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Hilfer Fractional Quantum Derivative and Boundary Value Problems

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Abstract: In this paper, we introduce an extension of the Hilfer fractional derivative, the "Hilfer fractional quantum derivative", and establish some of its properties. Then, we introduce and discuss initial and boundary value problems involving the Hilfer fractional quantum derivative. The existence of a unique solution of the considered problems is established via Banach's contraction mapping principle. Examples illustrating the obtained results are also presented.

Keywords: quantum calculus; Hilfer derivative; existence and uniqueness

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1. Introduction

Quantum calculus, also known as q-calculus, is a branch of mathematics that study calculus without the notion of limits. Results on q-calculus can be traced back to the works by Euler. Jackson [1] was the first to establish the notions of the q-derivative and the definite q-integral. Quantum calculus has found its applications in many area of mathematics, including orthogonal polynomials, hypergeometric functions, number theory, and combinatorics, as well as topics in physics, such as mechanics, relativity theory, and quantum theory [2,3]. In [4], the author applied the neutrix limit to generalize q-integrals. The fractional q-difference calculus, which further generalizes the idea of q-derivatives and q-integrals with non-integer orders, was developed in the works by Al-Salam [5] and Agarwal [6]. Recent treatment on such material can be found in [7]. Research on the topic has yield variety of new results in recent years, as seen in [8–23] and references therein.

Tariboon and Ntouyas [24] introduced the idea of quantum calculus on finite intervals, in which they obtained q-analogues of several well-known mathematical objects and opened a new avenue of research. Further details can be obtained from the recent monograph [25].

The Riemann–Liouville fractional q-derivative and q-integral on an interval $[a,b]$ were introduced in [26]. In the present paper, we introduce the Hilfer fractional quantum derivative, which generalizes the Hilfer fractional derivative created by R. Hilfer in [27], and establish some of its properties. It is well known that the Hilfer fractional derivative reduces to both Riemann–Liouville and Caputo fractional derivatives in special cases. In addition, we present and discuss some examples of initial and boundary value problems with the Hilfer fractional quantum derivative.

The remaining part of the paper is organized as follows. In Section 2, we recall some preliminaries and lemmas from q-calculus and fractional q-difference calculus. In Section 3, we introduce the Hilfer fractional quantum derivative and prove some basic results. Section 4 is devoted to applications, where existence and uniqueness results are established for initial and boundary value problems involving the Hilfer fractional quantum derivative.

2. Preliminaries

Let $[n]_q$ be the quantum number defined by

$$[n]_q = \frac{1-q^n}{1-q}, \tag{1}$$

where $0 < q < 1$ and $n \in \mathbb{N}$. For example, $[3]_q = (1-q^3)/(1-q) = 1+q+q^2$, and $[3]_{\frac{1}{2}} = 7/4$.

Definition 1. *Let $f : [a,b] \to \mathbb{R}$ be a given function. The quantum derivative (q-difference) starting at a point "a" is defined by*

$$_aD_qf(t) = \frac{f(t) - f(qt+(1-q)a)}{(1-q)(t-a)}, \quad t \in (a,b] \tag{2}$$

and $_aD_qf(a) = \lim_{t\to a}\left({}_aD_qf\right)(t)$.

If $a = 0$, then for $t \in (0,b]$, we have

$$D_qf(t) = \frac{f(t)-f(qt)}{(1-q)t}, \tag{3}$$

which is the Jackson q-derivative. Observe that the quantum derivative of Jackson sense [1] in Equation (3) is around the point "0" while Tariboon and Ntouyas sense [24], in Equation (2), is related to any point "a". If $f(t) = (t-a)^\alpha$ is the power function, then the quantum derivative is expressed by

$$_aD_q(t-a)^\alpha = [\alpha]_q(t-a)^{\alpha-1}, \quad \text{for } \alpha \geq 0,\ t > a. \tag{4}$$

Definition 2 ([24]). *The q-integration of a function, f, on an interval, $[a,b]$, is defined by*

$$\begin{aligned} _aI_qf(t) &= \int_a^t f(s)\,{}_ad_qs \\ &= (1-q)(t-a)\sum_{n=0}^{\infty} q^n f(q^nt+(1-q^n)a). \end{aligned} \tag{5}$$

If $a = 0$, then the above definition is reduced to the Jackson quantum integration [1], as follows:

$$\int_0^t f(s)d_qs = (1-q)t\sum_{n=0}^{\infty} q^n f(q^nt). \tag{6}$$

For example, if we let $f(t) = (t-a)^\alpha$, then we obtain the q-analogue of integral formula for a power function:

$$_aI_q(t-a)^\alpha = \int_a^t (s-a)^\alpha\,{}_ad_qs = \frac{1}{[\alpha+1]_q}(t-a)^{\alpha+1}, \tag{7}$$

for $\alpha > -1$. In [24], the authors proved the following:

$$\int_a^t \left(\int_a^s f(r)\,{}_ad_qr\right){}_ad_qs = \int_a^t [t-(qs+(1-q)a)]f(s)\,{}_ad_qs, \tag{8}$$

which reduces a double integration to a single one.

In [26], the authors defined a power function by

$$ {}_a(t-s)_q^{(k)} = \prod_{i=0}^{k-1}(t - {}_a\Phi_q^i(s)), \quad k \in \mathbb{N} \cup \{\infty\}, \tag{9} $$

and ${}_a(t-s)_q^{(0)} = 1$, where ${}_a\Phi_q^i(t) = q^i t + (1-q^i)a$. If α is any real number, then it is defined by setting

$$ {}_a(t-s)_q^{(\alpha)} = \prod_{i=0}^{\infty} \frac{(t - {}_a\Phi_q^i(s))}{(t - {}_a\Phi_q^{i+\alpha}(s))}. \tag{10} $$

For example, if $\alpha = 3/2$, then we have

$$ {}_a(t-s)_q^{(3/2)} = \frac{(t-s)(t - {}_a\Phi_q(s))(t - {}_a\Phi_q^2(s))\ldots}{(t - {}_a\Phi_q^{3/2}(s))(t - {}_a\Phi_q^{5/2}(s))(t - {}_a\Phi_q^{7/2}(s))\ldots}. \tag{11} $$

Let us now recall the Riemann–Liouville fractional q-derivative and q-integral on an interval $[a,b]$, and present some of their properties from [26].

