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On Some Constrained Optimization Problems

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Abstract: By using appropriate methods of variational analysis, the necessary conditions of optimality are established for new classes of constrained optimization problems involving multiple and curvilinear integral functionals. Additionally, two illustrative examples are provided to support the main results formulated in this paper.

Keywords: multi-time Lagrangian; multiple integral; Euler–Lagrange equations; curvilinear integral

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1. Introduction

By considering the three major approaches (variational calculus, the Pontryagin maximum principle, and dynamic programming) associated with the optimal control theory, many researchers have investigated certain controlled processes in nature using some functionals with ODE/PDE or mixed constraints. In this regard, Treanţă [1–3], Jayswal et al. [4], and Mititelu and Treanţă [5] studied some classes of optimization problems defined by integral functionals of multiple and/or path-independent curvilinear type, having various constraints involving first-order partial differential equations and inequations. Schmitendorf [6], by transforming the considered control problems into the standard form and then using Pontryagin's principle, formulated necessary conditions of optimality for a class of control problems subjected to isoperimetric constraints. Later, Forster and Long [7] derived an alternative transformation technique for obtaining the necessary optimality conditions for the control problem considered in Schmitendorf [6] (see also Schmitendorf [8]). On the other hand, Benner et al. [9] studied bang-bang control strategies for a control problem with isoperimetric constraints. For other studies on this subject, we direct the reader to Batista [10], Caputo [11], Enache and Philippin [12], Takahashi [13], or Sabermahani and Ordokhani [14].

Nevertheless, let us consider the following real problem: *Under the action of gravity, let us find a homogeneous chain of length l_0, which is fixed at its ends*. If we consider the chain represented by the graph of the function $x = x(\tau)$, fixed at its ends $x(\tau_0) = x_0,\ x(\tau_1) = x_1$, then the shape of the considered chain is given by the stipulation that the potential energy is minimal. Consequently, we have to extremize the following simple integral

$$I(x(\cdot)) = \int_{\tau_0}^{\tau_1} x(\tau)\sqrt{1 + \dot{x}^2(\tau)}d\tau$$

subject to

$$T(x(\cdot)) = \int_{\tau_0}^{\tau_1} \sqrt{1 + \dot{x}^2(\tau)}d\tau = l_0$$

$$x(\tau_0) = x_0, \quad x(\tau_1) = x_1.$$

Keeping in mind the form of the previous optimization problem and motivated and inspired by continued research in this field (see, for instance, Hestenes [15], Lee [16], Schmitendorf [6], Udrişte and Tevy [17], and Treanţă [2]) in the present paper, we study new classes of variational control problems with isoperimetric constraints defined by multiple and curvilinear integrals. More precisely, without limiting our investigation to convex costs as in Lee [16], the current framework is more comprehensive than in Schmitendorf [6], Hestenes [15], Udrişte and Ţevy [17], or Treanţă [2] both by the inclusion of integral functionals of multiple and curvilinear type as constraints and by the inclusion of new proofs. Additionally, compared with a very recent research paper (see Treanţă [18]), the present paper takes into account the isoperimetric constraints defined by multiple integral functionals (see Section 2.2). Moreover, due to the very important physical applications of the functionals used (for example, mechanical work), this paper is a very good starting point for researchers in the field of applied mathematics that deal with the design, theory, and applications of mathematics, management science, operations research, optimal control science, and economics.

The rest of the paper is structured as follows: Section 2 deals with the optimization of a multiple integral functional with constraints given by curvilinear and multiple integrals. Two main results are formulated and proven (see Theorems 1 and 2), and two illustrative examples are given. This section is concluded with an algorithm that highlights the steps for solving such problems. Section 3 provides the conclusions of this paper and formulates further developments.

