, then it coincides with $\{J_t^k S_0(t) : t > 0\}$ due to the uniqueness of the Laplace transform.

By the conditions of the theorem for every $\theta \in (\pi/2, \theta_0)$, $a > a_0$, for all $\lambda \in S_{\theta,a}$

$$\|\widehat{S}_0(\lambda)\|_{\mathcal{L}(\mathcal{Z})} = \|\lambda^{m-1}(W(\lambda)I - A)^{-1}\|_{\mathcal{L}(\mathcal{Z})} \leq \frac{K(\theta,a)}{|\lambda - a|^{1-\beta}}.$$

Consequently, for all $\theta \in (\pi/2, \theta_0)$, $a > a_1$, $\lambda \in S_{\theta,a}$

$$\|\widehat{J_t^k S_0}(\lambda)\|_{\mathcal{L}(\mathcal{Z})} = \|\lambda^{m-1-k}(W(\lambda)I - A)^{-1}\|_{\mathcal{L}(\mathcal{Z})} \leq \frac{K(\theta,a)}{|\lambda|^k|\lambda - a|^{1-\beta}} \leq \frac{K(\theta,a)(1 + a/|\lambda|)^{\beta}}{|\lambda|^{k-\beta}|\lambda - a|} \leq \frac{K_1(\theta,a)}{|\lambda - a|},$$

if we take $a_1 > 0$, $a_1 \geq a_0$. □

Remark 1. *Due to Proposition 1 further we will not write about the type of k-resolving families, $k = 1, 2, \ldots, m-1$.*

Denote by $\rho(A)$ the resolvent set of an operator A. Let an operator $A \in \mathcal{C}l(\mathcal{Z})$ satisfy the following conditions:

(1) there exist such $\theta_0 \in (\pi/2, \pi]$, $a_0 \geq 0$, that for $\lambda \in S_{\theta_0,a_0}$ we have $W(\lambda) \in \rho(A)$;
(2) there exists $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$ for every $\theta \in (\pi/2, \theta_0)$, $a > a_0$ there exists such $K(\theta, a, \varepsilon) > 0$, that for all $\lambda \in S_{\theta_0,a_0}$

$$\|(W(\lambda)I - A)^{-1}\|_{\mathcal{L}(\mathcal{Z})} \leq \frac{K(\theta,a,\varepsilon)}{|\lambda|^{m-1}|\lambda - a|^{c+1-m-\varepsilon}}.$$

Then we will say that the operator A belongs to the class $\mathcal{A}^R_{c,\varepsilon}(\theta_0, a_0)$. Here, as before, $m - 1 < c \leq m \in \mathbb{N}$, c is the upper limit of the integration in the definition of W.

If condition 2) is valid for $\varepsilon = 0$, we will denote such class as $\mathcal{A}^R_c(\theta_0, a_0)$. Obviously, $\mathcal{A}^R_c(\theta_0, a_0) \subset \mathcal{A}^R_{c,\varepsilon}(\theta_0, a_0)$.

If $A \in \mathcal{A}^R_{c,\varepsilon}(\theta_0, a_0)$, the operators,

$$Z_k(t) := \frac{1}{2\pi i}\int_{\Gamma} \lambda^{m-1-k}(W(\lambda)I - A)^{-1}e^{\lambda t}d\lambda, \quad k = 0, 1, \ldots, m-1,$$

are defined at $t > 0$. Here $\Gamma := \Gamma_+ \cup \Gamma_- \cup \Gamma_0$, $\Gamma_\pm := \{\lambda \in \mathbb{C} : \lambda = a + re^{\pm i\theta},\ r \in (\delta, \infty)\}$, $\Gamma_0 := \{\lambda \in \mathbb{C} : \lambda = a + \delta e^{i\varphi},\ \varphi \in (-\theta, \theta)\}$ for some $\delta > 0$, $a > a_0$, $\theta \in (\pi/2, \theta_0)$.

Theorem 2. *Let $-\infty < b \leq 0 \leq m-1 < c \leq m \in \mathbb{N}$, $\theta_0 \in (\pi/2, \pi]$, $a_0 \geq 0$, $\omega \in L_1(b, c)$.*

(i) *If there exists an analytic 0-resolving family of operators of the type $(\theta_0 - \pi/2, a_0, m + \varepsilon - c)$ for every $\varepsilon \in (0, \varepsilon_0)$ for Equation (4), then $A \in \mathcal{A}^R_{c,\varepsilon}(\theta_0, a_0)$.*

(ii) *If $A \in \mathcal{A}^R_{c,\varepsilon}(\theta_0, a_0)$, then there exist an analytic 0-resolving family of operators $\{S_0(t) \in \mathcal{L}(\mathcal{Z}) : t > 0\}$ of the type $(\theta_0 - \pi/2, a_0, m + \varepsilon - c)$ at every $\varepsilon \in (0, \varepsilon_0)$ and analytic k-resolving families of operators $\{S_k(t) \in \mathcal{L}(\mathcal{Z}) : t > 0\}$, $k = 1, 2, \ldots, m-1$, for Equation (4). In this case, for every $k = 0, 1, \ldots, m-1$ a k-resolving family of operators is unique, $S_k(t) \equiv Z_k(t) \equiv J_t^k Z_0(t)$, $t > 0$, and at any $z_0, z_1, \ldots, z_{m-1} \in D_A$ the function:*

$$z(t) = \sum_{k=0}^{m-1} Z_k(t) z_k$$

is a unique solution of problem (4), (6) in the space $\widehat{\mathcal{Z}}$.

Proof. Let $A \in \mathcal{A}^R_{c,\varepsilon}(\theta_0, a_0)$, $R > \delta$,

$$\Gamma_R := \bigcup_{k=1}^{4} \Gamma_{k,R}, \quad \Gamma_{1,R} := \Gamma_0, \quad \Gamma_{2,R} := \{\lambda \in \mathbb{C} : \lambda = a + Re^{i\varphi},\ \varphi \in (-\theta, \theta)\},$$

$$\Gamma_{3,R} := \{\lambda \in \mathbb{C} : \lambda = a + re^{i\theta},\ r \in [\delta, R]\}, \quad \Gamma_{4,R} := \{\lambda \in \mathbb{C} : \lambda = a + re^{-i\theta},\ r \in [\delta, R]\},$$

Γ_R is the positively oriented closed loop,

$$\Gamma_{5,R} := \{\lambda \in \mathbb{C} : \lambda = a + re^{i\theta},\ r \in [R, \infty)\}, \quad \Gamma_{6,R} := \{\lambda \in \mathbb{C} : \lambda = a + re^{-i\theta},\ r \in [R, \infty)\},$$

then $\Gamma = \Gamma_{5,R} \cup \Gamma_{6,R} \cup \Gamma_R \setminus \Gamma_{2,R}$.

For $A \in \mathcal{A}^R_{c,\varepsilon}(\theta_0, a_0)$ by Theorem 1 with $\mathcal{X} = \mathcal{L}(\mathcal{Z})$ the operator family $\{Z_0(t) \in \mathcal{L}(\mathcal{Z}) : t > 0\}$ is analytic of the type $(\theta_0 - \pi/2, a_0, m + \varepsilon - c)$ at every $\varepsilon \in (0, \varepsilon_0)$, $\|Z_0(t)\|_{\mathcal{L}(\mathcal{Z})} \le C(\theta, a, \varepsilon)|t|^{c-m-\varepsilon}e^{a\mathrm{Re}t}$ for all $\theta \in (\pi/2, \theta_0)$, $a > a_0$, $\varepsilon \in (0, \varepsilon_0)$. Then for $k = 0, 1, \ldots, m-1$ there exist the Laplace transforms at$\mathrm{Re}\lambda > a_0$ $\widehat{Z}_k(\lambda) = \lambda^{m-1-k}(W(\lambda)I - A)^{-1}$, $\mathrm{Lap}[J^{m-\alpha}_t Z_k](\lambda) = \lambda^{\alpha-1-k}(W(\lambda)I - A)^{-1}$, $\alpha < m$, therefore, $Z_k(t) = J^k_t Z_0(t)$.