Definition 3 ([26])**.** *The fractional q-derivative of Riemann–Liouville type of order $\alpha \geq 0$ on an interval $[a,b]$ is defined by*

$$ \begin{aligned} ({}_a^{RL}D_q^\alpha f)(t) &= ({}_aD_q^n {}_aI_q^{n-\alpha} f)(t) \\ &= \frac{1}{\Gamma_q(n-\alpha)} \left({}_aD_q^n\right) \int_a^t {}_a(t - {}_a\Phi_q(s))_q^{(n-\alpha-1)} f(s)\,{}_ad_qs, \ \ \alpha > 0, \end{aligned} $$

and $({}_a^{RL}D_q^0 f)(t) = f(t)$, where n is the least integer greater than or equal to α and the $\Gamma_q(u)$ is defined by

$$ \Gamma_q(u) = \frac{{}_0(1-q)_q^{(u-1)}}{(1-q)^{u-1}}, \ \ u \in \mathbb{R} \backslash \{0, -1, -2, \ldots\}. $$

Definition 4. *Let $\alpha \geq 0$ and f be a function defined on an interval $[a,b]$. The fractional q-integral of Riemann–Liouville type is given by*

$$ ({}_aI_q^\alpha f)(t) = \frac{1}{\Gamma_q(\alpha)} \int_a^t {}_a(t - {}_a\Phi_q(s))_q^{(\alpha-1)} f(s)\,{}_ad_qs, \ \ \alpha > 0, \ \ t \in [a,b]. $$

and $({}_aI_q^0 f)(t) = f(t)$, provided that the right hand side exists.

Lemma 1 ([26])**.** *Let α, $\beta \in \mathbb{R}^+$ and f be a continuous function on an interval $[a,b]$, $a \geq 0$. The Riemann–Liouville fractional q-integral satisfies the following semi-group property:*

$$ {}_aI_q^\beta\left({}_aI_q^\alpha f\right)(t) = {}_aI_q^\alpha\left({}_aI_q^\beta f\right)(t) = {}_aI_q^{\alpha+\beta} f(t). \tag{12} $$

Lemma 2 ([26])**.** *Let f be a q-integrable function on an interval $[a,b]$. Then, the following equality holds:*

$$ {}_a^{RL}D_q^\alpha\left({}_aI_q^\alpha f\right)(t) = f(t), \qquad \text{for } \ \alpha > 0, \ t \in [a,b]. $$

Lemma 3 ([26])**.** *Let $\alpha > 0$ and n be a positive integer. Then, for any $t \in [a,b]$ the following equality holds:*

$$ {}_aI_q^\alpha\left({}_aD_q^n f\right)(t) = {}_aD_q^n\left({}_aI_q^\alpha f\right)(t) - \sum_{k=0}^{n-1} \frac{(t-a)^{\alpha+k-n}}{\Gamma_q(\alpha+k-n+1)}\left({}_aD_q^k f\right)(a). $$

In addition, from [26], we have the following formulas:

$$ {}_{a}^{RL}D_q^{\alpha}(t-a)^{\beta} = \frac{\Gamma_q(\beta+1)}{\Gamma_q(\beta-\alpha+1)}(t-a)^{\beta-\alpha}, \tag{13} $$

$$ {}_{a}I_q^{\alpha}(t-a)^{\beta} = \frac{\Gamma_q(\beta+1)}{\Gamma_q(\beta+\alpha+1)}(t-a)^{\beta+\alpha}. \tag{14} $$

From [28], the Caputo fractional q-derivative was defined as follows:

Definition 5. *The fractional q-derivative of Caputo type of order $\alpha \geq 0$ on an interval $[a,b]$ is defined by*

$$ \begin{aligned} ({}_{a}^{C}D_q^{\alpha}f)(t) &= {}_{a}I_q^{n-\alpha}\left({}_{a}D_q^{n}f\right)(t), \\ &= \frac{1}{\Gamma_q(n-\alpha)}\int_a^t {}_{a}(t-{}_{a}\Phi_q(s))_q^{(n-\alpha-1)}\left({}_{a}D_q^{n}f\right)(s)\,{}_{a}d_qs, \ \ \alpha > 0, \end{aligned} $$

and $({}_{a}^{C}D_q^{0}f)(t) = f(t)$, where n is the least integer greater than or equal to α.

Lemma 4 ([28])**.** *Let $\alpha > 0$ and n be the least integer greater than or equal to α. Then, for any $t \in [a,b]$, the following holds*

$$ {}_{a}I_q^{\alpha}\left({}_{a}^{C}D_q^{\alpha}f\right)(t) = f(t) - \sum_{k=0}^{n-1}\frac{(t-a)^k}{\Gamma_q(k+1)}\left({}_{a}D_q^{k}f\right)(a). $$

Lemma 5 ([29])**.** *Let $u, v > 0$ and $0 < p, q < 1$. Then, for $\pi \in [a,b]$, we have*

$$ {}_{a}I_p^{u}\,{}_{a}I_q^{v}(1)(\pi) = \frac{\Gamma_p(v+1)}{\Gamma_p(u+v+1)\Gamma_q(v+1)}(\pi-a)^{u+v}. $$

We now present the proof of the following lemma, which might not yet appeared in the literature.

Lemma 6. *Let $\alpha > 0$ and n be the least integer greater than or equal to α. Then, for any $t \in [a,b]$, we have*

$$ {}_{a}I_q^{\alpha}\left({}_{a}^{RL}D_q^{\alpha}f\right)(t) = f(t) - \sum_{j=1}^{n}\frac{(t-a)^{\alpha-j}}{\Gamma_q(\alpha-j+1)}\left({}_{a}^{RL}D_q^{\alpha-j}f\right)(a), \tag{15} $$

where $n-1 < \alpha < n$.