2. Constrained Optimization Problem with Multiple Integral Objective Functional

The following class of problems is motivated by generalized Dieudonné–Rashevski type problems, which are seen as isoperimetric constrained variational problems and occur when we talk about resources (see Pitea [19]). For this purpose, we start with a function $\Theta(x(\tau), \vartheta(\tau), \tau)$ of C^1-class, called *multi-time Lagrangian*, where $\tau = (\tau^\gamma) = (\tau^1, \cdots, \tau^m) \in \Omega_{\tau_0,\tau_1} \subset R^m_+$, $x = (x^i) = \left(x^1, \cdots, x^n\right) : \Omega_{\tau_0,\tau_1} \to R^n$ is a C^2-class function (called the *state variable)* and $\vartheta = (\vartheta^\alpha) = \left(\vartheta^1, \cdots, \vartheta^k\right) : \Omega_{\tau_0,\tau_1} \to R^k$ is a piecewise continuous function (called the *control variable*). Additionally, denote $x_\gamma(\tau) := \dfrac{\partial x}{\partial \tau^\gamma}(\tau)$, $\gamma \in \{1, \cdots, m\}$, and consider $\Omega_{\tau_0,\tau_1} = [\tau_0, \tau_1]$, a *multi-time interval* in R^m_+.

Isoperimetric constrained control problem. *Find (x^*, ϑ^*) that provides the minimum for the following multiple integral objective functional*

$$T(x(\cdot), \vartheta(\cdot)) = \int_{\Omega_{\tau_0,\tau_1}} \Theta(x(\tau), \vartheta(\tau), \tau) d\tau^1 \cdots d\tau^m \tag{1}$$

among all of the pair functions (x, ϑ) with

$$x(\tau_0) = x_0, \quad x(\tau_1) = x_1,$$

or

$$x(\tau)|_{\partial\Omega_{\tau_0,\tau_1}} = given$$

and satisfying the following isoperimetric constraints:

2.1. Curvilinear Integrals as Isoperimetric Constraints

We begin with the next constraints

$$\int_{\Gamma_{\tau_0,\tau_1}} g^a_\zeta(x(\tau), x_\gamma(\tau), \vartheta(\tau), \tau) d\tau^\zeta = l^a, \quad a = 1, 2, \cdots, r \le n,$$

where Γ_{τ_0,τ_1} represents a piecewise smooth curve, which is contained in Ω_{τ_0,τ_1} and join $\tau_0,\ \tau_1 \in R^m_+$. The C^1-class functions

$$g^a_\zeta(x(\tau), x_\gamma(\tau), \vartheta(\tau), \tau)d\tau^\zeta, \quad a = 1, 2, \cdots, r$$

are considered complete integrable differential 1-forms, more exactly, $D_\gamma g_\zeta = D_\zeta g_\gamma,\ \gamma \neq \zeta,\ \gamma, \zeta \in \{1, \cdots, m\}$, with $D_\gamma := \dfrac{\partial}{\partial \tau^\gamma},\ \gamma \in \{1, \cdots, m\}$.

This kind of problem arises when we have in mind the optimization problems associated with convex bodies and the geometrical constraints. More precisely, the maximization of a surface with fixed width and diameter.

For the study of the aforementioned variational control problem (1), associated with the above constraints, we consider $\Gamma_{\tau_0,\tau} \subset \Gamma_{\tau_0,\tau_1}$ and the auxiliary variables

$$\Psi^a(\tau) = \int_{\Gamma_{\tau_0,\tau}} g^a_\zeta(x(s), x_\gamma(s), \vartheta(s), s)ds^\zeta, \quad a = 1, 2, \cdots, r,$$

satisfying $\Psi^a(\tau_0) = 0,\ \Psi^a(\tau_1) = l^a$. In other words, the functions Ψ^a are solutions for the next first-order complete integrable PDEs

$$\frac{\partial \Psi^a}{\partial \tau^\zeta}(\tau) = g^a_\zeta(x(\tau), x_\gamma(\tau), \vartheta(\tau), \tau), \quad \Psi^a(\tau_1) = l^a.$$

