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3-Derivations and 3-Automorphisms on Lie Algebras

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Abstract: In this paper, first we establish the explicit relation between 3-derivations and 3-automorphisms of a Lie algebra using the differential and exponential map. More precisely, we show that the Lie algebra of 3-derivations is the Lie algebra of the Lie group of 3-automorphisms. Then we study the derivations and automorphisms of the standard embedding Lie algebra of a Lie triple system. We prove that derivations and automorphisms of a Lie triple system give rise to derivations and automorphisms of the corresponding standard embedding Lie algebra. Finally we compute the 3-derivations and 3-automorphisms of 3-dimensional real Lie algebras.

Keywords: Lie algebra; 3-derivation; 3-automorphism; exponential map

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1. Introduction

Lie algebras were introduced to study infinitesimal transformations by Sophus Lie in the 1870s, and it is closely related to Lie groups. According to the classical Lie theory, Lie groups provide a natural model for the concept of continuous symmetry, and any Lie group gives rise to a Lie algebra (tangent space at the identity), which may be thought of as infinitesimal symmetry motions. Conversely, for any finite-dimensional Lie algebra over real or complex numbers, there is a corresponding connected Lie group unique up to finite coverings (Lie's third theorem). This correspondence allows one to study the structure and classification of Lie groups in terms of Lie algebras.

Derivations and automorphisms are important objects in the theory of Lie algebra [1]. The relationship between derivations and automorphisms are also important. For a Lie algebra $\mathfrak{g}$, the space of derivations on $\mathfrak{g}$ is a Lie algebra, which is the Lie algebra of the Lie group of automorphisms on $\mathfrak{g}$. This is an application of Lie's third theorem.

The notions of 3-derivations and 3-automorphisms on Lie algebras were introduced in [2] in the study of isometries of bi-invariant pseudo-Riemannian metrics on Lie groups. Recently 3-derivations and 3-automorphisms on different algebraic structure are widely studied, e.g., 3-derivations and 3-homomorphisms on perfect Lie algebras were studied in [3,4], 3-derivations and 3-homomorphisms on perfect Lie superalgebras were studied in [5], 3-derivations and 3-homomorphisms on von Neumann algebras were studied in [6,7], 3-derivations on TUHF algebras were studied in [8], 3-derivations on nest algebras were studied in [9,10], 3-derivations on the Lie algebra of strictly upper triangular matrix over a commutative ring were studied in [11], and 3-derivations on quaternion algebras were studied in [12].

The first purpose of this article is to find the relationship between 3-derivations and 3-automorphisms of a Lie algebra. We first show that the space of 3-derivations of a Lie algebra is a Lie algebra, and the space of 3-automorphisms is a Lie group. By using the differential and exponential map, we prove that the exponential of a 3-derivation is a 3-automorphism. Conversely, the differential at the identity of a curve going through the identity in the Lie group of 3-automorphisms is a 3-derivation. So we prove that for any Lie algebra $\mathfrak{g}$, the Lie algebra of 3-derivations is the Lie algebra of the Lie group of 3-automorphisms. We also compute 3-derivations and 3-automorphisms on eight 3-dimensional non-perfect real Lie algebras.

Lie triple systems were originate from the research of symmetric spaces [13]. Jacobson firstly studied this system algebraically and named it Lie triple system [14]. There is a close connection between Lie triple systems and Lie algebras, namely, a Lie algebra naturally gives rise to a Lie triple system and conversely a Lie triple system also gives rise to a Lie algebra which is called the standard embedding Lie algebra. Derivations and automorphisms on Lie triple systems were studied in [15,16]. In this article, we establish a relation between derivations (automorphisms) on a Lie triple system and derivations (automorphisms) on the corresponding standard embedding Lie algebra. More precisely, we construct derivations (automorphisms) on the standard embedding Lie algebra from derivations (automorphisms) on a Lie triple system.

We are the first to calculates the realizations of 3-derivations and 3-automorphisms. Furthermore, we believe that these objects are interesting when studying isometry groups of pseudo-Riemannian metrics on Lie groups and homogeneous spaces.

The paper is organized as follows. In Section 2, we construct the relationship between 3 and derivations and 3-automorphisms on Lie algebras. We reveal that 3-derivations is the Lie algebra of 3-automorphisms. In Section 3, we use derivations and automorphisms on a Lie triple system to construct derivations and automorphisms on the standard embedding Lie algebra. Therefore 3-derivations and 3-automorphisms on a Lie algebra give rise to derivations and automorphisms on the standard embedding Lie algebra of the Lie triple system which is generated by the Lie algebra. In Section 4, we compute 3-derivations and 3-automorphisms on 3-dimensional non-perfect real Lie algebras.

2. 3-Derivations and 3-Automorphisms on Lie Algebras

In this section, we study the 3-derivations and 3-automorphisms on Lie algebras, and prove that the Lie algebra of 3-derivations is the Lie algebra of the Lie group of 3-automorphisms. First, we give some basic definitions and propositions.

Definition 1 ([2])**.** *Let $(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}})$ be a Lie algebra. A linear map $D \in \mathfrak{gl}(\mathfrak{g})$ is called a* **3-derivation** *if*

$$D[[x,y]_{\mathfrak{g}},z]_{\mathfrak{g}} = [[Dx,y]_{\mathfrak{g}},z]_{\mathfrak{g}} + [[x,Dy]_{\mathfrak{g}},z]_{\mathfrak{g}} + [[x,y]_{\mathfrak{g}},Dz]_{\mathfrak{g}}, \quad \forall x,y,z \in \mathfrak{g}. \tag{1}$$

The set of all 3-derivations on $\mathfrak{g}$ is denoted by 3-Der$(\mathfrak{g})$.

Definition 2 ([2])**.** *Let $(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}})$ be a Lie algebra. An invertible linear map $\varphi \in GL(\mathfrak{g})$ is called a* **3-automorphism** *if*

$$\varphi[[x,y]_{\mathfrak{g}},z]_{\mathfrak{g}} = [[\varphi(x),\varphi(y)]_{\mathfrak{g}},\varphi(z)]_{\mathfrak{g}}, \quad \forall x,y,z \in \mathfrak{g}. \tag{2}$$

The set of all 3-automorphisms on $\mathfrak{g}$ is denoted by 3-Aut$(\mathfrak{g})$.

Obviously, 3-Aut$(\mathfrak{g})$ is a subgroup of $GL(\mathfrak{g})$, the identity element of 3-Aut$(\mathfrak{g})$ is the identity matrix.