Proof. By applying Lemma 3, we obtain the following:

$$ \begin{aligned} {}_{a}I_q^{\alpha}\left({}_{a}^{RL}D_q^{\alpha}f\right)(t) &= {}_{a}I_q^{\alpha}\left({}_{a}D_q^{n}\left({}_{a}I_q^{n-\alpha}f(t)\right)\right) \\ &= {}_{a}D_q^{n}\left({}_{a}I_q^{\alpha}\left({}_{a}I_q^{n-\alpha}f(t)\right)\right) - \sum_{k=0}^{n-1}\frac{(t-a)^{\alpha+k-n}}{\Gamma_q(\alpha+k-n+1)}\left({}_{a}D_q^{k}I^{n-\alpha}f(t)\right)(a) \\ &= {}_{a}D_q^{n}\left({}_{a}I_q^{n}f(t)\right) - \sum_{k=0}^{n-1}\frac{(t-a)^{\alpha+k-n}}{\Gamma_q(\alpha+k-n+1)}\left({}_{a}^{RL}D_q^{\alpha+k-n}f\right)(a), \end{aligned} $$

by Lemma 1. The relation (15) is obtained by setting $j = n-k$ for $k = 0, \ldots, n-1$, and Lemma 2. □

3. The Hilfer Fractional Quantum Derivative

The Hilfer fractional quantum derivative of order $\alpha > 0$, with parameter $\beta \in [0,1]$, of a function, f, on an interval, $[a,b]$, is defined by

$$ {}_a^H D_q^{\alpha,\beta} f(t) = {}_a I_q^{\beta(n-\alpha)} \, {}_a D_q^n \, {}_a I_q^{(1-\beta)(n-\alpha)} f(t), \tag{16} $$

where $n-1 < \alpha < n$, the quantum number $0 < q < 1$, and a variable $t > a$.

If $\beta = 0$, then we have

$$ {}_a^H D_q^{\alpha,0} f(t) = {}_a^{RL} D_q^{\alpha} f(t), $$

which is the fractional quantum Riemann–Liouville derivative. If $\beta = 1$, then we obtain the following:

$$ {}_a^H D_q^{\alpha,1} f(t) = {}_a^C D_q^{\alpha} f(t). $$

which is the fractional quantum Caputo derivative. If we set a constant $\gamma = \alpha + \beta(n-\alpha)$, then (16) can be rewritten as

$$ {}_a^H D_q^{\alpha,\beta} f(t) = {}_a I_q^{\gamma-\alpha} \left({}_a^{RL} D_q^{\gamma} f \right)(t), \quad t \in [a,b]. \tag{17} $$

Theorem 1. *Let $f \in C^n[a,b]$, $n-1 < \alpha < n$, $0 \le \beta \le 1$ and quantum number $0 < q < 1$. Then, we have*

(i) $\quad {}_a I_q^{\alpha} \left({}_a^H D_q^{\alpha,\beta} f \right)(t) = f(t) - \displaystyle\sum_{j=1}^{n} \frac{(t-a)^{\gamma-j}}{\Gamma_q(\gamma-j+1)} \left({}_a^{RL} D_q^{\gamma-j} f \right)(a),$

(ii) $\quad {}_a^H D_q^{\alpha,\beta} \left({}_a I_q^{\alpha} f \right)(t) = f(t),$

where $\gamma = \alpha + \beta(n-\alpha)$.

Proof. The Riemann–Liouville fractional q-integral of order $\alpha > 0$ acting on the Hilfer fractional derivative of order α, by (12) and (17), leads to

$$ \begin{aligned} {}_a I_q^{\alpha} \left({}_a^H D_q^{\alpha,\beta} f \right)(t) &= {}_a I_q^{\alpha} \left({}_a I_q^{\gamma-\alpha} \left({}_a^{RL} D_q^{\gamma} f \right) \right)(t) \\ &= {}_a I_q^{\gamma} \left({}_a^{RL} D_q^{\gamma} f \right)(t). \end{aligned} $$

Applying (15), we obtain a requested relation in (i).

To prove (ii), from (17), we obtain the following:

$$ \begin{aligned} {}_a^H D_q^{\alpha,\beta} \left({}_a I_q^{\alpha} f \right)(t) &= {}_a I_q^{\gamma-\alpha} \left({}_a^{RL} D_q^{\gamma} \left({}_a I_q^{\alpha} f \right) \right)(t) \\ &= {}_a I_q^{\gamma-\alpha} \left({}_a^{RL} D_q^{\gamma-\alpha} f \right)(t), \end{aligned} $$

since $\gamma \in [\alpha, n]$. Using (15), then we have

$$ {}_a^H D_q^{\alpha,\beta} \left({}_a I_q^{\alpha} f \right)(t) = f(t) - \frac{(t-a)^{\gamma-\alpha-1}}{\Gamma_q(\gamma-\alpha)} \left({}_a I_q^{1+\alpha-\gamma} f \right)(a). $$

Since f is continuous, it follows that f is bounded, that is, $|f(t)| \le M$, for all $t \in [a,b]$, $M \ge 0$. Hence

$$ \begin{aligned} \left| \left({}_a I_q^{1+\alpha-\gamma} f \right)(a) \right| &\le M \lim_{t \to a} \left| \left({}_a I_q^{1+\alpha-\gamma}(1) \right)(t) \right| \\ &= \frac{M}{\Gamma_q(2+\alpha-\gamma)} \lim_{t \to a} (t-a)^{1+\alpha-\gamma} \\ &= 0. \end{aligned} $$

This implies that equation in (ii) holds, as desired. □

Remark 1. *The notation* ${}_a^{RL}D_q^{-\phi}$ *is replaced by* ${}_aI_q^{\phi}$*, where* $\phi > 0$*. Since* $\beta \in [0,1]$*, then* $\gamma \in [\alpha, n]$*, and consequently some terms in Theorem 1 are given by* ${}_a^{RL}D_q^{\gamma-n}f(a) = {}_aI_q^{n-\gamma}f(a)$ *and* ${}_a^{RL}D_q^{\gamma-\alpha-j}f = {}_aI_q^{\alpha+j-\gamma}f(a)$*.*

Lemma 7. *Let* $\lambda > \alpha - 1, 0 \le \beta \le 1$ *be constants. Then, we have*

$$ {}_a^HD_q^{\alpha,\beta}(t-a)^{\lambda} = \frac{\Gamma_q(\lambda+1)}{\Gamma_q(\lambda-\alpha+1)}(t-a)^{\lambda-\alpha}. \tag{18} $$

Proof. To claim this, the Formulas (13) and (14) are used as

$$ \begin{aligned} {}_a^HD_q^{\alpha,\beta}(t-a)^{\lambda} &= {}_aI_q^{\gamma-\alpha}\left({}_a^{RL}D_q^{\gamma}(t-a)^{\lambda}\right) \\ &= \frac{\Gamma_q(\lambda+1)}{\Gamma_q(\lambda-\alpha+1)}{}_aI_q^{\gamma-\alpha}(t-a)^{\lambda-\gamma} \\ &= \frac{\Gamma_q(\lambda+1)}{\Gamma_q(\lambda-\alpha+1)}(t-a)^{\lambda-\alpha}, \end{aligned} $$

and this completes the proof. □

4. Applications

In this section, we use the newly established Hilfer quantum fractional derivative in (16) to introduce and study two new boundary value problems of orders $0 < \alpha < 1$ and $1 < \alpha < 2$, respectively. Banach's contraction mapping principle helped us to guarantee the existence and uniqueness of the solutions for the investigated problems.