Introducing $p = \left(p^\zeta_a(\tau)\right)$ (Lagrange multiplier) and considering $\Psi = (\Psi^a(\tau))$, we build a new Lagrangian

$$\Theta_1\big(x(\tau), x_\gamma(\tau), \vartheta(\tau), \Psi(\tau), \Psi_\zeta(\tau), p(\tau), \tau\big) = \Theta(x(\tau), \vartheta(\tau), \tau)$$
$$+ p^\zeta_a(\tau)\left(g^a_\zeta(x(\tau), x_\gamma(\tau), \vartheta(\tau), \tau) - \frac{\partial \Psi^a}{\partial \tau^\zeta}(\tau)\right)$$

that modifies the original problem (with constraints of isoperimetric type) into an unconstrained optimization problem

$$\min_{x(\cdot),\, \vartheta(\cdot),\, \Psi(\cdot),\, p(\cdot)} \int_{\Omega_{\tau_0,\tau_1}} \Theta_1\big(x(\tau), x_\gamma(\tau), \vartheta(\tau), \Psi(\tau), \Psi_\zeta(\tau), p(\tau), \tau\big)d\tau^1 \cdots d\tau^m \tag{2}$$

$$x(\tau_0) = x_0, \quad x(\tau_1) = x_1$$
$$\Psi(\tau_0) = 0, \quad \Psi(\tau_1) = l.$$

An extreme pair function of (1) can be found among the extreme pair functions of (2). The next theorem is the first main result. It establishes the necessary optimality conditions of the considered optimization problem.

Theorem 1. *Consider* $(x^*(\cdot),\ \vartheta^*(\cdot),\ \Psi^*(\cdot),\ p^*(\cdot))$ *solves* (2). *Then,*

$$(x^*(\cdot),\ \vartheta^*(\cdot),\ \Psi^*(\cdot),\ p^*(\cdot))$$

solves the following Euler–Lagrange PDEs

$$\frac{\partial \Theta_1}{\partial x^i} - D_\gamma \frac{\partial \Theta_1}{\partial x^i_\gamma} = 0, \quad i = \overline{1, n}$$

$$\frac{\partial \Theta_1}{\partial \vartheta^\alpha} - D_\gamma \frac{\partial \Theta_1}{\partial \vartheta^\alpha_\gamma} = 0, \quad \alpha = \overline{1, k}$$

$$\frac{\partial \Theta_1}{\partial \Psi^a} - D_\zeta \frac{\partial \Theta_1}{\partial \Psi^a_\zeta} = 0, \quad a = \overline{1, r}$$

$$\frac{\partial \Theta_1}{\partial p^\zeta_a} - D_\gamma \frac{\partial \Theta_1}{\partial p^\zeta_{a,\gamma}} = 0,$$

where $p^\zeta_{a,\gamma} := \dfrac{\partial p^\zeta_a}{\partial \tau^\gamma}$.

Proof. Consider that $(x(\tau), \vartheta(\tau), \Psi(\tau), p(\tau))$ solves (2) and $x(\tau) + \omega h(\tau),\ p(\tau) + \omega f(\tau)$, and $\vartheta(\tau) + \omega m(\tau),\ \Psi(\tau) + \omega n(\tau)$ are some variations of $x(\tau),\ p(\tau),\ \vartheta(\tau)$, and $\Psi(\tau)$, respectively, with $h(\tau)|_{\partial\Omega_{\tau_0,\tau_1}} = f(\tau)|_{\partial\Omega_{\tau_0,\tau_1}} = m(\tau)|_{\partial\Omega_{\tau_0,\tau_1}} = n(\tau)|_{\partial\Omega_{\tau_0,\tau_1}} = 0$. Therefore, the multiple integral functional turns into a function of ω, more precisely

$$I(\omega) = \int_{\Omega_{\tau_0,\tau_1}} \Theta_1(x(\tau) + \omega h(\tau), x_\gamma(\tau) + \omega h_\gamma(\tau), \vartheta(\tau) + \omega m(\tau),$$
$$\Psi(\tau) + \omega n(\tau), \Psi_\zeta(\tau) + \omega n_\zeta(\tau), p(\tau) + \omega f(\tau), \tau) d\tau^1 \cdots d\tau^m.$$