Proposition 1. *(3-Der$(\mathfrak{g}), [\cdot,\cdot]$) is a Lie subalgebra of $(\mathfrak{gl}(\mathfrak{g}), [\cdot,\cdot])$.*

Proof. We need to prove if $D_1, D_2 \in$ 3-Der$(\mathfrak{g})$, then $[D_1, D_2] \in$ 3-Der$(\mathfrak{g})$. Let $D_1, D_2 \in$ 3-Der$(\mathfrak{g})$, $x,y,z \in \mathfrak{g}$. Then we have

$$\begin{aligned}
& [D_1,D_2]([[x,y]_{\mathfrak{g}},z]_{\mathfrak{g}}) \\
= \; & (D_1D_2 - D_2D_1)([[x,y]_{\mathfrak{g}},z]_{\mathfrak{g}}) \\
= \; & D_1([[D_2(x),y]_{\mathfrak{g}},z]_{\mathfrak{g}} + [[x,D_2(y)]_{\mathfrak{g}},z]_{\mathfrak{g}} + [[x,y]_{\mathfrak{g}},D_2(z)]_{\mathfrak{g}}) \\
& -D_2([[D_1(x),y]_{\mathfrak{g}},z]_{\mathfrak{g}} + [[x,D_1(y)]_{\mathfrak{g}},z]_{\mathfrak{g}} + [[x,y]_{\mathfrak{g}},D_1(z)]_{\mathfrak{g}}) \\
= \; & [[D_1D_2(x),y]_{\mathfrak{g}},z]_{\mathfrak{g}} + [[D_2(x),D_1(y)]_{\mathfrak{g}},z]_{\mathfrak{g}} + [[D_2(x),y]_{\mathfrak{g}},D_1(z)]_{\mathfrak{g}} \\
& +[[D_1(x),D_2(y)]_{\mathfrak{g}},z]_{\mathfrak{g}} + [[x,D_1D_2(y)]_{\mathfrak{g}},z]_{\mathfrak{g}} + [[x,D_2(y)]_{\mathfrak{g}},D_1(z)]_{\mathfrak{g}} \\
& +[[D_1(x),y]_{\mathfrak{g}},D_2(z)]_{\mathfrak{g}} + [[x,D_1(y)]_{\mathfrak{g}},D_2(z)]_{\mathfrak{g}} + [[x,y]_{\mathfrak{g}},D_1D_2(z)]_{\mathfrak{g}} \\
& -[[D_2D_1(x),y]_{\mathfrak{g}},z]_{\mathfrak{g}} - [[D_1(x),D_2(y)]_{\mathfrak{g}},z]_{\mathfrak{g}} - [[D_1(x),y]_{\mathfrak{g}},D_2(z)]_{\mathfrak{g}} \\
& -[[D_2(x),D_1(y)]_{\mathfrak{g}},z]_{\mathfrak{g}} - [[x,D_2D_1(y)]_{\mathfrak{g}},z]_{\mathfrak{g}} - [[x,D_1(y)]_{\mathfrak{g}},D_2(z)]_{\mathfrak{g}} \\
& -[[D_2(x),y]_{\mathfrak{g}},D_1(z)]_{\mathfrak{g}} - [[x,D_2(y)]_{\mathfrak{g}},D_1(z)]_{\mathfrak{g}} - [[x,y]_{\mathfrak{g}},D_2D_1(z)]_{\mathfrak{g}} \\
= \; & [[[D_1,D_2](x),y]_{\mathfrak{g}},z]_{\mathfrak{g}} + [[x,[D_1,D_2](y)]_{\mathfrak{g}},z]_{\mathfrak{g}} + [[x,y]_{\mathfrak{g}},[D_1,D_2](z)]_{\mathfrak{g}}.
\end{aligned}$$

It means that $[D_1, D_2] \in$ 3-Der$(\mathfrak{g})$. Therefore (3-Der$(\mathfrak{g}), [\cdot,\cdot]$) is a Lie subalgebra of the Lie algebra $(\mathfrak{gl}(\mathfrak{g}), [\cdot,\cdot])$. □

Lemma 1. 3-Aut$(\mathfrak{g})$ *is a closed Lie subgroup of* $GL(\mathfrak{g})$*.*

Proof. Let A be an element of the derived set of 3-Aut$(\mathfrak{g})$. Let $A_n \in$ 3-Aut$(\mathfrak{g})$ satisfying $\lim\limits_{n\to+\infty} A_n \to A$. Then we have

$$A[[x,y]_{\mathfrak{g}},z]_{\mathfrak{g}} = \lim_{n\to+\infty} A_n[[x,y]_{\mathfrak{g}},z]_{\mathfrak{g}} = \lim_{n\to+\infty}[[A_nx,A_ny]_{\mathfrak{g}},A_nz]_{\mathfrak{g}} = [[Ax,Ay]_{\mathfrak{g}},Az]_{\mathfrak{g}}.$$

Therefore, according to Cartan's theorem, 3-Aut$(\mathfrak{g})$ is a closed Lie subgroup of $GL(\mathfrak{g})$. □

Next, we study the relations between 3-derivation and 3-automorphism. We will prove that 3-Der$(\mathfrak{g})$ is the Lie algebra of 3-Aut$(\mathfrak{g})$. Because 3-Der$(\mathfrak{g})$ is the subalgebra of $\mathfrak{gl}(\mathfrak{g})$ and 3-Aut$(\mathfrak{g})$ is a closed Lie subgroup of $GL(\mathfrak{g})$, we only need to prove that 3-Der$(\mathfrak{g}) = T_I$3-Aut$(\mathfrak{g})$.

Theorem 1. *Let* $(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}})$ *be a Lie algebra. Then* 3-Der$(\mathfrak{g})$ *is the Lie algebra of* 3-Aut$(\mathfrak{g})$*.*

First, we prove that 3-Der$(\mathfrak{g}) \supset T_I$3-Aut$(\mathfrak{g})$.

Proposition 2. *Let* $\varphi(t) \in$ 3-Aut$(\mathfrak{g})$ *and* $\varphi(0) = I$*. Then* $D = \frac{d}{dt}\mid_{t=0} \varphi(t)$ *is a 3-derivation of the Lie algebra* $\mathfrak{g}$*, i.e.,*

$$\text{3-Der}(\mathfrak{g}) \supset T_I\text{3-Aut}(\mathfrak{g}).$$

Proof. Due to φ_t is a 3-automorphism, then $\varphi(t)([[x,y]_{\mathfrak{g}},z]_{\mathfrak{g}}) = [[\varphi(t)(x),\varphi(t)(y)]_{\mathfrak{g}},\varphi(t)(z)]_{\mathfrak{g}}$. So