4.1. Boundary Value Problems of Order $0 < \alpha < 1$

Consider the following boundary value problem with the Hilfer quantum fractional derivative of order $0 < \alpha < 1$:

$$ \begin{cases} {}_a^HD_q^{\alpha,\beta}x(t) = f(t, x(t)), \quad 0 < \alpha < 1, \ a < t < b, \\ {}_aI_q^{1-\gamma}x(a) = \mu x(b) + \sum_{i=1}^{m} \lambda_i \left({}_aI_{q_i}^{\kappa_i}x\right)(\eta_i), \ \gamma = \alpha + \beta(1-\alpha), \end{cases} \tag{19} $$

where ${}_a^HD_q^{\alpha,\beta}$ is the Hilfer quantum fractional derivative of order α with parameter $0 < \beta < 1$ and quantum number $0 < q < 1$, staring at a point $a \ge 0$, $f : [a,b] \times \mathbb{R} \to \mathbb{R}$ is a nonlinear function, μ, λ_i are given constants, ${}_aI_{q_i}^{\kappa_i}$ is the quantum fractional integral of order $\kappa_i > 0$, $0 < q_i < 1$, and $\eta_i \in [a,b]$, $i = 1, \ldots, m$.

Before proceeding to the main result, we transform the problem into an integral equation. The following lemma concerns a linear variant of the boundary value problem (19), and is the basic tool to transforming the given problem into a fixed-point problem.

Lemma 8. *Let* $h : [a,b] \to \mathbb{R}$ *be a continuous function and* $\Omega \neq 0$*. Then, the boundary value problem*

$$ \begin{cases} {}_a^HD_q^{\alpha,\beta}x(t) = h(t), \quad 0 < \alpha < 1, \ a < t < b, \\ {}_aI_q^{1-\gamma}x(a) = \mu x(b) + \sum_{i=1}^{m} \lambda_i \left({}_aI_{q_i}^{\kappa_i}x\right)(\eta_i), \ \gamma = \alpha + \beta(1-\alpha), \end{cases} \tag{20} $$

is equivalent to the integral equation

$$x(t) = \frac{(t-a)^{\gamma-1}}{\Omega\Gamma_q(\gamma)}\left[\mu_a I_q^\alpha h(b) + \sum_{i=1}^m \lambda_i\left({}_aI_{q_i}^{\kappa_i}\,{}_aI_q^\alpha h\right)(\eta_i)\right] + {}_aI_q^\alpha h(t), \tag{21}$$

where

$$\Omega = 1 - \frac{\mu}{\Gamma_q(\gamma)}(b-a)^{\gamma-1} - \frac{1}{\Gamma_q(\gamma)}\sum_{i=1}^m \frac{\lambda_i\Gamma_{q_i}(\gamma)}{\Gamma_{q_i}(\gamma+\kappa_i)}(\eta_i-a)^{\gamma+\kappa_i-1}. \tag{22}$$

Proof. By applying the formula (i) of Theorem 1 for $n=1$, to the equation of (19), we have

$$x(t) = \frac{(t-a)^{\gamma-1}}{\Gamma_q(\gamma)}A + {}_aI_q^\alpha h(t), \tag{23}$$

where $A = \left({}_aI_q^{1-\gamma}x\right)(a)$ is a constant and consequently,

$$x(b) = \frac{(b-a)^{\gamma-1}}{\Gamma_q(\gamma)}A + {}_aI_q^\alpha h(b),$$

and

$$\begin{aligned}\sum_{i=1}^m \lambda_i\left({}_aI_{q_i}^{\kappa_i}x\right)(\eta_i) &= \frac{A}{\Gamma_q(\gamma)}\left[\sum_{i=1}^m \lambda_i\,{}_aI_{q_i}^{\kappa_i}(t-a)^{\gamma-1}\right]_{t=\eta_i} + \sum_{i=1}^m \lambda_i\left({}_aI_{q_i}^{\kappa_i}\,{}_aI_q^\alpha h\right)(\eta_i)\\ &= \frac{A}{\Gamma_q(\gamma)}\sum_{i=1}^m \frac{\lambda_i\Gamma_{q_i}(\gamma)}{\Gamma_{q_i}(\gamma+\kappa_i)}(\eta_i-a)^{\gamma+\kappa_i-1} + \sum_{i=1}^m \lambda_i\left({}_aI_{q_i}^{\kappa_i}\,{}_aI_q^\alpha h\right)(\eta_i).\end{aligned}$$

From the boundary condition of (19), we obtain the following:

$$A = \frac{1}{\Omega}\left[\mu_a I_q^\alpha h(b) + \sum_{i=1}^m \lambda_i\left({}_aI_{q_i}^{\kappa_i}\,{}_aI_q^\alpha h\right)(\eta_i)\right].$$

The integral Equation (21) is obtained by substituting the constant A in (23).