By our hypothesis, we must have

$$0 = \frac{d}{d\omega} I(\omega)|_{\omega=0}$$
$$= \int_{\Omega_{\tau_0,\tau_1}} \left(\frac{\partial \Theta_1}{\partial x^j} h^j + \frac{\partial \Theta_1}{\partial x^j_\gamma} h^j_\gamma + \frac{\partial \Theta_1}{\partial \vartheta^\alpha} m^\alpha + \frac{\partial \Theta_1}{\partial \Psi^a} n^a + \frac{\partial \Theta_1}{\partial \Psi^a_\zeta} n^a_\zeta + \frac{\partial \Theta_1}{\partial p^\zeta_a} f^a_\zeta \right) d\tau^1 \cdots d\tau^m$$
$$= BT + \int_{\Omega_{\tau_0,\tau_1}} \left(\frac{\partial \Theta_1}{\partial x^j} - D_\gamma \frac{\partial \Theta_1}{\partial x^j_\gamma} \right) h^j d\tau^1 \cdots d\tau^m$$
$$+ \int_{\Omega_{\tau_0,\tau_1}} \left(\frac{\partial \Theta_1}{\partial \Psi^a} - D_\zeta \frac{\partial \Theta_1}{\partial \Psi^a_\zeta} \right) n^a d\tau^1 \cdots d\tau^m$$
$$+ \int_{\Omega_{\tau_0,\tau_1}} \left(\frac{\partial \Theta_1}{\partial \vartheta^\alpha} - D_\gamma \frac{\partial \Theta_1}{\partial \vartheta^\alpha_\gamma} \right) m^\alpha d\tau^1 \cdots d\tau^m$$
$$+ \int_{\Omega_{\tau_0,\tau_1}} \left(\frac{\partial \Theta_1}{\partial p^\zeta_a} - D_\gamma \frac{\partial \Theta_1}{\partial p^\zeta_{a,\gamma}} \right) f^a_\zeta d\tau^1 \cdots d\tau^m.$$

By using the following equalities

$$\frac{\partial \Theta_1}{\partial x^j_\gamma} h^j_\gamma = -h^j D_\gamma \frac{\partial \Theta_1}{\partial x^j_\gamma} + D_\gamma \left(\frac{\partial \Theta_1}{\partial x^j_\gamma} h^j \right),$$
$$\frac{\partial \Theta_1}{\partial \Psi^a_\zeta} n^a_\zeta = -n^a D_\zeta \frac{\partial \Theta_1}{\partial \Psi^a_\zeta} + D_\zeta \left(\frac{\partial \Theta_1}{\partial \Psi^a_\zeta} n^a \right)$$

and the formula of divergence, the boundary terms BT (see below) disappear (see $n^\xi(\tau)$ as the normal vector associated with $\partial\Omega_{\tau_0,\tau_1}$, and $\delta_{\nu\xi}$ as the symbol of Kronecker),

$$\int_{\Omega_{\tau_0,\tau_1}} D_\gamma \left(\frac{\partial \Theta_1}{\partial x^j_\gamma} h^j \right) d\tau^1 \cdots d\tau^m = \int_{\partial\Omega_{\tau_0,\tau_1}} \delta_{\nu\xi} \frac{\partial \Theta_1}{\partial x^j_\nu} h^j n^\xi d\sigma$$
$$\int_{\Omega_{\tau_0,\tau_1}} D_\zeta \left(\frac{\partial \Theta_1}{\partial \Psi^a_\zeta} n^a \right) = \int_{\partial\Omega_{\tau_0,\tau_1}} \delta_{\nu\xi} \frac{\partial \Theta_1}{\partial \Psi^a_\nu} n^a n^\xi d\sigma.$$

Now, by using a fundamental lemma of variational calculus ("If $\int_{x_0}^{x_1} f(x)\phi(x)dx = 0$, for all C^2-class functions satisfying $f(x_0) = f(x_1) = 0$, then $\phi(x) = 0$."), the proof is complete. □

Remark 1. *We notice that the above PDEs of Euler–Lagrange type can be formulated as follows*

$$\frac{\partial \Theta_1}{\partial x^i} - D_\gamma \frac{\partial \Theta_1}{\partial x^i_\gamma} = 0, \quad i = \overline{1,n}$$

$$\frac{\partial \Theta_1}{\partial \vartheta^\alpha} - D_\gamma \frac{\partial \Theta_1}{\partial \vartheta^\alpha_\gamma} = 0, \quad \alpha = \overline{1,k}$$

$$\frac{\partial p^\zeta_a}{\partial \tau^\zeta} = 0, \quad a = \overline{1,r}, \quad \zeta \in \{1,2,\cdots,m\}$$

$$\frac{\partial \Psi^a}{\partial \tau^\zeta}(\tau) = g^a_\zeta(x(\tau), x_\gamma(\tau), \vartheta(\tau), \tau).$$