$$\begin{aligned}
D[[x,y]_{\mathfrak{g}},z]_{\mathfrak{g}} &= \frac{d}{dt}\mid_{t=0} \varphi(t)([[x,y]_{\mathfrak{g}},z]_{\mathfrak{g}}) \\
&= \frac{d}{dt}\mid_{t=0} [[\varphi(t)(x),\varphi(t)(y)]_{\mathfrak{g}},\varphi(t)(z)]_{\mathfrak{g}} \\
&= \lim_{\Delta t\to 0}\frac{1}{\Delta t}([[\varphi(\Delta t)(x),\varphi(\Delta t)(y)]_{\mathfrak{g}},\varphi(\Delta t)(z)]_{\mathfrak{g}} - [[\varphi(0)(x),\varphi(0)(y)]_{\mathfrak{g}},\varphi(0)(z)]_{\mathfrak{g}}) \\
&= \lim_{\Delta t\to 0}\frac{1}{\Delta t}([[\varphi(\Delta t)(x),\varphi(\Delta t)(y)]_{\mathfrak{g}},\varphi(\Delta t)(z)]_{\mathfrak{g}} - [[\varphi(\Delta t)(x),\varphi(\Delta t)(y)]_{\mathfrak{g}},\varphi(0)(z)]_{\mathfrak{g}} \\
&\quad +[[\varphi(\Delta t)(x),\varphi(\Delta t)(y)]_{\mathfrak{g}},\varphi(0)(z)]_{\mathfrak{g}} - [[\varphi(\Delta t)(x),\varphi(0)(y)]_{\mathfrak{g}},\varphi(0)(z)]_{\mathfrak{g}} \\
&\quad +[[\varphi(\Delta t)(x),\varphi(0)(y)]_{\mathfrak{g}},\varphi(0)(z)]_{\mathfrak{g}} - [[\varphi(0)(x),\varphi(0)(y)]_{\mathfrak{g}},\varphi(0)(z)]_{\mathfrak{g}}) \\
&= \lim_{\Delta t\to 0}\frac{1}{\Delta t}([[\varphi(\Delta t)(x),\varphi(\Delta t)(y)]_{\mathfrak{g}},(\varphi(\Delta t)-\varphi(0))(z)]_{\mathfrak{g}} \\
&\quad +[[\varphi(\Delta t)(x),(\varphi(\Delta t)-\varphi(0))(y)]_{\mathfrak{g}},\varphi(0)(z)]_{\mathfrak{g}} \\
&\quad +[[(\varphi(\Delta t)-\varphi(0))(x),\varphi(0)(y)]_{\mathfrak{g}},\varphi(0)(z)]_{\mathfrak{g}}) \\
&= [[\frac{d}{dt}\mid_{t=0} \varphi(t)(x),\varphi(0)(y)]_{\mathfrak{g}},\varphi(0)(z)]_{\mathfrak{g}} \\
&\quad +[[\varphi(0)(x),\frac{d}{dt}\mid_{t=0} \varphi(t)(y)]_{\mathfrak{g}},\varphi(0)(z)]_{\mathfrak{g}} \\
&\quad +[[\varphi(0)(x),\varphi(0)(y)]_{\mathfrak{g}},\frac{d}{dt}\mid_{t=0} \varphi(t)(z)]_{\mathfrak{g}} \\
&= [[Dx,y]_{\mathfrak{g}},z]_{\mathfrak{g}} + [[x,Dy]_{\mathfrak{g}},z]_{\mathfrak{g}} + [[x,y]_{\mathfrak{g}},Dz]_{\mathfrak{g}}.
\end{aligned}$$

Therefore, D is a 3-derivation. □

Second, we prove that 3-Der$(\mathfrak{g}) \subset T_I$3-Aut$(\mathfrak{g})$.

Lemma 2. *For any positive integers i, j, n satisfied $i + j \leq n$, we have*

$$C_{n+1}^{i} C_{n+1-i}^{j} = C_n^i C_{n-i}^{j} + C_n^i C_{n-i}^{j-1} + C_n^{i-1} C_{n+1-i}^{j}$$

where C_n^k is the the binomial coefficient given by $C_n^k = \frac{n!}{k!(n-k)!}$.

Proof. Due to $C_{n-i}^{j} + C_{n-i}^{j-1} = C_{n-i+1}^{j}$, we have

$$\begin{aligned} C_n^i C_{n-i}^{j} + C_n^i C_{n-i}^{j-1} + C_n^{i-1} C_{n+1-i}^{j} =& C_n^i C_{n-i+1}^{j} + C_n^{i-1} C_{n+1-i}^{j} \\ =& C_{n+1}^{i} C_{n+1-i}^{j}, \end{aligned}$$

□

Lemma 3. *Let $D \in$ 3-Der$(\mathfrak{g})$. Then for any $x, y, z \in \mathfrak{g}$, we have*

$$D^n[[x,y]_{\mathfrak{g}}, z]_{\mathfrak{g}} = \sum_{i=0}^{n} \sum_{j=0}^{n-i} C_n^i C_{n-i}^{j} [[D^{n-i-j}x, D^j y]_{\mathfrak{g}}, D^i z]_{\mathfrak{g}}. \tag{3}$$

Proof. By induction on n, obviously the equation holds for $n = 1$. Assume that the equation holds for n, then for $n + 1$, we have

$$\begin{aligned} D^{n+1}[[x,y]_{\mathfrak{g}}, z]_{\mathfrak{g}} =& D \sum_{i=0}^{n} \sum_{j=0}^{n-i} C_n^i C_{n-i}^{j} [[D^{n-i-j}x, D^j y]_{\mathfrak{g}}, D^i z]_{\mathfrak{g}} \\ =& \sum_{i=0}^{n} \sum_{j=0}^{n-i} C_n^i C_{n-i}^{j} [[D^{n+1-i-j}x, D^j y]_{\mathfrak{g}}, D^i z]_{\mathfrak{g}} \\ &+ \sum_{i=0}^{n} \sum_{j=0}^{n-i} C_n^i C_{n-i}^{j} [[D^{n-i-j}x, D^{j+1} y]_{\mathfrak{g}}, D^i z]_{\mathfrak{g}} \\ &+ \sum_{i=0}^{n} \sum_{j=0}^{n-i} C_n^i C_{n-i}^{j} [[D^{n-i-j}x, D^j y]_{\mathfrak{g}}, D^{i+1} z]_{\mathfrak{g}} \\ =& \sum_{i=0}^{n} \sum_{j=0}^{n-i} C_n^i C_{n-i}^{j} [[D^{n+1-i-j}x, D^j y]_{\mathfrak{g}}, D^i z]_{\mathfrak{g}} \\ &+ \sum_{i=0}^{n} \sum_{j=1}^{n+1-i} C_n^i C_{n-i}^{j-1} [[D^{n+1-i-j}x, D^j y]_{\mathfrak{g}}, D^i z]_{\mathfrak{g}} \\ &+ \sum_{i=1}^{n+1} \sum_{j=0}^{n+1-i} C_n^{i-1} C_{n+1-i}^{j} [[D^{n+1-i-j}x, D^j y]_{\mathfrak{g}}, D^i z]_{\mathfrak{g}} \\ =& \sum_{j=0}^{n} C_n^j [[D^{n+1-j}x, D^j y]_{\mathfrak{g}}, z]_{\mathfrak{g}} + \sum_{j=1}^{n+1} C_n^{j-1} [[D^{n+1-j}x, D^j y]_{\mathfrak{g}}, z]_{\mathfrak{g}} \\ &+ \sum_{i=1}^{n} C_n^i [[D^{n+1-i}x, y]_{\mathfrak{g}}, D^i z]_{\mathfrak{g}} + \sum_{i=1}^{n+1} C_n^{i-1} [[D^{n+1-i}x, y]_{\mathfrak{g}}, D^i z]_{\mathfrak{g}} \\ &+ \sum_{i=1}^{n-1} \sum_{j=1}^{n-i} C_n^i C_{n-i}^{j} [[D^{n+1-i-j}x, D^j y]_{\mathfrak{g}}, D^i z]_{\mathfrak{g}} \\ &+ \sum_{i=1}^{n} \sum_{j=1}^{n+1-i} C_n^i C_{n-i}^{j-1} [[D^{n+1-i-j}x, D^j y]_{\mathfrak{g}}, D^i z]_{\mathfrak{g}} \\ &+ \sum_{i=1}^{n} \sum_{j=1}^{n+1-i} C_n^{i-1} C_{n+1-i}^{j} [[D^{n+1-i-j}x, D^j y]_{\mathfrak{g}}, D^i z]_{\mathfrak{g}}. \end{aligned} \tag{4}$$