Conversely, taking the Riemann–Liouville quantum fractional derivative of order γ to both sides of (21), we have

$$\begin{aligned}{}_a^{RL}D_q^\gamma x(t) &= \frac{1}{\Omega\Gamma_q(\gamma)}\left[\mu_a I_q^\alpha h(b) + \sum_{i=1}^m \lambda_i\left({}_aI_{q_i}^{\kappa_i}\,{}_aI_q^\alpha h(\eta_i)\right)\right]{}_a^{RL}D_q^\gamma(t-a)^{\gamma-1}\\ &\quad + {}_a^{RL}D_q^\gamma\left({}_aI_q^\alpha h(t)\right)\\ &= {}_a^{RL}D_q^{\beta(1-\alpha)}h(t).\end{aligned}$$

Applying the Riemann–Liouville fractional q-integral of order $\beta(1-\alpha)$ to both sides of the above equation, we deduce from (17) that

$$\begin{aligned}{}_aI_q^{\beta(1-\alpha)}\left({}_a^{RL}D_q^\gamma x\right)(t) &= {}_a^{H}D_q^{\alpha,\beta}x(t)\\ &= {}_aI_q^{\beta(1-\alpha)}\left({}_a^{RL}D_q^{\beta(1-\alpha)}h(t)\right)\\ &= h(t) - \frac{(t-a)^{\beta(1-\alpha)-1}}{\Gamma_q(\beta(1-\alpha))}\,{}_aI_q^{1-\beta(1-\alpha)}h(a)\\ &= h(t).\end{aligned}$$

Therefore x in (21) satisfies the first equation of (20). By direct computation, we can also show that x in (21) satisfies the boundary condition in (20). The proof is completed. $\square$

Let $C([a,b],\mathbb{R})$ be the Banach space of all continuous functions from an interval $[a,b]$ to $\mathbb{R}$ with norm $\|x\| = \sup_{t\in[a,b]}|x(t)|$. We define the weighted space of continuous functions $C_{1-\gamma}([a,b],\mathbb{R}) = \{x : [a,b] \to \mathbb{R} : (t-a)^{1-\gamma}x(t) \in C([a,b],\mathbb{R})\}$ with the norm $\|x\|_{1-\gamma} = \sup_{t\in[a,b]}\{|(t-a)^{1-\gamma}x(t)|\}$. Clearly, $C_{1-\gamma}([a,b],\mathbb{R})$ is a Banach space.

We will show the existence and uniqueness of the solution of a Hilfer quantum boundary value problem (19) by using the Banach contraction mapping principle.

Theorem 2. *Assume that the nonlinear function $f : [a,b]\times\mathbb{R} \to \mathbb{R}$ satisfies*

$$|f(t,x) - f(t,y)| \le L|x-y|, \tag{24}$$

where $L > 0$, $\forall t \in [a,b]$ and $x, y \in \mathbb{R}$. If $L\Omega_1 < 1$, where

$$\begin{aligned}\Omega_1 &= \frac{1}{|\Omega|\Gamma_q(\gamma+\alpha)}\left[|\mu|(b-a)^{\gamma+\alpha-1} + \sum_{i=1}^{m}\frac{|\lambda_i|\Gamma_{q_i}(\gamma+\alpha)}{\Gamma_{q_i}(\gamma+\alpha+\kappa_i)}(\eta_i-a)^{\gamma+\alpha+\kappa_i-1}\right] \\ &\quad + \frac{\Gamma_q(\gamma)}{\Gamma_q(\gamma+\alpha)}(b-a)^{\alpha},\end{aligned}$$

then the boundary value problem (19), containing the Hilfer quantum fractional derivative, has a unique solution in $C_{1-\gamma}([a,b],\mathbb{R})$.

Proof. To use the Banach contraction mapping principle, in view of Lemma 8, we define an operator $\mathcal{K} : C_{1-\gamma}([a,b],\mathbb{R}) \to C_{1-\gamma}([a,b],\mathbb{R})$ by

$$\mathcal{K}x(t) = \frac{(t-a)^{\gamma-1}}{\Omega\Gamma_q(\gamma)}\left[\mu\,{}_aI_q^{\alpha}f(b,x(b)) + \sum_{i=1}^{m}\lambda_i\left({}_aI_{q_i}^{\kappa_i}\,{}_aI_q^{\alpha}f(\eta_i,x(\eta_i))\right)\right] + {}_aI_q^{\alpha}f(t,x(t)),$$

where Ω is defined by (22) and we will prove that $\mathcal{K}$ has the unique fixed point, which is the unique solution of boundary value problem (19). Setting a constant $M = \sup_{t\in[a,b]}|f(t,0)|$, we now define a ball, B_r, with radius, r, such that

$$r \ge \frac{M\Omega_2}{1-L\Omega_1},$$

where

$$\Omega_2 = \frac{1}{|\Omega|\Gamma_q(\gamma)}\left[\frac{|\mu|(b-a)^{\alpha}}{\Gamma_q(\alpha+1)} + \sum_{i=1}^{m}\frac{|\lambda_i|\Gamma_{q_i}(\alpha+1)}{\Gamma_{q_i}(\alpha+\kappa_i+1)\Gamma_q(\alpha+1)}(\eta_i-a)^{\alpha+\kappa_i}\right] + \frac{1}{\Gamma_q(\alpha+1)}.$$

Then, for any $x \in B_r$, from (24) that $|f(t,x)| \le |f(t,x)-f(t,0)| + |f(t,0)| \le L|x| + M$ and from Lemma 5, we have

$$
\begin{aligned}
|(t-a)^{1-\gamma}\mathcal{K}x(t)| \leq\; & \frac{1}{|\Omega|\Gamma_q(\gamma)}\left[|\mu|_a I_q^{\alpha}|f(b,x(b))| + \sum_{i=1}^{m}|\lambda_i|\left({}_a I_{q_i}^{\kappa_i}{}_a I_q^{\alpha}|f(\eta_i,x(\eta_i))|\right)\right] \\
& + (t-a)^{1-\gamma}{}_a I_q^{\alpha}|f(t,x(t))| \\
\leq\; & \frac{1}{|\Omega|\Gamma_q(\gamma)}\left[|\mu|\left\{L\|x\|_{1-\gamma}\,{}_a I_q^{\alpha}(t-a)^{\gamma-1}(b) + M\,{}_a I_q^{\alpha}(1)(b)\right\}\right. \\
& + L\|x\|_{1-\gamma}\sum_{i=1}^{m}|\lambda_i|\left({}_a I_{q_i}^{\kappa_i}{}_a I_q^{\alpha}(t-a)^{\gamma-1}\right)(\eta_i) \\
& \left. + M\sum_{i=1}^{m}|\lambda_i|\left({}_a I_{q_i}^{\kappa_i}{}_a I_q^{\alpha}(1)\right)(\eta_i)\right] \\
& + (b-a)^{1-\gamma}\left\{L\|x\|_{1-\gamma}\,{}_a I_q^{\alpha}(t-a)^{\gamma-1}(b) + M\,{}_a I_q^{\alpha}(1)(b)\right\} \\
=\; & \frac{1}{|\Omega|\Gamma_q(\gamma)}\left[|\mu|L\|x\|_{1-\gamma}\frac{\Gamma_q(\gamma)}{\Gamma_q(\gamma+\alpha)}(b-a)^{\gamma+\alpha-1} + M\frac{|\mu|(b-a)^{\alpha}}{\Gamma_q(\alpha+1)}\right. \\
& + L\|x\|_{1-\gamma}\sum_{i=1}^{m}\frac{|\lambda_i|\Gamma_q(\gamma)\Gamma_{q_i}(\gamma+\alpha)}{\Gamma_q(\gamma+\alpha)\Gamma_{q_i}(\gamma+\alpha+\kappa_i)}(\eta_i-a)^{\gamma+\alpha+\kappa_i-1} \\
& \left. + M\sum_{i=1}^{m}\frac{|\lambda_i|\Gamma_{q_i}(\alpha+1)}{\Gamma_{q_i}(\alpha+\kappa_i+1)\Gamma_q(\alpha+1)}(\eta_i-a)^{\alpha+\kappa_i}\right] \\
& + L\|x\|_{1-\gamma}\frac{\Gamma_q(\gamma)}{\Gamma_q(\gamma+\alpha)}(b-a)^{\alpha} + M\frac{(b-a)^{\alpha+1-\gamma}}{\Gamma_q(\alpha+1)} \\
\leq\; & Lr\Omega_1 + M\Omega_2 \leq r.
\end{aligned}
$$