For $t \in [0, 1]$, $\lambda \in \Gamma$, $\alpha \in (b, c)$,

$$\left\| e^{\lambda t}\lambda^{\alpha-1-k}(W(\lambda)I - A)^{-1} \right\|_{\mathcal{L}(\mathcal{Z})} \le \frac{e^{a+\delta}K(\theta, a)}{|\lambda|^{m-\alpha+k}|\lambda - a|^{c+1-m-\varepsilon}} \le \frac{C_1}{|\lambda|^{1+c-\alpha+k-\varepsilon}} \le \frac{C_1}{|\lambda|^{1+k-\varepsilon}}.$$

Hence, at $k = 1, 2, \ldots, m-1$ the integral $Z_k(t)$ converges uniformly on $t \in [0, 1]$ and

$$J^{m-\alpha}_t Z_k(0)z_k = \frac{1}{2\pi i}\int_\Gamma \lambda^{\alpha-1-k}(W(\lambda)I - A)^{-1}z_k d\lambda =$$

$$= \lim_{R\to\infty}\frac{1}{2\pi i}\left(\int_{\Gamma_R} - \int_{\Gamma_{2,R}} + \int_{\Gamma_{5,R}} + \int_{\Gamma_{6,R}}\right)\lambda^{\alpha-1-k}(W(\lambda)I - A)^{-1}z_k d\lambda = 0,$$

since by the Cauchy theorem

$$\int_{\Gamma_R} \lambda^{\alpha-1-k}(W(\lambda)I - A)^{-1}z_k d\lambda = 0,$$

for $t \in [0, 1]$, $\lambda \in \Gamma_{s,R}$

$$\left\| \int_{\Gamma_{s,R}} \lambda^{\alpha-1-k}(W(\lambda)I - A)^{-1}z_k d\lambda \right\|_{\mathcal{Z}} \le \frac{C_2}{R^{1-\varepsilon}}, \quad s = 2, 5, 6.$$

At the same time,

$$\int_{m-1}^{c} \omega(\alpha)J^{m-\alpha}_t Z_0(t)z_0 d\alpha = \frac{1}{2\pi i}\int_\Gamma\int_{m-1}^{c} \omega(\alpha)\lambda^{\alpha-1}d\alpha(W(\lambda)I - A)^{-1}z_0 e^{\lambda t}d\lambda =$$

$$= \frac{1}{2\pi i}\int_\Gamma \frac{W_{m-1}(\alpha) - W(\lambda)}{\lambda}(W(\lambda)I - A)^{-1}z_0 e^{\lambda t}d\lambda + \frac{1}{2\pi i}\int_\Gamma \frac{e^{\lambda t}}{\lambda}z_0 d\lambda +$$

$$+ \frac{1}{2\pi i}\int_\Gamma \frac{1}{\lambda}(W(\lambda)I - A)^{-1}Az_0 e^{\lambda t}d\lambda \to z_0,$$

as $t \to 0+$, since

$$\left\| \frac{W_{m-1}(\alpha) - W(\lambda)}{\lambda}(W(\lambda)I - A)^{-1} \right\|_{\mathcal{L}(\mathcal{Z})} \le \frac{C_1}{|\lambda|^{2+c-m-\varepsilon}}, \quad 2 + c - m - \varepsilon > 1,$$

$$\left\| \frac{1}{\lambda}(W(\lambda)I - A)^{-1}Az_0 \right\|_{\mathcal{Z}} \le \frac{C_2}{|\lambda|^{1+c-\varepsilon}}, \quad 1 + c - \varepsilon > 1.$$

We have at $c > 1$, $\alpha \in (m-1, c)$, $\lambda \in \Gamma$, $z_0 \in D_A$

$$\mathrm{Lap}[D_t^{\alpha-m+1} Z_0(t) z_0](\lambda) = \lambda^\alpha (W(\lambda)I - A)^{-1} z_0 - J_t^{m-\alpha} Z_0(0) z_0,$$

$$\int_{m-2}^{c} \omega(\alpha) D_t^{\alpha-m+1} Z_0(t) z_0 d\alpha = \frac{1}{2\pi i} \int_\Gamma \int_{m-2}^{c} \omega(\alpha)\lambda^\alpha d\alpha (W(\lambda)I - A)^{-1} e^{\lambda t} z_0 d\lambda -$$

$$-\frac{1}{2\pi i} \int_\Gamma \int_{m-1}^{c} \omega(\alpha) J_t^{m-\alpha} Z_0(0) z_0 d\alpha e^{\lambda t} d\lambda = \frac{1}{2\pi i} \int_\Gamma (W_{m-2}(\lambda) - W(\lambda))(W(\lambda)I - A)^{-1} e^{\lambda t} z_0 d\lambda +$$

$$+\frac{1}{2\pi i} \int_\Gamma e^{\lambda t} z_0 d\lambda + \frac{1}{2\pi i} \int_\Gamma (W(\lambda)I - A)^{-1} e^{\lambda t} A z_0 d\lambda - \frac{1}{2\pi i} \int_\Gamma e^{\lambda t} z_0 d\lambda,$$

$$\left\|(W_{m-2}(\lambda) - W(\lambda))(W(\lambda)I - A)^{-1}\right\|_{\mathcal{L}(\mathcal{Z})} \le \frac{C_1}{|\lambda|^{2+c-m-\varepsilon}}, \quad \left\|(W(\lambda)I - A)^{-1} A z_0\right\|_{\mathcal{Z}} \le \frac{C_2}{|\lambda|^{c-\varepsilon}},$$

hence,

$$\lim_{t\to 0+} \int_{m-2}^{c} \omega(\alpha) D_t^{\alpha-m+1} Z_0(t) z_0 d\alpha = 0.$$

At $z_1 \in D_A$, $\alpha \in (m-1, c)$, $\lambda \in \Gamma$

$$\mathrm{Lap}[D_t^{\alpha-m+1} Z_1](\lambda) = \lambda^{\alpha-1} (W(\lambda)I - A)^{-1},$$

$$\int_{m-2}^{c} \omega(\alpha) D_t^{\alpha-m+1} Z_1(t) z_1 d\alpha = \frac{1}{2\pi i} \int_\Gamma \int_{m-2}^{c} \omega(\alpha)\lambda^{\alpha-1} d\alpha (W(\lambda)I - A)^{-1} z_1 e^{\lambda t} d\lambda =$$

$$= \frac{1}{2\pi i} \int_\Gamma \frac{W_{m-2}(\lambda) - W(\lambda)}{\lambda} (W(\lambda)I - A)^{-1} z_1 e^{\lambda t} d\lambda + \frac{1}{2\pi i} \int_\Gamma \frac{e^{\lambda t} d\lambda}{\lambda} z_1 +$$

$$+\frac{1}{2\pi i} \int_\Gamma \frac{1}{\lambda} (W(\lambda)I - A)^{-1} A z_1 e^{\lambda t} d\lambda \to z_1$$

as $t \to 0+$, since

$$\left\|\frac{W_{m-2}(\lambda) - W(\lambda)}{\lambda} (W(\lambda)I - A)^{-1} z_1\right\|_{\mathcal{Z}} \le \frac{C_1}{|\lambda|^{3+c-m-\varepsilon}}, \quad \left\|\frac{1}{\lambda} (W(\lambda)I - A)^{-1} A z_1\right\|_{\mathcal{Z}} \le \frac{C_1}{|\lambda|^{1+c-\varepsilon}}.$$

For $k = 2, 3, \ldots, m-1$, $\alpha \in (m-1, c)$, $\lambda \in \Gamma$

$$\mathrm{Lap}[D_t^{\alpha-m+1} Z_k](\lambda) = \lambda^{\alpha-k} (W(\lambda)I - A)^{-1}, \quad \left\|\lambda^{\alpha-k} (W(\lambda)I - A)^{-1}\right\|_{\mathcal{L}(\mathcal{Z})} \le \frac{C_1}{|\lambda|^{k-\varepsilon}},$$

hence, $D_t^{\alpha-m+1} Z_k(0) = 0$.