Moreover, we have

$$\begin{aligned}
&\sum_{i=0}^{n+1}\sum_{j=0}^{n+1-i} C_{n+1}^{i}C_{n+1-i}^{j}[[D^{n+1-i-j}x, D^{j}y]_{\mathfrak{g}}, D^{i}z]_{\mathfrak{g}} \qquad (5)\\
&= \sum_{j=0}^{n+1} C_{n+1}^{j}[[D^{n+1-j}x, D^{j}y]_{\mathfrak{g}}, z]_{\mathfrak{g}} + \sum_{i=1}^{n+1}\sum_{j=0}^{n+1-i} C_{n+1}^{i}C_{n+1-i}^{j}[[D^{n+1-i-j}x, D^{j}y]_{\mathfrak{g}}, D^{i}z]_{\mathfrak{g}}\\
&= \sum_{j=0}^{n+1} C_{n+1}^{j}[[D^{n+1-j}x, D^{j}y]_{\mathfrak{g}}, z]_{\mathfrak{g}} + \sum_{i=1}^{n+1} C_{n+1}^{i}[[D^{n+1-i}x, y]_{\mathfrak{g}}, D^{i}z]_{\mathfrak{g}}\\
&\quad + \sum_{i=1}^{n}\sum_{j=1}^{n+1-i} C_{n+1}^{i}C_{n+1-i}^{j}[[D^{n+1-i-j}x, D^{j}y]_{\mathfrak{g}}, D^{i}z]_{\mathfrak{g}},
\end{aligned}$$

and

$$\begin{aligned}
&\sum_{j=0}^{n+1} C_{n+1}^{j}[[D^{n+1-j}x, D^{j}y]_{\mathfrak{g}}, z]_{\mathfrak{g}} \qquad (6)\\
&= \sum_{j=1}^{n}(C_{n}^{j} + C_{n}^{j-1})[[D^{n+1-j}x, D^{j}y]_{\mathfrak{g}}, z]_{\mathfrak{g}} + [[D^{n+1}x, y]_{\mathfrak{g}}, z]_{\mathfrak{g}} + [[x, D^{n+1}y]_{\mathfrak{g}}, z]_{\mathfrak{g}}\\
&= \sum_{j=0}^{n} C_{n}^{j}[[D^{n+1-j}x, D^{j}y]_{\mathfrak{g}}, z]_{\mathfrak{g}} + \sum_{j=1}^{n+1} C_{n}^{j-1}[[D^{n+1-j}x, D^{j}y]_{\mathfrak{g}}, z]_{\mathfrak{g}}.
\end{aligned}$$

Similarly,

$$\sum_{i=1}^{n+1} C_{n+1}^{i}[[D^{n+1-i}x, y]_{\mathfrak{g}}, D^{i}z]_{\mathfrak{g}} = \sum_{i=1}^{n} C_{n}^{i}[[D^{n+1-i}x, y]_{\mathfrak{g}}, D^{i}z]_{\mathfrak{g}} + \sum_{i=1}^{n+1} C_{n}^{i-1}[[D^{n+1-i}x, y]_{\mathfrak{g}}, D^{i}z]_{\mathfrak{g}}. \qquad (7)$$

By computing, we deduce that

$$\begin{aligned}
&\sum_{i=1}^{n}\sum_{j=1}^{n+1-i} C_{n}^{i}C_{n-i}^{j-1}[[D^{n+1-i-j}x, D^{j}y]_{\mathfrak{g}}, D^{i}z]_{\mathfrak{g}} \qquad (8)\\
&\quad + \sum_{i=1}^{n}\sum_{j=1}^{n+1-i} C_{n}^{i-1}C_{n+1-i}^{j}[[D^{n+1-i-j}x, D^{j}y]_{\mathfrak{g}}, D^{i}z]_{\mathfrak{g}}\\
&= \sum_{i=1}^{n-1}\sum_{j=1}^{n-i} C_{n}^{i}C_{n-i}^{j-1}[[D^{n+1-i-j}x, D^{j}y]_{\mathfrak{g}}, D^{i}z]_{\mathfrak{g}}\\
&\quad + \sum_{i=1}^{n} C_{n}^{i}C_{n-i}^{n-i}[[x, D^{n+1-i}y]_{\mathfrak{g}}, D^{i}z]_{\mathfrak{g}}\\
&\quad + \sum_{i=1}^{n-1}\sum_{j=1}^{n-i} C_{n}^{i-1}C_{n+1-i}^{j}[[D^{n+1-i-j}x, D^{j}y]_{\mathfrak{g}}, D^{i}z]_{\mathfrak{g}}\\
&\quad + \sum_{i=1}^{n} C_{n}^{i-1}C_{n+1-i}^{n+1-i}[[x, D^{n+1-i}y]_{\mathfrak{g}}, D^{i}z]_{\mathfrak{g}}\\
&= \sum_{i=1}^{n-1}\sum_{j=1}^{n-i} C_{n}^{i}C_{n-i}^{j-1}[[D^{n+1-i-j}x, D^{j}y]_{\mathfrak{g}}, D^{i}z]_{\mathfrak{g}}\\
&\quad + \sum_{i=1}^{n-1}\sum_{j=1}^{n-i} C_{n}^{i-1}C_{n+1-i}^{j}[[D^{n+1-i-j}x, D^{j}y]_{\mathfrak{g}}, D^{i}z]_{\mathfrak{g}}\\
&\quad + \sum_{i=1}^{n} C_{n+1}^{i}[[x, D^{n+1-i}y]_{\mathfrak{g}}, D^{i}z]_{\mathfrak{g}}.
\end{aligned}$$

By Lemma 2 and (6)–(8), we deduce that (3) holds for $n+1$. By induction, the equation holds for any n. $\square$

Proposition 3. *Let $D \in$ 3-Der$(\mathfrak{g})$. Then $\exp^{tD}$ is a curve in 3-Aut$(\mathfrak{g})$ and pass though the identity element, where* exp *is the exponential map, $t \in \mathbb{R}$ is parameter, i.e.,*

$$\text{3-Der}(\mathfrak{g}) \subset T_I\text{3-Aut}(\mathfrak{g}).$$

Proof. Obviously $\exp^{tD}$ is an invertible linear map and satisfies $\exp^{0D} = I$. By Lemma 3, we have

$$\begin{aligned}\exp^{tD}([[x,y]_{\mathfrak{g}},z]_{\mathfrak{g}}) &= \sum_{l=0}^{\infty}\frac{t^l}{l!}D^l[[x,y]_{\mathfrak{g}},z]_{\mathfrak{g}} \\ &= \sum_{l=0}^{\infty}\sum_{i=0}^{l}\sum_{j=0}^{l-i}\frac{t^l}{l!}C_l^iC_{l-i}^j[[D^{l-i-j}x,D^jy]_{\mathfrak{g}},D^iz]_{\mathfrak{g}} \\ &= \sum_{l=0}^{\infty}\sum_{i=0}^{l}\sum_{j=0}^{l-i}t^l[[\frac{1}{(l-i-j)!}D^{l-i-j}x,\frac{1}{j!}D^jy]_{\mathfrak{g}},\frac{1}{i!}D^iz]_{\mathfrak{g}} \\ &= \sum_{l=0}^{\infty}\sum_{i=0}^{l}\sum_{j=0}^{l-i}[[\frac{t^{l-i-j}}{(l-i-j)!}D^{l-i-j}x,\frac{t^j}{j!}D^jy]_{\mathfrak{g}},\frac{t^i}{i!}D^iz]_{\mathfrak{g}}.\end{aligned}$$