This means that $\|\mathcal{K}x\|_{1-\gamma} \leq r$, which implies $\mathcal{K}B_r \subseteq B_r$. Next, we will show that $\mathcal{K}$ is a contraction. For any $x, y \in B_r$, we have

$$
\begin{aligned}
& |(t-a)^{1-\gamma}\mathcal{K}x(t) - (t-a)^{1-\gamma}\mathcal{K}y(t)| \\
\leq\; & \frac{1}{|\Omega|\Gamma_q(\gamma)}\left[|\mu|_a I_q^{\alpha}|f(b,x(b)) - f(b,y(b))|\right. \\
& \left. + \sum_{i=1}^{m}|\lambda_i|\left({}_a I_{q_i}^{\kappa_i}{}_a I_q^{\alpha}|f(\eta_i,x(\eta_i)) - f(\eta_i,y(\eta_i))|\right)\right] \\
& + (t-a)^{1-\gamma}{}_a I_q^{\alpha}|f(t,x(t)) - f(t,y(t))| \\
\leq\; & \frac{L\|x-y\|_{1-\gamma}}{|\Omega|\Gamma_q(\gamma)}\left[|\mu|_a I_q^{\alpha}(t-a)^{\gamma-1}(b) + \sum_{i=1}^{m}|\lambda_i|\left({}_a I_{q_i}^{\kappa_i}{}_a I_q^{\alpha}(t-a)^{\gamma-1}\right)(\eta_i)\right] \\
& + L\|x-y\|_{1-\gamma\,a} I_q^{\alpha}(t-a)^{\gamma-1}(b) \\
=\; & L\Omega_1\|x-y\|_{1-\gamma}.
\end{aligned}
$$

This leads to $\|\mathcal{K}x - \mathcal{K}y\|_{1-\gamma} \leq L\Omega_1\|x-y\|_{1-\gamma}$, because $L\Omega_1 < 1$, $\mathcal{K}$ is a contraction operator. Therefore, the boundary value problem, containing the Hilfer quantum fractional derivative of order $0 < \alpha < 1$ in (19), has a unique solution on $[a, b]$. The proof is now completed. $\square$

Example 1. *Consider the following boundary value problem containing the Hilfer quantum fractional derivative:*

$$\begin{cases} {}_{1/4}^{H}D_{1/2}^{1/3,5/7}x(t) = \dfrac{7e^{-(4t-1)^2}}{2(8t+13)}\left(\dfrac{x^2(t)+2|x(t)|}{1+|x(t)|}\right) + \dfrac{1}{2}t^2 + 1, \quad \dfrac{1}{4} < t < \dfrac{3}{2}, \\ \left({}_{1/4}I_{1/2}^{4/21}x\right)\left(\dfrac{1}{4}\right) = \dfrac{1}{\pi}x\left(\dfrac{3}{2}\right) + \dfrac{1}{44}\left({}_{1/4}I_{1/3}^{1/2}x\right)\left(\dfrac{1}{2}\right) \\ \qquad + \dfrac{3}{55}\left({}_{1/4}I_{1/5}^{3/5}x\right)\left(\dfrac{3}{4}\right) + \dfrac{5}{66}\left({}_{1/4}I_{1/7}^{7/5}x\right)\left(\dfrac{5}{4}\right). \end{cases} \tag{25}$$

Here, $\alpha = 1/3$, $\beta = 5/7$, $q = 1/2$, $a = 1/4$, $b = 3/2$, $\gamma = 17/21$, $\mu = 1/\pi$, $\lambda_1 = 1/44$, $\lambda_2 = 3/55$, $\lambda_3 = 5/66$, $\kappa_1 = 1/2$, $\kappa_2 = 3/5$, $\kappa_3 = 7/5$, $q_1 = 1/3$, $q_2 = 1/5$, $q_3 = 1/7$, $\eta_1 = 1/2$, $\eta_2 = 3/4$, $\eta_3 = 5/4$. These constants leads to $\Omega \approx 0.598505$ and $\Omega_1 \approx 2.028477$. Choosing

$$f(t,x) = \frac{7e^{-(4t-1)^2}}{2(8t+13)}\left(\frac{x^2+2|x|}{1+|x|}\right) + \frac{1}{2}t^2 + 1,$$

then, we have

$$|f(t,x) - f(t,y)| \le \frac{7}{15}|x-y|, \quad \forall x,y \in \mathbb{R},\ t \in \left[\frac{1}{4}, \frac{3}{2}\right],$$

which implies that $L\Omega_1 \approx 0.946623 < 1$. Therefore, problem (25) has a unique solution—$x \in C_{\frac{4}{21}}[1/4, 3/2]$.