Consequently, we obtain that p has zero total divergence. Moreover, this multiplier is well defined only if the optimal pair function is not an extreme pair function for at least one of the following functionals $\int_{\Gamma_{\tau_0,\tau_1}} g^a_\zeta(x(\tau), x_\gamma(\tau), \vartheta(\tau), \tau)d\tau^\zeta,\ a = \overline{1,r}.$

Example 1. *Let us find the optimal pair function for the next objective functional*

$$I(x(\cdot), \vartheta(\cdot)) = \frac{1}{2}\int_\Omega \Big(x^2(\tau) + \vartheta^2(\tau)\Big) d\tau^1 d\tau^2$$

subject to $\int_\Gamma x_{\tau^1}(\tau)d\tau^1 + x_{\tau^2}(\tau)d\tau^2 = 0$ *and the boundary conditions* $x(0,0) = x(1,1) = 0$, *where* Γ *is a curve of* C^1*-class that is contained in* $\Omega = [0,1]^2$ *and join* $(0,0),\ (1,1)$.

Solution. The auxiliary Lagrangian is

$$\Theta = \frac{1}{2}\Big(x^2(\tau) + \vartheta^2(\tau)\Big) + p(\tau)(\Psi_{\tau^1}(\tau) - x_{\tau^1}(\tau)) + q(\tau)(\Psi_{\tau^2}(\tau) - x_{\tau^2}(\tau)).$$

The extreme pair functions are given by the following partial differential equations

$$\frac{\partial \Theta}{\partial x} - \frac{\partial}{\partial \tau^1}\left(\frac{\partial \Theta}{\partial x_{\tau^1}}\right) - \frac{\partial}{\partial \tau^2}\left(\frac{\partial \Theta}{\partial x_{\tau^2}}\right) = 0$$

$$\frac{\partial \Theta}{\partial \vartheta} - \frac{\partial}{\partial \tau^1}\left(\frac{\partial \Theta}{\partial \vartheta_{\tau^1}}\right) - \frac{\partial}{\partial \tau^2}\left(\frac{\partial \Theta}{\partial \vartheta_{\tau^2}}\right) = 0$$

$$\frac{\partial p}{\partial \tau^1} + \frac{\partial q}{\partial \tau^2} = 0, \quad \Psi_{\tau^1}(\tau) - x_{\tau^1}(\tau) = 0, \quad \Psi_{\tau^2}(\tau) - x_{\tau^2}(\tau) = 0,$$

or, equivalently,

$$x + \frac{\partial p}{\partial \tau^1} + \frac{\partial q}{\partial \tau^2} = 0$$

$$\vartheta = 0$$

$$\frac{\partial p}{\partial \tau^1} + \frac{\partial q}{\partial \tau^2} = 0, \quad \Psi_{\tau^1}(\tau) - x_{\tau^1}(\tau) = 0, \quad \Psi_{\tau^2}(\tau) - x_{\tau^2}(\tau) = 0,$$

which involves $(x^*, \vartheta^*) = (0,0)$.

2.2. Multiple Integrals as Isoperimetric Constraints

Let the following be constraints:

$$\int_{\Omega_{\tau_0,\tau_1}} g^a(x(\tau), x_\gamma(\tau), \vartheta(\tau), \tau) d\tau^1 \cdots d\tau^m = 0, \quad a = 1, 2, \cdots, r \leq n.$$

This type of problem arises when we characterize the torsion of prismatic bars in the elastic and elastic-plastic case (see Ting [20]).