Arguing as before, we obtain the equalities:

$$\int_{m-l-2}^{c} \omega(\alpha) D_t^{\alpha-m+l-1} Z_k(0) z_k d\alpha = 0, \quad k \in \{0, 1, \ldots, m-1\} \setminus \{l-1\},$$

$$\int_{m-l-2}^{c} \omega(\alpha) D_t^{\alpha-m+l-1} Z_{l-1}(0) z_{l-1} d\alpha = z_{l-1}.$$

Hence, for $c > l \in \{2, 3, \ldots, m-1\}$, $k \in \{0, 1, \ldots, l-1\}$, $\alpha \in (m-l, c)$, $\lambda \in \Gamma$, $z_k \in D_A$

$$\mathrm{Lap}[D_t^{\alpha-m+l} Z_k(t) z_k](\lambda) = \lambda^{\alpha+l-k-1} (W(\lambda)I - A)^{-1} z_k - \lambda^{l-1} D_t^{\alpha-m} Z_k(0) z_k -$$

$$-\lambda^{l-2}D_t^{\alpha-m+1}Z_k(0)z_k-\cdots-D_t^{\alpha-m+l-1}Z_k(0)z_k,$$

$$\int\limits_{m-l-1}^{c}\omega(\alpha)D_t^{\alpha-m+l}Z_k(t)z_kd\alpha=\frac{1}{2\pi i}\int\limits_{\Gamma}\int\limits_{m-l-1}^{c}\omega(\alpha)\lambda^{\alpha+l-k-1}d\alpha(W(\lambda)I-A)^{-1}z_ke^{\lambda t}d\lambda-$$

$$-\frac{1}{2\pi i}\int\limits_{\Gamma}\int\limits_{m-1}^{c}\omega(\alpha)\lambda^{l-1}D_t^{\alpha-m}Z_k(0)z_kd\alpha e^{\lambda t}d\lambda-\frac{1}{2\pi i}\int\limits_{\Gamma}\int\limits_{m-2}^{c}\omega(\alpha)\lambda^{l-2}D_t^{\alpha-m+1}Z_k(0)z_kd\alpha e^{\lambda t}d\lambda-$$

$$-\cdots-\frac{1}{2\pi i}\int\limits_{\Gamma}\int\limits_{m-l}^{c}\omega(\alpha)D_t^{\alpha-m+l-1}Z_k(0)z_kd\alpha e^{\lambda t}d\lambda=$$

$$=\frac{1}{2\pi i}\int\limits_{\Gamma}\lambda^{l-k-1}(W_{m-l-1}(\lambda)-W(\lambda))(W(\lambda)I-A)^{-1}z_ke^{\lambda t}d\lambda+$$

$$+\frac{1}{2\pi i}\int\limits_{\Gamma}\lambda^{l-k-1}z_ke^{\lambda t}d\lambda+\frac{1}{2\pi i}\int\limits_{\Gamma}\lambda^{l-k-1}(W(\lambda)I-A)^{-1}Az_ke^{\lambda t}d\lambda-\frac{1}{2\pi i}\int\limits_{\Gamma}\lambda^{l-k-1}z_ke^{\lambda t}d\lambda\to 0$$

as $t\to 0+$, since

$$\left\|\lambda^{l-k-1}(W_{m-l-1}(\lambda)-W(\lambda))(W(\lambda)I-A)^{-1}\right\|_{\mathcal{L}(\mathcal{Z})}\le\frac{C_1}{|\lambda|^{2+c-m-\varepsilon}},$$

$$\left\|\lambda^{l-k-1}(W(\lambda)I-A)^{-1}Az_0\right\|_{\mathcal{Z}}\le\frac{C_2}{|\lambda|^{2+c-m-\varepsilon}}.$$

If in these arguments $k=l$, $\alpha\in(m-k,c)$, $\lambda\in\Gamma$, $z_k\in D_A$, then,

$$\mathrm{Lap}[D_t^{\alpha-m+k}Z_k(t)z_k](\lambda)=\lambda^{\alpha-1}(W(\lambda)I-A)^{-1}z_k-\lambda^{k-1}D_t^{\alpha-m}Z_k(0)z_k-$$
$$-\lambda^{k-2}D_t^{\alpha-m+1}Z_k(0)z_k-\cdots-D_t^{\alpha-m+k-1}Z_k(0)z_k,$$

$$\int\limits_{m-k-1}^{c}\omega(\alpha)D_t^{\alpha-m+k}Z_k(t)z_kd\alpha=\frac{1}{2\pi i}\int\limits_{\Gamma}\int\limits_{m-k-1}^{c}\omega(\alpha)\lambda^{\alpha-1}d\alpha(W(\lambda)I-A)^{-1}z_ke^{\lambda t}d\lambda-$$

$$-\frac{1}{2\pi i}\int\limits_{\Gamma}\int\limits_{m-1}^{c}\omega(\alpha)\lambda^{k-1}D_t^{\alpha-m}Z_k(0)z_kd\alpha e^{\lambda t}d\lambda-\frac{1}{2\pi i}\int\limits_{\Gamma}\int\limits_{m-2}^{c}\omega(\alpha)\lambda^{k-2}D_t^{\alpha-m+1}Z_k(0)z_kd\alpha e^{\lambda t}d\lambda-$$

$$-\cdots-\frac{1}{2\pi i}\int\limits_{\Gamma}\int\limits_{m-k}^{c}\omega(\alpha)D_t^{\alpha-m+k-1}Z_k(0)z_kd\alpha e^{\lambda t}d\lambda=$$

$$=\frac{1}{2\pi i}\int\limits_{\Gamma}\frac{W_{m-k-1}(\lambda)-W(\lambda)}{\lambda}(W(\lambda)I-A)^{-1}z_ke^{\lambda t}d\lambda+\frac{1}{2\pi i}\int\limits_{\Gamma}\frac{e^{\lambda t}d\lambda}{\lambda}z_k+$$

$$+\frac{1}{2\pi i}\int\limits_{\Gamma}\frac{e^{\lambda t}}{\lambda}(W(\lambda)I-A)^{-1}Az_kd\lambda\to z_k$$

as $t\to 0+$. For $k\in\{l+1,l+2,\dots,m-1\}$, $\alpha\in(m-l,c)$, $\lambda\in\Gamma$, $z_k\in D_A$

$$\mathrm{Lap}[D_t^{\alpha-m+l}Z_k(t)z_k](\lambda)=\lambda^{\alpha+l-k-1}(W(\lambda)I-A)^{-1}z_k-\lambda^{l-1}D_t^{\alpha-m}Z_k(0)z_k-$$
$$-\lambda^{l-2}D_t^{\alpha-m+1}Z_k(0)z_k-\cdots-D_t^{\alpha-m+l-1}Z_k(0)z_k,$$

$$\int\limits_{m-l-1}^{c}\omega(\alpha)D_t^{\alpha-m+l}Z_k(0)z_kd\alpha=\frac{1}{2\pi i}\int\limits_{\Gamma}\int\limits_{m-l-1}^{c}\omega(\alpha)\lambda^{\alpha+l-k-1}d\alpha(W(\lambda)I-A)^{-1}z_kd\lambda=$$

$$= \frac{1}{2\pi i}\int_{\Gamma} \frac{W_{m-l-1}(\lambda) - W(\lambda)}{\lambda^{k+1-l}}(W(\lambda)I - A)^{-1} z_k d\lambda +$$

$$+ \frac{1}{2\pi i}\int_{\Gamma} \frac{d\lambda}{\lambda^{k+1-l}} z_k + \frac{1}{2\pi i}\int_{\Gamma} \frac{1}{\lambda^{k+1-l}}(W(\lambda)I - A)^{-1} A z_k d\lambda = 0.$$