On the other hand, we have

$$\begin{aligned}[[\exp^{tD}(x),\exp^{tD}(y)]_{\mathfrak{g}},\exp^{tD}(z)]_{\mathfrak{g}} &=[[\sum_{m=0}^{\infty}\frac{t^m}{m!}D^mx,\sum_{n=0}^{\infty}\frac{t^n}{n!}D^ny]_{\mathfrak{g}},\sum_{r=0}^{\infty}\frac{t^r}{r!}D^rz]_{\mathfrak{g}} \\ &= \sum_{l=0}^{\infty}\sum_{i=0}^{l}\sum_{j=0}^{l-i}[[\frac{t^{l-i-j}}{(l-i-j)!}D^{l-i-j}x,\frac{t^j}{j!}D^jy]_{\mathfrak{g}},\frac{t^i}{i!}D^iz]_{\mathfrak{g}}.\end{aligned}$$

Then we have

$$\exp^{tD}([[x,y]_{\mathfrak{g}},z]_{\mathfrak{g}}) = [[\exp^{tD}(x),\exp^{tD}(y)]_{\mathfrak{g}},\exp^{tD}(z)]_{\mathfrak{g}}.$$

i.e., $\exp^{tD}$ is a curve in 3-Aut$(\mathfrak{g})$ passing though the identity element. $\square$

Proof of Theorem 1. By Proposition 2, we have T_I3-Aut$(\mathfrak{g}) \subset$ 3-Der$(\mathfrak{g})$. By Proposition 3, we have 3-Der$(\mathfrak{g}) \subset T_I$3-Aut$(\mathfrak{g})$. So we have

$$\text{3-Der}(\mathfrak{g}) = T_I\text{3-Aut}(\mathfrak{g}).$$

i.e., 3-Der$(\mathfrak{g})$ is the Lie algebra of 3-Aut$(\mathfrak{g})$. $\square$

3. Lie Triple Systems and the Standard Embedding Lie Algebras

In this section, we construct derivations and automorphisms on the standard embedding Lie algebra by using derivations and automorphisms on the Lie triple system.

Definition 3 ([14])**.** *A **Lie triple system** is a vector space* T *with a trilinear map* $\{\cdot,\cdot,\cdot\}$: $\mathrm{T}\otimes\mathrm{T}\otimes\mathrm{T}\to\mathrm{T}$ *satisfying the identities*

$$\begin{aligned}\{x_1,x_2,x_3\}+\{x_2,x_1,x_3\} &= 0, &(9)\\ \{x_1,x_2,x_3\}+\{x_2,x_3,x_1\}+\{x_3,x_1,x_2\} &= 0, &(10)\\ \{x_1,x_2,\{x_3,x_4,x_5\}\} &= \{\{x_1,x_2,x_3\},x_4,x_5\}+\{x_3,\{x_1,x_2,x_4\},x_5\} &(11)\\ &\quad +\{x_3,x_4,\{x_1,x_2,x_5\}\}.\end{aligned}$$

for any $x_1,x_2,x_3,x_4,x_5 \in \mathrm{T}$.

Given a Lie algebra $(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}})$, it is easy to check that $\mathfrak{g}$ with the trilinear product $\{x,y,z\} := [[x,y]_{\mathfrak{g}}, z]_{\mathfrak{g}}$ is a Lie triple system. We denote this Lie triple system by $\mathrm{T}_{\mathfrak{g}}$.

Definition 4 ([17]). *Let $(\mathrm{T}, \{\cdot,\cdot,\cdot\})$ be a Lie triple system. A linear map $D \in \mathfrak{gl}(\mathrm{T})$ is called a **derivation** of T if*

$$D\{x,y,z\} = \{Dx,y,z\} + \{x,Dy,z\} + \{x,y,Dz\}, \quad \forall x,y,z \in \mathrm{T}. \tag{12}$$

The set of all derivations of T is denoted by $\mathrm{Der}(\mathrm{T})$.

For all $x,y,z \in \mathrm{T}$, define $\mathrm{ad} : \wedge^2\mathrm{T} \to \mathfrak{gl}(T)$ by $\mathrm{ad}_{x,y}(z) = \{x,y,z\}$. By the definition of Lie triple systems, $\mathrm{ad}_{x,y}$ is a derivation of the Lie triple system which is called an **inner derivation**. Denote the set of inner derivations by $\mathrm{InnDer}(\mathrm{T})$, i.e.,

$$\mathrm{InnDer}(\mathrm{T}) = \mathrm{span}\{\mathrm{ad}_{x,y} : x,y \in \mathrm{T}\}.$$

Proposition 4 ([18]). $\mathrm{InnDer}(\mathrm{T})$ *is an ideal of* $\mathrm{Der}(\mathrm{T})$, *and* (12) *can be written in the form*

$$[D, \mathrm{ad}_{x,y}] = \mathrm{ad}_{Dx,y} + \mathrm{ad}_{x,Dy}. \tag{13}$$

Definition 5 ([17]). *Let $(\mathrm{T}, \{\cdot,\cdot,\cdot\})$ be a Lie triple system. A linear map $\varphi \in GL(\mathrm{T})$ is called an **automorphism** on T if*

$$\varphi\{x,y,z\} = \{\varphi(x), \varphi(y), \varphi(z)\}, \quad \forall x,y,z \in \mathrm{T}. \tag{14}$$

The set of all automorphism of T is denoted by $\mathrm{Aut}(\mathrm{T})$.

Definition 6 ([19]). *A Lie algebra is said to be $\mathbb{Z}_2$**-graded** if $\mathfrak{g}$ is a direct sum of a pair of submodules $\mathfrak{g}_0$ and $\mathfrak{g}_1$ such that $[\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j}$ for any $i,j \in \mathbb{Z}_2 = \{0,1\}$. The decomposition $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ is called a $\mathbb{Z}_2$-grading of $\mathfrak{g}$.*

Let $\mathrm{Ste}(\mathrm{T}) = \mathrm{InnDer}(\mathrm{T}) \oplus \mathrm{T}$. Define a bilinear map $[\cdot,\cdot]_S : \mathrm{Ste}(\mathrm{T}) \otimes \mathrm{Ste}(\mathrm{T}) \to \mathrm{Ste}(\mathrm{T})$ by

$$[\mathrm{ad}_{x_1,y_1} + z_1, \mathrm{ad}_{x_2,y_2} + z_2]_S = ([\mathrm{ad}_{x_1,y_1}, \mathrm{ad}_{x_2,y_2}] + \mathrm{ad}_{z_1,z_2}) + (\mathrm{ad}_{x_1,y_1}(z_2) - \mathrm{ad}_{x_2,y_2}(z_1)). \tag{15}$$

for any $x_1,y_1,z_1,x_2,y_2,z_2 \in \mathrm{T}$.

Proposition 5 ([18]). *Let $(\mathrm{T}, \{\cdot,\cdot,\cdot\})$ be a Lie triple system. Then $(\mathrm{Ste}(\mathrm{T}), [\cdot,\cdot]_S)$ is a $\mathbb{Z}_2-$graded Lie algebra, called the standard embedding Lie algebra of the Lie triple system T.*

As we can construct a Lie triple system from a Lie algebra, and 3-derivations and 3-automorphisms of the Lie algebra correspond to derivations and automorphisms of the Lie triple system. We can also construct derivations and automorphisms of the standard embedding of Lie triple system by following method.