4.2. Boundary Value Problems of Order $1 < \alpha < 2$

Now, we consider the boundary value problem for the Hilfer quantum fractional derivative of the following form:

$$\begin{cases} {}_{a}^{H}D_{q}^{\alpha,\beta}x(t) = f(t,x(t)), \quad 1 < \alpha < 2,\ a < t < b, \\ x(a) = 0, \quad x(b) = \displaystyle\sum_{i=1}^{m}\lambda_i\left({}_aI_{q_i}^{\kappa_i}x\right)(\eta_i),\ \gamma = \alpha + \beta(2-\alpha), \end{cases} \tag{26}$$

where ${}_{a}^{H}D_{q}^{\alpha,\beta}$ is the Hilfer quantum fractional derivative of order $\alpha \in (1,2)$ with parameter $0 < \beta < 1$ and quantum number $0 < q < 1$, $f : [a,b] \times \mathbb{R} \to \mathbb{R}$, ${}_aI_{q_i}^{\kappa_i}$ is the quantum fractional integral of order $\kappa_i > 0$, $\lambda_i \in \mathbb{R}$, $0 < q_i < 1$, $\eta_i \in [a,b]$, $i = 1, \ldots, m$. Note that $\gamma \in (\alpha, 2)$.

Lemma 9. *Let $g : [a,b] \to \mathbb{R}$ be a continuous function and $\Phi \neq 0$. Then, the boundary value problem*

$$\begin{cases} {}_{a}^{H}D_{q}^{\alpha,\beta}x(t) = g(t), \quad 1 < \alpha < 2,\ a < t < b, \\ x(a) = 0, \quad x(b) = \displaystyle\sum_{i=1}^{m}\lambda_i\left({}_aI_{q_i}^{\kappa_i}x\right)(\eta_i),\ \gamma = \alpha + \beta(2-\alpha), \end{cases} \tag{27}$$

is equivalent to the following integral equation:

$$x(t) = \frac{(t-a)^{\gamma-1}}{\Phi}\left[\sum_{i=1}^{m}\lambda_i\left({}_aI_{q_i}^{\kappa_i}\,{}_aI_q^{\alpha}g\right)(\eta_i) - {}_aI_q^{\alpha}g(b)\right] + {}_aI_q^{\alpha}g(t), \tag{28}$$

where

$$\Phi = (b-a)^{\gamma-1} - \sum_{i=1}^{m}\frac{\lambda_i\Gamma_{q_i}(\gamma)}{\Gamma_{q_i}(\gamma+\kappa_i)}(\eta_i - a)^{\gamma+\kappa_i-1}. \tag{29}$$

Proof. The formula (i) of Theorem 1 for $n = 2$, implies

$$x(t) = \frac{(t-a)^{\gamma-1}}{\Gamma_q(\gamma)}C_1 + \frac{(t-a)^{\gamma-2}}{\Gamma_q(\gamma-1)}C_2 + {}_aI_q^{\alpha}g(t), \tag{30}$$

by taking the Riemann–Liouville fractional quantum integral of order α to both sides of the equation in (26), where $C_1 = \left({}_a^{RL}D_q^{\gamma-1}x\right)(a)$ and $C_2 = \left({}_aI_q^{2-\gamma}x\right)(a)$. From $\gamma \in (\alpha, 2)$ and the given first condition $x(a) = 0$, we have $C_2 = 0$. Therefore, (30) is reduced to

$$x(t) = \frac{(t-a)^{\gamma-1}}{\Gamma_q(\gamma)}C_1 + {}_aI_q^{\alpha}g(t).$$

To find the constant C_1, we use the second condition of (26) to above equation as

$$x(b) = \frac{(b-a)^{\gamma-1}}{\Gamma_q(\gamma)}C_1 + {}_aI_q^{\alpha}g(b),$$

and

$$\sum_{i=1}^{m}\lambda_i\left({}_aI_{q_i}^{\kappa_i}x\right)(\eta_i) = \frac{C_1}{\Gamma_q(\gamma)}\sum_{i=1}^{m}\frac{\lambda_i\Gamma_{q_i}(\gamma)}{\Gamma_{q_i}(\gamma+\kappa_i)}(\eta_i-a)^{\gamma+\kappa_i-1} + \sum_{i=1}^{m}\lambda_i\left({}_aI_{q_i}^{\kappa_i}{}_aI_q^{\alpha}g\right)(\eta_i),$$

which yields

$$C_1 = \frac{\Gamma_q(\gamma)}{\Phi}\left[\sum_{i=1}^{m}\lambda_i\left({}_aI_{q_i}^{\kappa_i}{}_aI_q^{\alpha}g\right)(\eta_i) - {}_aI_q^{\alpha}g(b)\right].$$

Maintaining a constant C_1, the integral Equation (28) is established. The converse can be proved easily by direct computation. The proof is now completed. □

The existence and uniqueness of the fixed point of the problem $x = \mathcal{G}x$ will be proved by defining the operator $\mathcal{G} : C([a,b],\mathbb{R}) \to C([a,b],\mathbb{R})$ by

$$\mathcal{G}x(t) = \frac{(t-a)^{\gamma-1}}{\Phi}\left[\sum_{i=1}^{m}\lambda_i\left({}_aI_{q_i}^{\kappa_i}{}_aI_q^{\alpha}f(\eta_i, x(\eta_i))\right) - {}_aI_q^{\alpha}f(b, x(b))\right] + {}_aI_q^{\alpha}f(t, x(t)),$$

in view of Lemma 9, where Φ is defined in (29), and using the Banach contraction mapping principle.

Theorem 3. *Assume that (24) holds. If $L\Phi_1 < 1$, then the boundary value problem (26) has the unique solution on an interval $[a,b]$, where*

$$\Phi_1 = \frac{(b-a)^{\gamma-1}}{|\Phi|}\left[\sum_{i=1}^{m}\frac{|\lambda_i|\Gamma_{q_i}(\alpha+1)}{\Gamma_{q_i}(\alpha+\kappa_i+1)\Gamma_q(\alpha+1)}(\eta_i-a)^{\alpha+\kappa_i} + \frac{(b-a)^{\alpha}}{\Gamma_q(\alpha+1)}\right] + \frac{(b-a)^{\alpha}}{\Gamma_q(\alpha+1)}.$$