Consequently, the function $z(t) := \sum\limits_{k=0}^{m-1} Z_k(t) z_k$ satisfies initial conditions (6). Since the operator A is closed and commutes with the operators $(W(\lambda)I - A)^{-1}$ on D_A, at $z_k \in D_A$, $k = 0, 1, \ldots, m-1$ the inclusions $AZ_k(\cdot)z_k = Z_k(\cdot)Az_k \in C(\mathbb{R}_+; \mathcal{Z})$ are fulfilled also, i. e. $z(\cdot) := \sum\limits_{k=0}^{m-1} Z_k(\cdot) z_k \in C(\mathbb{R}_+; D_A)$.

Using Formula (3) for the Laplace transform, we obtain for $\mathrm{Re}\lambda > a_0$

$$\mathrm{Lap}\left[\int_b^c \omega(\alpha) D_t^\alpha Z_k(t) z_k d\alpha\right](\lambda) = \lambda^{m-1-k} W(\lambda)(W(\lambda)I - A)^{-1} z_k - \lambda^{m-k-1} z_k =$$

$$= \lambda^{m-1-k}(W(\lambda)I - A)^{-1} A z_k = \mathrm{Lap}[AZ_k(t)z_k](\lambda).$$

We apply the inverse Laplace transform to both sides of the obtained equality and get equality (4) at all continuity points of the function $AZ_k(\cdot)z_k$, that is, for all $t > 0$. Hence, $\{Z_0(t) \in \mathcal{L}(\mathcal{Z}) : t > 0\}$ is an analytic 0-resolving family of operators of the type $(\theta_0 - \pi/2, a_0, m - c - \varepsilon)$ at every $\varepsilon \in (0, \varepsilon_0)$ for Equation (4) and $\{Z_k(t) \in \mathcal{L}(\mathcal{Z}) : t > 0\}$ are analytic k-resolving families of operators for Equation (4), $k = 1, 2, \ldots, m-1$.

Let $\theta_0 \in (\pi/2, \pi]$, $a_0 \geq 0$, $\beta \in [0, 1)$, there exists an analytic k-resolving family of operators $\{S_k(t) \in \mathcal{L}(\mathcal{Z}) : t > 0\}$, $k \in \{0, 1, \ldots, m-1\}$, of the type $(\theta_0 - \pi/2, a_0, m - c - \varepsilon)$ at every $\varepsilon \in (0, \varepsilon_0)$ at $k = 0$ and of the type $(\theta_0 - \pi/2, a_0, 0)$ at $k = 1, 2, \ldots, m-1$ for Equation (4). From Equation (4) due to condition (ii) of the k-resolving family definition we obtain at $z_k \in D_A$ equalities

$$\int_0^b \omega(\alpha) D_t^\alpha S_k(t) z_k d\alpha = AS_k(t) z_k = S_k(t) A z_k,$$

hence, due to the closedness of the operator A at $\lambda > a_0$ $\widehat{S}_k(\lambda)[D_A] \subset D_A$,

$$\mathrm{Lap}\left[\int_0^b \omega(\alpha) D_t^\alpha S_k(t) z_k d\alpha\right](\lambda) = W(\lambda)\widehat{S}_k(\lambda) z_k - \lambda^{m-1-k} z_k = A\widehat{S}_k(\lambda) z_k = \widehat{S}_k(\lambda) A z_k.$$

Therefore, the operator $W(\lambda)I - A : D_A \to \mathcal{Z}$ is bijective and $\widehat{S}_k(\lambda) = \lambda^{m-1-k}(W(\lambda)I - A)^{-1}$, $\lambda > a_0$. For $k = 0$ from Theorem 1 it follows that $A \in \mathcal{A}^R_{c,\varepsilon}(\theta_0, a_0)$; for all $k \in \{0, 1, \ldots, m-1\}$ we obtain $S_k(t) \equiv Z_k(t) \equiv J_t^k Z_0(t)$ by virtue of the uniqueness of the inverse Laplace transform.

If there exist two solutions y_1, y_2 of problem (4), (6) from the class $\widehat{\mathcal{Z}}$, then their difference $y = y_1 - y_2 \in \widehat{\mathcal{Z}}$ is a solution of Equation (4) and satisfies the initial conditions (6) with $z_k = 0$, $k = 0, 1, \ldots, m-1$. Performing the Laplace transform on both parts of Equation (4) and due to the initial conditions, we get the equality $W(\lambda)\widehat{y}(\lambda) = A\widehat{y}(\lambda)$. Since $A \in \mathcal{A}^R_{c,\varepsilon}(\theta_0, a_0)$, at $\lambda \in S_{\theta_0, a_0}$ we obtain the identity $\widehat{y}(\lambda) \equiv 0$. It means that $y \equiv 0$. Therefore, there exists a unique solution of problem (4), (6) in the space $\widehat{\mathcal{Z}}$. □

Corollary 1. *Let $-\infty < b \leq 0 \leq m-1 < c \leq m \in \mathbb{N}$, $\theta_0 \in (\pi/2, \pi]$, $a_0 \geq 0$, $\omega \in L_1(b, c)$. If there exists an analytic 0-resolving family of operators of the type $(\theta_0 - \pi/2, a_0, m + \varepsilon - c)$ at every $\varepsilon \in (0, \varepsilon_0)$ for Equation (4), there exist analytic k-resolving families of operators $\{S_k(t) \in$*

$\mathcal{L}(\mathcal{Z}) : t > 0\}$, $k = 1, 2, \ldots, m-1$, *for Equation* (4). *In this case, for every* $k = 0, 1, \ldots, m-1$ $S_k(t) \equiv J_t^k Z_0(t)$, $t > 0$.

Remark 2. *If we consider problem* (4), (6) *on a segment* $[0, T]$, *then we can continue the function* $y = y_1 - y_2$ *on* $[T, \infty)$ *by a continuous bounded way. Reasoning in the same way, we get the uniqueness of a solution on a segment not only in the space* $\widehat{\mathcal{Z}}$.

Remark 3. *In* [18] *it is shown that there exists a* 0*-resolving family of Equation* (4) *with the distributed Gerasimov–Caputo derivative, if and only if an operator* $A \in \mathcal{C}l(\mathcal{Z})$ *satisfy the next conditions:*

(1) *there exist such* $\theta_0 \in (\pi/2, \pi]$, $a_0 \geq 0$, *that for* $\lambda \in S_{\theta_0, a_0}$ *we have* $W(\lambda) \in \rho(A)$;
(2) *for every* $\theta \in (\pi/2, \theta_0)$, $a > a_0$ *there exists such* $K(\theta, a) > 0$, *that for all* $\lambda \in S_{\theta_0, a_0}$

$$\|(W(\lambda)I - A)^{-1}\|_{\mathcal{L}(\mathcal{Z})} \leq \frac{|\lambda| K(\theta, a)}{|W(\lambda)||\lambda - a|}.$$

The corresponding class of operators is denoted by $\mathcal{A}_W(\theta_0, a_0)$. *It is easy to show that, if* $\omega \in L_1(b, c)$ *and is continuous from the left at the point* c, $\lim\limits_{\alpha \to c-} \omega(\alpha) \neq 0$, *then due to Lemmas* 2 *and* 3 $\mathcal{A}_c^R(\theta_0, a_0) \subset \mathcal{A}_W(\theta_0, a_0) \subset \mathcal{A}_{c,\varepsilon}^R(\theta_0, a_0)$.