Derivations of the standard embedding Lie algebra of Lie triple system can be constructed as follows:

Theorem 2. *Let T be a Lie triple system, and $\mathrm{Ste}(\mathrm{T})$ be the standard embedding Lie algebra of the Lie triple system. Let $D \in \mathrm{Der}(\mathrm{T})$. Then $\mathfrak{D} = \begin{pmatrix} \mathrm{ad}_D & 0 \\ 0 & D \end{pmatrix}$ is a derivation of $\mathrm{Ste}(\mathrm{T})$.*

Proof. For all $x_1, y_1, z_1, x_2, y_2, z_2 \in \mathrm{T}$, we only need to verify $\mathfrak{D}$ satisfies the condition of derivations when it acts on $[\mathrm{ad}_{x_1,y_1}, \mathrm{ad}_{x_2,y_2}]_S, [\mathrm{ad}_{x_1,y_1}, z_1]_S, [z_1, z_2]_S$ these three items. By (15) and the Jacobi identity we have

$$\begin{aligned}\mathfrak{D}[\mathrm{ad}_{x_1,y_1}, \mathrm{ad}_{x_2,y_2}]_S =&[D, [\mathrm{ad}_{x_1,y_1}, \mathrm{ad}_{x_2,y_2}]] \\ =&[[D, \mathrm{ad}_{x_1,y_1}], \mathrm{ad}_{x_2,y_2}]] + [\mathrm{ad}_{x_1,y_1}, [D, \mathrm{ad}_{x_2,y_2}]] \\ =&[\mathfrak{D}\mathrm{ad}_{x_1,y_1}, \mathrm{ad}_{x_2,y_2}]_S + [\mathrm{ad}_{x_1,y_1}, \mathfrak{D}\mathrm{ad}_{x_2,y_2}]_S.\end{aligned}$$

By (15) and (13) we have

$$\begin{aligned}\mathfrak{D}[\mathrm{ad}_{x_1,y_1}, z_1]_S =&D\{x_1, y_1, z_1\} \\ =&\{Dx_1, y_1, z_1\} + \{x_1, Dy_1, z_1\} + \{x_1, y_1, Dz_1\} \\ =&[\mathrm{ad}_{Dx_1,y_1} + \mathrm{ad}_{x_1,Dy_1}, z_1]_S + [\mathrm{ad}_{x_1,y_1}, Dz_1]_S \\ =&[[D, \mathrm{ad}_{x_1,y_1}], z_1]_S + [\mathrm{ad}_{x_1,y_1}, Dz_1]_S \\ =&[\mathfrak{D}\mathrm{ad}_{x_1,y_1}, z_1]_S + [\mathrm{ad}_{x_1,y_1}, \mathfrak{D}z_1]_S.\end{aligned}$$

By (15) and (13) we have

$$\mathfrak{D}[z_1, z_2]_S = [D, \mathrm{ad}_{z_1,z_2}] = \mathrm{ad}_{Dx_1,y_1} + \mathrm{ad}_{x_1,Dy_1} = [\mathfrak{D}z_1, z_2]_S + [z_1, \mathfrak{D}z_2]_S.$$

Therefore $\mathfrak{D} \in \mathrm{Der}(\mathrm{Ste}(\mathrm{T}))$. □

Corollary 1. *Let $(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}})$ be a Lie algebra, $D \in 3\text{-}\mathrm{Der}(\mathfrak{g})$. Then $\mathfrak{D} = \begin{pmatrix} \mathrm{ad}_D & 0 \\ 0 & D \end{pmatrix}$ is a derivation of the standard embedding Lie algebra $\mathrm{Ste}(\mathrm{T}_{\mathfrak{g}})$ of the Lie triple system $\mathrm{T}_{\mathfrak{g}}$.*

Automorphisms of the standard embedding Lie algebra of Lie triple system can be constructed as follows:

Theorem 3. *Let T be a Lie triple system, and $\varphi \in \mathrm{Aut}(\mathrm{T})$. Then $\Phi = \begin{pmatrix} \mathrm{Ad}_\varphi & 0 \\ 0 & \varphi \end{pmatrix}$ is an automorphism of the standard embedding Lie algebra $\mathrm{Ste}(\mathrm{T})$.*

Proof. For any $x, y, z \in \mathrm{T}$ we have

$$\mathrm{Ad}_\varphi \mathrm{ad}_{x,y}(z) = \varphi \mathrm{ad}_{x,y} \varphi^{-1}(z) = \varphi\{x, y, \varphi^{-1}(z)\} = \{\varphi(x), \varphi(y), z\} = \mathrm{ad}_{\varphi(x),\varphi(y)}(z).$$

So

$$\mathrm{Ad}_\varphi \mathrm{ad}_{x,y} = \mathrm{ad}_{\varphi(x),\varphi(y)}. \tag{16}$$

Then for any $x_1, y_1, z_1, x_2, y_2, z_2 \in \mathrm{T}$, we only need to verify Φ satisfies the condition of automorphisms when it acts on $[\mathrm{ad}_{x_1,y_1}, \mathrm{ad}_{x_2,y_2}]_S, [\mathrm{ad}_{x_1,y_1}, z_1]_S, [z_1, z_2]_S$ these three items. By (13)–(16) and we have

$$\begin{aligned}\Phi[\mathrm{ad}_{x_1,y_1}, \mathrm{ad}_{x_2,y_2}]_S =&\mathrm{ad}_{\{\varphi(x_1),\varphi(y_1),\varphi(x_2)\},\varphi(y_2)} + \mathrm{ad}_{\varphi(x_2),\{\varphi(x_1),\varphi(y_1),\varphi(y_2)\}} \\ =&[\Phi\mathrm{ad}_{x_1,y_1}, \Phi\mathrm{ad}_{x_2,y_2}]_S.\end{aligned}$$

By (14)–(16) we have

$$\Phi[\mathrm{ad}_{x_1,y_1}, z_1]_S = \varphi\{x_1, y_1, z_1\} = \{\varphi(x_1), \varphi(y_1), \varphi(z_1)\} = [\Phi\mathrm{ad}_{x_1,y_1}, \Phi z_1]_S.$$

By (15) and (16) we have

$$\Phi[z_1, z_2]_S = \mathrm{Ad}_\varphi \mathrm{ad}_{z_1,z_2} = \mathrm{ad}_{\varphi(z_1),\varphi(z_2)} = [\Phi z_1, \Phi z_2]_S.$$

Therefore $\Phi \in \mathrm{Aut}(\mathrm{Ste}(\mathrm{T}))$. □

Corollary 2. *Let* $(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}})$ *be a Lie algebra,* $\varphi \in$ 3-Aut$(\mathfrak{g})$*. Then* $\Phi = \begin{pmatrix} \mathrm{Ad}_\varphi & 0 \\ 0 & \varphi \end{pmatrix}$ *is an automorphism of the standard embedding Lie algebra* $\mathrm{Ste}(\mathrm{T}_{\mathfrak{g}})$ *of the Lie triple system* $\mathrm{T}_{\mathfrak{g}}$*.*

4. 3-Derivations and 3-Automorphisms on 3-Dimensional Non-Perfect Real Lie Algebras

In this section, we study 3-derivations and 3-automorphisms on ten 3-dimensional real Lie algebras according to Mubarakzyanov's classification [20], and compare 3-derivations and 3-automorphisms with derivations and automorphisms. Derivations and automorphisms of 3-dimensional real Lie algebras were also investigated by Popovych [21,22]. Because $\mathfrak{sl}(2,\mathbb{R})$ and $\mathfrak{so}(3)$ are perfect Lie algebra, according following theorem which gives a sufficient condition of when a 3-derivation of a Lie algebra is a derivation.