By introducing the variations $\hat{x}(\tau) = x(\tau) + \omega_1 h(\tau)$ and $\hat{\vartheta}(\tau) = \vartheta(\tau) + \omega_2 m(\tau)$ associated with $x(\tau)$ and $\vartheta(\tau)$, respectively, we convert the multiple integral functionals in functions depending on (ω_1, ω_2). Furthermore, we introduce the constant vector multiplier $p = (p_a)$ and the auxiliary Lagrangian

$$\Theta_2(x(\tau), x_\gamma(\tau), \vartheta(\tau), p, \tau) = \Theta(x(\tau), \vartheta(\tau), \tau)$$
$$+ p_a g^a(x(\tau), x_\gamma(\tau), \vartheta(\tau), \tau).$$

In this way, we modify the original constrained optimization problem (1) (with isoperimetric constraints) in a new optimization problem (without constraints of isoperimetric type)

$$\min_{x(\cdot),\, \vartheta(\cdot),\, p} \int_{\Omega_{\tau_0,\tau_1}} \Theta_2(x(\tau), x_\gamma(\tau), \vartheta(\tau), p, \tau) d\tau^1 \cdots d\tau^m \tag{3}$$

subject to

$$x(\tau_0) = x_0, \quad x(\tau_1) = x_1$$

or

$$x(\tau)|_{\partial\Omega_{\tau_0,\tau_1}} = \text{given}.$$

An extreme pair function of (1) is found among the extreme pair functions of (3). The next theorem is the second main result. It formulates the necessary optimality conditions of the considered constrained optimization problem.

Theorem 2. *Consider that $(x^*(\cdot),\ \vartheta^*(\cdot))$ solves $(Equation(1))$. Then, $(x^*(\cdot),\ \vartheta^*(\cdot))$ solves the following system:*

$$\frac{\partial \Theta_2}{\partial x^i} - D_\gamma \frac{\partial \Theta_2}{\partial x^i_\gamma} = 0, \quad i = \overline{1, n}$$
$$\frac{\partial \Theta_2}{\partial \vartheta^\alpha} - D_\gamma \frac{\partial \Theta_2}{\partial \vartheta^\alpha_\gamma} = 0, \quad \alpha = \overline{1, k}.$$

Proof. As mentioned above, by introducing the variations $x(\tau) + \omega_1 h(\tau)$ and $\vartheta(\tau) + \omega_2 m(\tau)$ associated with $x(\tau)$ and $\vartheta(\tau)$, respectively, we convert the multiple integral functionals in functions of (ω_1, ω_2), namely

$$F(\omega_1, \omega_2) = \int_{\Omega_{\tau_0,\tau_1}} \Theta(x(\tau) + \omega_1 h(\tau), \vartheta(\tau) + \omega_2 m(\tau), \tau) d\tau^1 \cdots d\tau^m$$

$$G^a(\omega_1, \omega_2) = \int_{\Omega_{\tau_0,\tau_1}} g^a(x(\tau) + \omega_1 h(\tau), x_\gamma(\tau) + \omega_1 h_\gamma(\tau), \vartheta(\tau) + \omega_2 m(\tau), \tau) d\tau^1 \cdots d\tau^m,$$

for $a = 1, 2, \cdots, r \leq n$.

Let $(x(\cdot), \vartheta(\cdot))$ be an optimal solution for (1). In consequence, $(0, 0)$ is the solution for the following optimization problem

$$\min_{\omega_1, \omega_2} F(\omega_1, \omega_2)$$

subject to

$$G^a(\omega_1, \omega_2) = 0, \quad a = 1, 2, \cdots, r \le n$$

$$h|_{\partial\Omega_{\tau_0,\tau_1}} = m|_{\partial\Omega_{\tau_0,\tau_1}} = 0,$$

and there exists the constant vector $p = (p_a)$ such that the following Karush–Kuhn–Tucker conditions at $(0, 0)$ are fulfilled

$$\nabla F(0,0) + p_a \nabla G^a(0,0) = 0, \tag{4}$$

where $\nabla f(x, \Psi)$ represents the gradient of f at (x, Ψ). The relation (4) can be rewritten as follows:

$$\int_{\Omega_{\tau_0,\tau_1}} \left(\frac{\partial \Theta}{\partial x^i} h^i + p_a \frac{\partial g^a}{\partial x^i} h^i + p_a \frac{\partial g^a}{\partial x^i_\gamma} h^i_\gamma \right) d\tau^1 \cdots d\tau^m = 0$$