Theorem 3. *Let* $-\infty < b \leq 0 \leq m-1 < c \leq m \in \mathbb{N}$, $\theta_0 \in (\pi/2, \pi]$, $a_0 \geq 0$, $\omega \in L_1(b, c)$, W *satisfies* (7), $A \in \mathcal{A}_{c,\varepsilon}^R(\theta_0, a_0)$. *There exists a limit,*

$$\lim_{t \to 0+} \int_{m-1-k}^{c} \omega(\alpha) D_t^{\alpha - m + k} Z_k(t) d\alpha = I, \tag{8}$$

in the norm of $\mathcal{L}(\mathcal{Z})$ *at some* $k \in \{0, 1, \ldots, m-1\}$, *if and only if* $A \in \mathcal{L}(\mathcal{Z})$.

Proof. Due to the proof of Theorem 2, if there exists limit in (8), then it equals the identical operator, since it is so on D_A. Let the function

$$\eta(t) = \left\| \int_{m-1-k}^{c} \omega(\alpha) D_t^{\alpha - m + k} Z_k(t) d\alpha - I \right\|_{\mathcal{L}(\mathcal{Z})}$$

is continuous on the segment $[0, 1]$ and $\eta(0) = 0$. Therefore, the function η is bounded on $[0, 1]$. Due to the proof of Theorem 2 and Lemma 3 for all $t > 1$

$$\eta(t) = \left\| \int_{m-1-k}^{c} \omega(\alpha) \frac{1}{2\pi i} \int_{\Gamma} \lambda^{\alpha - 1} (W(\lambda)I - A)^{-1} e^{\lambda t} d\lambda d\alpha - I \right\|_{\mathcal{L}(\mathcal{Z})} =$$

$$= \left\| \frac{1}{2\pi i} \int_{\Gamma} \frac{W_{m-1-k}(\lambda)}{\lambda} (W(\lambda)I - A)^{-1} e^{\lambda t} d\lambda - I \right\|_{\mathcal{L}(\mathcal{Z})} \leq$$

$$\leq C_1 e^{at} \int_{\delta}^{\infty} r^{\varepsilon - 1} e^{tr\cos\theta} dr + C_2 e^{at} \int_{-\theta}^{\theta} e^{t\delta \cos\varphi} d\varphi + 1 \leq C_3 e^{at} t^{-\varepsilon} + C_4 e^{(a+\delta)t} + 1 \leq C_5 e^{(a+\delta)t}.$$

Take $\mathrm{Re}\lambda > a + \delta > a_0$. Then we obtain, as in the proof of Theorem 2,

$$\int_0^{\infty} e^{-\lambda t} \left(\int_{m-1-k}^{c} \omega(\alpha) D_t^{\alpha - m + k} Z_k(t) d\alpha - I \right) dt =$$

$$= \int_{m-1-k}^{c} \omega(\alpha)\lambda^{\alpha-1}(W(\lambda)I - A)^{-1}d\alpha - \frac{I}{\lambda} = \frac{W_{m-1-k}(\lambda)}{\lambda}(W(\lambda)I - A)^{-1} - \frac{I}{\lambda} =$$

$$= \frac{W_{m-1-k}(\lambda) - W(\lambda)}{\lambda}(W(\lambda)I - A)^{-1} + \frac{W(\lambda)}{\lambda}(W(\lambda)I - A)^{-1} - \frac{I}{\lambda}. \quad (9)$$

For any $\chi \in (0,1)$ take $\rho > 0$ such that $\eta(t) \le \chi$ for all $t \in [0,\rho]$. Then due to Lemma 3 and equality (9)

$$\left\| \frac{W(\lambda)}{\lambda}(W(\lambda)I - A)^{-1} - \frac{I}{\lambda} \right\|_{\mathcal{L}(\mathcal{Z})} \le \int_0^{\rho} e^{-\lambda t}\eta(t)dt + \int_{\rho}^{\infty} e^{-\lambda t}\eta(t)dt +$$

$$+ \left\| \frac{W_{m-1-k}(\lambda) - W(\lambda)}{\lambda}(W(\lambda)I - A)^{-1} \right\|_{\mathcal{L}(\mathcal{Z})} \le \frac{\chi}{\lambda} + o\left(\frac{1}{\lambda}\right)$$

as $\mathrm{Re}\lambda \to +\infty$. Consequently, for large enough $\mathrm{Re}\lambda$

$$\left\| \frac{W(\lambda)}{\lambda}(W(\lambda)I - A)^{-1} - \frac{I}{\lambda} \right\|_{\mathcal{L}(\mathcal{Z})} < 1;$$

hence, the operator $\lambda^{-1}W(\lambda)(W(\lambda)I - A)^{-1}$ is continuously invertible,

$$[\lambda^{-1}W(\lambda)(W(\lambda)I - A)^{-1}]^{-1} = \lambda W(\lambda)^{-1}(W(\lambda)I - A) \in \mathcal{L}(\mathcal{Z}).$$

Thus, $A \in \mathcal{L}(\mathcal{Z})$.

Now let $A \in \mathcal{L}(\mathcal{Z})$, then for $t > 0$

$$\int_{m-1-k}^{c} \omega(\alpha) D_t^{\alpha-m+k} Z_k(t)d\alpha = \frac{1}{2\pi i}\int_{\Gamma} \frac{W_{m-1-k}(\lambda)}{\lambda}(W(\lambda)I - A)^{-1}e^{\lambda t}d\lambda =$$

$$= \frac{1}{2\pi i}\int_{\Gamma} \frac{W_{m-1-k}(\lambda) - W(\lambda)}{\lambda}(W(\lambda)I - A)^{-1}e^{\lambda t}d\lambda + I + \frac{1}{2\pi i}\int_{\Gamma} \sum_{n=1}^{\infty} \frac{A^n e^{\lambda t}d\lambda}{\lambda W(\lambda)^n},$$

$$\left\| \frac{W_{m-1-k}(\lambda) - W(\lambda)}{\lambda}(W(\lambda)I - A)^{-1} \right\|_{\mathcal{L}(\mathcal{Z})} \le \frac{C_1}{|\lambda|^{2+c-m-\varepsilon+k}},$$

$$\left\| \sum_{n=1}^{\infty} \frac{A^n}{\lambda W(\lambda)^n} \right\|_{\mathcal{L}(\mathcal{Z})} \le \sum_{n=1}^{\infty} \frac{C\|A\|^n_{\mathcal{L}(\mathcal{Z})}}{|\lambda|^{(c-\varepsilon)n+1}} \le \frac{C\|A\|_{\mathcal{L}(\mathcal{Z})}}{|\lambda|^{c-\varepsilon+1}\left(1 - \frac{\|A\|_{\mathcal{L}(\mathcal{Z})}}{|\lambda|^{c-\varepsilon}}\right)} \le \frac{2C\|A\|_{\mathcal{L}(\mathcal{Z})}}{|\lambda|^{c-\varepsilon+1}}$$