Proof. If $x \in B_\rho$, then we have

$$\begin{aligned}
|\mathcal{G}x(t)| &\leq \frac{(t-a)^{\gamma-1}}{|\Phi|}\left[\sum_{i=1}^{m}|\lambda_i|\left({}_aI_{q_i}^{\kappa_i}{}_aI_q^{\alpha}|f(\eta_i, x(\eta_i))|\right) + {}_aI_q^{\alpha}|f(b, x(b))|\right] \\
&\quad + {}_aI_q^{\alpha}|f(t, x(t))| \\
&\leq \frac{(b-a)^{\gamma-1}}{|\Phi|}\left[\sum_{i=1}^{m}|\lambda_i|\left({}_aI_{q_i}^{\kappa_i}{}_aI_q^{\alpha}(1)\right)(\eta_i) + {}_aI_q^{\alpha}(1)(b)\right](L\rho + M) \\
&\quad + {}_aI_q^{\alpha}(1)(b)(L\rho + M) \\
&\leq L\rho\Phi_1 + M\Phi_1 \leq \rho,
\end{aligned}$$

where $M = \sup_{t\in[a,b]} |f(t,0)|$ and $B_\rho = \{x \in C[a,b] : \|x\| \le \rho\}$ with ρ satisfying $\rho \ge \dfrac{M\Phi_1}{1-L\Phi_1}$. Consequently $\|\mathcal{G}x\| \le \rho$ which means that $\mathcal{G}B_\rho \subseteq B_\rho$.

To show that the operator $\mathcal{G}$ is a contraction, for any $x, y \in B_\rho$ we obtain

$$\begin{aligned}
& |\mathcal{G}x(t) - \mathcal{G}y(t)| \\
\le\ & \frac{(b-a)^{\gamma-1}}{|\Phi|}\left[\sum_{i=1}^{m}|\lambda_i|\left({}_aI_{q_i}^{\kappa_i}{}_aI_q^{\alpha}|f(\eta_i,x(\eta_i)) - f(\eta_i,y(\eta_i))|\right)\right. \\
& \left. + {}_aI_q^{\alpha}|f(b,x(b)) - f(b,y(b))|\right] + {}_aI_q^{\alpha}|f(t,x(t)) - f(t,y(t))| \\
\le\ & \frac{(b-a)^{\gamma-1}}{|\Phi|}\left[\sum_{i=1}^{m}|\lambda_i|\left({}_aI_{q_i}^{\kappa_i}{}_aI_q^{\alpha}(1)\right)(\eta_i) + {}_aI_q^{\alpha}(1)(b)\right]L\|x-y\| \\
& + {}_aI_q^{\alpha}(1)(b)L\|x-y\| \\
\le\ & L\Phi_1\|x-y\|,
\end{aligned}$$

which yields $\|\mathcal{G}x - \mathcal{G}y\| \le L\Phi_1\|x-y\|$. Since $L\Phi_1 < 1$, we conclude that $\mathcal{G}$ is a contraction operator, and thus has the unique fixed point in B_ρ. This implies that the boundary value problem (26) with the Hilfer fractional quantum derivative of order $1 < \alpha < 2$ has the unique solution x, such that $\|x\| \le \rho$ on $[a,b]$. The proof is now completed. □

Example 2. *Consider the boundary value problem containing the Hilfer fractional quantum derivative of the following form:*

$$\begin{cases}
{}^{H}_{1/8}D_{2/3}^{3/2,4/7}x(t) = \dfrac{5\cos^2 \pi t}{13(8t+3)}\left(\dfrac{3x^2(t)+4|x(t)|}{1+|x(t)|}\right) + \dfrac{1}{3}\sin^2 t + \dfrac{1}{4},\ \dfrac{1}{8} < t < \dfrac{11}{8}, \\
x\left(\dfrac{1}{8}\right) = 0, \\
x\left(\dfrac{11}{8}\right) = \dfrac{2}{33}\left({}_{1/8}I_{1/2}^{2/3}x\right)\left(\dfrac{3}{8}\right) + \dfrac{4}{55}\left({}_{1/8}I_{1/4}^{3/4}x\right)\left(\dfrac{5}{8}\right) \\
\qquad + \dfrac{6}{77}\left({}_{1/8}I_{1/6}^{6/5}x\right)\left(\dfrac{7}{8}\right) + \dfrac{8}{99}\left({}_{1/8}I_{1/8}^{8/7}x\right)\left(\dfrac{9}{8}\right).
\end{cases} \tag{31}$$

Here, $\alpha = 3/2$, $\beta = 4/7$, $q = 2/3$, $a = 1/8$, $b = 11/8$, $\lambda_1 = 2/33$, $\lambda_2 = 4/55$, $\lambda_3 = 6/77$, $\lambda_4 = 8/99$, $\kappa_1 = 2/3$, $\kappa_2 = 3/4$, $\kappa_3 = 6/5$, $\kappa_4 = 8/7$, $q_1 = 1/2$, $q_2 = 1/4$, $q_3 = 1/6$, $q_4 = 1/8$, $\eta_1 = 3/8$, $\eta_2 = 5/8$, $\eta_3 = 7/8$, $\eta_4 = 9/8$. From the given data, we obtain $\Phi \approx 1.0551$ and $\Phi_1 \approx 2.5027$. Setting

$$f(t,x) = \frac{5\cos^2 \pi t}{13(8t+3)}\left(\frac{3x^2+4|x|}{1+|x|}\right) + \frac{1}{3}\sin^2 t + \frac{1}{4},$$

then we have $|f(t,x) - f(t,y)| \le (5/13)|x-y|$ for all $x, y \in \mathbb{R}$ and $t \in [1/8, 11/8]$. Applying Theorem 3, the boundary value problem (31) has the unique solution on $[1/8, 11/8]$.

5. Conclusions

Two of the most popular fractional derivatives are those in the sense of Riemann–Liouville and Caputo. In [27], R. Hilfer created a generalization that combined both aspects of Riemann–Liouville and Caputo fractional derivatives, known as the Hilfer fractional derivative of order $\alpha \in (0,1)$ and a type $\beta \in [0,1]$, which can be reduced to the Riemann–Liouville derivative when $\beta = 0$, and to the Caputo fractional derivative when $\beta = 1$. On the other hand, in [26], the fractional q-derivatives of Riemann–Liouville and Caputo type were introduced. In the present paper, we have introduced the notion of "the Hilfer fractional quantum derivative", and we have studied some of its basic properties. For applications of this new notion, we studied one initial value problem of order in $(0,1)$ and a boundary value problem of order in $(1,2)$, involving the Hilfer fractional quantum derivative. Using a linear variant of the considered problems, we transformed them into fixed-point problems, in which we can apply Banach's contraction mapping principle and proved the existence and uniqueness of the solutions in both problems. By constructing numerical examples, we demonstrated the applicability of our theoretical results.

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