Theorem 4 ([3])**.** *Let* $(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}})$ *be a Lie algebra over field* $\mathbb{K}$, $\mathrm{char}(\mathbb{K}) \neq 2$*. If* $\mathfrak{g}$ *is perfect (i.e.,* $[\mathfrak{g},\mathfrak{g}]_{\mathfrak{g}} = \mathfrak{g}$*) and has zero center, then we have* 3-Der$(\mathfrak{g}) = \mathrm{Der}(\mathfrak{g})$*.*

We know that their 3-derivations are same with derivations. So we mainly study other eight 3-dimensional non-perfect real Lie algebras.

We will give multiplication tables of eight 3-dimensional non-perfect real Lie algebras except $\mathfrak{sl}(2,\mathbb{R})$ and $\mathfrak{so}(3)$, and multiplication tables of corresponding Lie triple system and the matrix form of their 3-derivations and 3-automorphisms Table 1.

Table 1. Eight 3-dimensional non-perfect real Lie algebras and corresponding Lie triple systems.

Lie Algebra	**Lie Triple System**
$A_{2.1} \oplus A_1$ $[e_1,e_2] = e_1$	$\mathrm{T}_{A_{2.1}\oplus A_1}$ $[e_1,e_2,e_2] = e_1$
$A_{3.1}$ $[e_2,e_3] = e_1$	$\mathrm{T}_{A_{3.1}}$ $[\cdot,\cdot,\cdot] = 0$
$A_{3.2}$ $[e_1,e_3] = e_1, [e_2,e_3] = e_1 + e_2$	$\mathrm{T}_{A_{3.2}}$ $[e_1,e_3,e_3] = e_1, [e_2,e_3,e_3] = 2e_1 + e_2$
$A_{3.3}$ $[e_1,e_3] = e_1, [e_2,e_3] = e_2$	$\mathrm{T}_{A_{3.3}}$ $[e_1,e_3,e_3] = e_1, [e_2,e_3,e_3] = e_2$
$A_{3.4}^{-1}$ $[e_1,e_3] = e_1, [e_2,e_3] = -e_2$	$\mathrm{T}_{A_{3.4}^{-1}}$ $[e_1,e_3,e_3] = e_1, [e_2,e_3,e_3] = e_2$
$A_{3.4}^{a}, 0 < \lvert a \rvert < 1$ $[e_1,e_3] = e_1, [e_2,e_3] = ae_2$	$\mathrm{T}_{A_{3.4}^{a}}, 0 < \lvert a \rvert < 1$ $[e_1,e_3,e_3] = e_1, [e_2,e_3,e_3] = a^2e_2$
$A_{3.5}^{0}$ $[e_1,e_3] = -e_2, [e_2,e_3] = e_1$	$\mathrm{T}_{A_{3.5}^{0}}$ $[e_1,e_3,e_3] = -e_1, [e_2,e_3,e_3] = -e_2$
$A_{3.5}^{b}, b > 0$ $[e_1,e_3] = be_1 - e_2,$ $[e_2,e_3] = e_1 + be_2$	$\mathrm{T}_{A_{3.5}^{b}}, b > 0$ $[e_1,e_3,e_3] = (b^2-1)e_1 - 2be_2,$ $[e_2,e_3,e_3] = 2be_1 + (b^2-1)e_2$

Theorem 5. *Derivations, automorphisms, 3-derivations and 3-automorphisms of above eight 3-dimensional non-perfect real Lie algebras are given as follows:*

(1) $\boldsymbol{A_{2.1} \oplus A_1}$**:**

$$\mathrm{Der}(A_{2.1} \oplus A_1) = \left\{ \begin{pmatrix} c_{11} & c_{12} & 0 \\ 0 & 0 & 0 \\ 0 & c_{32} & c_{33} \end{pmatrix} \right\},$$

$$\mathrm{Aut}(A_{2.1} \oplus A_1) = \left\{ \begin{pmatrix} c_{11} & c_{12} & 0 \\ 0 & 1 & 0 \\ 0 & c_{32} & c_{33} \end{pmatrix} \middle| c_{11}c_{33} \neq 0 \right\},$$

$$\text{3-Der}(A_{2.1} \oplus A_1) = \left\{ \begin{pmatrix} c_{11} & c_{12} & 0 \\ 0 & 0 & 0 \\ 0 & c_{32} & c_{33} \end{pmatrix} \right\},$$

$$\text{3-Aut}(A_{2.1} \oplus A_1) = \left\{ \begin{pmatrix} c_{11} & c_{12} & 0 \\ 0 & \pm 1 & 0 \\ 0 & c_{32} & c_{33} \end{pmatrix} \middle| c_{11}c_{33} \neq 0 \right\}.$$

(2) $\boldsymbol{A_{3.1}}$**:**

$$\text{Der}(A_{3.1}) = \left\{ \begin{pmatrix} c_{22} + c_{33} & c_{12} & c_{13} \\ 0 & c_{22} & c_{23} \\ 0 & c_{32} & c_{33} \end{pmatrix} \right\},$$

$$\text{Aut}(A_{3.1}) = \left\{ \begin{pmatrix} c_{22}c_{33} - c_{23}c_{32} & c_{12} & c_{13} \\ 0 & c_{22} & c_{23} \\ 0 & c_{32} & c_{33} \end{pmatrix} \middle| c_{22}c_{33} - c_{23}c_{32} \neq 0 \right\},$$

$$\text{3-Der}(A_{3.1}) = \left\{ \begin{pmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{pmatrix} \right\},$$

$$\text{3-Aut}(A_{3.1}) = \left\{ \begin{pmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{pmatrix} \in GL(3) \right\}.$$

(3) $\boldsymbol{A_{3.2}}$**:**

$$\text{Der}(A_{3.2}) = \left\{ \begin{pmatrix} c_{22} & c_{12} & c_{13} \\ 0 & c_{22} & c_{23} \\ 0 & 0 & 0 \end{pmatrix} \right\},$$

$$\text{Aut}(A_{3.2}) = \left\{ \begin{pmatrix} c_{22} & c_{12} & c_{13} \\ 0 & c_{22} & c_{23} \\ 0 & 0 & 1 \end{pmatrix} \middle| c_{22} \neq 0 \right\},$$

$$\text{3-Der}(A_{3.2}) = \left\{ \begin{pmatrix} c_{22} & c_{12} & c_{13} \\ 0 & c_{22} & c_{23} \\ 0 & 0 & 0 \end{pmatrix} \right\},$$

$$\text{3-Aut}(A_{3.2}) = \left\{ \begin{pmatrix} c_{22} & c_{12} & c_{13} \\ 0 & c_{22} & c_{23} \\ 0 & 0 & \pm 1 \end{pmatrix} \middle| c_{22} \neq 0 \right\}.$$

(4) $\boldsymbol{A_{3.3}}$**:**

$$\text{Der}(A_{3.3}) = \left\{ \begin{pmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ 0 & 0 & 0 \end{pmatrix} \right\},$$

$$\text{Aut}(A_{3.3}) = \left\{ \begin{pmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ 0 & 0 & 1 \end{pmatrix} \middle| c_{11}c_{22} - c_{12}c_{21} \neq 0 \right\},$$

$$\text{3-Der}(A_{3.3}) = \left\{ \begin{pmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ 0 & 0 & 0 \end{pmatrix} \right\},$$

$$\text{3-Aut}(A_{3.3}) = \left\{ \begin{pmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ 0 & 0 & \pm 1 \end{pmatrix} \middle| c_{11}c_{22} - c_{12}c_{21} \neq 0 \right\}.$$