for small enough $\varepsilon > 0$ and $|\lambda| > (2\|A\|_{\mathcal{L}(\mathcal{Z})})^{\frac{1}{c-\varepsilon}}$. Take small $t > 0$ and $R = 1/t > (2\|A\|_{\mathcal{L}(\mathcal{Z})})^{\frac{1}{c-\varepsilon}}$, then

$$\left\| \int_{\Gamma} \frac{W_{m-1-k}(\lambda) - W(\lambda)}{\lambda}(W(\lambda)I - A)^{-1}e^{\lambda t}d\lambda \right\|_{\mathcal{L}(\mathcal{Z})} =$$

$$= \left\| \left(\int_{\Gamma_R} - \int_{\Gamma_{2,R}} + \int_{\Gamma_{5,R}} + \int_{\Gamma_{6,R}} \right) \frac{W_{m-1-k}(\lambda) - W(\lambda)}{\lambda}(W(\lambda)I - A)^{-1}e^{\lambda t}d\lambda \right\|_{\mathcal{L}(\mathcal{Z})} \le$$

$$\le \sum_{s=2,5,6} \left\| \int_{\Gamma_{s,R}} \frac{W_{m-1-k}(\lambda) - W(\lambda)}{\lambda}(W(\lambda)I - A)^{-1}e^{\lambda t}d\lambda \right\|_{\mathcal{L}(\mathcal{Z})} \le \frac{C_2}{R^{1+c-m-\varepsilon}} = C_2 t^{1+c-m-\varepsilon}.$$

Analogously, we obtain:

$$\left\|\int_\Gamma \sum_{n=1}^{\infty} \frac{A^n e^{\lambda t} d\lambda}{\lambda W(\lambda)^n}\right\|_{\mathcal{L}(\mathcal{Z})} \leq \frac{C_3}{R^{c-\varepsilon}} = C_3 t^{c-\varepsilon}, \quad \left\|\int_{m-1-k}^{c} \omega(\alpha) D_t^{\alpha-m+k} Z_k(t) d\alpha - I\right\|_{\mathcal{L}(\mathcal{Z})} \leq C_4 t^{c-\varepsilon} \to 0$$

as $t \to 0+$. □

Remark 4. *Reasoning as in the proof of the previous theorem we can show that if at some* $k, l \in \{0, \dots, m-1\}$ *the family* $\{D_t^{m-\alpha+l} S_k(t) \in \mathcal{L}(\mathcal{Z}) : t > 0\}$ *is continuous at* $t = 0$ *in the norm of* $\mathcal{L}(\mathcal{Z})$*, then* $A \in \mathcal{L}(\mathcal{Z})$*.*

Remark 5. *For* $A \in \mathcal{L}(\mathcal{Z})$ *the k-resolving operators of Equation (4) have the form:*

$$Z_k(t) = \frac{1}{2\pi i} \int_\Gamma \lambda^{m-k-1} \sum_{n=0}^{\infty} \frac{A^n e^{\lambda t}}{W(\lambda)^{n+1}} d\lambda = \sum_{n=0}^{\infty} a_n(t) A^n, \quad a_n(t) = \frac{1}{2\pi i} \int_\gamma \frac{\lambda^{m-k-1} e^{\lambda t} d\lambda}{W(\lambda)^{n+1}}, \; n \in \mathbb{N}_0,$$

$\gamma = \{Re^{i\varphi} : \varphi \in (-\pi, \pi)\} \cup \{xe^{i\pi}, x \in (-R, -\infty)\} \cup \{xe^{-i\pi}, x \in (-\infty, -R)\}$ *at* $R > 0$ *large enough. For equation* $D_t^\alpha z(t) = Az(t)$ *we have* $W(\lambda) = \lambda^\alpha$*, and we obtain using the Hankel representation for the Euler gamma function that, for every* $n \in \mathbb{N}_0$*,*

$$a_n(t) = \frac{1}{2\pi i} \int_\gamma \frac{e^{\lambda t} d\lambda}{\lambda^{\alpha n+\alpha-m+k+1}} = \frac{t^{\alpha n+\alpha-m+k}}{2\pi i} \int_{t\gamma} \frac{e^{\mu} d\mu}{\mu^{\alpha n+\alpha-m+k+1}} = \frac{t^{\alpha n+\alpha-m+k}}{\Gamma(\alpha n+\alpha-m+k+1)}.$$

Thus,

$$Z_k(t) = \sum_{n=0}^{\infty} \frac{t^{\alpha n+\alpha-m+k} A^n}{\Gamma(\alpha n+\alpha-m+k+1)} = t^{\alpha-m+k} E_{\alpha,\alpha-m+k+1}(t^\alpha A),$$

where $E_{\beta,\gamma}$ *is the Mittag-Leffler function.*

5. Inhomogeneous Equation

A solution of initial problem,

$$\int_{m-1-k}^{c} \omega(\alpha) D_t^{\alpha-m+k} z(0) d\alpha = z_k, \quad k = 0, 1, \dots, m-1, \tag{10}$$

for the inhomogeneous equation

$$\int_b^c \omega(\alpha) D_t^\alpha z(t) d\alpha = Az(t) + g(t), \;\; t \in (0, T), \tag{11}$$

where $-\infty < b \leq 0 \leq m-1 < c \leq m \in \mathbb{N}$, $\omega : (b, c) \to \mathbb{C}$, $T > 0$, $g \in C([0, T]; \mathcal{Z})$, is a function $z \in C((0, T); D_A)$, such that there exist $\int_a^b \omega(\alpha) D_t^\alpha z(t) d\alpha \in C((0, T); \mathcal{Z})$, $\lim_{t\to 0+} \int_{m-1-k}^{c} \omega(\alpha) D_t^{\alpha-m+k} z(t) d\alpha$ and equalities (10) and (11) are fulfilled.

Lemma 4. *Let* $-\infty < b \leq 0 \leq m-1 < c \leq m \in \mathbb{N}$, $\omega \in L_1(b, c)$, $\theta_0 \in (\pi/2, \pi]$, $a_0 \geq 0$, $A \in \mathcal{A}_{c,\varepsilon}^R(\theta_0, a_0)$, $g \in C([0, T]; D_A)$*. Then the function:*

$$z_g(t) = \int_0^t Z_{m-1}(t-s) g(s) ds$$

is a unique solution of problem (10), (11) *with* $z_k = 0$, $k = 0, 1, \dots, m-1$.

Proof. Due to Theorem 2 and Proposition 1, $Z_{m-1}(t)$ has an analytic extension to $\Sigma_{\theta_0 - \pi/2}$ and for every $\theta \in (\pi/2, \theta_0)$, $a > a_0$ there exists $C(\theta, a)$ such that $\|Z_{m-1}(t)\|_{\mathcal{L}(\mathcal{Z})} \le C(\theta, a)e^{at}$. Define $g(t) = 0$ at $t \ge T$; then $z_g = Z_{m-1} * g$ is the convolution, $\widehat{z}_g(\lambda) = \widehat{Z}_{m-1}(\lambda)\widehat{g}(\lambda)$. In the proof of Theorem 2 it was shown that $\widehat{Z}_{m-1}(\lambda) = (W(\lambda)I - A)^{-1}$, $\mathrm{Lap}[D_t^{\alpha-m+k}Z_{m-1}](\lambda) = \lambda^{\alpha-m+k}(W(\lambda)I - A)^{-1}$, at $\alpha \in (m-1-k, c)$, $t \in (0, 1]$

$$\|D_t^{\alpha-m+k}Z_{m-1}(t)\|_{\mathcal{L}(\mathcal{Z})} \le C_1 \int_\Gamma \frac{ds}{|\lambda|^{m-k-\varepsilon}}, \quad k = 0, 1, \dots, m-2,$$

hence, $D_t^{\alpha-m+k}Z_{m-1}(0) = 0$. Therefore, $\widehat{z}_g(\lambda) = (W(\lambda)I - A)^{-1}\widehat{g}(\lambda)$, $\mathrm{Lap}[J_t^{m-\alpha}z_g](\lambda) = \lambda^{\alpha-m}(W(\lambda)I - A)^{-1}\widehat{g}(\lambda)$ at $\alpha < c$, by the mean value theorem:

$$\|J_t^{m-\alpha}z_g(t)\|_{\mathcal{Z}} \le \int_0^t \|J_t^{m-\alpha}Z_{m-1}(t-s)\|_{\mathcal{L}(\mathcal{Z})}\|g(s)\|_{\mathcal{Z}}ds = \|J_t^{m-\alpha}Z_{m-1}(t-\xi)\|_{\mathcal{L}(\mathcal{Z})}\int_0^t \|g(s)\|_{\mathcal{Z}}ds \to 0$$

as $t \to 0+$, since $\xi \in (0, t)$, $t - \xi \to 0$.