$$\int_{\Omega_{\tau_0,\tau_1}} \left(\frac{\partial \Theta}{\partial \vartheta^\alpha} m^\alpha + p_a \frac{\partial g^a}{\partial \vartheta^\alpha} m^\alpha \right) d\tau^1 \cdots d\tau^m = 0,$$

or, equivalently,

$$\int_{\Omega_{\tau_0,\tau_1}} \left(\frac{\partial \Theta}{\partial x^i} + p_a \frac{\partial g^a}{\partial x^i} - p_a D_\gamma \frac{\partial g^a}{\partial x^i_\gamma} \right) h^i d\tau^1 \cdots d\tau^m = 0,$$

$$\int_{\Omega_{\tau_0,\tau_1}} \left(\frac{\partial \Theta}{\partial \vartheta^\alpha} + p_a \frac{\partial g^a}{\partial \vartheta^\alpha} \right) m^\alpha d\tau^1 \cdots d\tau^m = 0.$$

Furthermore, by using the conditions $h|_{\partial\Omega_{\tau_0,\tau_1}} = m|_{\partial\Omega_{\tau_0,\tau_1}} = 0$, it follows that

$$\frac{\partial \Theta}{\partial x^i} + p_a \frac{\partial g^a}{\partial x^i} - p_a D_\gamma \frac{\partial g^a}{\partial x^i_\gamma} = 0, \quad i = \overline{1, n}$$

$$\frac{\partial \Theta}{\partial \vartheta^\alpha} + p_a \frac{\partial g^a}{\partial \vartheta^\alpha} = 0, \quad \alpha = \overline{1, k},$$

or, equivalently,

$$\frac{\partial \Theta_2}{\partial x^i} - D_\gamma \frac{\partial \Theta_2}{\partial x^i_\gamma} = 0, \quad i = \overline{1, n}$$

$$\frac{\partial \Theta_2}{\partial \vartheta^\alpha} - D_\gamma \frac{\partial \Theta_2}{\partial \vartheta^\alpha_\gamma} = 0, \quad \alpha = \overline{1, k}$$

and this completes the proof. □

Remark 2. *We notice that the Lagrange multiplier p is well defined only if the optimal pair function is not an extreme pair function for at least one of the following functionals*

$$\int_{\Omega_{\tau_0,\tau_1}} g^a(x(\tau), x_\gamma(\tau), \vartheta(\tau), \tau) d\tau^1 \cdots d\tau^m, \ a = \overline{1, r}.$$

Example 2. *For* $\Omega = [0,1]^2$*, let us compute the extreme pair functions associated with the following functional*

$$I(x(\cdot), \vartheta(\cdot)) = \frac{1}{2} \int_\Omega \left(x^2(\tau) + \vartheta^2(\tau) \right) d\tau^1 d\tau^2$$

subject to $\int_\Omega x_{\tau^1}(\tau) x_{\tau^2}(\tau) d\tau^1 d\tau^2 = 0$ *and* $x(0,0) = 0, \ x(1,1) = 1.$

Solution. The auxiliary Lagrangian is

$$\Theta = \frac{1}{2} \left(x^2 + \vartheta^2 \right) + p x_{\tau^1} x_{\tau^2}.$$

The extreme pair functions are provided by the following partial differential equations

$$\frac{\partial\Theta}{\partial x}-\frac{\partial}{\partial\tau^1}\left(\frac{\partial\Theta}{\partial x_{\tau^1}}\right)-\frac{\partial}{\partial\tau^2}\left(\frac{\partial\Theta}{\partial x_{\tau^2}}\right)=0$$

$$\frac{\partial\Theta}{\partial\vartheta}-\frac{\partial}{\partial\tau^1}\left(\frac{\partial\Theta}{\partial\vartheta_{\tau^1}}\right)-\frac{\partial}{\partial\tau^2}\left(\frac{\partial\Theta}{\partial\vartheta_{\tau^2}}\right)=0,$$

or, equivalently,

$$x-2px_{\tau^1\tau^2}=0$$

$$\vartheta=0.$$

Furthermore, the optimal pair function is given by the above second-order partial differential equation associated with the boundary conditions $x(0,0)=0,\ x(1,1)=1$*. The constant* p *is determined by imposing the isoperimetric condition, as well.*

Taking into account the aforementioned applications and the theory presented in this paper, we establish the following algorithm. Its main purpose is to highlight the steps used to solve a problem such as the one formulated here.