(5) $\boldsymbol{A_{3.4}^{-1}}$**:**

$$\text{Der}(A_{3.4}^{-1}) = \left\{ \begin{pmatrix} c_{11} & 0 & c_{13} \\ 0 & c_{22} & c_{23} \\ 0 & 0 & 0 \end{pmatrix} \right\},$$

$$\text{Aut}(A_{3.4}^{-1}) = \left\{ \begin{pmatrix} c_{11} & 0 & c_{13} \\ 0 & c_{22} & c_{23} \\ 0 & 0 & 1 \end{pmatrix} \middle| c_{11}c_{22} \neq 0 \right\} \bigcup \left\{ \begin{pmatrix} 0 & c_{12} & c_{13} \\ c_{21} & 0 & c_{23} \\ 0 & 0 & -1 \end{pmatrix} \middle| c_{12}c_{21} \neq 0 \right\},$$

$$\text{3-Der}(A_{3.4}^{-1}) = \left\{ \begin{pmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ 0 & 0 & 0 \end{pmatrix} \right\},$$

$$\text{3-Aut}(A_{3.4}^{-1}) = \left\{ \begin{pmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ 0 & 0 & \pm 1 \end{pmatrix} \middle| c_{11}c_{22} - c_{12}c_{21} \neq 0 \right\}.$$

(6) $\boldsymbol{A_{3.4}^a}$:

$$\text{Der}(A_{3.4}^a) = \left\{ \begin{pmatrix} c_{11} & 0 & c_{13} \\ 0 & c_{22} & c_{23} \\ 0 & 0 & 0 \end{pmatrix} \right\},$$

$$\text{Aut}(A_{3.4}^a) = \left\{ \begin{pmatrix} c_{11} & 0 & c_{13} \\ 0 & c_{22} & c_{23} \\ 0 & 0 & 1 \end{pmatrix} \middle| c_{11}c_{22} \neq 0 \right\},$$

$$\text{3-Der}(A_{3.4}^a) = \left\{ \begin{pmatrix} c_{11} & 0 & c_{13} \\ 0 & c_{22} & c_{23} \\ 0 & 0 & 0 \end{pmatrix} \right\},$$

$$\text{3-Aut}(A_{3.4}^a) = \left\{ \begin{pmatrix} c_{11} & 0 & c_{13} \\ 0 & c_{22} & c_{23} \\ 0 & 0 & \pm 1 \end{pmatrix} \middle| c_{11}c_{22} \neq 0 \right\}.$$

(7) $\boldsymbol{A_{3.5}^0}$:

$$\text{Der}(A_{3.5}^0) = \left\{ \begin{pmatrix} c_{11} & -c_{12} & c_{13} \\ c_{12} & c_{11} & c_{23} \\ 0 & 0 & 0 \end{pmatrix} \right\},$$

$$\text{Aut}(A_{3.5}^0) = \left\{ \begin{pmatrix} c_{11} & -c_{12} & c_{13} \\ c_{12} & c_{11} & c_{23} \\ 0 & 0 & 1 \end{pmatrix} \middle| c_{11}^2 + c_{22}^2 \neq 0 \right\} \bigcup \left\{ \begin{pmatrix} -c_{11} & c_{12} & c_{13} \\ c_{12} & c_{11} & c_{23} \\ 0 & 0 & -1 \end{pmatrix} \middle| c_{11}^2 + c_{22}^2 \neq 0 \right\},$$

$$\text{3-Der}(A_{3.5}^0) = \left\{ \begin{pmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ 0 & 0 & 0 \end{pmatrix} \right\},$$

$$\text{3-Aut}(A_{3.5}^0) = \left\{ \begin{pmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ 0 & 0 & \pm 1 \end{pmatrix} \middle| c_{11}c_{22} - c_{12}c_{21} \neq 0 \right\}.$$

(8) $\boldsymbol{A_{3.5}^b}$:

$$\text{Der}(A_{3.5}^b) = \left\{ \begin{pmatrix} c_{11} & -c_{12} & c_{13} \\ c_{12} & c_{11} & c_{23} \\ 0 & 0 & 0 \end{pmatrix} \right\},$$

$$\text{Aut}(A_{3.5}^b) = \left\{ \begin{pmatrix} c_{11} & -c_{12} & c_{13} \\ c_{12} & c_{11} & c_{23} \\ 0 & 0 & 1 \end{pmatrix} \middle| c_{11}^2 + c_{22}^2 \neq 0 \right\},$$

$$\text{3-Der}(A_{3.5}^b) = \left\{ \begin{pmatrix} c_{11} & -c_{12} & c_{13} \\ c_{12} & c_{11} & c_{23} \\ 0 & 0 & 0 \end{pmatrix} \right\},$$

$$\text{3-Aut}(A_{3.5}^b) = \left\{ \begin{pmatrix} c_{11} & -c_{12} & c_{13} \\ c_{12} & c_{11} & c_{23} \\ 0 & 0 & \pm 1 \end{pmatrix} \middle| c_{11}^2 + c_{22}^2 \neq 0 \right\} \bigcup$$

$$\left\{ \begin{pmatrix} c_{11} & \frac{b^2-1}{2b}c_{11} & c_{13} \\ -\frac{b^2-1}{2b}c_{11} & c_{11} & c_{23} \\ 0 & 0 & \pm 1 \end{pmatrix} \middle| c_{11} \neq 0 \right\} \bigcup \left\{ \begin{pmatrix} \frac{b^2-1}{2b}c_{11} & -c_{11} & c_{13} \\ c_{11} & \frac{b^2-1}{2b}c_{11} & c_{23} \\ 0 & 0 & \pm 1 \end{pmatrix} \middle| c_{11} \neq 0 \right\}.$$

We deduce that these five Lie algebras $A_{2.1} \oplus A_1$, $A_{3.2}$, $A_{3.3}$, $A_{3.4}^a$, $A_{3.5}^b$ have same derivations and 3-derivations, and these three Lie algebras $A_{3.1}$, $A_{3.4}^{-1}$, $A_{3.5}^0$ have different derivations and 3-derivations, and for all these eight Lie algebras, 3-automorphism groups have more connected components than automorphism groups.

5. Conclusions

We study 3-derivations and 3-automorphisms on Lie algebras using the differential and exponential map. The main results we obtained are the Lie algebra of 3-derivations is the Lie algebra of the Lie group of 3-automorphisms and derivations (automorphisms) of a

Lie triple system give rise to derivations(automorphisms) of the corresponding standard embedding Lie algebra. Our prove is basically technical and we also compute specific example of 3-derivations and 3-automorphisms of 3-dimensional real Lie algebras by Mathematica program on computer.

We plan to extend this study by finding more conceptual results through the results of classifying realizations, and to describe more algebraic properties of low dimensional Lie algebras and Lie triple systems. We have also begun investigations into a complete description of 3-derivations and 3-automorphism groups of low dimensional Lie algebras for all the constructed realizations. Furthermore, we want to find applications of the obtained results on the theory of homogeneous spaces, Lie groups and differential equations. We also hope to solve the analogous problem for higher-dimensional algebras in the near future.

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