Further on, we have: $\mathrm{Lap}[D_t^{\alpha-m+1}z_g](\lambda) = \lambda^{\alpha-m+1}(W(\lambda)I - A)^{-1}\widehat{g}(\lambda)$ at $\alpha > m-1$,

$$\|D_t^{\alpha-m+1}z_g(t)\|_{\mathcal{Z}} \le \|D_t^{\alpha-m+1}Z_{m-1}(t-\xi)\|_{\mathcal{L}(\mathcal{Z})}\int_0^t \|g(s)\|_{\mathcal{Z}}ds \to 0$$

as $t \to 0+$. Reasoning in the same way we obtain $D_t^{\alpha-m+k}z_g(0) = 0$ for $\alpha > m-k$, $k = 2, 3, \dots, m-2$.

Finally,
$\mathrm{Lap}[D_t^{\alpha-1}z_g](\lambda) = \lambda^{\alpha-1}(W(\lambda)I - A)^{-1}\widehat{g}(\lambda)$ at $\alpha > 1$,

$$D_t^{\alpha-1}z_g(t) = \int_0^t D_t^{\alpha-1}Z_{m-1}(t-s)g(s)ds,$$

$$\int_0^c \omega(\alpha)D_t^{\alpha-1}z_g(t)d\alpha = \frac{1}{2\pi i}\int_0^c \omega(\alpha)\int_\Gamma \lambda^{\alpha-1}(W(\lambda)I - A)^{-1}\widehat{g}(\lambda)e^{\lambda t}d\lambda d\alpha =$$

$$= \frac{1}{2\pi i}\int_\Gamma \frac{W_0(\lambda) - W(\lambda)}{\lambda}(W(\lambda)I - A)^{-1}\widehat{g}(\lambda)e^{\lambda t}d\lambda + \frac{1}{2\pi i}\int_\Gamma \frac{W(\lambda)}{\lambda}(W(\lambda)I - A)^{-1}\widehat{g}(\lambda)e^{\lambda t}d\lambda \to 0$$

as $t \to 0+$, since

$$\|\widehat{g}(\lambda)\|_{\mathcal{Z}} \le \max_{t\in[0,T]}\|g(t)\|_{\mathcal{Z}}\int_0^T e^{-\lambda t}dt \le \frac{C_1}{|\lambda|},$$

$$\left\|\frac{W_0(\lambda) - W(\lambda)}{\lambda}(W(\lambda)I - A)^{-1}\right\|_{\mathcal{L}(\mathcal{Z})} \le \frac{C_2}{|\lambda|^{1+c-\varepsilon}}, \quad \left\|\frac{W(\lambda)}{\lambda}(W(\lambda)I - A)^{-1}\widehat{g}(\lambda)\right\|_{\mathcal{L}(\mathcal{Z})} \le \frac{C_3}{|\lambda|^{2-\varepsilon}}.$$

Thus, the function z_g satisfies initial conditions (10).

We have:

$$\mathrm{Lap}\left[\int_b^c \omega(\alpha)D_t^\alpha z_g d\alpha\right](\mu) = W(\mu)(W(\mu)I - A)^{-1}\widehat{g}(\mu) = \widehat{g}(\mu) + A(W(\mu)I - A)^{-1}\widehat{g}(\mu).$$

Applying the inverse Laplace transform on the both sides of this equality, we get

$$\int_b^c \omega(\alpha) D_t^\alpha z_g(t) d\alpha = g(t) + A(Z_{m-1} * g)(t) = g(t) + A z_g(t),$$

since $g \in C((0,T];D_A)$ and due to the closedness of A the values $A(Z_{m-1} * g)(t) = Z_{m-1} * Ag(t)$, $t \in (0,T]$, are defined.

The proof of the uniqueness of the problem solution can be found in Remark 2. □

From Theorem 2 and Lemma 4, we get the following result.

Theorem 4. *Let* $-\infty < b \le 0 \le m-1 < c \le m \in \mathbb{N}$, $\omega \in L_1(b,c)$, $\theta_0 \in (\pi/2, \pi]$, $a_0 \ge 0$, $A \in \mathcal{A}^R_{c,\varepsilon}(\theta_0, a_0)$, $g \in C([0,T];D_A)$, $z_k \in D_A$, $k = 0,1,\dots,m-1$. *Then the function*

$$z(t) = \sum_{k=0}^{m-1} Z_k(t) z_k + \int_0^t Z_{m-1}(t-s) g(s) ds$$

is a unique solution of problem (10), (11).

6. Application to a Class of Initial-Boundary Value Problems

Consider polynomials $P_n(\lambda) = \sum_{j=0}^{n} c_j \lambda^j \not\equiv 0$, $Q_n(\lambda) = \sum_{j=0}^{n} d_j \lambda^j$, where $c_j, d_j \in \mathbb{R}$, $j = 0,1,\dots,n$, $d_n \ne 0$. Let $\Omega \subset \mathbb{R}^d$ are a bounded region with a smooth boundary $\partial\Omega$, an operator pencil $\Lambda, B_1, B_2, \dots, B_p$ be regularly elliptic [24], where

$$(\Lambda u)(s) = \sum_{|q| \le 2p} a_q(s) \frac{\partial^{|q|} u(s)}{\partial s_1^{q_1} \partial s_2^{q_2} \dots \partial s_d^{q_d}}, \quad a_q \in C^\infty(\overline{\Omega}),$$

$$(B_l u)(s) = \sum_{|q| \le p_l} b_{lq}(s) \frac{\partial^{|q|} u(s)}{\partial s_1^{q_1} \partial s_2^{q_2} \dots \partial s_d^{q_d}}, \quad b_{lq} \in C^\infty(\partial\Omega),\ l = 1,2,\dots,p,$$

$q = (q_1, q_2, \dots, q_d) \in \mathbb{N}_0^d$, $|q| = q_1 + \dots + q_d$. Define an operator $\Lambda_1 \in \mathcal{C}l(L_2(\Omega))$ with a domain $D_{\Lambda_1} = H^{2p}_{\{B_l\}}(\Omega)$ [24] by the equality $\Lambda_1 u = \Lambda u$. Let the operator Λ_1 is self-adjoint, then the spectrum $\sigma(\Lambda_1)$ of the operator Λ_1 is real and discrete [24]. Suppose, moreover, $\sigma(\Lambda_1)$ is bounded from the right and does not contain the origin, $\{\varphi_k : k \in \mathbb{N}\}$ is an orthonormal in $L_2(\Omega)$ system of eigenfunctions of the operator Λ_1, numbered according to the non-increase of the corresponding eigenvalues $\{\lambda_k : k \in \mathbb{N}\}$, taking into account their multiplicities.