More precisely, for a cost functional of multiple integral type, and a set of isoperimetric and boundary restrictions, together with self and/or normal data, our purpose is to find $(x^\star,\vartheta^\star)$ such that $T(x^\star,\vartheta^\star)\le T(x,\vartheta)$, for all (x,ϑ). For this, we consider a feasible solution (x,ϑ). If it satisfies the necessary conditions of optimality in Theorem 1 (or Theorem 2), then the "Stage of Generating" is verified and we move on to the next stage; otherwise, the algorithm stops. If the self/normal data set is satisfied, then we move on to the next stage, namely "Stage of Deciding"; otherwise, the algorithm stops. For $(x^\star,\vartheta^\star)$ derived in "Stage of Detecting", if the inequality $T(x^\star,\vartheta^\star)\le T(x,\vartheta)$ is true for all feasible solutions (x,ϑ), then $(x^\star,\vartheta^\star)$ is an optimal solution; otherwise, the algorithm stops (see Algorithm 1).

Algorithm 1: for new classes of constrained optimization problems involving multiple and curvilinear integral functionals

`DATA:`

• `the objective functional of multiple integral type`

$$\min_{(x,\vartheta)}T(x,\vartheta)=\int_{\Omega_{\tau_0,\tau_1}}\Theta(x(\tau),\vartheta(\tau),\tau)d\tau^1\cdots d\tau^m;$$

• `the constraint set`

$$\int_{\Gamma_{\tau_0,\tau_1}}g^a_\zeta(x(\tau),x_\gamma(\tau),\vartheta(\tau),\tau)d\tau^\zeta=l^a,\quad a=1,2,\cdots,r\le n,$$

or

$$\int_{\Omega_{\tau_0,\tau_1}}g^a(x(\tau),x_\gamma(\tau),\vartheta(\tau),\tau)d\tau^1\cdots d\tau^m=0,\quad a=1,2,\cdots,r\le n$$

$$x(\tau_0)=x_0,\quad x(\tau_1)=x_1,$$

or

$$x(\tau)|_{\partial\Omega_{\tau_0,\tau_1}}=\text{given};$$

• `the self/normal data set`

- $g=\left(g^a_\zeta\right)$ fulfils the complete integrability conditions;

`RESULT:`

$$\mathsf{S}=\{(x^\star,\vartheta^\star)|T(x^\star,\vartheta^\star)\le T(x,\vartheta),$$

$$\text{with }(x^\star,\vartheta^\star)\text{ fulfiling the constraint and self/normal data set}\};$$

Algorithm 1: *Cont.*

```
BEGIN

• Stage of Generating:  let (x, ϑ) be a feasible solution
if the necessary optimality conditions
are incompatible with respect to (x, ϑ)
then STOP
else GO to the next stage

• Stage of Detecting:  the analysis of Lagrange multipliers
if the self/normal data set is not satisfied
then STOP
else GO to the next stage

• Stage of Deciding:  let (x*, ϑ*) is obtained in Stage of Detecting
if T(x, ϑ) ≥ T(x*, ϑ*) is true for all (x, ϑ)
then (x*, ϑ*) is an optimal solution
else STOP

END
```

3. Conclusions and Further Developments

We studied two classes of optimization problems with isoperimetric constraints involving multiple and path-independent curvilinear integrals. More precisely, by using some tools of variational analysis, necessary conditions of optimality have been established for the considered problems. In order to illustrate the mathematical development derived in the paper, two examples were provided as well. Additionally, to synthesize the concrete steps in order to solve an optimization problem such as those analyzed in the paper, an algorithm was formulated.

As further developments associated with this paper, we mention the study of multidimensional variational problems with deviating arguments.

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