Consider the initial-boundary value problem:

$$\int_1^c \omega(\alpha) D_t^{\alpha-2} u(s,t) d\alpha = u_0(s), \quad \int_0^c \omega(\alpha) D_t^{\alpha-1} u(s,t) d\alpha = u_1(s), \ s \in \Omega, \tag{12}$$

$$B_l \Lambda^k u(s,t) = 0, \ \ k = 0,1,\dots,n-1, \ \ l = 1,2,\dots,p, \ \ (s,t) \in \partial\Omega \times (0,T), \tag{13}$$

$$\int_0^c \omega(\alpha) D_t^\alpha P_n(\Lambda) u(s,t) d\alpha = Q_n(\Lambda) u(s,t) + f(s,t), \ \ (s,t) \in \Omega \times (0,T), \tag{14}$$

where $1 < c < 2$, $\omega : [0,c] \to \mathbb{R}$. Denote $n_0 := \max\{j \in \{0,1,\dots,n\} : c_j \ne 0\}$,

$$\mathcal{X} = \{v \in H^{2rn_0}(\Omega) : B_l \Lambda^k v(s) = 0, k = 0,1,\dots,n_0-1, l = 1,2,\dots,p, x \in \partial\Omega\}, \tag{15}$$

$$\mathcal{Y} = L_2(\Omega), \ \ L = P_n(\Lambda), \ \ M = Q_n(\Lambda), \tag{16}$$

$$D_M = \{v \in H^{2rn}(\Omega) : B_l \Lambda^k v(s) = 0,\, k = 0, 1, \dots, n-1,\, l = 1, 2, \dots, p,\, x \in \partial\Omega\}. \quad (17)$$

Then $L \in \mathcal{L}(\mathcal{X}; \mathcal{Y})$, $M \in \mathcal{C}l(\mathcal{X}; \mathcal{Y})$ (if $n_0 = n$, that is, $c_n \neq 0$, then $M \in \mathcal{L}(\mathcal{X}; \mathcal{Y})$). Let $P_n(\lambda_k) \neq 0$ for all $k \in \mathbb{N}$, then there exists an inverse operator $L^{-1} \in \mathcal{L}(\mathcal{Y}; \mathcal{X})$ and problem (12)–(14) can be represented as problem (10), (11), where $\mathcal{Z} = \mathcal{X}$, $A = L^{-1}M \in \mathcal{C}l(\mathcal{Z})$, $D_A = D_M$, $z_0 = u_0(\cdot)$, $z_1 = u_1(\cdot)$, $g(t) = f(\cdot, t)$.

Lemma 5. *Let $c \in (0, 2)$, $\omega \in C([0, c]; \mathbb{R})$, $\omega(\alpha) \geq 0$ for $\alpha \in [0, c]$, $\omega(c) > 0$, the spectrum $\sigma(\Lambda_1)$ do not contain the origin and zeros of the polynomial $P_n(\lambda)$, and designations (15)–(17) be valid. Then $L^{-1}M \in \mathcal{A}^R_{c,\varepsilon}(\theta_0, a_0)$.*

Proof. Denote $\mu_k = Q_n(\lambda_k)/P_n(\lambda_k)$ for $k \in \mathbb{N}$. Since $(-1)^{n-n_0} d_n / c_{n_0} < 0$, then $\lim\limits_{k\to\infty} \mu_k = -\infty$ and there exists $\max\limits_{k\in\mathbb{N}} \mu_k$. Due to Lemma 1 the function $W(\lambda) := \int_0^c \omega(\alpha)\lambda^\alpha d\alpha$ is analytic on $S_{\pi,0}$. At $c \in (1, 2)$ take some $\theta_0 \in (\pi/2, \pi/c)$, then for some large enough $a_0 > 0$ and every $\lambda \in S_{\theta_0,a_0}$ we have $|W(\lambda)| \geq C(\varepsilon)|\lambda|^{c-\varepsilon} \geq \max\limits_{k\in\mathbb{N}} \mu_k$ and $W(\lambda) \in S_{c\theta_0,a_0}$, since $\omega(\alpha) \geq 0$ on $[0, c]$.

For any $\theta \in (\pi/2, \theta_0)$, $a > a_0$, $\lambda \in S_{\theta,a}$, $v \in \mathcal{X}$

$$\|(W(\lambda)I - A)^{-1}v\|^2_{\mathcal{X}} = \|(W(\lambda)I - P_n(\Lambda)^{-1}Q_n(\Lambda))^{-1}\|^2_{\mathcal{X}} = \sum_{k=1}^{\infty} \frac{(1 + \lambda_k^{2n_0})|\langle v, \varphi_k\rangle|^2}{|W(\lambda) - \frac{Q_n(\lambda_k)}{P_n(\lambda_k)}|^2} \leq$$

$$\leq C_1 \sum_{k=1}^{\infty} \frac{(1 + \lambda_k^{2n_0})|\langle v, \varphi_k\rangle|^2}{|W(\lambda)|^2 \sin^2(c\theta)} \leq \frac{C(\varepsilon)\|v\|^2_{\mathcal{X}}}{|\lambda|^{2(c-\varepsilon)}}$$

for every small enough $\varepsilon > 0$ by Lemma 2. Therefore, $L^{-1}M \in \mathcal{A}^R_{c,\varepsilon}(\theta_0, a_0)$.

Finally, when $c \in (0, 1]$, a similar arguments may be used to get the written conclusion. □

Lemma 5 and Theorem 4 implies the next unique solvability theorem.

Theorem 5. *Let $c \in (1, 2)$, $\omega \in C([0, c]; \mathbb{R})$, $\omega(\alpha) \geq 0$ for $\alpha \in [0, c]$, $\omega(c) > 0$, the spectrum $\sigma(\Lambda_1)$ do not contain the origin and zeros of the polynomial $P_n(\lambda)$, $u_0, u_1 \in D_M$, $f \in C([0, T]; D_M)$. Then there exists a unique solution of problem (12)–(14).*

Remark 6. *For $c \in (0, 1]$ instead of (12) the initial condition has the form*

$$\int_0^c \omega(\alpha) D_t^{\alpha-1} u(s, t) d\alpha = u_0(s), \ s \in \Omega.$$

Example 1. *Take $P_1(\lambda) \equiv 1$, $Q_1(\lambda) = \lambda$, $Au = \Delta u$, $p = 1$, $B_1 = I$. Then, (12)–(14) is the initial-boundary value problem for the modified equation of the ultraslow diffusion [14]*

$$\int_0^c \omega(\alpha) D_t^\alpha u(s, t) d\alpha = \Delta u(s, t), \ \ (s, t) \in \Omega \times (0, T),$$

$$u(s, t) = 0, \ \ (s, t) \in \partial\Omega \times (0, T),$$

$$\int_1^c \omega(\alpha) D_t^{\alpha-2} u(s, t) d\alpha = u_0(s), \ \int_0^c \omega(\alpha) D_t^{\alpha-1} u(s, t) d\alpha = u_1(s), \ s \in \Omega.$$

7. Conclusions

Linear differential equations in a Banach space with a distributed Riemann–Liouville derivative and with a closed operator in the right-hand side are studied. It is shown that a natural initial value problem for this equation is a problem with given values of special form distributed derivatives of a solution at initial time. A theorem on the generation of analytics in a sector resolving families of operators for such equations is proved. It gives necessary and sufficient conditions on the closed operator in the equation for the existence of the resolving family. This result allows us to study the unique solvability of the mentioned initial problem to the corresponding inhomogeneous equation. The abstract results of the work are applied to the research of the unique solvability for initial boundary value problems for a class of partial differential equations with a distributed Riemann–Liouville derivative in time.

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